Boolean algebra

Identity/Law	Expression
Identity Law for OR	$A + 0 = A$
Identity Law for AND	$A \cdot 1 = A$
Domination Law for OR	$A + 1 = 1$
Domination Law for AND	$A \cdot 0 = 0$
Idempotent Law for OR	$A + A = A$
Idempotent Law for AND	$A \cdot A = A$
Complement Law for OR	$A + \overline{A} = 1$
Complement Law for AND	$A \cdot \overline{A} = 0$
Double Negation Law	$\overline{\overline{A}} = A$
Distributive Law (AND over OR)	$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$
Distributive Law (OR over AND)	$A + (B \cdot C) = (A + B) \cdot (A + C)$
Absorption Law (OR)	$A + (A \cdot B) = A$
Absorption Law (AND)	$A \cdot (A + B) = A$
De Morgan's Law (OR)	$\overline{A + B} = \overline{A} \cdot \overline{B}$
De Morgan's Law (AND)	$\overline{A \cdot B} = \overline{A} + \overline{B}$

These are the essential Boolean algebra identities and laws in a clear and organized format.

Logical operations

Conjunction (AND)

P	Q	$P \wedge Q$
True	True	True
True	False	False
False	True	False
False	False	False

Disjunction (OR)

P	Q	$P \lor Q$
True	True	True
True	False	True
False	True	True
False	False	False

Conditional (Implication)

P	Q	$P \to Q$
True	True	True
True	False	False
False	True	True
False	False	True

Biconditional (If and only if)

P	Q	$P \leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

Remember:

- **Conjunction (AND):** Both conditions must be true for the result to be true.
- **Disjunction (OR):** At least one condition must be true for the result to be true.
- **Conditional (Implication):** If the first condition is true and the second is false, the result is false; otherwise, it's true.
- **Biconditional (If and only if):** Both conditions must have the same truth value for the result to be true.

Proofs

Direct Proof

- **Idea:** Start with the assumption and use logical steps to reach the conclusion.
- **Example:**
  - **Claim:** If a and b are even integers, then $a + b$ is also an even integer.

- **Proof:**
  - Assume $a = 2k$ and $b = 2m$ for some integers k and m.
  - $a + b = 2k + 2m = 2(k + m)$.
  - Let $n = k + m$, then $a + b = 2n$, which is even.
  - Therefore, if a and b are even, $a + b$ is even.

Contraposition Proof

- **Idea:** Prove the contrapositive of the statement.
- **Example:**
  - **Claim:** If n^2 is an odd integer, then n is also an odd integer.
  - **Proof:**
    - Contrapositive: If n is not an odd integer, then n^2 is not an odd integer.
    - Assume n is even, so $n = 2k$ for some integer k.
    - $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even.
    - Therefore, if n^2 is odd, n is odd.

Contradiction Proof

- **Idea:** Assume the negation of what you want to prove and derive a contradiction.
- **Example:**
  - **Claim:** $\sqrt{2}$ is irrational.
  - **Proof:**
    - Assume $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{a}{b}$ where a and b have no common factors.
    - Squaring both sides: $2 = \frac{a^2}{b^2}$.
    - $2b^2 = a^2$, which means a^2 is even.
    - This implies a is even (a^2 even only if a is even).
    - If a is even, $a = 2k$ for some integer k.
    - Substituting back: $2b^2 = (2k)^2 = 4k^2$.
    - $b^2 = 2k^2$, which means b^2 is also even.
    - This contradicts the assumption that a and b have no common factors.
    - Therefore, $\sqrt{2}$ is irrational.

These examples illustrate the basic techniques of direct proof, contraposition, and contradiction. Adapt these methods to suit the specific structure of the statement you are trying to prove.

Proof of Equivalence

1. **Idea:**

- To prove two statements P and Q are equivalent, you need to show $P \Rightarrow Q$ and $Q \Rightarrow P$.

2. **Proof Structure:**
   - **Step 1:** Prove $P \Rightarrow Q$.
   - **Step 2:** Prove $Q \Rightarrow P$.
   - **Conclusion:** Conclude that P and Q are equivalent.
3. **Example:**
   - **Claim:** For any real number x, $x^2 = 4$ if and only if $x = 2$ or $x = -2$.
   - **Proof:**
     - **(a)** If $x = 2$ or $x = -2$, then $x^2 = 4$.
     - **(b)** If $x^2 = 4$, then $x = 2$ or $x = -2$ (by taking the square root).
     - **Conclusion:** $x^2 = 4$ if and only if $x = 2$ or $x = -2$.

