SERC v2.1 – Mathematical Foundations

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Stability of Relational Dynamics on the Simplex

Abstract

This document provides the mathematical foundations of the SERC framework. We analyze the geometry of the 3-simplex, the Gram metric $G = 4I - J$, the projected gradient flow, and prove global asymptotic stability of the relational equilibrium $P_0 = \frac{1}{4}\mathbf{1}$. We also derive the exponential relaxation rate and interpret the results in the context of LLM guardrails.

1 Stability of Relational Dynamics on the Simplex

1.1 Setup

Let $\Delta^3 \subset \mathbb{R}^4$ denote the standard 3-simplex:

$$\Delta^3 = \{Z \in \mathbb{R}^4_{\geq 0} \mid \mathbf{1}^\top Z = 1\}. \tag{1}$$

The tangent space at any interior point is

$$T\Delta^3 = \{v \in \mathbb{R}^4 \mid \mathbf{1}^\top v = 0\}, \tag{2}$$

with orthogonal projector onto $T\Delta^3$:

$$P = I_4 - \tfrac{1}{4}\,\mathbf{1}\mathbf{1}^\top. \tag{3}$$

The *relational tension functional* is

$$\Omega(Z) = \tfrac{1}{2}\,Z^\top G Z, \qquad G = 4I_4 - J_4, \tag{4}$$

where $J_4 = \mathbf{1}\mathbf{1}^\top$. Its gradient is $\nabla\Omega(Z) = GZ$.

1.2 Spectral Analysis

Lemma 1 (Spectrum of G on $T\Delta^3$)**.** *The matrix $G = 4I_4 - J_4$ has eigenvalues $\{0, 4, 4, 4\}$. The zero eigenvalue corresponds to eigenvector $v_0 = \frac{1}{2}\mathbf{1}$, which is orthogonal to $T\Delta^3$. The restriction $G|_{T\Delta^3}$ is positive definite with $\lambda_{\min}(G|_{T\Delta^3}) = 4$.*

Proof. Direct computation: $G\mathbf{1} = (4I_4 - J_4)\mathbf{1} = 4\mathbf{1} - 4\mathbf{1} = 0$. For any $v \perp \mathbf{1}$ (i.e. $v \in T\Delta^3$): $Gv = 4v - J_4 v = 4v - (\mathbf{1}^\top v)\mathbf{1} = 4v$. Hence $G|_{T\Delta^3} = 4I|_{T\Delta^3}$. □

1.3 Projected Gradient Flow

Define the *projected gradient flow* on Δ^3:

$$\dot{Z} = -P\,\nabla\Omega(Z) = -PGZ. \tag{5}$$

Since P projects onto $T\Delta^3$ and $\mathbf{1}^\top Z(0) = 1$, we have $\mathbf{1}^\top \dot{Z} = 0$ for all t, so $Z(t) \in \Delta^3$ for all $t \geq 0$.

Theorem 1 (Lyapunov Stability of P_0)**.** *The point $P_0 = \frac{1}{4}\mathbf{1}$ is the unique equilibrium of* (5) *in Δ^3, and is globally asymptotically stable. Ω is a Lyapunov function for this flow.*

Proof. **Step 1: Ω is non-increasing.**

$$\frac{d}{dt}\Omega(Z(t)) = \nabla\Omega(Z)^\top \dot{Z} = (GZ)^\top(-PGZ) = -(GZ)^\top P(GZ). \tag{6}$$

Let $w = GZ$. Since P is an orthogonal projector, $w^\top Pw = \|Pw\|^2 \geq 0$. Thus $\dot{\Omega} \leq 0$, with equality iff $Pw = 0$.

Step 2: Unique equilibrium. $\dot{Z} = 0$ iff $PGZ = 0$, i.e. $GZ = c\mathbf{1}$. Write $Z = P_0 + v$ with $v \in T\Delta^3$. Then $GZ = Gv$ and $Gv = c\mathbf{1}$. But $Gv \in T\Delta^3$ and $\mathbf{1} \perp T\Delta^3$, so $c = 0$ and $Gv = 0$. Since $G|_{T\Delta^3} = 4I$, we get $v = 0$.

Step 3: Asymptotic stability. $\Omega(Z) \geq 0$ with equality only at P_0. LaSalle's invariance principle gives global asymptotic stability. $\square$

Lemma 2 (Quadratic lower bound)**.** *For $Z \in \Delta^3$ and $v = Z - P_0 \in T\Delta^3$:*

$$\|PGZ\|^2 = 16\|v\|^2 = 32\,\Omega(Z). \tag{7}$$

Proof. Since $Gv = 4v$ on $T\Delta^3$, we have $\|Gv\|^2 = 16\|v\|^2$. Also $\Omega(Z) = \frac{1}{2}v^\top Gv = 2\|v\|^2$. $\square$

Corollary 1 (Exponential Relaxation Rate)**.** *Along trajectories of* (5)*:*

$$\Omega(Z(t)) \leq \Omega(Z(0))\,e^{-32t}. \tag{8}$$

Proof. From the previous lemma:

$$\dot{\Omega} = -\|PGZ\|^2 = -32\,\Omega(Z). \tag{9}$$

Integrating gives the result. $\square$

1.4 Interpretation for LLM Guardrails

Under the ideal flow, $\Omega(Z(t))$ decreases monotonically to zero. Actual LLM trajectories follow a discrete perturbed update:

$$Z_{t+1} = Z_t + \Delta_t. \tag{10}$$

Define the relational anomaly:

$$a_t = \Omega(Z_{t+1}) - \Omega(Z_t). \tag{11}$$

Healthy dynamics: $a_t < 0$. Pathology: sustained $a_t > 0$.

Guardrail condition $\Omega_{\text{mean}}(W) > \theta$ detects windows where $\sum a_t > 0$.