Stochastic Calculus for Finance, Volume I and II

Solution of Exercise Problems

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Contents

This is a solution manual for the two-volume textbook *Stochastic calculus for finance*, by Steven Shreve. If you have any comments or find any typos/errors, please email me at yz44@cornell.edu.

The current version omits the following problems. Volume I: 1.5, 3.3, 3.4, 5.7; Volume II: 3.9, 7.1, 7.2, 7.5–7.9, 10.8, 10.9, 10.10.

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Chapter 1

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model

1.1 The Binomial No-Arbitrage Pricing Model

1.1.

Proof. If we get the up sate, then $X_1 = X_1(H) = \Delta_0 uS_0 + (1+r)(X_0 - \Delta_0 S_0)$; if we get the down state, then $X_1 = X_1(T) = \Delta_0 dS_0 + (1+r)(X_0 - \Delta_0 S_0)$. If X_1 has a positive probability of being strictly positive, then we must either have $X_1(H) > 0$ or $X_1(T) > 0$.

(i) If $X_1(H) > 0$, then $\Delta_0 uS_0 + (1+r)(X_0 - \Delta_0 S_0) > 0$. Plug in $X_0 = 0$, we get $u\Delta_0 > (1+r)\Delta_0$. By condition $d < 1+r < u$, we conclude $\Delta_0 > 0$. In this case, $X_1(T) = \Delta_0 dS_0 + (1+r)(X_0 - \Delta_0 S_0) = \Delta_0 S_0[d - (1+r)] < 0$.

(ii) If $X_1(T) > 0$, then we can similarly deduce $\Delta_0 < 0$ and hence $X_1(H) < 0$.

So we cannot have X_1 strictly positive with positive probability unless X_1 is strictly negative with positive probability as well, regardless the choice of the number Δ_0.

Remark: Here the condition $X_0 = 0$ is not essential, as far as a property definition of arbitrage for arbitrary X_0 can be given. Indeed, for the one-period binomial model, we can define arbitrage as a trading strategy such that $P(X_1 \geq X_0(1+r)) = 1$ and $P(X_1 > X_0(1+r)) > 0$. First, this is a generalization of the case $X_0 = 0$; second, it is "proper" because it is comparing the result of an arbitrary investment involving money and stock markets with that of a safe investment involving only money market. This can also be seen by regarding X_0 as borrowed from money market account. Then at time 1, we have to pay back $X_0(1+r)$ to the money market account. In summary, arbitrage is a trading strategy that beats "safe" investment.

Accordingly, we revise the proof of Exercise 1.1. as follows. If X_1 has a positive probability of being strictly larger than $X_0(1+r)$, the either $X_1(H) > X_0(1+r)$ or $X_1(T) > X_0(1+r)$. The first case yields $\Delta_0 S_0(u-1-r) > 0$, i.e. $\Delta_0 > 0$. So $X_1(T) = (1+r)X_0 + \Delta_0 S_0(d-1-r) < (1+r)X_0$. The second case can be similarly analyzed. Hence we cannot have X_1 strictly greater than $X_0(1+r)$ with positive probability unless X_1 is strictly smaller than $X_0(1+r)$ with positive probability as well.

Finally, we comment that the above formulation of arbitrage is equivalent to the one in the textbook. For details, see Shreve [7], Exercise 5.7. □

1.2.

Proof. $X_1(u) = \Delta_0 \times 8 + \Gamma_0 \times 3 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = 3\Delta_0 + 1.5\Gamma_0$, and $X_1(d) = \Delta_0 \times 2 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = -3\Delta_0 - 1.5\Gamma_0$. That is, $X_1(u) = -X_1(d)$. So if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative.

Remark: Note the above relation $X_1(u) = -X_1(d)$ is not a coincidence. In general, let V_1 denote the payoff of the derivative security at time 1. Suppose $\bar{X}_0$ and $\bar{\Delta}_0$ are chosen in such a way that V_1 can be replicated: $(1+r)(\bar{X}_0 - \bar{\Delta}_0 S_0) + \bar{\Delta}_0 S_1 = V_1$. Using the notation of the problem, suppose an agent begins with 0 wealth and at time zero buys Δ_0 shares of stock and Γ_0 options. He then puts his cash position $-\Delta_0 S_0 - \Gamma_0 \bar{X}_0$ in a money market account. At time one, the value of the agent's portfolio of stock, option and money market assets is

$$X_1 = \Delta_0 S_1 + \Gamma_0 V_1 - (1+r)(\Delta_0 S_0 + \Gamma_0 \bar{X}_0).$$

Plug in the expression of V_1 and sort out terms, we have

$$X_1 = S_0(\Delta_0 + \bar{\Delta}_0 \Gamma_0)(\frac{S_1}{S_0} - (1+r)).$$

Since $d < (1+r) < u$, $X_1(u)$ and $X_1(d)$ have opposite signs. So if the price of the option at time zero is $\bar{X}_0$, then there will no arbitrage. $\square$

1.3.

Proof. $V_0 = \frac{1}{1+r}\left[\frac{1+r-d}{u-d}S_1(H) + \frac{u-1-r}{u-d}S_1(T)\right] = \frac{S_0}{1+r}\left[\frac{1+r-d}{u-d}u + \frac{u-1-r}{u-d}d\right] = S_0$. This is not surprising, since this is exactly the cost of replicating S_1.

Remark: This illustrates an important point. The "fair price" of a stock cannot be determined by the risk-neutral pricing, as seen below. Suppose $S_1(H)$ and $S_1(T)$ are given, we could have two current prices, S_0 and S_0'. Correspondingly, we can get u, d and u', d'. Because they are determined by S_0 and S_0', respectively, it's not surprising that risk-neutral pricing formula always holds, in both cases. That is,

$$S_0 = \frac{\frac{1+r-d}{u-d}S_1(H) + \frac{u-1-r}{u-d}S_1(T)}{1+r}, \; S_0' = \frac{\frac{1+r-d'}{u'-d'}S_1(H) + \frac{u'-1-r}{u'-d'}S_1(T)}{1+r}.$$

Essentially, this is because risk-neutral pricing relies on *fair price=replication cost*. Stock as a replicating component cannot determine its own "fair" price via the risk-neutral pricing formula. $\square$

1.4.

Proof.

$$\begin{aligned}
X_{n+1}(T) &= \Delta_n d S_n + (1+r)(X_n - \Delta_n S_n) \\
&= \Delta_n S_n (d-1-r) + (1+r)V_n \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}(d-1-r) + (1+r)\frac{\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)}{1+r} \\
&= \tilde{p}(V_{n+1}(T) - V_{n+1}(H)) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= \tilde{p}V_{n+1}(T) + \tilde{q}V_{n+1}(T) \\
&= V_{n+1}(T).
\end{aligned}$$

$\square$

1.6.

Proof. The bank's trader should set up a replicating portfolio whose payoff is the opposite of the option's payoff. More precisely, we solve the equation

$$(1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1 = -(S_1 - K)^+.$$

Then $X_0 = -1.20$ and $\Delta_0 = -\frac{1}{2}$. This means the trader should sell short 0.5 share of stock, put the income 2 into a money market account, and then transfer 1.20 into a separate money market account. At time one, the portfolio consisting of a short position in stock and $0.8(1+r)$ in money market account will cancel out with the option's payoff. Therefore we end up with $1.20(1+r)$ in the separate money market account.

Remark: This problem illustrates why we are interested in hedging a long position. In case the stock price goes down at time one, the option will expire without any payoff. The initial money 1.20 we paid at time zero will be wasted. By hedging, we convert the option back into liquid assets (cash and stock) which guarantees a sure payoff at time one. Also, cf. page 7, paragraph 2. As to why we hedge a short position (as a writer), see Wilmott [8], page 11-13.

□

1.7.

Proof. The idea is the same as Problem 1.6. The bank's trader only needs to set up the reverse of the replicating trading strategy described in Example 1.2.4. More precisely, he should short sell 0.1733 share of stock, invest the income 0.6933 into money market account, and transfer 1.376 into a separate money market account. The portfolio consisting a short position in stock and 0.6933-1.376 in money market account will replicate the opposite of the option's payoff. After they cancel out, we end up with $1.376(1+r)^3$ in the separate money market account.

□

1.8. (i)

Proof. $v_n(s,y) = \frac{2}{5}(v_{n+1}(2s, y+2s) + v_{n+1}(\frac{s}{2}, y+\frac{s}{2}))$. □

(ii)

Proof. 1.696. □

(iii)

Proof.

$$\delta_n(s,y) = \frac{v_{n+1}(us, y+us) - v_{n+1}(ds, y+ds)}{(u-d)s}.$$

□

1.9. (i)

Proof. Similar to Theorem 1.2.2, but replace r, u and d everywhere with r_n, u_n and d_n. More precisely, set $\widetilde{p}_n = \frac{1+r_n-d_n}{u_n-d_n}$ and $\widetilde{q}_n = 1-\widetilde{p}_n$. Then

$$V_n = \frac{\widetilde{p}_n V_{n+1}(H) + \widetilde{q}_n V_{n+1}(T)}{1+r_n}.$$

□

(ii)

Proof. $\Delta_n = \frac{V_{n+1}(H)-V_{n+1}(T)}{S_{n+1}(H)-S_{n+1}(T)} = \frac{V_{n+1}(H)-V_{n+1}(T)}{(u_n-d_n)S_n}$. □

(iii)

Proof. $u_n = \frac{S_{n+1}(H)}{S_n} = \frac{S_n+10}{S_n} = 1+\frac{10}{S_n}$ and $d_n = \frac{S_{n+1}(T)}{S_n} = \frac{S_n-10}{S_n} = 1-\frac{10}{S_n}$. So the risk-neutral probabilities at time n are $\tilde{p}_n = \frac{1-d_n}{u_n-d_n} = \frac{1}{2}$ and $\tilde{q}_n = \frac{1}{2}$. Risk-neutral pricing implies the price of this call at time zero is 9.375. □

1.2 Probability Theory on Coin Toss Space

2.1. (i)

Proof. $P(A^c) + P(A) = \sum_{\omega \in A^c} P(\omega) + \sum_{\omega \in A} P(\omega) = \sum_{\omega \in \Omega} P(\omega) = 1$. $\square$

(ii)

Proof. By induction, it suffices to work on the case $N = 2$. When A_1 and A_2 are disjoint, $P(A_1 \cup A_2) = \sum_{\omega \in A_1 \cup A_2} P(\omega) = \sum_{\omega \in A_1} P(\omega) + \sum_{\omega \in A_2} P(\omega) = P(A_1) + P(A_2)$. When A_1 and A_2 are arbitrary, using the result when they are disjoint, we have $P(A_1 \cup A_2) = P((A_1 - A_2) \cup A_2) = P(A_1 - A_2) + P(A_2) \leq P(A_1) + P(A_2)$. $\square$

2.2. (i)

Proof. $\widetilde{P}(S_3 = 32) = \widetilde{p}^3 = \frac{1}{8}$, $\widetilde{P}(S_3 = 8) = 3\widetilde{p}^2\widetilde{q} = \frac{3}{8}$, $\widetilde{P}(S_3 = 2) = 3\widetilde{p}\widetilde{q}^2 = \frac{3}{8}$, and $\widetilde{P}(S_3 = 0.5) = \widetilde{q}^3 = \frac{1}{8}$. $\square$

(ii)

Proof. $\widetilde{E}[S_1] = 8\widetilde{P}(S_1 = 8) + 2\widetilde{P}(S_1 = 2) = 8\widetilde{p} + 2\widetilde{q} = 5$, $\widetilde{E}[S_2] = 16\widetilde{p}^2 + 4 \cdot 2\widetilde{p}\widetilde{q} + 1 \cdot \widetilde{q}^2 = 6.25$, and $\widetilde{E}[S_3] = 32 \cdot \frac{1}{8} + 8 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 0.5 \cdot \frac{1}{8} = 7.8125$. So the average rates of growth of the stock price under $\widetilde{P}$ are, respectively: $\widetilde{r}_0 = \frac{5}{4} - 1 = 0.25$, $\widetilde{r}_1 = \frac{6.25}{5} - 1 = 0.25$ and $\widetilde{r}_2 = \frac{7.8125}{6.25} - 1 = 0.25$.

$\square$

(iii)

Proof. $P(S_3 = 32) = (\frac{2}{3})^3 = \frac{8}{27}$, $P(S_3 = 8) = 3 \cdot (\frac{2}{3})^2 \cdot \frac{1}{3} = \frac{4}{9}$, $P(S_3 = 2) = 2 \cdot \frac{1}{9} = \frac{2}{9}$, and $P(S_3 = 0.5) = \frac{1}{27}$.

Accordingly, $E[S_1] = 6$, $E[S_2] = 9$ and $E[S_3] = 13.5$. So the average rates of growth of the stock price under P are, respectively: $r_0 = \frac{6}{4} - 1 = 0.5$, $r_1 = \frac{9}{6} - 1 = 0.5$, and $r_2 = \frac{13.5}{9} - 1 = 0.5$. $\square$

2.3.

Proof. Apply conditional Jensen's inequality. $\square$

2.4. (i)

Proof. $E_n[M_{n+1}] = M_n + E_n[X_{n+1}] = M_n + E[X_{n+1}] = M_n$. $\square$

(ii)

Proof. $E_n[\frac{S_{n+1}}{S_n}] = E_n[e^{\sigma X_{n+1}} \frac{2}{e^{\sigma} + e^{-\sigma}}] = \frac{2}{e^{\sigma} + e^{-\sigma}} E[e^{\sigma X_{n+1}}] = 1$. $\square$

2.5. (i)

Proof. $2I_n = 2\sum_{j=0}^{n-1} M_j(M_{j+1} - M_j) = 2\sum_{j=0}^{n-1} M_j M_{j+1} - \sum_{j=1}^{n-1} M_j^2 - \sum_{j=1}^{n-1} M_j^2 = 2\sum_{j=0}^{n-1} M_j M_{j+1} + M_n^2 - \sum_{j=0}^{n-1} M_{j+1}^2 - \sum_{j=0}^{n-1} M_j^2 = M_n^2 - \sum_{j=0}^{n-1}(M_{j+1} - M_j)^2 = M_n^2 - \sum_{j=0}^{n-1} X_{j+1}^2 = M_n^2 - n$. $\square$

(ii)

Proof. $E_n[f(I_{n+1})] = E_n[f(I_n + M_n(M_{n+1} - M_n))] = E_n[f(I_n + M_n X_{n+1})] = \frac{1}{2}[f(I_n + M_n) + f(I_n - M_n)] = g(I_n)$, where $g(x) = \frac{1}{2}[f(x + \sqrt{2x+n}) + f(x - \sqrt{2x+n})]$, since $\sqrt{2I_n + n} = |M_n|$. $\square$

2.6.

Proof. $E_n[I_{n+1} - I_n] = E_n[\Delta_n(M_{n+1} - M_n)] = \Delta_n E_n[M_{n+1} - M_n] = 0$. $\square$

2.7.

Proof. We denote by X_n the result of n-th coin toss, where Head is represented by $X = 1$ and Tail is represented by $X = -1$. We also suppose $P(X = 1) = P(X = -1) = \frac{1}{2}$. Define $S_1 = X_1$ and $S_{n+1} = S_n + b_n(X_1, \cdots, X_n)X_{n+1}$, where $b_n(\cdot)$ is a bounded function on $\{-1, 1\}^n$, to be determined later on. Clearly $(S_n)_{n\geq 1}$ is an adapted stochastic process, and we can show it is a martingale. Indeed, $E_n[S_{n+1} - S_n] = b_n(X_1, \cdots, X_n)E_n[X_{n+1}] = 0$.

For any arbitrary function f, $E_n[f(S_{n+1})] = \frac{1}{2}[f(S_n + b_n(X_1, \cdots, X_n)) + f(S_n - b_n(X_1, \cdots, X_n))]$. Then intuitively, $E_n[f(S_{n+1}]$ cannot be solely dependent upon S_n when b_n's are properly chosen. Therefore in general, $(S_n)_{n\geq 1}$ cannot be a Markov process.