Proof of Existence

1. **Idea:**
   - To prove the existence of something, provide an example or show that there is at least one instance satisfying the given conditions.
2. **Proof Structure:**
   - **Step 1:** Specify the object or condition you want to show exists.
   - **Step 2:** Present an example or provide a method to construct an example.
   - **Conclusion:** Conclude that at least one instance exists.
3. **Example:**
   - **Claim:** There exists a real number x such that $x^2 - 3x + 2 = 0$.
   - **Proof:**
     - **(a)** Consider the equation $x^2 - 3x + 2 = 0$.
     - **(b)** Factor the equation as $(x - 1)(x - 2) = 0$.
     - **(c)** Solutions are $x = 1$ and $x = 2$.
     - **Conclusion:** There exists a real number x (e.g., $x = 1$) satisfying the equation.

These are basic structures for proving equivalence and existence. Adapt them according to the specific nature of the statements or conditions you are dealing with.

Disproof by Counterexample

1. **Idea:**
   - To disprove a universal statement (for all cases), provide a single example where the statement is false.
2. **Structure:**

- **Step 1:** Identify the universal statement you want to disprove.
- **Step 2:** Provide a specific example that contradicts the statement.
- **Conclusion:** Conclude that the universal statement is false based on the counterexample.

3. **Example:**
   - **Claim:** "All prime numbers are odd."
   - **Counterexample:** 2 is a prime number, but it is even.
   - **Conclusion:** The statement "All prime numbers are odd" is false because 2 is a counterexample.

Note

- A counterexample doesn't prove that a statement is always false; it just shows that the statement is not universally true.
- Make sure the counterexample is valid and adheres to the conditions specified in the original statement.

The idea is to find a single instance that goes against the generalization, thereby disproving the universal claim.

Proof by Mathematical Induction

1. **Idea:**
   - Prove a statement is true for all positive integers by establishing its truth for a base case and showing that if it holds for an arbitrary k, it also holds for $k+1$.
2. **Structure:**
   - **Basis Step:**
     - Prove the statement for the base case (usually $n=1$ or $n=0$).
   - **Inductive Step:**
     - Assume the statement is true for an arbitrary k.
     - Prove that if it is true for k, it must also be true for $k+1$.
   - **Conclusion:**
     - Conclude that the statement is true for all positive integers.
3. **Example:**
   - **Claim:** For all positive integers n, $1+2+3+\ldots+n=\frac{n(n+1)}{2}$.
   - **Basis Step** $(n=1)$**:**
     - $1=\frac{1(1+1)}{2}$ is true.
   - **Inductive Step:**
     - Assume the formula holds for k: $1+2+3+\ldots+k=\frac{k(k+1)}{2}$.

- Prove it for $k+1$:
  $1+2+3+\ldots+k+(k+1)=\frac{(k+1)(k+2)}{2}$
  - Add $(k+1)$ to both sides of the assumed formula.
- **Conclusion:**
  - By mathematical induction, the statement is true for all positive integers.

Note

- Ensure the base case is proven.
- Be explicit about the assumption in the inductive step.
- Carefully perform the inductive step, ensuring that the formula holds for $k+1$ based on the assumption for k.

This method is used to prove statements about all positive integers, relying on the idea that if a statement holds for a specific integer, it must hold for the next integer as well.

Counting

Combinations

1. **Number of Combinations:**
   - $C(n,k)=\binom{n}{k}=\frac{n!}{k!(n-k)!}$
   - Choose k elements from a set of n distinct elements without considering the order.
2. **Combination Formula:**
   - The number of ways to choose k elements from a set of n elements.

Combinations with Repetition

1. **Combinations with Repetition:**
   - $C(n+k-1,k)=\binom{n+k-1}{k}$
   - The number of ways to choose k elements from a set of n distinct elements with replacement.

Permutations

1. **Number of Permutations:**
   - $P(n,k)=\frac{n!}{(n-k)!}$
   - Arrange k elements from a set of n distinct elements in a specific order.
2. **Permutation Formula:**

- The number of ways to arrange k elements from a set of n elements in a specific order.

Permutations with Repetition

1. **Permutations with Repetition:**
   - $P(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$
   - The number of permutations of a multiset with n total elements, where n_1 elements are of one type, n_2 of another, and so on.

Rule of Addition

1. **Idea:**
   - If a task can be done in a ways and for each way of doing the first task, there are b ways of doing the second task (with no overlap), then the total number of ways to do either the first or second task is $a + b$.
2. **Formula:**
   - If there are a ways to do one task and b ways to do another task (with no overlap), then there are $a + b$ ways to either do the first task or do the second task.
3. **Example:**
   - **Task 1:** There are 3 ways to choose a book.
   - **Task 2:** There are 5 ways to choose a magazine.
   - **Total ways to choose reading material:** $3 + 5 = 8$ ways.