Remark 1. *If X_n is regarded as the gain/loss of n-th bet in a gambling game, then S_n would be the wealth at time n. b_n is therefore the wager for the (n+1)-th bet and is devised according to past gambling results.*

□

2.8. (i)

Proof. Note $M_n = E_n[M_N]$ and $M'_n = E_n[M'_N]$. □

(ii)

Proof. In the proof of Theorem 1.2.2, we proved by induction that $X_n = V_n$ where X_n is defined by (1.2.14) of Chapter 1. In other words, the sequence $(V_n)_{0\leq n\leq N}$ can be realized as the value process of a portfolio, which consists of stock and money market accounts. Since $(\frac{X_n}{(1+r)^n})_{0\leq n\leq N}$ is a martingale under $\widetilde{P}$ (Theorem 2.4.5), $(\frac{V_n}{(1+r)^n})_{0\leq n\leq N}$ is a martingale under $\widetilde{P}$. □

(iii)

Proof. $\frac{V'_n}{(1+r)^n} = E_n\left[\frac{V_N}{(1+r)^N}\right]$, so V'_0, $\frac{V'_1}{1+r}$, $\cdots$, $\frac{V'_{N-1}}{(1+r)^{N-1}}$, $\frac{V_N}{(1+r)^N}$ is a martingale under $\widetilde{P}$. □

(iv)

Proof. Combine (ii) and (iii), then use (i). □

2.9. (i)

Proof. $u_0 = \frac{S_1(H)}{S_0} = 2$, $d_0 = \frac{S_1(H)}{S_0} = \frac{1}{2}$, $u_1(H) = \frac{S_2(HH)}{S_1(H)} = 1.5$, $d_1(H) = \frac{S_2(HT)}{S_1(H)} = 1$, $u_1(T) = \frac{S_2(TH)}{S_1(T)} = 4$ and $d_1(T) = \frac{S_2(TT)}{S_1(T)} = 1$.

So $\widetilde{p}_0 = \frac{1+r_0-d_0}{u_0-d_0} = \frac{1}{2}$, $\widetilde{q}_0 = \frac{1}{2}$, $\widetilde{p}_1(H) = \frac{1+r_1(H)-d_1(H)}{u_1(H)-d_1(H)} = \frac{1}{2}$, $\widetilde{q}_1(H) = \frac{1}{2}$, $\widetilde{p}_1(T) = \frac{1+r_1(T)-d_1(T)}{u_1(T)-d_1(T)} = \frac{1}{6}$, and $\widetilde{q}_1(T) = \frac{5}{6}$.

Therefore $\widetilde{P}(HH) = \widetilde{p}_0\widetilde{p}_1(H) = \frac{1}{4}$, $\widetilde{P}(HT) = \widetilde{p}_0\widetilde{q}_1(H) = \frac{1}{4}$, $\widetilde{P}(TH) = \widetilde{q}_0\widetilde{p}_1(T) = \frac{1}{12}$ and $\widetilde{P}(TT) = \widetilde{q}_0\widetilde{q}_1(T) = \frac{5}{12}$.

The proofs of Theorem 2.4.4, Theorem 2.4.5 and Theorem 2.4.7 still work for the random interest rate model, with proper modifications (i.e. $\widetilde{P}$ would be constructed according to *conditional probabilities* $\widetilde{P}(\omega_{n+1} = H|\omega_1, \cdots, \omega_n) := \widetilde{p}_n$ and $\widetilde{P}(\omega_{n+1} = T|\omega_1, \cdots, \omega_n) := \widetilde{q}_n$. Cf. notes on page 39.). So the time-zero value of an option that pays off V_2 at time two is given by the risk-neutral pricing formula $V_0 = \widetilde{E}\left[\frac{V_2}{(1+r_0)(1+r_1)}\right]$. □

(ii)

Proof. $V_2(HH) = 5$, $V_2(HT) = 1$, $V_2(TH) = 1$ and $V_2(TT) = 0$. So $V_1(H) = \frac{\widetilde{p}_1(H)V_2(HH)+\widetilde{q}_1(H)V_2(HT)}{1+r_1(H)} = 2.4$, $V_1(T) = \frac{\widetilde{p}_1(T)V_2(TH)+\widetilde{q}_1(T)V_2(TT)}{1+r_1(T)} = \frac{1}{9}$, and $V_0 = \frac{\widetilde{p}_0V_1(H)+\widetilde{q}_0V_1(T)}{1+r_0} \approx 1$. □

(iii)

Proof. $\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)} = \frac{2.4-\frac{1}{9}}{8-2} = 0.4 - \frac{1}{54} \approx 0.3815$. □

(iv)

Proof. $\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{S_2(HH)-S_2(HT)} = \frac{5-1}{12-8} = 1.$ □

2.10. (i)

Proof. $\widetilde{E}_n[\frac{X_{n+1}}{(1+r)^{n+1}}] = \widetilde{E}_n[\frac{\Delta_n Y_{n+1}S_n}{(1+r)^{n+1}} + \frac{(1+r)(X_n-\Delta_n S_n)}{(1+r)^{n+1}}] = \frac{\Delta_n S_n}{(1+r)^{n+1}}\widetilde{E}_n[Y_{n+1}] + \frac{X_n-\Delta_n S_n}{(1+r)^n} = \frac{\Delta_n S_n}{(1+r)^{n+1}}(u\widetilde{p} + d\widetilde{q}) + \frac{X_n-\Delta_n S_n}{(1+r)^n} = \frac{\Delta_n S_n + X_n - \Delta_n S_n}{(1+r)^n} = \frac{X_n}{(1+r)^n}.$ □

(ii)

Proof. From (2.8.2), we have

$$\begin{cases}\Delta_n u S_n + (1+r)(X_n - \Delta_n S_n) = X_{n+1}(H)\\ \Delta_n d S_n + (1+r)(X_n - \Delta_n S_n) = X_{n+1}(T).\end{cases}$$

So $\Delta_n = \frac{X_{n+1}(H)-X_{n+1}(T)}{uS_n - dS_n}$ and $X_n = \widetilde{E}_n[\frac{X_{n+1}}{1+r}]$. To make the portfolio replicate the payoff at time N, we must have $X_N = V_N$. So $X_n = \widetilde{E}_n[\frac{X_N}{(1+r)^{N-n}}] = \widetilde{E}_n[\frac{V_N}{(1+r)^{N-n}}]$. Since $(X_n)_{0\leq n\leq N}$ is the value process of the unique replicating portfolio (uniqueness is guaranteed by the uniqueness of the solution to the above linear equations), the no-arbitrage price of V_N at time n is $V_n = X_n = \widetilde{E}_n[\frac{V_N}{(1+r)^{N-n}}]$. □

(iii)

Proof.

$$\begin{aligned}\widetilde{E}_n[\frac{S_{n+1}}{(1+r)^{n+1}}] &= \frac{1}{(1+r)^{n+1}}\widetilde{E}_n[(1-A_{n+1})Y_{n+1}S_n]\\ &= \frac{S_n}{(1+r)^{n+1}}[\widetilde{p}(1-A_{n+1}(H))u + \widetilde{q}(1-A_{n+1}(T))d]\\ &< \frac{S_n}{(1+r)^{n+1}}[\widetilde{p}u + \widetilde{q}d]\\ &= \frac{S_n}{(1+r)^n}.\end{aligned}$$

If A_{n+1} is a constant a, then $\widetilde{E}_n[\frac{S_{n+1}}{(1+r)^{n+1}}] = \frac{S_n}{(1+r)^{n+1}}(1-a)(\widetilde{p}u+\widetilde{q}d) = \frac{S_n}{(1+r)^n}(1-a)$. So $\widetilde{E}_n[\frac{S_{n+1}}{(1+r)^{n+1}(1-a)^{n+1}}] = \frac{S_n}{(1+r)^n(1-a)^n}$. □

2.11. (i)

Proof. $F_N + P_N = S_N - K + (K - S_N)^+ = (S_N - K)^+ = C_N$. □

(ii)

Proof. $C_n = \widetilde{E}_n[\frac{C_N}{(1+r)^{N-n}}] = \widetilde{E}_n[\frac{F_N}{(1+r)^{N-n}}] + \widetilde{E}_n[\frac{P_N}{(1+r)^{N-n}}] = F_n + P_n$. □

(iii)

Proof. $F_0 = \widetilde{E}[\frac{F_N}{(1+r)^N}] = \frac{1}{(1+r)^N}\widetilde{E}[S_N - K] = S_0 - \frac{K}{(1+r)^N}$. □

(iv)

Proof. At time zero, the trader has $F_0 = S_0$ in money market account and one share of stock. At time N, the trader has a wealth of $(F_0 - S_0)(1+r)^N + S_N = -K + S_N = F_N$. □

(v)

Proof. By (ii), $C_0 = F_0 + P_0$. Since $F_0 = S_0 - \frac{(1+r)^N S_0}{(1+r)^N} = 0$, $C_0 = P_0$. $\square$

(vi)

Proof. By (ii), $C_n = P_n$ if and only if $F_n = 0$. Note $F_n = \widetilde{E}_n[\frac{S_N - K}{(1+r)^{N-n}}] = S_n - \frac{(1+r)^N S_0}{(1+r)^{N-n}} = S_n - S_0(1+r)^n$. So F_n is not necessarily zero and $C_n = P_n$ is not necessarily true for $n \geq 1$. $\square$

2.12.

Proof. First, the no-arbitrage price of the chooser option at time m must be $\max(C, P)$, where

$$C = \widetilde{E}\left[\frac{(S_N - K)^+}{(1+r)^{N-m}}\right], \text{ and } P = \widetilde{E}\left[\frac{(K - S_N)^+}{(1+r)^{N-m}}\right].$$

That is, C is the no-arbitrage price of a call option at time m and P is the no-arbitrage price of a put option at time m. Both of them have maturity date N and strike price K. Suppose the market is liquid, then the chooser option is equivalent to receiving a payoff of $\max(C, P)$ at time m. Therefore, its current no-arbitrage price should be $\widetilde{E}[\frac{\max(C,P)}{(1+r)^m}]$.

By the put-call parity, $C = S_m - \frac{K}{(1+r)^{N-m}} + P$. So $\max(C, P) = P + (S_m - \frac{K}{(1+r)^{N-m}})^+$. Therefore, the time-zero price of a chooser option is

$$\widetilde{E}\left[\frac{P}{(1+r)^m}\right] + \widetilde{E}\left[\frac{(S_m - \frac{K}{(1+r)^{N-m}})^+}{(1+r)^m}\right] = \widetilde{E}\left[\frac{(K - S_N)^+}{(1+r)^N}\right] + \widetilde{E}\left[\frac{(S_m - \frac{K}{(1+r)^{N-m}})^+}{(1+r)^m}\right].$$

The first term stands for the time-zero price of a put, expiring at time N and having strike price K, and the second term stands for the time-zero price of a call, expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$.

If we feel unconvinced by the above argument that the chooser option's no-arbitrage price is $\widetilde{E}[\frac{\max(C,P)}{(1+r)^m}]$, due to the economical argument involved (like "the chooser option is equivalent to receiving a payoff of $\max(C, P)$ at time m"), then we have the following mathematically rigorous argument. First, we can construct a portfolio $\Delta_0, \cdots, \Delta_{m-1}$, whose payoff at time m is $\max(C, P)$. Fix ω, if $C(\omega) > P(\omega)$, we can construct a portfolio $\Delta'_m, \cdots, \Delta'_{N-1}$ whose payoff at time N is $(S_N - K)^+$; if $C(\omega) < P(\omega)$, we can construct a portfolio $\Delta''_m, \cdots, \Delta''_{N-1}$ whose payoff at time N is $(K - S_N)^+$. By defining $(m \leq k \leq N-1)$

$$\Delta_k(\omega) = \begin{cases} \Delta'_k(\omega) & \text{if } C(\omega) > P(\omega) \\ \Delta''_k(\omega) & \text{if } C(\omega) < P(\omega), \end{cases}$$

we get a portfolio $(\Delta_n)_{0 \leq n \leq N-1}$ whose payoff is the same as that of the chooser option. So the no-arbitrage price process of the chooser option must be equal to the value process of the replicating portfolio. In particular, $V_0 = X_0 = \widetilde{E}[\frac{X_m}{(1+r)^m}] = \widetilde{E}[\frac{\max(C,P)}{(1+r)^m}]$. $\square$

2.13. (i)

Proof. Note under both actual probability P and risk-neutral probability $\widetilde{P}$, coin tosses ω_n's are i.i.d.. So without loss of generality, we work on P. For any function g, $E_n[g(S_{n+1}, Y_{n+1})] = E_n[g(\frac{S_{n+1}}{S_n} S_n, Y_n + \frac{S_{n+1}}{S_n} S_n)] = pg(uS_n, Y_n + uS_n) + qg(dS_n, Y_n + dS_n)$, which is a function of (S_n, Y_n). So $(S_n, Y_n)_{0 \leq n \leq N}$ is Markov under P. $\square$

(ii)

Proof. Set $v_N(s, y) = f(\frac{y}{N+1})$. Then $v_N(S_N, Y_N) = f(\frac{\sum_{n=0}^N S_n}{N+1}) = V_N$. Suppose v_{n+1} is given, then $V_n = \widetilde{E}_n[\frac{V_{n+1}}{1+r}] = \widetilde{E}_n[\frac{v_{n+1}(S_{n+1}, Y_{n+1})}{1+r}] = \frac{1}{1+r}[\widetilde{p} v_{n+1}(uS_n, Y_n + uS_n) + \widetilde{q} v_{n+1}(dS_n, Y_n + dS_n)] = v_n(S_n, Y_n)$, where

$$v_n(s, y) = \frac{\widetilde{v}_{n+1}(us, y + us) + \widetilde{v}_{n+1}(ds, y + ds)}{1+r}.$$

$\square$

2.14. (i)

Proof. For $n \leq M$, $(S_n, Y_n) = (S_n, 0)$. Since coin tosses ω_n's are i.i.d. under $\widetilde{P}$, $(S_n, Y_n)_{0\leq n\leq M}$ is Markov under $\widetilde{P}$. More precisely, for any function h, $\widetilde{E}_n[h(S_{n+1})] = \widetilde{p}h(uS_n) + \widetilde{h}(dS_n)$, for $n = 0, 1, \cdots, M-1$.

For any function g of two variables, we have $\widetilde{E}_M[g(S_{M+1}, Y_{M+1})] = \widetilde{E}_M[g(S_{M+1}, S_{M+1})] = \widetilde{p}g(uS_M, uS_M) + \widetilde{q}g(dS_M, dS_M)$. And for $n \geq M+1$, $\widetilde{E}_n[g(S_{n+1}, Y_{n+1})] = \widetilde{E}_n[g(\frac{S_{n+1}}{S_n}S_n, Y_n + \frac{S_{n+1}}{S_n}S_n)] = \widetilde{p}g(uS_n, Y_n + uS_n) + \widetilde{q}g(dS_n, Y_n + dS_n)$, so $(S_n, Y_n)_{0\leq n\leq N}$ is Markov under $\widetilde{P}$. □

(ii)

Proof. Set $v_N(s, y) = f(\frac{y}{N-M})$. Then $v_N(S_N, Y_N) = f(\frac{\sum_{K=M+1}^N S_k}{N-M}) = V_N$. Suppose v_{n+1} is already given.

a) If $n > M$, then $\widetilde{E}_n[v_{n+1}(S_{n+1}, Y_{n+1})] = \widetilde{p}v_{n+1}(uS_n, Y_n + uS_n) + \widetilde{q}v_{n+1}(dS_n, Y_n + dS_n)$. So $v_n(s, y) = \widetilde{p}v_{n+1}(us, y + us) + \widetilde{q}v_{n+1}(ds, y + ds)$.

b) If $n = M$, then $\widetilde{E}_M[v_{M+1}(S_{M+1}, Y_{M+1})] = \widetilde{p}v_{M+1}(uS_M, uS_M) + \widetilde{v}_{n+1}(dS_M, dS_M)$. So $v_M(s) = \widetilde{p}v_{M+1}(us, us) + \widetilde{q}v_{M+1}(ds, ds)$.

c) If $n < M$, then $\widetilde{E}_n[v_{n+1}(S_{n+1})] = \widetilde{p}v_{n+1}(uS_n) + \widetilde{q}v_{n+1}(dS_n)$. So $v_n(s) = \widetilde{p}v_{n+1}(us) + \widetilde{q}v_{n+1}(ds)$. □

1.3 State Prices

3.1.

Proof. Note $\widetilde{Z}(\omega) := \frac{P(\omega)}{\widetilde{P}(\omega)} = \frac{1}{Z(\omega)}$. Apply Theorem 3.1.1 with P, $\widetilde{P}$, Z replaced by $\widetilde{P}$, P, $\widetilde{Z}$, we get the analogous of properties (i)-(iii) of Theorem 3.1.1. □

3.2. (i)

Proof. $\widetilde{P}(\Omega) = \sum_{\omega\in\Omega} \widetilde{P}(\omega) = \sum_{\omega\in\Omega} Z(\omega)P(\omega) = E[Z] = 1$. □

(ii)

Proof. $\widetilde{E}[Y] = \sum_{\omega\in\Omega} Y(\omega)\widetilde{P}(\omega) = \sum_{\omega\in\Omega} Y(\omega)Z(\omega)P(\omega) = E[YZ]$. □

(iii)

Proof. $\widetilde{P}(A) = \sum_{\omega\in A} Z(\omega)P(\omega)$. Since $P(A) = 0$, $P(\omega) = 0$ for any $\omega \in A$. So $\widetilde{P}(A) = 0$. □

(iv)

Proof. If $\widetilde{P}(A) = \sum_{\omega\in A} Z(\omega)P(\omega) = 0$, by $P(Z > 0) = 1$, we conclude $P(\omega) = 0$ for any $\omega \in A$. So $P(A) = \sum_{\omega\in A} P(\omega) = 0$. □

(v)

Proof. $P(A) = 1 \iff P(A^c) = 0 \iff \widetilde{P}(A^c) = 0 \iff \widetilde{P}(A) = 1$. □

(vi)

Proof. Pick ω_0 such that $P(\omega_0) > 0$, define $Z(\omega) = \begin{cases} 0, & \text{if } \omega \neq \omega_0 \\ \frac{1}{P(\omega_0)}, & \text{if } \omega = \omega_0. \end{cases}$ Then $P(Z \geq 0) = 1$ and $E[Z] = \frac{1}{P(\omega_0)} \cdot P(\omega_0) = 1$.

Clearly $\widetilde{P}(\Omega \setminus \{\omega_0\}) = E[Z 1_{\Omega\setminus\{\omega_0\}}] = \sum_{\omega\neq\omega_0} Z(\omega)P(\omega) = 0$. But $P(\Omega \setminus \{\omega_0\}) = 1 - P(\omega_0) > 0$ if $P(\omega_0) < 1$. Hence in the case $0 < P(\omega_0) < 1$, P and $\widetilde{P}$ are not equivalent. If $P(\omega_0) = 1$, then $E[Z] = 1$ if and only if $Z(\omega_0) = 1$. In this case $\widetilde{P}(\omega_0) = Z(\omega_0)P(\omega_0) = 1$. And $\widetilde{P}$ and P have to be equivalent.

In summary, if we can find ω_0 such that $0 < P(\omega_0) < 1$, then Z as constructed above would induce a probability $\widetilde{P}$ that is not equivalent to P. □

3.5. (i)

Proof. $Z(HH) = \frac{9}{16}$, $Z(HT) = \frac{9}{8}$, $Z(TH) = \frac{3}{8}$ and $Z(TT) = \frac{15}{4}$. □

(ii)

Proof. $Z_1(H) = E_1[Z_2](H) = Z_2(HH)P(\omega_2 = H|\omega_1 = H) + Z_2(HT)P(\omega_2 = T|\omega_1 = H) = \frac{3}{4}$. $Z_1(T) = E_1[Z_2](T) = Z_2(TH)P(\omega_2 = H|\omega_1 = T) + Z_2(TT)P(\omega_2 = T|\omega_1 = T) = \frac{3}{2}$. □

(iii)

Proof.

$$V_1(H) = \frac{[Z_2(HH)V_2(HH)P(\omega_2 = H|\omega_1 = H) + Z_2(HT)V_2(HT)P(\omega_2 = T|\omega_1 = T)]}{Z_1(H)(1 + r_1(H))} = 2.4,$$

$$V_1(T) = \frac{[Z_2(TH)V_2(TH)P(\omega_2 = H|\omega_1 = T) + Z_2(TT)V_2(TT)P(\omega_2 = T|\omega_1 = T)]}{Z_1(T)(1 + r_1(T))} = \frac{1}{9},$$

and

$$V_0 = \frac{Z_2(HH)V_2(HH)}{(1+\frac{1}{4})(1+\frac{1}{4})}P(HH) + \frac{Z_2(HT)V_2(HT)}{(1+\frac{1}{4})(1+\frac{1}{4})}P(HT) + \frac{Z_2(TH)V_2(TH)}{(1+\frac{1}{4})(1+\frac{1}{2})}P(TH) + 0 \approx 1.$$

□

3.6.

Proof. $U'(x) = \frac{1}{x}$, so $I(x) = \frac{1}{x}$. (3.3.26) gives $E[\frac{Z}{(1+r)^N}\frac{(1+r)^N}{\lambda Z}] = X_0$. So $\lambda = \frac{1}{X_0}$. By (3.3.25), we have $X_N = \frac{(1+r)^N}{\lambda Z} = \frac{X_0}{Z}(1+r)^N$. Hence $X_n = \widetilde{E}_n[\frac{X_N}{(1+r)^{N-n}}] = \widetilde{E}_n[\frac{X_0(1+r)^n}{Z}] = X_0(1+r)^n\widetilde{E}_n[\frac{1}{Z}] = X_0(1+r)^n\frac{1}{Z_n}E_n[Z\cdot\frac{1}{Z}] = \frac{X_0}{\xi_n}$, where the second to last "=" comes from Lemma 3.2.6. □

3.7.

Proof. $U'(x) = x^{p-1}$ and so $I(x) = x^{\frac{1}{p-1}}$. By (3.3.26), we have $E[\frac{Z}{(1+r)^N}(\frac{\lambda Z}{(1+r)^N})^{\frac{1}{p-1}}] = X_0$. Solve it for λ, we get

$$\lambda = \left(\frac{X_0}{E\left[\frac{Z^{\frac{p}{p-1}}}{(1+r)^{\frac{Np}{p-1}}}\right]}\right)^{p-1} = \frac{X_0^{p-1}(1+r)^{Np}}{(E[Z^{\frac{p}{p-1}}])^{p-1}}.$$

So by (3.3.25), $X_N = (\frac{\lambda Z}{(1+r)^N})^{\frac{1}{p-1}} = \frac{\lambda^{\frac{1}{p-1}}Z^{\frac{1}{p-1}}}{(1+r)^{\frac{N}{p-1}}} = \frac{X_0(1+r)^{\frac{Np}{p-1}}}{E[Z^{\frac{p}{p-1}}]}\frac{Z^{\frac{1}{p-1}}}{(1+r)^{\frac{N}{p-1}}} = \frac{(1+r)^N X_0 Z^{\frac{1}{p-1}}}{E[Z^{\frac{p}{p-1}}]}$. □

3.8. (i)

Proof. $\frac{d}{dx}(U(x) - yx) = U'(x) - y$. So $x = I(y)$ is an extreme point of $U(x) - yx$. Because $\frac{d^2}{dx^2}(U(x) - yx) = U''(x) \leq 0$ (U is concave), $x = I(y)$ is a maximum point. Therefore $U(x) - y(x) \leq U(I(y)) - yI(y)$ for every x. □

(ii)

Proof. Following the hint of the problem, we have

$$E[U(X_N)] - E[X_N \frac{\lambda Z}{(1+r)^N}] \le E[U(I(\frac{\lambda Z}{(1+r)^N}))] - E[\frac{\lambda Z}{(1+r)^N} I(\frac{\lambda Z}{(1+r)^N})],$$

i.e. $E[U(X_N)] - \lambda X_0 \le E[U(X_N^*)] - \widetilde{E}[\frac{\lambda}{(1+r)^N} X_N^*] = E[U(X_N^*)] - \lambda X_0$. So $E[U(X_N)] \le E[U(X_N^*)]$. $\square$

3.9. (i)

Proof. $X_n = \widetilde{E}_n[\frac{X_N}{(1+r)^{N-n}}]$. So if $X_N \ge 0$, then $X_n \ge 0$ for all n. $\square$

(ii)

Proof. a) If $0 \le x < \gamma$ and $0 < y \le \frac{1}{\gamma}$, then $U(x) - yx = -yx \le 0$ and $U(I(y)) - yI(y) = U(\gamma) - y\gamma = 1 - y\gamma \ge 0$. So $U(x) - yx \le U(I(y)) - yI(y)$.

b) If $0 \le x < \gamma$ and $y > \frac{1}{\gamma}$, then $U(x) - yx = -yx \le 0$ and $U(I(y)) - yI(y) = U(0) - y \cdot 0 = 0$. So $U(x) - yx \le U(I(y)) - yI(y)$.

c) If $x \ge \gamma$ and $0 < y \le \frac{1}{\gamma}$, then $U(x) - yx = 1 - yx$ and $U(I(y)) - yI(y) = U(\gamma) - y\gamma = 1 - y\gamma \ge 1 - yx$. So $U(x) - yx \le U(I(y)) - yI(y)$.

d) If $x \ge \gamma$ and $y > \frac{1}{\gamma}$, then $U(x) - yx = 1 - yx < 0$ and $U(I(y)) - yI(y) = U(0) - y \cdot 0 = 0$. So $U(x) - yx \le U(I(y)) - yI(y)$. $\square$

(iii)

Proof. Using (ii) and set $x = X_N$, $y = \frac{\lambda Z}{(1+r)^N}$, where X_N is a random variable satisfying $\widetilde{E}[\frac{X_N}{(1+r)^N}] = X_0$, we have

$$E[U(X_N)] - E[\frac{\lambda Z}{(1+r)^N} X_N] \le E[U(X_N^*)] - E[\frac{\lambda Z}{(1+r)^N} X_N^*].$$

That is, $E[U(X_N)] - \lambda X_0 \le E[U(X_N^*)] - \lambda X_0$. So $E[U(X_N)] \le E[U(X_N^*)]$. $\square$

(iv)

Proof. Plug p_m and ξ_m into (3.6.4), we have

$$X_0 = \sum_{m=1}^{2^N} p_m \xi_m I(\lambda \xi_m) = \sum_{m=1}^{2^N} p_m \xi_m \gamma 1_{\{\lambda \xi_m \le \frac{1}{\gamma}\}}.$$

So $\frac{X_0}{\gamma} = \sum_{m=1}^{2^N} p_m \xi_m 1_{\{\lambda \xi_m \le \frac{1}{\gamma}\}}$. Suppose there is a solution λ to (3.6.4), note $\frac{X_0}{\gamma} > 0$, we then can conclude $\{m : \lambda \xi_m \le \frac{1}{\gamma}\} \neq \emptyset$. Let $K = \max\{m : \lambda \xi_m \le \frac{1}{\gamma}\}$, then $\lambda \xi_K \le \frac{1}{\gamma} < \lambda \xi_{K+1}$. So $\xi_K < \xi_{K+1}$ and $\frac{X_0}{\gamma} = \sum_{m=1}^K p_m \xi_m$ (Note, however, that K could be 2^N. In this case, ξ_{K+1} is interpreted as ∞. Also, note we are looking for positive solution $\lambda > 0$). Conversely, suppose there exists some K so that $\xi_K < \xi_{K+1}$ and $\sum_{m=1}^K \xi_m p_m = \frac{X_0}{\gamma}$. Then we can find $\lambda > 0$, such that $\xi_K < \frac{1}{\lambda\gamma} < \xi_{K+1}$. For such λ, we have

$$E[\frac{Z}{(1+r)^N} I(\frac{\lambda Z}{(1+r)^N})] = \sum_{m=1}^{2^N} p_m \xi_m 1_{\{\lambda \xi_m \le \frac{1}{\gamma}\}} \gamma = \sum_{m=1}^{K} p_m \xi_m \gamma = X_0.$$

Hence (3.6.4) has a solution. $\square$

(v)

Proof. $X_N^*(\omega^m) = I(\lambda \xi_m) = \gamma 1_{\{\lambda \xi_m \le \frac{1}{\gamma}\}} = \begin{cases} \gamma, & \text{if } m \le K \\ 0, & \text{if } m \ge K+1 \end{cases}.$ $\square$

1.4 American Derivative Securities

Before proceeding to the exercise problems, we first give a brief summary of pricing American derivative securities as presented in the textbook. We shall use the notation of the book.

From the buyer's perspective: At time n, if the derivative security has not been exercised, then the buyer can choose a policy τ with $\tau \in \mathcal{S}_n$. The valuation formula for cash flow (Theorem 2.4.8) gives a fair price for the derivative security exercised according to τ:

$$V_n(\tau) = \sum_{k=n}^{N} \widetilde{E}_n\left[1_{\{\tau=k\}}\frac{1}{(1+r)^{k-n}}G_k\right] = \widetilde{E}_n\left[1_{\{\tau\le N\}}\frac{1}{(1+r)^{\tau-n}}G_\tau\right].$$

The buyer wants to consider all the possible τ's, so that he can find the least upper bound of security value, which will be the maximum price of the derivative security acceptable to him. This is the price given by Definition 4.4.1: $V_n = \max_{\tau\in\mathcal{S}_n}\widetilde{E}_n[1_{\{\tau\le N\}}\frac{1}{(1+r)^{\tau-n}}G_\tau]$.

From the seller's perspective: A price process $(V_n)_{0\le n\le N}$ is acceptable to him if and only if at time n, he can construct a portfolio at cost V_n so that (i) $V_n \ge G_n$ and (ii) he needs no further investing into the portfolio as time goes by. Formally, the seller can find $(\Delta_n)_{0\le n\le N}$ and $(C_n)_{0\le n\le N}$ so that $C_n \ge 0$ and $V_{n+1} = \Delta_n S_{n+1} + (1+r)(V_n - C_n - \Delta_n S_n)$. Since $(\frac{S_n}{(1+r)^n})_{0\le n\le N}$ is a martingale under the risk-neutral measure $\widetilde{P}$, we conclude

$$\widetilde{E}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right] - \frac{V_n}{(1+r)^n} = -\frac{C_n}{(1+r)^n} \le 0,$$

i.e. $(\frac{V_n}{(1+r)^n})_{0\le n\le N}$ is a supermartingale. This inspired us to check if the converse is also true. This is exactly the content of Theorem 4.4.4. So $(V_n)_{0\le n\le N}$ is the value process of a portfolio that needs no further investing if and only if $\left(\frac{V_n}{(1+r)^n}\right)_{0\le n\le N}$ is a supermartingale under $\widetilde{P}$ (note this is independent of the requirement $V_n \ge G_n$). In summary, a price process $(V_n)_{0\le n\le N}$ is acceptable to the seller if and only if (i) $V_n \ge G_n$; (ii) $\left(\frac{V_n}{(1+r)^n}\right)_{0\le n\le N}$ is a supermartingale under $\widetilde{P}$.

Theorem 4.4.2 shows the buyer's upper bound is the seller's lower bound. So it gives the price acceptable to both. Theorem 4.4.3 gives a specific algorithm for calculating the price, Theorem 4.4.4 establishes the one-to-one correspondence between super-replication and supermartingale property, and finally, Theorem 4.4.5 shows how to decide on the optimal exercise policy.

4.1. (i)

Proof. $V_2^P(HH) = 0$, $V_2^P(HT) = V_2^P(TH) = 0.8$, $V_2^P(TT) = 3$, $V_1^P(H) = 0.32$, $V_1^P(T) = 2$, $V_0^P = 9.28$. □

(ii)

Proof. $V_0^C = 5$. □

(iii)

Proof. $g_S(s) = |4-s|$. We apply Theorem 4.4.3 and have $V_2^S(HH) = 12.8$, $V_2^S(HT) = V_2^S(TH) = 2.4$, $V_2^S(TT) = 3$, $V_1^S(H) = 6.08$, $V_1^S(T) = 2.16$ and $V_0^S = 3.296$. □

(iv)

Proof. First, we note the simple inequality

$$\max(a_1, b_1) + \max(a_2, b_2) \ge \max(a_1 + a_2, b_1 + b_2).$$

“$>$” holds if and only if $b_1 > a_1$, $b_2 < a_2$ or $b_1 < a_1$, $b_2 > a_2$. By induction, we can show

$$\begin{aligned}
V_n^S &= \max\left\{g_S(S_n), \frac{\widetilde{p}V_{n+1}^S + \widetilde{V}_{n+1}^S}{1+r}\right\} \\
&\leq \max\left\{g_P(S_n) + g_C(S_n), \frac{\widetilde{p}V_{n+1}^P + \widetilde{V}_{n+1}^P}{1+r} + \frac{\widetilde{p}V_{n+1}^C + \widetilde{V}_{n+1}^C}{1+r}\right\} \\
&\leq \max\left\{g_P(S_n), \frac{\widetilde{p}V_{n+1}^P + \widetilde{V}_{n+1}^P}{1+r}\right\} + \max\left\{g_C(S_n), \frac{\widetilde{p}V_{n+1}^C + \widetilde{V}_{n+1}^C}{1+r}\right\} \\
&= V_n^P + V_n^C.
\end{aligned}$$

As to when “$<$” holds, suppose $m = \max\{n : V_n^S < V_n^P + V_n^C\}$. Then clearly $m \leq N-1$ and it is possible that $\{n : V_n^S < V_n^P + V_n^C\} = \emptyset$. When this set is not empty, m is characterized as $m = \max\{n : g_P(S_n) < \frac{\widetilde{p}V_{n+1}^P + \widetilde{q}V_{n+1}^P}{1+r}$ and $g_C(S_n) > \frac{\widetilde{p}V_{n+1}^C + \widetilde{q}V_{n+1}^C}{1+r}$ *or* $g_P(S_n) > \frac{\widetilde{p}V_{n+1}^P + \widetilde{q}V_{n+1}^P}{1+r}$ and $g_C(S_n) < \frac{\widetilde{p}V_{n+1}^C + \widetilde{q}V_{n+1}^C}{1+r}\}$. □

4.2.