Rule of Difference

1. **Idea:**
   - If a task can be done in a ways and for each way of doing the first task, there are b ways of avoiding the second task (with no overlap), then the total number of ways to either do the first task or avoid the second task is $a + b$.
2. **Formula:**
   - If there are a ways to do one task and b ways to avoid doing another task (with no overlap), then there are $a + b$ ways to either do the first task or avoid the second task.
3. **Example:**
   - **Task 1:** There are 6 ways to choose a movie.
   - **Task 2:** There are 3 ways to decide not to watch TV.
   - **Total ways to choose entertainment:** $6 + 3 = 9$ ways.

Rule of Subtraction

1. **Idea:**
   - The Rule of Subtraction is used when you want to count the number of outcomes where at least one condition is satisfied by subtracting the outcomes where the condition is not satisfied from the total outcomes.
2. **Formula:**
   - If there are a ways to do a task, and for each of those ways, there are b ways to avoid another task, then the total number of ways to do the first task while avoiding the second task is $a - b$.
3. **Example:**
   - **Task 1:** There are 10 ways to choose a book.
   - **Task 2:** There are 3 ways you wouldn't read a magazine.
   - **Total ways to choose reading material without reading a magazine:** $10 - 3 = 7$ ways.

Application

Suppose you have 10 different books (Task 1) and you want to choose a book without reading a magazine (Task 2). According to the Counting Principle - Rule of Subtraction:

[\text{Total ways to choose a book without reading a magazine} = \text{Ways to choose a book} - \text{Ways to read a magazine} = 10 - 3 = 7]

Therefore, you have 7 different ways to choose a book without reading a magazine.

Note

- The Rule of Subtraction is useful when you want to count the number of outcomes that satisfy a condition by subtracting the outcomes that do not satisfy the condition from the total outcomes.
- This principle is often applied in scenarios where you have a total number of possibilities, and you want to find the number of possibilities that meet certain criteria.

Rule of Multiplication

1. **Idea:**
   - If a task can be done in a ways and for each way of doing the first task, there are b ways of doing the second task, then the total number of ways to do both tasks is $a \times b$.
2. **Formula:**
   - If there are a ways to do one task and b ways to do another task, then there are $a \times b$ ways to do both tasks.
3. **Example:**

- **Task 1:** There are 4 ways to choose a shirt.
- **Task 2:** For each shirt, there are 3 ways to choose pants.
- **Total ways to choose outfit:** $4 \times 3 = 12$ ways.

Rule of Division

1. **Idea:**
   - If a task can be done in $a \times b$ ways and for each way of doing the first task, there are b ways of doing another task, then the total number of ways to do the first task is $\frac{a \times b}{b} = a$.
2. **Formula:**
   - If there are $a \times b$ ways to do a task, and for each of those ways, there are b ways to do another task, then the total number of ways to do the first task is $\frac{a \times b}{b} = a$.
3. **Example:**
   - **Task 1:** There are 15 total outfits.
   - **Task 2:** For each outfit, there are 5 choices of shoes.
   - **Total ways to choose an outfit:** $\frac{15}{5} = 3$ ways.

Summary:

- **Rule of Addition:** Counting possibilities that can occur in different ways.
- **Rule of Difference:** Counting possibilities where you either perform one task or avoid another.
- **Rule of Multiplication:** Counting possibilities involving multiple steps where choices in one step do not affect choices in another step.
- **Rule of Division:** Counting possibilities involving multiple steps where the total number of outcomes is known and you want to distribute them based on certain conditions.

Factorial

1. **Factorial:**
   - $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
   - The product of all positive integers up to n.
   - $0! = 1$
   - $1! = 1$
   - $2! = 2$

Binomial Coefficient Identity

1. **Binomial Coefficient Identity:**

- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
- The number of ways to choose k elements from a set of n elements is equal to the sum of choosing $k-1$ elements and choosing k elements from a set of $n-1$ elements.

Summary

- **Combinations:** Choosing without replacement.
- **Permutations:** Arrangements with respect to order.
- **Multiplication Rule:** Counting possibilities involving multiple steps.
- **Addition Rule:** Counting possibilities that can occur in different ways.
- **Factorial:** Product of positive integers up to a given number.
- **Binomial Coefficient Identity:** A relationship between binomial coefficients.
- **Permutations with Repetition:** Arrangements allowing repeated elements.
- **Combinations with Repetition:** Choosing with replacement.

Remember, these principles are fundamental in solving problems related to counting and arrangements. Adapt them based on the specific context of the problem you are working on.

Graphs

Term	Requirements/Properties
Walk	Sequence of edges and vertices where edges can repeat.
Path	Walk with no repeated vertices.
Trail	Walk with no repeated consecutive edges.
Closed Walk	Walk that starts and ends at the same vertex.
Circuit	Closed walk with no repeated edges (except start/end).
Simple Circuit	Circuit with no repeated vertices (except start/end).
Euler Path	Walk that traverses every edge exactly once.
Euler Circuit	Circuit that traverses every edge exactly once.