Proof. For this problem, we need Figure 4.2.1, Figure 4.4.1 and Figure 4.4.2. Then

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = -\frac{1}{12},\ \Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = -1,$$

and

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \approx -0.433.$$

The optimal exercise time is $\tau = \inf\{n : V_n = G_n\}$. So

$$\tau(HH) = \infty,\ \tau(HT) = 2,\ \tau(TH) = \tau(TT) = 1.$$

Therefore, the agent borrows 1.36 at time zero and buys the put. At the same time, to hedge the long position, he needs to borrow again and buy 0.433 shares of stock at time zero.

At time one, if the result of coin toss is tail and the stock price goes down to 2, the value of the portfolio is $X_1(T) = (1+r)(-1.36 - 0.433S_0) + 0.433S_1(T) = (1+\frac{1}{4})(-1.36 - 0.433\times 4) + 0.433\times 2 = -3$. The agent should exercise the put at time one and get 3 to pay off his debt.

At time one, if the result of coin toss is head and the stock price goes up to 8, the value of the portfolio is $X_1(H) = (1+r)(-1.36 - 0.433S_0) + 0.433S_1(H) = -0.4$. The agent should borrow to buy $\frac{1}{12}$ shares of stock. At time two, if the result of coin toss is head and the stock price goes up to 16, the value of the portfolio is $X_2(HH) = (1+r)(X_1(H) - \frac{1}{12}S_1(H)) + \frac{1}{12}S_2(HH) = 0$, and the agent should let the put expire. If at time two, the result of coin toss is tail and the stock price goes down to 4, the value of the portfolio is $X_2(HT) = (1+r)(X_1(H) - \frac{1}{12}S_1(H)) + \frac{1}{12}S_2(HT) = -1$. The agent should exercise the put to get 1. This will pay off his debt. □

4.3.

Proof. We need Figure 1.2.2 for this problem, and calculate the intrinsic value process and price process of the put as follows.

For the intrinsic value process, $G_0 = 0$, $G_1(T) = 1$, $G_2(TH) = \frac{2}{3}$, $G_2(TT) = \frac{5}{3}$, $G_3(THT) = 1$, $G_3(TTH) = 1.75$, $G_3(TTT) = 2.125$. All the other outcomes of G is negative.

For the price process, $V_0 = 0.4$, $V_1(T) = 1$, $V_1(TH) = \frac{2}{3}$, $V_1(TT) = \frac{5}{3}$, $V_3(THT) = 1$, $V_3(TTH) = 1.75$, $V_3(TTT) = 2.125$. All the other outcomes of V is zero.

Therefore the time-zero price of the derivative security is 0.4 and the optimal exercise time satisfies

$$\tau(\omega) = \begin{cases} \infty & \text{if } \omega_1 = H, \\ 1 & \text{if } \omega_1 = T. \end{cases}$$

□

4.4.

Proof. 1.36 is the cost of super-replicating the American derivative security. It enables us to construct a portfolio sufficient to pay off the derivative security, no matter when the derivative security is exercised. So to hedge our short position after selling the put, there is no need to charge the insider more than 1.36. $\square$

4.5.

Proof. The stopping times in $\mathcal{S}_0$ are
(1) $\tau \equiv 0$;
(2) $\tau \equiv 1$;
(3) $\tau(HT) = \tau(HH) = 1$, $\tau(TH)$, $\tau(TT) \in \{2, \infty\}$ (4 different ones);
(4) $\tau(HT)$, $\tau(HH) \in \{2, \infty\}$, $\tau(TH) = \tau(TT) = 1$ (4 different ones);
(5) $\tau(HT), \tau(HH), \tau(TH), \tau(TT) \in \{2, \infty\}$ (16 different ones).

When the option is out of money, the following stopping times do not exercise
(i) $\tau \equiv 0$;
(ii) $\tau(HT) \in \{2, \infty\}$, $\tau(HH) = \infty$, $\tau(TH), \tau(TT) \in \{2, \infty\}$ (8 different ones);
(iii) $\tau(HT) \in \{2, \infty\}$, $\tau(HH) = \infty$, $\tau(TH) = \tau(TT) = 1$ (2 different ones).

For (i), $\widetilde{E}[1_{\{\tau \le 2\}}(\frac{4}{5})^\tau G_\tau] = G_0 = 1$. For (ii), $\widetilde{E}[1_{\{\tau \le 2\}}(\frac{4}{5})^\tau G_\tau] \le \widetilde{E}[1_{\{\tau^* \le 2\}}(\frac{4}{5})^{\tau^*} G_{\tau^*}]$, where $\tau^*(HT) = 2$, $\tau^*(HH) = \infty$, $\tau^*(TH) = \tau^*(TT) = 2$. So $\widetilde{E}[1_{\{\tau^* \le 2\}}(\frac{4}{5})^{\tau^*} G_{\tau^*}] = \frac{1}{4}[(\frac{4}{5})^2 \cdot 1 + (\frac{4}{5})^2(1+4)] = 0.96$. For (iii), $\widetilde{E}[1_{\{\tau \le 2\}}(\frac{4}{5})^\tau G_\tau]$ has the biggest value when τ satisfies $\tau(HT) = 2$, $\tau(HH) = \infty$, $\tau(TH) = \tau(TT) = 1$. This value is 1.36. $\square$

4.6. (i)

Proof. The value of the put at time N, if it is not exercised at previous times, is $K - S_N$. Hence $V_{N-1} = \max\{K - S_{N-1}, \widetilde{E}_{N-1}[\frac{V_N}{1+r}]\} = \max\{K - S_{N-1}, \frac{K}{1+r} - S_{N-1}\} = K - S_{N-1}$. The second equality comes from the fact that discounted stock price process is a martingale under risk-neutral probability. By induction, we can show $V_n = K - S_n$ $(0 \le n \le N)$. So by Theorem 4.4.5, the optimal exercise policy is to sell the stock at time zero and the value of this derivative security is $K - S_0$.

Remark 2. *We cheated a little bit by using American algorithm and Theorem 4.4.5, since they are developed for the case where τ is allowed to be ∞. But intuitively, results in this chapter should still hold for the case $\tau \le N$, provided we replace "$\max\{G_n, 0\}$" with "G_n".*

$\square$

(ii)

Proof. This is because at time N, if we have to exercise the put and $K - S_N < 0$, we can exercise the European call to set off the negative payoff. In effect, throughout the portfolio's lifetime, the portfolio has intrinsic values greater than that of an American put stuck at K with expiration time N. So, we must have $V_0^{AP} \le V_0 + V_0^{EC} \le K - S_0 + V_0^{EC}$. $\square$

(iii)

Proof. Let V_0^{EP} denote the time-zero value of a European put with strike K and expiration time N. Then

$$V_0^{AP} \ge V_0^{EP} = V_0^{EC} - \widetilde{E}[\frac{S_N - K}{(1+r)^N}] = V_0^{EC} - S_0 + \frac{K}{(1+r)^N}.$$

$\square$

4.7.

Proof. $V_N = S_N - K$, $V_{N-1} = \max\{S_{N-1} - K, \widetilde{E}_{N-1}[\frac{V_N}{1+r}]\} = \max\{S_{N-1} - K, S_{N-1} - \frac{K}{1+r}\} = S_{N-1} - \frac{K}{1+r}$. By induction, we can prove $V_n = S_n - \frac{K}{(1+r)^{N-n}}$ $(0 \le n \le N)$ and $V_n > G_n$ for $0 \le n \le N-1$. So the time-zero value is $S_0 - \frac{K}{(1+r)^N}$ and the optimal exercise time is N. $\square$

1.5 Random Walk

5.1. (i)

Proof. $E[\alpha^{\tau_2}] = E[\alpha^{(\tau_2-\tau_1)+\tau_1}] = E[\alpha^{(\tau_2-\tau_1)}]E[\alpha^{\tau_1}] = E[\alpha^{\tau_1}]^2$. $\square$

(ii)

Proof. If we define $M_n^{(m)} = M_{n+\tau_m} - M_{\tau_m}$ $(m = 1, 2, \cdots)$, then $(M_\cdot^{(m)})_m$ as random functions are i.i.d. with distributions the same as that of M. So $\tau_{m+1} - \tau_m = \inf\{n : M_n^{(m)} = 1\}$ are i.i.d. with distributions the same as that of τ_1. Therefore

$$E[\alpha^{\tau_m}] = E[\alpha^{(\tau_m-\tau_{m-1})+(\tau_{m-1}-\tau_{m-2})+\cdots+\tau_1}] = E[\alpha^{\tau_1}]^m.$$

$\square$

(iii)

Proof. Yes, since the argument of (ii) still works for asymmetric random walk. $\square$

5.2. (i)

Proof. $f'(\sigma) = pe^\sigma - qe^{-\sigma}$, so $f'(\sigma) > 0$ if and only if $\sigma > \frac{1}{2}(\ln q - \ln p)$. Since $\frac{1}{2}(\ln q - \ln p) < 0$, $f(\sigma) > f(0) = 1$ for all $\sigma > 0$. $\square$

(ii)

Proof. $E_n[\frac{S_{n+1}}{S_n}] = E_n[e^{\sigma X_{n+1}}\frac{1}{f(\sigma)}] = pe^\sigma\frac{1}{f(\sigma)} + qe^{-\sigma}\frac{1}{f(\sigma)} = 1$. $\square$

(iii)

Proof. By optional stopping theorem, $E[S_{n\wedge\tau_1}] = E[S_0] = 1$. Note $S_{n\wedge\tau_1} = e^{\sigma M_{n\wedge\tau_1}}(\frac{1}{f(\sigma)})^{n\wedge\tau_1} \le e^{\sigma\cdot 1}$, by bounded convergence theorem, $E[1_{\{\tau_1<\infty\}}S_{\tau_1}] = E[\lim_{n\to\infty} S_{n\wedge\tau_1}] = \lim_{n\to\infty} E[S_{n\wedge\tau_1}] = 1$, that is, $E[1_{\{\tau_1<\infty\}}e^\sigma(\frac{1}{f(\sigma)})^{\tau_1}] = 1$. So $e^{-\sigma} = E[1_{\{\tau_1<\infty\}}(\frac{1}{f(\sigma)})^{\tau_1}]$. Let $\sigma \downarrow 0$, again by bounded convergence theorem, $1 = E[1_{\{\tau_1<\infty\}}(\frac{1}{f(0)})^{\tau_1}] = P(\tau_1 < \infty)$. $\square$

(iv)

Proof. Set $\alpha = \frac{1}{f(\sigma)} = \frac{1}{pe^\sigma+qe^{-\sigma}}$, then as σ varies from 0 to ∞, α can take all the values in $(0, 1)$. Write σ in terms of α, we have $e^\sigma = \frac{1\pm\sqrt{1-4pq\alpha^2}}{2p\alpha}$ (note $4pq\alpha^2 < 4(\frac{p+q}{2})^2 \cdot 1^2 = 1$). We want to choose $\sigma > 0$, so we should take $\sigma = \ln(\frac{1+\sqrt{1-4pq\alpha^2}}{2p\alpha})$. Therefore $E[\alpha^{\tau_1}] = \frac{2p\alpha}{1+\sqrt{1-4pq\alpha^2}} = \frac{1-\sqrt{1-4pq\alpha^2}}{2q\alpha}$. $\square$

(v)

Proof. $\frac{\partial}{\partial\alpha}E[\alpha^{\tau_1}] = E[\frac{\partial}{\partial\alpha}\alpha^{\tau_1}] = E[\tau_1\alpha^{\tau_1-1}]$, and

$$\begin{aligned}
&\left(\frac{1-\sqrt{1-4pq\alpha^2}}{2q\alpha}\right)' \\
=\ & \frac{1}{2q}\left[(1-\sqrt{1-4pq\alpha^2})\alpha^{-1}\right]' \\
=\ & \frac{1}{2q}[-\frac{1}{2}(1-4pq\alpha^2)^{-\frac{1}{2}}(-4pq2\alpha)\alpha^{-1} + (1-\sqrt{1-4pq\alpha^2})(-1)\alpha^2].
\end{aligned}$$

So $E[\tau_1] = \lim_{\alpha\uparrow 1}\frac{\partial}{\partial\alpha}E[\alpha^{\tau_1}] = \frac{1}{2q}[-\frac{1}{2}(1-4pq)^{-\frac{1}{2}}(-8pq) - (1-\sqrt{1-4pq})] = \frac{1}{2p-1}$. $\square$

5.3. (i)

Proof. Solve the equation $pe^{\sigma}+qe^{-\sigma}=1$ and a positive solution is $\ln\frac{1+\sqrt{1-4pq}}{2p}=\ln\frac{1-p}{p}=\ln q-\ln p$. Set $\sigma_0=\ln q-\ln p$, then $f(\sigma_0)=1$ and $f'(\sigma)>0$ for $\sigma>\sigma_0$. So $f(\sigma)>1$ for all $\sigma>\sigma_0$. $\square$

(ii)

Proof. As in Exercise 5.2, $S_n=e^{\sigma M_n}(\frac{1}{f(\sigma)})^n$ is a martingale, and $1=E[S_0]=E[S_{n\wedge\tau_1}]=E[e^{\sigma M_{n\wedge\tau_1}}(\frac{1}{f(\sigma)})^{\tau_1\wedge n}]$. Suppose $\sigma>\sigma_0$, then by bounded convergence theorem,

$$1=E[\lim_{n\to\infty}e^{\sigma M_{n\wedge\tau_1}}(\frac{1}{f(\sigma)})^{n\wedge\tau_1}]=E[1_{\{\tau_1<\infty\}}e^{\sigma}(\frac{1}{f(\sigma)})^{\tau_1}].$$

Let $\sigma\downarrow\sigma_0$, we get $P(\tau_1<\infty)=e^{-\sigma_0}=\frac{p}{q}<1$. $\square$

(iii)

Proof. From (ii), we can see $E[1_{\{\tau_1<\infty\}}(\frac{1}{f(\sigma)})^{\tau_1}]=e^{-\sigma}$, for $\sigma>\sigma_0$. Set $\alpha=\frac{1}{f(\alpha)}$, then $e^{\sigma}=\frac{1\pm\sqrt{1-4pq\alpha^2}}{2p\alpha}$. We need to choose the root so that $e^{\sigma}>e^{\sigma_0}=\frac{q}{p}$, so $\sigma=\ln(\frac{1+\sqrt{1-4pq\alpha^2}}{2p\alpha})$, then $E[\alpha^{\tau_1}1_{\{\tau_1<\infty\}}]=\frac{1-\sqrt{1-4pq\alpha^2}}{2q\alpha}$. $\square$

(iv)

Proof. $E[\tau_1 1_{\{\tau_1<\infty\}}]=\frac{\partial}{\partial\alpha}E[\alpha^{\tau_1}1_{\{\tau_1<\infty\}}]|_{\alpha=1}=\frac{1}{2q}[\frac{4pq}{\sqrt{1-4pq}}-(1-\sqrt{1-4pq})]=\frac{1}{2q}[\frac{4pq}{2q-1}-1+2q-1]=\frac{p}{q}\frac{1}{2q-1}$. $\square$

5.4. (i)

Proof. $E[\alpha^{\tau_2}]=\sum_{k=1}^{\infty}P(\tau_2=2k)\alpha^{2k}=\sum_{k=1}^{\infty}(\frac{\alpha}{2})^{2k}P(\tau_2=2k)4^k$. So $P(\tau_2=2k)=\frac{(2k)!}{4^k(k+1)!k!}$. $\square$

(ii)

Proof. $P(\tau_2=2)=\frac{1}{4}$. For $k\geq 2$, $P(\tau_2=2k)=P(\tau_2\leq 2k)-P(\tau_2\leq 2k-2)$.

$$\begin{aligned}P(\tau_2\leq 2k) &= P(M_{2k}=2)+P(M_{2k}\geq 4)+P(\tau_2\leq 2k, M_{2k}\leq 0)\\ &= P(M_{2k}=2)+2P(M_{2k}\geq 4)\\ &= P(M_{2k}=2)+P(M_{2k}\geq 4)+P(M_{2k}\leq -4)\\ &= 1-P(M_{2k}=-2)-P(M_{2k}=0).\end{aligned}$$

Similarly, $P(\tau_2\leq 2k-2)=1-P(M_{2k-2}=-2)-P(M_{2k-2}=0)$. So

$$\begin{aligned}P(\tau_2=2k) &= P(M_{2k-2}=-2)+P(M_{2k-2}=0)-P(M_{2k}=-2)-P(M_{2k}=0)\\ &= (\frac{1}{2})^{2k-2}\left[\frac{(2k-2)!}{k!(k-2)!}+\frac{(2k-2)!}{(k-1)!(k-1)!}\right]-(\frac{1}{2})^{2k}\left[\frac{(2k)!}{(k+1)!(k-1)!}+\frac{(2k)!}{k!k!}\right]\\ &= \frac{(2k)!}{4^k(k+1)!k!}\left[\frac{4}{2k(2k-1)}(k+1)k(k-1)+\frac{4}{2k(2k-1)}(k+1)k^2-k-(k+1)\right]\\ &= \frac{(2k)!}{4^k(k+1)!k!}\left[\frac{2(k^2-1)}{2k-1}+\frac{2(k^2+k)}{2k-1}-\frac{4k^2-1}{2k-1}\right]\\ &= \frac{(2k)!}{4^k(k+1)!k!}.\end{aligned}$$

$\square$

5.5. (i)

Proof. For every path that crosses level m by time n and resides at b at time n, there corresponds a reflected path that resides at time $2m-b$. So

$$P(M_n^* \geq m,\ M_n = b) = P(M_n = 2m-b) = (\frac{1}{2})^n \frac{n!}{(m+\frac{n-b}{2})!(\frac{n+b}{2}-m)!}.$$

□

(ii)

Proof.

$$P(M_n^* \geq m,\ M_n = b) = \frac{n!}{(m+\frac{n-b}{2})!(\frac{n+b}{2}-m)!} p^{m+\frac{n-b}{2}} q^{\frac{n+b}{2}-m}.$$

□

5.6.

Proof. On the infinite coin-toss space, we define $M_n = \{$stopping times that takes values $0, 1, \cdots, n, \infty\}$ and $M_\infty = \{$stopping times that takes values $0, 1, 2, \cdots\}$. Then the time-zero value V^* of the perpetual American put as in Section 5.4 can be defined as $\sup_{\tau \in M_\infty} \widetilde{E}[1_{\{\tau<\infty\}} \frac{(K-S_\tau)^+}{(1+r)^\tau}]$. For an American put with the same strike price K that expires at time n, its time-zero value $V^{(n)}$ is $\max_{\tau \in M_n} \widetilde{E}[1_{\{\tau<\infty\}} \frac{(K-S_\tau)^+}{(1+r)^\tau}]$. Clearly $(V^{(n)})_{n\geq 0}$ is nondecreasing and $V^{(n)} \leq V^*$ for every n. So $\lim_n V^{(n)}$ exists and $\lim_n V^{(n)} \leq V^*$.

For any given $\tau \in M_\infty$, we define $\tau^{(n)} = \begin{cases} \infty, & \text{if } \tau = \infty \\ \tau \wedge n, & \text{if } \tau < \infty \end{cases}$, then $\tau^{(n)}$ is also a stopping time, $\tau^{(n)} \in M_n$ and $\lim_{n\to\infty} \tau^{(n)} = \tau$. By bounded convergence theorem,

$$\widetilde{E}\left[1_{\{\tau<\infty\}} \frac{(K-S_\tau)^+}{(1+r)^\tau}\right] = \lim_{n\to\infty} \widetilde{E}\left[1_{\{\tau^{(n)}<\infty\}} \frac{(K-S_{\tau^{(n)}})^+}{(1+r)^{\tau^{(n)}}}\right] \leq \lim_{n\to\infty} V^{(n)}.$$

Take sup at the left hand side of the inequality, we get $V^* \leq \lim_{n\to\infty} V^{(n)}$. Therefore $V^* = \lim_n V^{(n)}$.

Remark: In the above proof, rigorously speaking, we should use $(K-S_\tau)$ in the places of $(K-S_\tau)^+$. So this needs some justification.

□

5.8. (i)

Proof. $v(S_n) = S_n \geq S_n - K = g(S_n)$. Under risk-neutral probabilities, $\frac{1}{(1+r)^n} v(S_n) = \frac{S_n}{(1+r)^n}$ is a martingale by Theorem 2.4.4. □

(ii)

Proof. If the purchaser chooses to exercises the call at time n, then the discounted risk-neutral expectation of her payoff is $\widetilde{E}\left[\frac{S_n-K}{(1+r)^n}\right] = S_0 - \frac{K}{(1+r)^n}$. Since $\lim_{n\to\infty}\left[S_0 - \frac{K}{(1+r)^n}\right] = S_0$, the value of the call at time zero is at least $\sup_n \left[S_0 - \frac{K}{(1+r)^n}\right] = S_0$. □

(iii)

Proof. $\max\left\{g(s), \frac{\widetilde{p}v(us)+\widetilde{q}v(ds)}{1+r}\right\} = \max\{s-K, \frac{\widetilde{p}u+\widetilde{q}v}{1+r}s\} = \max\{s-K, s\} = s = v(s)$, so equation (5.4.16) is satisfied. Clearly $v(s) = s$ also satisfies the boundary condition (5.4.18). □

(iv)

Proof. Suppose τ is an optimal exercise time, then $\widetilde{E}\left[\frac{S_\tau - K}{(1+r)^\tau}1_{\{\tau<\infty\}}\right] \geq S_0$. Then $P(\tau < \infty) \neq 0$ and $\widetilde{E}\left[\frac{K}{(1+r)^\tau}1_{\{\tau<\infty\}}\right] > 0$. So $\widetilde{E}\left[\frac{S_\tau - K}{(1+r)^\tau}1_{\{\tau<\infty\}}\right] < \widetilde{E}\left[\frac{S_\tau}{(1+r)^\tau}1_{\{\tau<\infty\}}\right]$. Since $\left(\frac{S_n}{(1+r)^n}\right)_{n\geq 0}$ is a martingale under risk-neutral measure, by Fatou's lemma, $\widetilde{E}\left[\frac{S_\tau}{(1+r)^\tau}1_{\{\tau<\infty\}}\right] \leq \liminf_{n\to\infty}\widetilde{E}\left[\frac{S_{\tau\wedge n}}{(1+r)^{\tau\wedge n}}1_{\{\tau<\infty\}}\right] = \liminf_{n\to\infty}\widetilde{E}\left[\frac{S_{\tau\wedge n}}{(1+r)^{\tau\wedge n}}\right] = \liminf_{n\to\infty}\widetilde{E}[S_0] = S_0$. Combined, we have $S_0 \leq \widetilde{E}\left[\frac{S_\tau - K}{(1+r)^\tau}1_{\{\tau<\infty\}}\right] < S_0$. Contradiction. So there is no optimal time to exercise the perpetual American call. Simultaneously, we have shown $\widetilde{E}\left[\frac{S_\tau - K}{(1+r)^\tau}1_{\{\tau<\infty\}}\right] < S_0$ for any stopping time τ. Combined with (ii), we conclude S_0 is the least upper bound for all the prices acceptable to the buyer. □

5.9. (i)

Proof. Suppose $v(s) = s^p$, then we have $s^p = \frac{2}{5}2^p s^p + \frac{2}{5}\frac{s^p}{2^p}$. So $1 = \frac{2^{p+1}}{5} + \frac{2^{1-p}}{5}$. Solve it for p, we get $p = 1$ or $p = -1$. □

(ii)

Proof. Since $\lim_{s\to\infty} v(s) = \lim_{s\to\infty}(As + \frac{B}{s}) = 0$, we must have $A = 0$. □

(iii)

Proof. $f_B(s) = 0$ if and only if $B + s^2 - 4s = 0$. The discriminant $\Delta = (-4)^2 - 4B = 4(4-B)$. So for $B \leq 4$, the equation has roots and for $B > 4$, this equation does not have roots. □

(iv)

Proof. Suppose $B \leq 4$, then the equation $s^2 - 4s + B = 0$ has solution $2 \pm \sqrt{4-B}$. By drawing graphs of $4 - s$ and $\frac{B}{s}$, we should choose $B = 4$ and $s_B = 2 + \sqrt{4-B} = 2$. □

(v)

Proof. To have continuous derivative, we must have $-1 = -\frac{B}{s_B^2}$. Plug $B = s_B^2$ back into $s_B^2 - 4s_B + B = 0$, we get $s_B = 2$. This gives $B = 4$. □

1.6 Interest-Rate-Dependent Assets

6.2.

Proof. $X_k = S_k - E_k[D_m(S_m - K)]D_k^{-1} - \frac{S_n}{B_{n,m}}B_{k,m}$ for $n \leq k \leq m$. Then

$$\begin{aligned}
E_{k-1}[D_kX_k] &= E_{k-1}[D_kS_k - E_k[D_m(S_m-K)] - \frac{S_n}{B_{n,m}}B_{k,m}D_k] \\
&= D_{k-1}S_{k-1} - E_{k-1}[D_m(S_m-K)] - \frac{S_n}{B_{n,m}}E_{k-1}[E_k[D_m]] \\
&= D_{k-1}[S_{k-1} - E_{k-1}[D_m(S_m-K)]D_{k-1}^{-1} - \frac{S_n}{B_{n,m}}B_{k-1,m}] \\
&= D_{k-1}X_{k-1}.
\end{aligned}$$

□

6.3.

Proof.

$$\frac{1}{D_n}\widetilde{E}_n[D_{m+1}R_m] = \frac{1}{D_n}\widetilde{E}_n[D_m(1+R_m)^{-1}R_m] = \widetilde{E}_n[\frac{D_m - D_{m+1}}{D_n}] = B_{n,m} - B_{n,m+1}.$$

□

6.4.(i)

Proof. $D_1V_1 = E_1[D_3V_3] = E_1[D_2V_2] = D_2E_1[V_2]$. So $V_1 = \frac{D_2}{D_1}E_1[V_2] = \frac{1}{1+R_1}E_1[V_2]$. In particular, $V_1(H) = \frac{1}{1+R_1(H)}V_2(HH)P(w_2 = H|w_1 = H) = \frac{4}{21}$, $V_1(T) = 0$. □

(ii)

Proof. Let $X_0 = \frac{2}{21}$. Suppose we buy Δ_0 shares of the maturity two bond, then at time one, the value of our portfolio is $X_1 = (1+R_0)(X_0 - \Delta B_{0,2}) + \Delta_0 B_{1,2}$. To replicate the value V_1, we must have

$$\begin{cases} V_1(H) = (1+R_0)(X_0 - \Delta_0 B_{0,2}) + \Delta_0 B_{1,2}(H) \\ V_1(T) = (1+R_0)(X_0 - \Delta_0 B_{0,2}) + \Delta_0 B_{1,2}(T). \end{cases}$$

So $\Delta_0 = \frac{V_1(H)-V_1(T)}{B_{1,2}(H)-B_{1,2}(T)} = \frac{4}{3}$. The hedging strategy is therefore to borrow $\frac{4}{3}B_{0,2} - \frac{2}{21} = \frac{20}{21}$ and buy $\frac{4}{3}$ share of the maturity two bond. The reason why we do not invest in the maturity three bond is that $B_{1,3}(H) = B_{1,3}(T)(= \frac{4}{7})$ and the portfolio will therefore have the same value at time one regardless the outcome of first coin toss. This makes impossible the replication of V_1, since $V_1(H) \neq V_1(T)$. □

(iii)

Proof. Suppose we buy Δ_1 share of the maturity three bond at time one, then to replicate V_2 at time two, we must have $V_2 = (1+R_1)(X_1 - \Delta_1 B_{1,3}) + \Delta_1 B_{2,3}$. So $\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{B_{2,3}(HH)-B_{2,3}(HT)} = -\frac{2}{3}$, and $\Delta_1(T) = \frac{V_2(TH)-V_2(TT)}{B_{2,3}(TH)-B_{2,3}(TT)} = 0$. So the hedging strategy is as follows. If the outcome of first coin toss is T, then we do nothing. If the outcome of first coin toss is H, then short $\frac{2}{3}$ shares of the maturity three bond and invest the income into the money market account. We do not invest in the maturity two bond, because at time two, the value of the bond is its face value and our portfolio will therefore have the same value regardless outcomes of coin tosses. This makes impossible the replication of V_2. □

6.5. (i)

Proof. Suppose $1 \leq n \leq m$, then

$$\begin{aligned}
\widetilde{E}^{m+1}_{n-1}[F_{n,m}] &= \widetilde{E}_{n-1}[B^{-1}_{n,m+1}(B_{n,m} - B_{n,m+1})Z_{n,m+1}Z^{-1}_{n-1,m+1}] \\
&= \widetilde{E}_{n-1}\left[\left(\frac{B_{n,m}}{B_{n,m+1}} - 1\right)\frac{B_{n,m+1}D_n}{B_{n-1,m+1}D_{n-1}}\right] \\
&= \frac{D_n}{B_{n-1,m+1}D_{n-1}}\widetilde{E}_{n-1}[D_n^{-1}\widetilde{E}_n[D_m] - D_n^{-1}\widetilde{E}_n[D_{m+1}]] \\
&= \frac{\widetilde{E}_{n-1}[D_m - D_{m+1}]}{B_{n-1,m_1}D_{n-1}} \\
&= \frac{B_{n-1,m} - B_{n-1,m+1}}{B_{n-1,m+1}} \\
&= F_{n-1,m}.
\end{aligned}$$

□

6.6. (i)

Proof. The agent enters the forward contract at no cost. He is obliged to buy certain asset at time m at the strike price $K = For_{n,m} = \frac{S_n}{B_{n,m}}$. At time $n+1$, the contract has the value $\widetilde{E}_{n+1}[D_m(S_m - K)] = S_{n+1} - KB_{n+1,m} = S_{n+1} - \frac{S_n B_{n+1,m}}{B_{n,m}}$. So if the agent sells this contract at time $n+1$, he will receive a cash flow of $S_{n+1} - \frac{S_n B_{n+1,m}}{B_{n,m}}$ □

(ii)

Proof. By (i), the cash flow generated at time $n+1$ is

$$
\begin{aligned}
& (1+r)^{m-n-1}\left(S_{n+1} - \frac{S_n B_{n+1,m}}{B_{n,m}}\right) \\
= \; & (1+r)^{m-n-1}\left(S_{n+1} - \frac{\frac{S_n}{(1+r)^{m-n-1}}}{\frac{1}{(1+r)^{m-n}}}\right) \\
= \; & (1+r)^{m-n-1}S_{n+1} - (1+r)^{m-n}S_n \\
= \; & (1+r)^m \widetilde{E}_{n_1}[\frac{S_m}{(1+r)^m}] + (1+r)^m \widetilde{E}_n[\frac{S_m}{(1+r)^m}] \\
= \; & Fut_{n+1,m} - Fut_{n,m}.
\end{aligned}
$$

□

6.7.