Remember, these definitions assume an undirected graph unless otherwise specified. Additionally, an Euler circuit or path is defined for graphs with certain conditions (e.g., all vertices having even degree for Euler circuits). Adjustments may be needed based on specific graph characteristics.

Walk

Imagine you're taking a stroll around the playground, moving from one spot to another, visiting different places. In a graph, a walk is like that – it's a series of edges (connections) that take you from one vertex (point) to another, and you can visit the same vertex more than once.

Path

Now, think of a path as a walk where you don't repeat any playground equipment. So, if you visit the swing once, you don't swing on it again during the same path. In a graph, a path is a walk where you don't repeat vertices.

Trail

A trail is like a path, but you're allowed to repeat edges (connections) as long as you don't revisit the same vertex consecutively. In our playground, you can go from the slide to the monkey bars and back to the slide, but you can't go directly from the slide to the slide without going somewhere else first.

In summary, a walk is a general movement between vertices, a path is a walk without repeating vertices, and a trail is a walk without repeating consecutive edges.

Closed Walk

As mentioned before, a closed walk is like taking a journey around the playground and ending up back where you started. You can visit any vertices and use any edges multiple times.

Circuit

A circuit is a closed walk where you don't repeat any edges, except for the starting and ending edge. In the playground, it's like going from the slide to the swing, then the monkey bars, and finally back to the slide without using any path or edge twice (except for the initial and final edges).

Simple Circuit

A simple circuit is a circuit that doesn't repeat any vertices other than the starting and ending vertex. In the playground, this would mean you can't visit the same equipment more than once while still forming a closed loop.

In summary, a closed walk is a general concept of returning to the starting point, a circuit is a closed walk without repeating edges (except for the starting and ending edge), and a simple circuit is a circuit without repeating any vertices (except for the starting and ending vertex).

Euler Circuit

Imagine you want to visit every piece of equipment in the playground, starting and ending at the same place, and using each path exactly once. An Euler Circuit is a closed walk that includes every edge in the graph exactly once. In the playground, you'd go down the slide, climb the ladder, swing, use the monkey bars, and return to the starting point, making sure you've used every connection.

Euler Path

Now, let's say you want to visit every piece of equipment, but you're okay with starting and ending at different places. An Euler Path is a walk that includes every edge exactly once. In the playground, you might go down the slide, climb the ladder, swing, use the monkey bars, and finish at a different spot. You've still used every connection but started and ended at different vertices.

So, in summary, an Euler Circuit is an Euler Path that forms a closed loop by starting and ending at the same place. An Euler Path is a walk that includes every edge exactly once but may start and end at different vertices.

To quickly identify whether a graph has an Euler circuit or an Euler path, you can use the following rules:

1. **Euler Circuit:**
   - Every vertex in the graph must have an even degree (number of edges incident to it).
   - The graph must be connected.
2. **Euler Path:**
   - The graph can have at most two vertices with an odd degree.
   - The graph must be connected.

If these conditions are met, the graph has an Euler circuit or an Euler path, depending on whether there are zero or two vertices with an odd degree. If there are exactly two vertices with an odd degree, the Euler path starts at one of them and ends at the other. If there are no vertices with an odd degree, the graph has an Euler circuit.

By quickly checking the degrees of the vertices, you can determine the possibility of an Euler circuit or an Euler path in a graph.

Connected graph

A connected graph is a graph in which there is a path between every pair of vertices. In other words, you can reach any vertex from any other vertex by following the edges of the graph. There are no isolated or disconnected parts; everything is reachable.

Imagine a city with streets as edges and intersections as vertices. If you can travel from any part of the city to any other part by following the streets, the city is like a connected graph. If there are areas you can't reach without leaving the city, it's not connected.

In mathematical terms, a connected graph has only one connected component, meaning there is a path between every pair of vertices within that component.

In a connected graph, every vertex is indirectly connected to every other vertex. While not every vertex is directly connected by a single edge, there exists at least one path (a sequence of edges) between any pair of vertices.

Think of it like navigating through a city. In a connected graph, even if two locations are not directly connected by a single road, you can still reach one from the other by traveling through a series of roads and intersections. This indirect connection ensures that there is a path between any pair of vertices in a connected graph.

Complete graph

A graph in which every vertex is directly connected to every other vertex is called a **complete graph**. In a complete graph, there is an edge between every pair of distinct vertices. If the graph has n vertices, then it has $\frac{n\cdot(n-1)}{2}$ edges, making it fully connected. The complete graph on n vertices is often denoted as K_n, where n represents the number of vertices.

It's like having a city where there is a direct road connecting every pair of locations, ensuring that you can travel directly from any place to any other place.