Proof.

$$
\begin{aligned}
\psi_{n+1}(0) &= \widetilde{E}[D_{n+1}V_{n+1}(0)] \\
&= \widetilde{E}[\frac{D_n}{1+r_n(0)}1_{\{\#H(\omega_1\cdots\omega_{n+1})=0\}}] \\
&= \widetilde{E}[\frac{D_n}{1+r_n(0)}1_{\{\#H(\omega_1\cdots\omega_n)=0\}}1_{\{\omega_{n+1}=T\}}] \\
&= \frac{1}{2}\widetilde{E}[\frac{D_n}{1+r_n(0)}] \\
&= \frac{\psi_n(0)}{2(1+r_n(0))}.
\end{aligned}
$$

For $k = 1, 2, \cdots, n$,

$$
\begin{aligned}
\psi_{n+1}(k) &= \widetilde{E}\left[\frac{D_n}{1+r_n(\#H(\omega_1\cdots\omega_n))}1_{\{\#H(\omega_1\cdots\omega_{n+1})=k\}}\right] \\
&= \widetilde{E}\left[\frac{D_n}{1+r_n(k)}1_{\{\#H(\omega_1\cdots\omega_n)=k\}}1_{\{\omega_{n+1}=T\}}\right] + \widetilde{E}\left[\frac{D_n}{1+r_n(k-1)}1_{\{\#H(\omega_1\cdots\omega_n)=k\}}1_{\{\omega_{n+1}=H\}}\right] \\
&= \frac{1}{2}\frac{\widetilde{E}[D_nV_n(k)]}{1+r_n(k)} + \frac{1}{2}\frac{\widetilde{E}[D_nV_n(k-1)]}{1+r_n(k-1)} \\
&= \frac{\psi_n(k)}{2(1+r_n(k))} + \frac{\psi_n(k-1)}{2(1+r_n(k-1))}.
\end{aligned}
$$

Finally,

$$
\psi_{n+1}(n+1) = \widetilde{E}[D_{n+1}V_{n+1}(n+1)] = \widetilde{E}\left[\frac{D_n}{1+r_n(n)}1_{\{\#H(\omega_1\cdots\omega_n)=n\}}1_{\{\omega_{n+1}=H\}}\right] = \frac{\psi_n(n)}{2(1+r_n(n))}.
$$

Remark 3. *In the above proof, we have used the independence of* ω_{n+1} *and* $(\omega_1, \cdots, \omega_n)$. *This is guaranteed by the assumption that* $\widetilde{p} = \widetilde{q} = \frac{1}{2}$ *(note* $\xi \perp \eta$ *if and only if* $E[\xi|\eta] = constant$*). In case the binomial model has stochastic up- and down-factor* u_n *and* d_n*, we can use the fact that* $\widetilde{P}(\omega_{n+1} = H|\omega_1, \cdots, \omega_n) = p_n$ *and* $\widetilde{P}(\omega_{n+1} = T|\omega_1, \cdots, \omega_n) = q_n$*, where* $p_n = \frac{1+r_n-d_n}{u_n-d_n}$ *and* $q_n = \frac{u-1-r_n}{u_n-d_n}$ *(cf. solution of Exercise 2.9 and notes on page 39). Then for any* $X \in \mathcal{F}_n = \sigma(\omega_1, \cdots, \omega_n)$*, we have* $\widetilde{E}[Xf(\omega_{n+1})] = \widetilde{E}[X\widetilde{E}[f(\omega_{n+1})|\mathcal{F}_n]] = \widetilde{E}[X(p_n f(H) + q_n f(T))]$.

□

Chapter 2

Stochastic Calculus for Finance II: Continuous-Time Models

2.1 General Probability Theory

1.1. (i)

Proof. $P(B) = P((B - A) \cup A) = P(B - A) + P(A) \geq P(A)$. $\square$

(ii)

Proof. $P(A) \leq P(A_n)$ implies $P(A) \leq \lim_{n\to\infty} P(A_n) = 0$. So $0 \leq P(A) \leq 0$, which means $P(A) = 0$. $\square$

1.2. (i)

Proof. We define a mapping ϕ from A to Ω as follows: $\phi(\omega_1\omega_2\cdots) = \omega_1\omega_3\omega_5\cdots$. Then ϕ is one-to-one and onto. So the cardinality of A is the same as that of Ω, which means in particular that A is uncountably infinite. $\square$

(ii)

Proof. Let $A_n = \{\omega = \omega_1\omega_2\cdots : \omega_1 = \omega_2, \cdots, \omega_{2n-1} = \omega_{2n}\}$. Then $A_n \downarrow A$ as $n \to \infty$. So

$$P(A) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} [P(\omega_1 = \omega_2)\cdots P(\omega_{2n-1} = \omega_{2n})] = \lim_{n\to\infty} (p^2 + (1-p)^2)^n.$$

Since $p^2 + (1-p)^2 < 1$ for $0 < p < 1$, we have $\lim_{n\to\infty}(p^2 + (1-p)^2)^n = 0$. This implies $P(A) = 0$. $\square$

1.3.

Proof. Clearly $P(\emptyset) = 0$. For any A and B, if both of them are finite, then $A \cup B$ is also finite. So $P(A \cup B) = 0 = P(A) + P(B)$. If at least one of them is infinite, then $A \cup B$ is also infinite. So $P(A \cup B) = \infty = P(A) + P(B)$. Similarly, we can prove $P(\cup_{n=1}^N A_n) = \sum_{n=1}^N P(A_n)$, even if A_n's are not disjoint.

To see countable additivity property doesn't hold for P, let $A_n = \{\frac{1}{n}\}$. Then $A = \cup_{n=1}^\infty A_n$ is an infinite set and therefore $P(A) = \infty$. However, $P(A_n) = 0$ for each n. So $P(A) \neq \sum_{n=1}^\infty P(A_n)$. $\square$

1.4. (i)

Proof. By Example 1.2.5, we can construct a random variable X on the coin-toss space, which is uniformly distributed on $[0,1]$. For the strictly increasing and continuous function $N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi$, we let $Z = N^{-1}(X)$. Then $P(Z \leq a) = P(X \leq N(a)) = N(a)$ for any real number a, i.e. Z is a standard normal random variable on the coin-toss space $(\Omega_\infty, \mathcal{F}_\infty, P)$. $\square$

(ii)

Proof. Define $X_n = \sum_{i=1}^n \frac{Y_i}{2^i}$, where

$$Y_i(\omega) = \begin{cases} 1, & \text{if } \omega_i = H \\ 0, & \text{if } \omega_i = T. \end{cases}$$

Then $X_n(\omega) \to X(\omega)$ for every $\omega \in \Omega_\infty$ where X is defined as in (i). So $Z_n = N^{-1}(X_n) \to Z = N^{-1}(X)$ for every ω. Clearly Z_n depends only on the first n coin tosses and $\{Z_n\}_{n\geq 1}$ is the desired sequence. □

1.5.

Proof. First, by the information given by the problem, we have

$$\int_\Omega \int_0^\infty 1_{[0,X(\omega))}(x)dxdP(\omega) = \int_0^\infty \int_\Omega 1_{[0,X(\omega))}(x)dP(\omega)dx.$$

The left side of this equation equals to

$$\int_\Omega \int_0^{X(\omega)} dxdP(\omega) = \int_\Omega X(\omega)dP(\omega) = E\{X\}.$$

The right side of the equation equals to

$$\int_0^\infty \int_\Omega 1_{\{x<X(\omega)\}}dP(\omega)dx = \int_0^\infty P(x < X)dx = \int_0^\infty (1 - F(x))dx.$$

So $E\{X\} = \int_0^\infty (1 - F(x))dx$. □

1.6. (i)

Proof.

$$\begin{aligned} E\{e^{uX}\} &= \int_{-\infty}^\infty e^{ux} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2-2\sigma^2 ux}{2\sigma^2}} dx \\ &= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(\mu+\sigma^2 u)]^2-(2\sigma^2 u\mu+\sigma^4 u^2)}{2\sigma^2}} dx \\ &= e^{u\mu+\frac{\sigma^2 u^2}{2}} \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[x-(\mu+\sigma^2 u)]^2}{2\sigma^2}} dx \\ &= e^{u\mu+\frac{\sigma^2 u^2}{2}} \end{aligned}$$

□

(ii)

Proof. $E\{\phi(X)\} = E\{e^{uX}\} = e^{u\mu+\frac{u^2\sigma^2}{2}} \geq e^{u\mu} = \phi(E\{X\})$. □

1.7. (i)

Proof. Since $|f_n(x)| \leq \frac{1}{\sqrt{2n\pi}}$, $f(x) = \lim_{n\to\infty} f_n(x) = 0$. □

(ii)

Proof. By the change of variable formula, $\int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$. So we must have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = 1.$$

$\square$

(iii)

Proof. This is not contradictory with the Monotone Convergence Theorem, since $\{f_n\}_{n\geq 1}$ doesn't increase to 0. $\square$

1.8. (i)

Proof. By (1.9.1), $|Y_n| = \left|\frac{e^{tX}-e^{s_nX}}{t-s_n}\right| = |Xe^{\theta X}| = Xe^{\theta X} \leq Xe^{2tX}$. The last inequality is by $X \geq 0$ and the fact that θ is between t and s_n, and hence smaller than $2t$ for n sufficiently large. So by the Dominated Convergence Theorem, $\varphi'(t) = \lim_{n\to\infty} E\{Y_n\} = E\{\lim_{n\to\infty} Y_n\} = E\{Xe^{tX}\}$. $\square$

(ii)

Proof. Since $E[e^{tX^+}1_{\{X\geq 0\}}] + E[e^{-tX^-}1_{\{X<0\}}] = E[e^{tX}] < \infty$ for every $t \in \mathbb{R}$, $E[e^{t|X|}] = E[e^{tX^+}1_{\{X\geq 0\}}] + E[e^{-(-t)X^-}1_{\{X<0\}}] < \infty$ for every $t \in \mathbb{R}$. Similarly, we have $E[|X|e^{t|X|}] < \infty$ for every $t \in \mathbb{R}$. So, similar to (i), we have $|Y_n| = |Xe^{\theta X}| \leq |X|e^{2t|X|}$ for n sufficiently large, So by the Dominated Convergence Theorem, $\varphi'(t) = \lim_{n\to\infty} E\{Y_n\} = E\{\lim_{n\to\infty} Y_n\} = E\{Xe^{tX}\}$. $\square$

1.9.

Proof. If $g(x)$ is of the form $1_B(x)$, where B is a Borel subset of $\mathbb{R}$, then the desired equality is just (1.9.3). By the linearity of Lebesgue integral, the desired equality also holds for simple functions, i.e. g of the form $g(x) = \sum_{i=1}^{n} 1_{B_i}(x)$, where each B_i is a Borel subset of $\mathbb{R}$. Since any nonnegative, Borel-measurable function g is the limit of an increasing sequence of simple functions, the desired equality can be proved by the Monotone Convergence Theorem. $\square$

1.10. (i)

Proof. If $\{A_i\}_{i=1}^{\infty}$ is a sequence of disjoint Borel subsets of $[0,1]$, then by the Monotone Convergence Theorem, $\widetilde{P}(\cup_{i=1}^{\infty} A_i)$ equals to

$$\int 1_{\cup_{i=1}^{\infty} A_i} ZdP = \int \lim_{n\to\infty} 1_{\cup_{i=1}^{n} A_i} ZdP = \lim_{n\to\infty} \int 1_{\cup_{i=1}^{n} A_i} ZdP = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} ZdP = \sum_{i=1}^{\infty} \widetilde{P}(A_i).$$

Meanwhile, $\widetilde{P}(\Omega) = 2P([\frac{1}{2}, 1]) = 1$. So $\widetilde{P}$ is a probability measure. $\square$

(ii)

Proof. If $P(A) = 0$, then $\widetilde{P}(A) = \int_A ZdP = 2\int_{A\cap[\frac{1}{2},1]} dP = 2P(A \cap [\frac{1}{2}, 1]) = 0$. $\square$

(iii)

Proof. Let $A = [0, \frac{1}{2})$. $\square$

1.11.

Proof.

$$\widetilde{E}\{e^{uY}\} = E\{e^{uY}Z\} = E\{e^{uX+u\theta}e^{-\theta X-\frac{\theta^2}{2}}\} = e^{u\theta-\frac{\theta^2}{2}}E\{e^{(u-\theta)X}\} = e^{u\theta-\frac{\theta^2}{2}}e^{\frac{(u-\theta)^2}{2}} = e^{\frac{u^2}{2}}.$$

$\square$

1.12.

Proof. First, $\hat{Z} = e^{\theta Y - \frac{\theta^2}{2}} = e^{\theta(X+\theta)-\frac{\theta^2}{2}} = e^{\frac{\theta^2}{2}+\theta X} = Z^{-1}$. Second, for any $A \in \mathcal{F}$, $\hat{P}(A) = \int_A \hat{Z} d\widetilde{P} = \int (1_A \hat{Z}) Z dP = \int 1_A dP = P(A)$. So $P = \hat{P}$. In particular, X is standard normal under $\hat{P}$, since it's standard normal under P. □

1.13. (i)

Proof. $\frac{1}{\epsilon}P(X \in B(x,\epsilon)) = \frac{1}{\epsilon}\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$ is approximately $\frac{1}{\epsilon}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \cdot \epsilon = \frac{1}{\sqrt{2\pi}}e^{-\frac{X^2(\bar{\omega})}{2}}$. □

(ii)

Proof. Similar to (i). □

(iii)

Proof. $\{X \in B(x,\epsilon)\} = \{X \in B(y-\theta,\epsilon)\} = \{X+\theta \in B(y,\epsilon)\} = \{Y \in B(y,\epsilon)\}$. □

(iv)

Proof. By (i)-(iii), $\frac{\widetilde{P}(A)}{P(A)}$ is approximately

$$\frac{\frac{\epsilon}{\sqrt{2\pi}}e^{-\frac{Y^2(\bar{\omega})}{2}}}{\frac{\epsilon}{\sqrt{2\pi}}e^{-\frac{X^2(\bar{\omega})}{2}}} = e^{-\frac{Y^2(\bar{\omega})-X^2(\bar{\omega})}{2}} = e^{-\frac{(X(\bar{\omega})+\theta)^2-X^2(\bar{\omega})}{2}} = e^{-\theta X(\bar{\omega})-\frac{\theta^2}{2}}.$$

□

1.14. (i)

Proof.

$$\widetilde{P}(\Omega) = \int \frac{\widetilde{\lambda}}{\lambda} e^{-(\widetilde{\lambda}-\lambda)X} dP = \frac{\widetilde{\lambda}}{\lambda}\int_0^\infty e^{-(\widetilde{\lambda}-\lambda)x}\lambda e^{-\lambda x}dx = \int_0^\infty \widetilde{\lambda} e^{-\widetilde{\lambda}x}dx = 1.$$

□

(ii)

Proof.

$$\widetilde{P}(X \le a) = \int_{\{X\le a\}} \frac{\widetilde{\lambda}}{\lambda} e^{-(\widetilde{\lambda}-\lambda)X} dP = \int_0^a \frac{\widetilde{\lambda}}{\lambda} e^{-(\widetilde{\lambda}-\lambda)x}\lambda e^{-\lambda x}dx = \int_0^a \widetilde{\lambda} e^{-\widetilde{\lambda}x}dx = 1 - e^{-\widetilde{\lambda}a}.$$

□

1.15. (i)

Proof. Clearly $Z \ge 0$. Furthermore, we have

$$E\{Z\} = E\left\{\frac{h(g(X))g'(X)}{f(X)}\right\} = \int_{-\infty}^\infty \frac{h(g(x))g'(x)}{f(x)} f(x)dx = \int_{-\infty}^\infty h(g(x))dg(x) = \int_{-\infty}^\infty h(u)du = 1.$$

□

(ii)

Proof.

$$\widetilde{P}(Y \le a) = \int_{\{g(X)\le a\}} \frac{h(g(X))g'(X)}{f(X)} dP = \int_{-\infty}^{g^{-1}(a)} \frac{h(g(x))g'(x)}{f(x)} f(x)dx = \int_{-\infty}^{g^{-1}(a)} h(g(x))dg(x).$$

By the change of variable formula, the last equation above equals to $\int_{-\infty}^a h(u)du$. So Y has density h under $\widetilde{P}$. □

2.2 Information and Conditioning

2.1.

Proof. For any real number a, we have $\{X \leq a\} \in \mathcal{F}_0 = \{\emptyset, \Omega\}$. So $P(X \leq a)$ is either 0 or 1. Since $\lim_{a\to\infty} P(X \leq a) = 1$ and $\lim_{a\to\infty} P(X \leq a) = 0$, we can find a number x_0 such that $P(X \leq x_0) = 1$ and $P(X \leq x) = 0$ for any $x < x_0$. So

$$P(X = x_0) = \lim_{n\to\infty} P(x_0 - \frac{1}{n} < X \leq x_0) = \lim_{n\to\infty}(P(X \leq x_0) - P(X \leq x_0 - \frac{1}{n})) = 1.$$

□

2.2. (i)

Proof. $\sigma(X) = \{\emptyset, \Omega, \{HT, TH\}, \{TT, HH\}\}$. □

(ii)

Proof. $\sigma(S_1) = \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\}$. □

(iii)

Proof. $\widetilde{P}(\{HT, TH\} \cap \{HH, HT\}) = \widetilde{P}(\{HT\}) = \frac{1}{4}$, $\widetilde{P}(\{HT, TH\}) = \widetilde{P}(\{HT\}) + \widetilde{P}(\{TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $\widetilde{P}(\{HH, HT\}) = \widetilde{P}(\{HH\}) + \widetilde{P}(\{HT\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. So we have

$$\widetilde{P}(\{HT, TH\} \cap \{HH, HT\}) = \widetilde{P}(\{HT, TH\})\widetilde{P}(\{HH, HT\}).$$

Similarly, we can work on other elements of $\sigma(X)$ and $\sigma(S_1)$ and show that $\widetilde{P}(A \cap B) = \widetilde{P}(A)\widetilde{P}(B)$ for any $A \in \sigma(X)$ and $B \in \sigma(S_1)$. So $\sigma(X)$ and $\sigma(S_1)$ are independent under $\widetilde{P}$. □

(iv)

Proof. $P(\{HT, TH\} \cap \{HH, HT\}) = P(\{HT\}) = \frac{2}{9}$, $P(\{HT, TH\}) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$ and $P(\{HH, HT\}) = \frac{4}{9} + \frac{2}{9} = \frac{6}{9}$. So

$$P(\{HT, TH\} \cap \{HH, HT\}) \neq P(\{HT, TH\})P(\{HH, HT\}).$$

Hence $\sigma(X)$ and $\sigma(S_1)$ are not independent under P. □

(v)

Proof. Because S_1 and X are not independent under the probability measure P, knowing the value of X will affect our opinion on the distribution of S_1. □

2.3.

Proof. We note (V, W) are jointly Gaussian, so to prove their independence it suffices to show they are uncorrelated. Indeed, $E\{VW\} = E\{-X^2 \sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta + Y^2\sin\theta\cos\theta\} = -\sin\theta\cos\theta + 0 + 0 + \sin\theta\cos\theta = 0$. □

2.4. (i)

Proof.

$$\begin{aligned}
E\{e^{uX+vY}\} &= E\{e^{uX+vXZ}\} \\
&= E\{e^{uX+vXZ}|Z=1\}P(Z=1) + E\{e^{uX+vXZ}|Z=-1\}P(Z=-1) \\
&= \frac{1}{2}E\{e^{uX+vX}\} + \frac{1}{2}E\{e^{uX-vX}\} \\
&= \frac{1}{2}[e^{\frac{(u+v)^2}{2}} + e^{\frac{(u-v)^2}{2}}] \\
&= e^{\frac{u^2+v^2}{2}}\frac{e^{uv}+e^{-uv}}{2}.
\end{aligned}$$

□

(ii)

Proof. Let $u=0$. □

(iii)

Proof. $E\{e^{uX}\} = e^{\frac{u^2}{2}}$ and $E\{e^{vY}\} = e^{\frac{v^2}{2}}$. So $E\{e^{uX+vY}\} \neq E\{e^{uX}\}E\{e^{vY}\}$. Therefore X and Y cannot be independent. □

2.5.

Proof. The density $f_X(x)$ of X can be obtained by

$$f_X(x) = \int f_{X,Y}(x,y)dy = \int_{\{y\geq -|x|\}} \frac{2|x|+y}{\sqrt{2\pi}}e^{-\frac{(2|x|+y)^2}{2}}dy = \int_{\{\xi\geq|x|\}} \frac{\xi}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

The density $f_Y(y)$ of Y can be obtained by

$$\begin{aligned}
f_Y(y) &= \int f_{XY}(x,y)dx \\
&= \int 1_{\{|x|\geq -y\}}\frac{2|x|+y}{\sqrt{2\pi}}e^{-\frac{(2|x|+y)^2}{2}}dx \\
&= \int_{0\vee(-y)}^{\infty} \frac{2x+y}{\sqrt{2\pi}}e^{-\frac{(2x+y)^2}{2}}dx + \int_{-\infty}^{0\wedge y} \frac{-2x+y}{\sqrt{2\pi}}e^{-\frac{(-2x+y)^2}{2}}dx \\
&= \int_{0\vee(-y)}^{\infty} \frac{2x+y}{\sqrt{2\pi}}e^{-\frac{(2x+y)^2}{2}}dx + \int_{\infty}^{0\vee(-y)} \frac{2x+y}{\sqrt{2\pi}}e^{-\frac{(2x+y)^2}{2}}d(-x) \\
&= 2\int_{|y|}^{\infty} \frac{\xi}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d(\frac{\xi}{2}) \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}.
\end{aligned}$$

So both X and Y are standard normal random variables. Since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent. However, if we set $F(t)=\int_t^\infty \frac{u^2}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du$, we have

$$
\begin{aligned}
E\{XY\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy1_{\{y\geq -|x|\}}\frac{2|x|+y}{\sqrt{2\pi}}e^{-\frac{(2|x|+y)^2}{2}}dxdy \\
&= \int_{-\infty}^{\infty} xdx\int_{-|x|}^{\infty} y\frac{2|x|+y}{\sqrt{2\pi}}e^{-\frac{(2|x|+y)^2}{2}}dy \\
&= \int_{-\infty}^{\infty} xdx\int_{|x|}^{\infty}(\xi-2|x|)\frac{\xi}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi \\
&= \int_{-\infty}^{\infty} xdx(\int_{|x|}^{\infty}\frac{\xi^2}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi - 2|x|\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}) \\
&= \int_0^{\infty} x\int_x^{\infty}\frac{\xi^2}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi dx + \int_{-\infty}^0 x\int_{-x}^{\infty}\frac{\xi^2}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi dx \\
&= \int_0^{\infty} xF(x)dx + \int_{-\infty}^0 xF(-x)dx.
\end{aligned}
$$

So $E\{XY\}=\int_0^\infty xF(x)dx-\int_0^\infty uF(u)du=0$.

□

2.6. (i)

Proof. $\sigma(X)=\{\emptyset,\Omega,\{a,b\},\{c,d\}\}$. □

(ii)

Proof.

$$E\{Y|X\}=\sum_{\alpha\in\{a,b,c,d\}}\frac{E\{Y1_{\{X=\alpha\}}\}}{P(X=\alpha)}1_{\{X=\alpha\}}.$$

□

(iii)

Proof.

$$E\{Z|X\}=X+E\{Y|X\}=X+\sum_{\alpha\in\{a,b,c,d\}}\frac{E\{Y1_{\{X=\alpha\}}\}}{P(X=\alpha)}1_{\{X=\alpha\}}.$$

□

(iv)

Proof. $E\{Z|X\}-E\{Y|X\}=E\{Z-Y|X\}=E\{X|X\}=X$. □

2.7.

Proof. Let $\mu=E\{Y-X\}$ and $\xi=E\{Y-X-\mu|\mathcal{G}\}$. Note ξ is $\mathcal{G}$-measurable, we have

$$
\begin{aligned}
Var(Y-X) &= E\{(Y-X-\mu)^2\} \\
&= E\{[(Y-E\{Y|\mathcal{G}\})+(E\{Y|\mathcal{G}\}-X-\mu)]^2\} \\
&= Var(Err)+2E\{(Y-E\{Y|\mathcal{G}\})\xi\}+E\{\xi^2\} \\
&= Var(Err)+2E\{Y\xi-E\{Y|\mathcal{G}\}\xi\}+E\{\xi^2\} \\
&= Var(Err)+E\{\xi^2\} \\
&\geq Var(Err).
\end{aligned}
$$

□

2.8.

Proof. It suffices to prove the more general case. For any $\sigma(X)$-measurable random variable ξ, $E\{Y_2\xi\} = E\{(Y - E\{Y|X\})\xi\} = E\{Y\xi - E\{Y|X\}\xi\} = E\{Y\xi\} - E\{Y\xi\} = 0$. □

2.9. (i)

Proof. Consider the dice-toss space similar to the coin-toss space. Then a typical element ω in this space is an infinite sequence $\omega_1\omega_2\omega_3\cdots$, with $\omega_i \in \{1, 2, \cdots, 6\}$ $(i \in \mathbb{N})$. We define $X(\omega) = \omega_1$ and $f(x) = 1_{\{\text{odd integers}\}}(x)$. Then it's easy to see

$$\sigma(X) = \{\emptyset, \Omega, \{\omega : \omega_1 = 1\}, \cdots, \{\omega : \omega_1 = 6\}\}$$

and $\sigma(f(X))$ equals to

$$\{\emptyset, \Omega, \{\omega : \omega_1 = 1\} \cup \{\omega : \omega_1 = 3\} \cup \{\omega : \omega_1 = 5\}, \{\omega : \omega_1 = 2\} \cup \{\omega : \omega_1 = 4\} \cup \{\omega : \omega_1 = 6\}\}.$$

So $\{\emptyset, \Omega\} \subset \sigma(f(X)) \subset \sigma(X)$, and each of these containment is strict. □

(ii)

Proof. No. $\sigma(f(X)) \subset \sigma(X)$ is always true. □

2.10.

Proof.

$$\begin{aligned}
\int_A g(X)dP &= E\{g(X)1_B(X)\} \\
&= \int_{-\infty}^{\infty} g(x)1_B(x)f_X(x)dx \\
&= \int\int \frac{yf_{X,Y}(x,y)}{f_X(x)}dy1_B(x)f_X(x)dx \\
&= \int\int y1_B(x)f_{X,Y}(x,y)dxdy \\
&= E\{Y1_B(X)\} \\
&= E\{YI_A\} \\
&= \int_A YdP.
\end{aligned}$$

□

2.11. (i)

Proof. We can find a sequence $\{W_n\}_{n\geq 1}$ of $\sigma(X)$-measurable simple functions such that $W_n \uparrow W$. Each W_n can be written in the form $\sum_{i=1}^{K_n} a_i^n 1_{A_i^n}$, where A_i^n's belong to $\sigma(X)$ and are disjoint. So each A_i^n can be written as $\{X \in B_i^n\}$ for some Borel subset B_i^n of $\mathbb{R}$, i.e. $W_n = \sum_{i=1}^{K_n} a_i^n 1_{\{X\in B_i^n\}} = \sum_{i=1}^{K_n} a_i^n 1_{B_i^n}(X) = g_n(X)$, where $g_n(x) = \sum_{i=1}^{K_n} a_i^n 1_{B_i^n}(x)$. Define $g = \limsup g_n$, then g is a Borel function. By taking upper limits on both sides of $W_n = g_n(X)$, we get $W = g(X)$. □

(ii)

Proof. Note $E\{Y|X\}$ is $\sigma(X)$-measurable. By (i), we can find a Borel function g such that $E\{Y|X\} = g(X)$. □

2.3 Brownian Motion

3.1.

Proof. We have $\mathcal{F}_t \subset \mathcal{F}_{u_1}$ and $W_{u_2} - W_{u_1}$ is independent of $\mathcal{F}_{u_1}$. So in particular, $W_{u_2} - W_{u_1}$ is independent of $\mathcal{F}_t$. □

3.2.

Proof. $E[W_t^2 - W_s^2|\mathcal{F}_s] = E[(W_t - W_s)^2 + 2W_tW_s - 2W_s^2|\mathcal{F}_s] = t - s + 2W_sE[W_t - W_s|\mathcal{F}_s] = t - s.$ □

3.3.

Proof. $\varphi^{(3)}(u) = 2\sigma^4 ue^{\frac{1}{2}\sigma^2u^2} + (\sigma^2+\sigma^4u^2)\sigma^2ue^{\frac{1}{2}\sigma^2u^2} = e^{\frac{1}{2}\sigma^2u^2}(3\sigma^4u + \sigma^4u^2)$, and $\varphi^{(4)}(u) = \sigma^2ue^{\frac{1}{2}\sigma^2u^2}(3\sigma^4u + \sigma^4u^2) + e^{\frac{1}{2}\sigma^2u^2}(3\sigma^4 + 2\sigma^4u)$. So $E[(X-\mu)^4] = \varphi^{(4)}(0) = 3\sigma^4$. □

3.4. (i)

Proof. Assume there exists $A \in \mathcal{F}$, such that $P(A) > 0$ and for every $\omega \in A$, $\lim_n \sum_{j=0}^{n-1}|W_{t_{j+1}} - W_{t_j}|(\omega) < \infty$. Then for every $\omega \in A$, $\sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})^2(\omega) \le \max_{0\le k\le n-1}|W_{t_{k+1}} - W_{t_k}|(\omega)\sum_{j=0}^{n-1}|W_{t_{j+1}} - W_{t_j}|(\omega) \to 0$, since $\lim_{n\to\infty}\max_{0\le k\le n-1}|W_{t_{k+1}} - W_{t_k}|(\omega) = 0$. This is a contradiction with $\lim_{n\to\infty}\sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})^2 = T$ a.s.. □

(ii)

Proof. Note $\sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})^3 \le \max_{0\le k\le n-1}|W_{t_{k+1}} - W_{t_k}|\sum_{j=0}^{n-1}(W_{t_{j+1}} - W_{t_j})^2 \to 0$ as $n \to \infty$, by an argument similar to (i). □

3.5.

Proof.

$$\begin{aligned}
& E[e^{-rT}(S_T - K)^+] \\
= \;& e^{-rT}\int_{\frac{1}{\sigma}(\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T)}^{\infty}(S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma x} - K)\frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}}dx \\
= \;& e^{-rT}\int_{\frac{1}{\sigma\sqrt{T}}(\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T)}^{\infty}(S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}y} - K)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy \\
= \;& S_0e^{-\frac{1}{2}\sigma^2T}\int_{\frac{1}{\sigma\sqrt{T}}(\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}+\sigma\sqrt{T}y}dy - Ke^{-rT}\int_{\frac{1}{\sigma\sqrt{T}}(\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\
= \;& S_0\int_{\frac{1}{\sigma\sqrt{T}}(\ln\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T)-\sigma\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi - Ke^{-rT}N\left(\frac{1}{\sigma\sqrt{T}}(\ln\frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T)\right) \\
= \;& Ke^{-rT}N(d_+(T,S_0)) - Ke^{-rT}N(d_-(T,S_0)).
\end{aligned}$$

□

3.6. (i)

Proof.

$$\begin{aligned}
E[f(X_t)|\mathcal{F}_t] &= E[f(W_t - W_s + a)|\mathcal{F}_s]|_{a=W_s+\mu t} = E[f(W_{t-s} + a)]|_{a=W_s+\mu t} \\
&= \int_{-\infty}^{\infty} f(x + W_s + \mu t) \frac{e^{-\frac{x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} dx \\
&= \int_{-\infty}^{\infty} f(y) \frac{e^{-\frac{(y-W_s-\mu s-\mu(t-s))^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} dy \\
&= g(X_s).
\end{aligned}$$

So $E[f(X_t)|\mathcal{F}_s] = \int_{-\infty}^{\infty} f(y)p(t-s, X_s, y)dy$ with $p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x-\mu\tau)^2}{2\tau}}$. □

(ii)

Proof. $E[f(S_t)|\mathcal{F}_s] = E[f(S_0 e^{\sigma X_t})|\mathcal{F}_s]$ with $\mu = \frac{v}{\sigma}$. So by (i),

$$\begin{aligned}
E[f(S_t)|\mathcal{F}_s] &= \int_{-\infty}^{\infty} f(S_0 e^{\sigma y}) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-X_s-\mu(t-s))^2}{2(t-s)}} dy \\
&\overset{S_0 e^{\sigma y}=z}{=} \int_0^{\infty} f(z) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(\frac{1}{\sigma}\ln\frac{z}{S_0} - \frac{1}{\sigma}\ln\frac{S_s}{S_0} - \mu(t-s))^2}{2}} \frac{dz}{\sigma z} \\
&= \int_0^{\infty} f(z) \frac{e^{-\frac{(\ln\frac{z}{S_s} - v(t-s))^2}{2\sigma^2(t-s)}}}{\sigma z\sqrt{2\pi(t-s)}} dz \\
&= \int_0^{\infty} f(z)p(t-s, S_s, z)dz \\
&= g(S_s).
\end{aligned}$$

□

3.7. (i)

Proof. $E\left[\frac{Z_t}{Z_s}|\mathcal{F}_s\right] = E[\exp\{\sigma(W_t - W_s) + \sigma\mu(t-s) - (\sigma\mu + \frac{\sigma^2}{2})(t-s)\}] = 1.$ □

(ii)

Proof. By optional stopping theorem, $E[Z_{t\wedge\tau_m}] = E[Z_0] = 1$, that is, $E[\exp\{\sigma X_{t\wedge\tau_m} - (\sigma\mu + \frac{\sigma^2}{2})t\wedge\tau_m\}] = 1$. □

(iii)

Proof. If $\mu \geq 0$ and $\sigma > 0$, $Z_{t\wedge\tau_m} \leq e^{\sigma m}$. By bounded convergence theorem,

$$E[1_{\{\tau_m<\infty\}} Z_{\tau_m}] = E[\lim_{t\to\infty} Z_{t\wedge\tau_m}] = \lim_{t\to\infty} E[Z_{t\wedge\tau_m}] = 1,$$

since on the event $\{\tau_m = \infty\}$, $Z_{t\wedge\tau_m} \leq e^{\sigma m - \frac{1}{2}\sigma^2 t} \to 0$ as $t \to \infty$. Therefore, $E[e^{\sigma m - (\sigma\mu + \frac{\sigma^2}{2})\tau_m}] = 1$. Let $\sigma \downarrow 0$, by bounded convergence theorem, we have $P(\tau_m < \infty) = 1$. Let $\sigma\mu + \frac{\sigma^2}{2} = \alpha$, we get

$$E[e^{-\alpha\tau_m}] = e^{-\sigma m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}}.$$

□

(iv)

Proof. We note for $\alpha > 0$, $E[\tau_m e^{-\alpha\tau_m}] < \infty$ since $xe^{-\alpha x}$ is bounded on $[0,\infty)$. So by an argument similar to Exercise 1.8, $E[e^{-\alpha\tau_m}]$ is differentiable and

$$\frac{\partial}{\partial\alpha}E[e^{-\alpha\tau_m}] = -E[\tau_m e^{-\alpha\tau_m}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}}\frac{-m}{\sqrt{2\alpha+\mu^2}}.$$

Let $\alpha \downarrow 0$, by monotone increasing theorem, $E[\tau_m] = \frac{m}{\mu} < \infty$ for $\mu > 0$. □

(v)

Proof. By $\sigma > -2\mu > 0$, we get $\sigma\mu + \frac{\sigma^2}{2} > 0$. Then $Z_{t\wedge\tau_m} \leq e^{\sigma m}$ and on the event $\{\tau_m = \infty\}$, $Z_{t\wedge\tau_m} \leq e^{\sigma m - (\frac{\sigma^2}{2}+\sigma\mu)t} \to 0$ as $t \to \infty$. Therefore,

$$E[e^{\sigma m - (\sigma\mu + \frac{\sigma^2}{2})\tau_m} 1_{\{\tau_m < \infty\}}] = E[\lim_{t\to\infty} Z_{t\wedge\tau_m}] = \lim_{t\to\infty} E[Z_{t\wedge\tau_m}] = 1.$$

Let $\sigma \downarrow -2\mu$, then we get $P(\tau_m < \infty) = e^{2\mu m} = e^{-2|\mu|m} < 1$. Set $\alpha = \sigma\mu + \frac{\sigma^2}{2}$. So we get

$$E[e^{-\alpha\tau_m}] = E[e^{-\alpha\tau_m}1_{\{\tau_m<\infty\}}] = e^{-\sigma m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}}.$$

□

3.8. (i)

Proof.

$$\begin{aligned}
\varphi_n(u) &= \widetilde{E}[e^{u\frac{1}{\sqrt{n}}M_{nt,n}}] = (\widetilde{E}[e^{\frac{u}{\sqrt{n}}X_{1,n}}])^{nt} = (e^{\frac{u}{\sqrt{n}}}\widetilde{p}_n + e^{-\frac{u}{\sqrt{n}}}\widetilde{q}_n)^{nt} \\
&= \left[e^{\frac{u}{\sqrt{n}}}\left(\frac{\frac{r}{n} + 1 - e^{-\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}} - e^{-\frac{\sigma}{\sqrt{n}}}}\right) + e^{-\frac{u}{\sqrt{n}}}\left(\frac{-\frac{r}{n} - 1 + e^{\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}} - e^{-\frac{\sigma}{\sqrt{n}}}}\right)\right]^{nt}.
\end{aligned}$$

□

(ii)

Proof.

$$\varphi_{\frac{1}{x^2}}(u) = \left[e^{ux}\left(\frac{rx^2 + 1 - e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}}\right) - e^{-ux}\left(\frac{rx^2 + 1 - e^{\sigma x}}{e^{\sigma x} - e^{-\sigma x}}\right)\right]^{\frac{t}{x^2}}.$$

So,

$$\begin{aligned}
\ln\varphi_{\frac{1}{x^2}}(u) &= \frac{t}{x^2}\ln\left[\frac{(rx^2+1)(e^{ux} - e^{-ux}) + e^{(\sigma-u)x} - e^{-(\sigma-u)x}}{e^{\sigma x} - e^{-\sigma x}}\right] \\
&= \frac{t}{x^2}\ln\left[\frac{(rx^2+1)\sinh ux + \sinh(\sigma-u)x}{\sinh\sigma x}\right] \\
&= \frac{t}{x^2}\ln\left[\frac{(rx^2+1)\sinh ux + \sinh\sigma x\cosh ux - \cosh\sigma x\sinh ux}{\sinh\sigma x}\right] \\
&= \frac{t}{x^2}\ln\left[\cosh ux + \frac{(rx^2 + 1 - \cosh\sigma x)\sinh ux}{\sinh\sigma x}\right].
\end{aligned}$$

□

(iii)

Proof.

$$
\begin{aligned}
& \cosh ux + \frac{(rx^2+1-\cosh\sigma x)\sinh ux}{\sinh\sigma x} \\
= \quad & 1 + \frac{u^2x^2}{2} + O(x^4) + \frac{(rx^2+1-1-\frac{\sigma^2x^2}{2}+O(x^4))(ux+O(x^3))}{\sigma x + O(x^3)} \\
= \quad & 1 + \frac{u^2x^2}{2} + \frac{(r-\frac{\sigma^2}{2})ux^3 + O(x^5)}{\sigma x + O(x^3)} + O(x^4) \\
= \quad & 1 + \frac{u^2x^2}{2} + \frac{(r-\frac{\sigma^2}{2})ux^3(1+O(x^2))}{\sigma x(1+O(x^2))} + O(x^4) \\
= \quad & 1 + \frac{u^2x^2}{2} + \frac{rux^2}{\sigma} - \frac{1}{2}\sigma ux^2 + O(x^4).
\end{aligned}
$$

□

(iv)

Proof.

$$
\ln\varphi_{\frac{1}{x^2}} = \frac{t}{x^2}\ln(1+\frac{u^2x^2}{2}+\frac{ru}{\sigma}x^2-\frac{\sigma ux^2}{2}+O(x^4)) = \frac{t}{x^2}(\frac{u^2x^2}{2}+\frac{ru}{\sigma}x^2-\frac{\sigma ux^2}{2}+O(x^4)).
$$

So $\lim_{x\downarrow 0}\ln\varphi_{\frac{1}{x^2}}(u) = t(\frac{u^2}{2}+\frac{ru}{\sigma}-\frac{\sigma u}{2})$, and $\widetilde{E}[e^{u\frac{1}{\sqrt{n}}M_{nt,n}}] = \varphi_n(u) \to \frac{1}{2}tu^2 + t(\frac{r}{\sigma}-\frac{\sigma}{2})u$. By the one-to-one correspondence between distribution and moment generating function, $(\frac{1}{\sqrt{n}}M_{nt,n})_n$ converges to a Gaussian random variable with mean $t(\frac{r}{\sigma}-\frac{\sigma}{2})$ and variance t. Hence $(\frac{\sigma}{\sqrt{n}}M_{nt,n})_n$ converges to a Gaussian random variable with mean $t(r-\frac{\sigma^2}{2})$ and variance $\sigma^2 t$. □

2.4 Stochastic Calculus

4.1.

Proof. Fix t and for any $s<t$, we assume $s\in[t_m,t_{m+1})$ for some m.

Case 1. $m=k$. Then $I(t)-I(s) = \Delta_{t_k}(M_t-M_{t_k}) - \Delta_{t_k}(M_s-M_{t_k}) = \Delta_{t_k}(M_t-M_s)$. So $E[I(t)-I(s)|\mathcal{F}_t] = \Delta_{t_k}E[M_t-M_s|\mathcal{F}_s] = 0$.

Case 2. $m<k$. Then $t_m \le s < t_{m+1} \le t_k \le t < t_{k+1}$. So

$$
\begin{aligned}
I(t)-I(s) &= \sum_{j=m}^{k-1}\Delta_{t_j}(M_{t_{j+1}}-M_{t_j}) + \Delta_{t_k}(M_s-M_{t_k}) - \Delta_{t_m}(M_s-M_{t_m}) \\
&= \sum_{j=m+1}^{k-1}\Delta_{t_j}(M_{t_{j+1}}-M_{t_j}) + \Delta_{t_k}(M_t-M_{t_k}) + \Delta_{t_m}(M_{t_{m+1}}-M_s).
\end{aligned}
$$

Hence

$$
\begin{aligned}
& E[I(t)-I(s)|\mathcal{F}_s] \\
= \quad & \sum_{j=m+1}^{k-1}E[\Delta_{t_j}E[M_{t_{j+1}}-M_{t_j}|\mathcal{F}_{t_j}]|\mathcal{F}_s] + E[\Delta_{t_k}E[M_t-M_{t_k}|\mathcal{F}_{t_k}]|\mathcal{F}_s] + \Delta_{t_m}E[M_{t_{m+1}}-M_s|\mathcal{F}_s] \\
= \quad & 0.
\end{aligned}
$$

Combined, we conclude $I(t)$ is a martingale. □

4.2. (i)

Proof. We follow the simplification in the hint and consider $I(t_k) - I(t_l)$ with $t_l < t_k$. Then $I(t_k) - I(t_l) = \sum_{j=l}^{k-1} \Delta_{t_j}(W_{t_{j+1}} - W_{t_j})$. Since Δ_t is a non-random process and $W_{t_{j+1}} - W_{t_j} \perp \mathcal{F}_{t_j} \supset \mathcal{F}_{t_l}$ for $j \geq l$, we must have $I(t_k) - I(t_l) \perp \mathcal{F}_{t_l}$. □

(ii)

Proof. We use the notation in (i) and it is clear $I(t_k) - I(t_l)$ is normal since it is a linear combination of independent normal random variables. Furthermore, $E[I(t_k) - I(t_l)] = \sum_{j=l}^{k-1} \Delta_{t_j} E[W_{t_{j+1}} - W_{t_j}] = 0$ and $Var(I(t_k) - I(t_l)) = \sum_{j=l}^{k-1} \Delta_{t_j}^2 Var(W_{t_{j+1}} - W_{t_j}) = \sum_{j=l}^{k-1} \Delta_{t_j}^2 (t_{j+1} - t_j) = \int_{t_l}^{t_k} \Delta_u^2 du$. □

(iii)

Proof. $E[I(t) - I(s)|\mathcal{F}_s] = E[I(t) - I(s)] = 0$, for $s < t$. □

(iv)

Proof. For $s < t$,

$$\begin{aligned}
& E[I^2(t) - \int_0^t \Delta_u^2 du - (I^2(s) - \int_0^s \Delta_u^2 du)|\mathcal{F}_s] \\
= \; & E[I^2(t) - I^2(s) - \int_s^t \Delta_u^2 du|\mathcal{F}_s] \\
= \; & E[(I(t) - I(s))^2 + 2I(t)I(s) - 2I^2(s)|\mathcal{F}_s] - \int_s^t \Delta_u^2 du \\
= \; & E[(I(t) - I(s))^2] + 2I(s)E[I(t) - I(s)|\mathcal{F}_s] - \int_s^t \Delta_u^2 du \\
= \; & \int_s^t \Delta_u^2 du + 0 - \int_s^t \Delta_u^2 du \\
= \; & 0.
\end{aligned}$$

□

4.3.

Proof. $I(t) - I(s) = \Delta_0(W_{t_1} - W_0) + \Delta_{t_1}(W_{t_2} - W_{t_1}) - \Delta_0(W_{t_1} - W_0) = \Delta_{t_1}(W_{t_2} - W_{t_1}) = W_s(W_t - W_s)$.

(i) $I(t) - I(s)$ is not independent of $\mathcal{F}_s$, since $W_s \in \mathcal{F}_s$.

(ii) $E[(I(t) - I(s))^4] = E[W_s^4]E[(W_t - W_s)^4] = 3s \cdot 3(t - s) = 9s(t - s)$ and $3E[(I(t) - I(s))^2] = 3E[W_s^2]E[(W_t - W_s)^2] = 3s(t - s)$. Since $E[(I(t) - I(s))^4] \neq 3E[(I(t) - I(s))^2]$, $I(t) - I(s)$ is not normally distributed.

(iii) $E[I(t) - I(s)|\mathcal{F}_s] = W_s E[W_t - W_s|\mathcal{F}_s] = 0$.

(iv)

$$\begin{aligned}
& E[I^2(t) - \int_0^t \Delta_u^2 du - (I^2(s) - \int_0^s \Delta_u^2 du)|\mathcal{F}_s] \\
= \; & E[(I(t) - I(s))^2 + 2I(t)I(s) - 2I^2(s) - \int_s^t \Delta_u^2 du|\mathcal{F}_s] \\
= \; & E[(I(t) - I(s))^2 + 2I(s)(I(t) - I(s)) - \int_s^t W_s^2 1_{(s,t]}(u) du|\mathcal{F}_s] \\
= \; & E[W_s^2(W_t - W_s)^2 + 2\Delta_0 W_s^2(W_t - W_s) - W_s^2(t - s)|\mathcal{F}_s] \\
= \; & W_s^2 E[(W_t - W_s)^2] + 2\Delta_0 W_s^2 E[W_t - W_s|\mathcal{F}_s] - W_s^2(t - s) \\
= \; & W_s^2(t - s) - W_s^2(t - s) \\
= \; & 0.
\end{aligned}$$

$\square$

4.4.

Proof. (Cf. Øksendal [3], Exercise 3.9.) We first note that

$$\begin{aligned}
&\sum_j B_{\frac{t_j+t_{j+1}}{2}}(B_{t_{j+1}} - B_{t_j}) \\
= &\sum_j \left[B_{\frac{t_j+t_{j+1}}{2}}(B_{t_{j+1}} - B_{\frac{t_j+t_{j+1}}{2}}) + B_{t_j}(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j})\right] + \sum_j (B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j})^2.
\end{aligned}$$

The first term converges in $L^2(P)$ to $\int_0^T B_t dB_t$. For the second term, we note

$$\begin{aligned}
&E\left[\left(\sum_j \left(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j}\right)^2 - \frac{t}{2}\right)^2\right] \\
= \; & E\left[\left(\sum_j \left(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j}\right)^2 - \sum_j \frac{t_{j+1}-t_j}{2}\right)^2\right] \\
= \; & \sum_{j,\,k} E\left[\left(\left(B_{\frac{t_j+t_{j+1}}{2}} - B_{t_j}\right)^2 - \frac{t_{j+1}-t_j}{2}\right)\left(\left(B_{\frac{t_k+t_{k+1}}{2}} - B_{t_k}\right)^2 - \frac{t_{k+1}-t_k}{2}\right)\right] \\
= \; & \sum_j E\left[\left(B^2_{\frac{t_{j+1}-t_j}{2}} - \frac{t_{j+1}-t_j}{2}\right)^2\right] \\
= \; & \sum_j 2\cdot\left(\frac{t_{j+1}-t_j}{2}\right)^2 \\
\leq \; & \frac{T}{2}\max_{1\leq j\leq n}|t_{j+1}-t_j| \to 0,
\end{aligned}$$

since $E[(B_t^2-t)^2] = E[B_t^4 - 2tB_t^2 + t^2] = 3E[B_t^2]^2 - 2t^2 + t^2 = 2t^2$. So

$$\sum_j B_{\frac{t_j+t_{j+1}}{2}}(B_{t_{j+1}} - B_{t_j}) \to \int_0^T B_t dB_t + \frac{T}{2} = \frac{1}{2}B_T^2 \quad \text{in } L^2(P).$$

$\square$

4.5. (i)

Proof.

$$d\ln S_t = \frac{dS_t}{S_t} - \frac{1}{2}\frac{d\langle S\rangle_t}{S_t^2} = \frac{2S_t dS_t - d\langle S\rangle_t}{2S_t^2} = \frac{2S_t(\alpha_t S_t dt + \sigma_t S_t dW_t) - \sigma_t^2 S_t^2 dt}{2S_t^2} = \sigma_t dW_t + (\alpha_t - \frac{1}{2}\sigma_t^2)dt.$$

$\square$

(ii)

Proof.

$$\ln S_t = \ln S_0 + \int_0^t \sigma_s dW_s + \int_0^t (\alpha_s - \frac{1}{2}\sigma_s^2)ds.$$

So $S_t = S_0 \exp\{\int_0^t \sigma_s dW_s + \int_0^t (\alpha_s - \frac{1}{2}\sigma_s^2)ds\}$. $\square$

4.6.

Proof. Without loss of generality, we assume $p \neq 1$. Since $(x^p)' = px^{p-1}$, $(x^p)'' = p(p-1)x^{p-2}$, we have

$$\begin{aligned} d(S_t^p) &= pS_t^{p-1}dS_t + \frac{1}{2}p(p-1)S_t^{p-2}d\langle S\rangle_t \\ &= pS_t^{p-1}(\alpha S_t dt + \sigma S_t dW_t) + \frac{1}{2}p(p-1)S_t^{p-2}\sigma^2 S_t^2 dt \\ &= S_t^p[p\alpha dt + p\sigma dW_t + \frac{1}{2}p(p-1)\sigma^2 dt] \\ &= S_t^p p[\sigma dW_t + (\alpha + \frac{p-1}{2}\sigma^2)dt]. \end{aligned}$$

□

4.7. (i)

Proof. $dW_t^4 = 4W_t^3 dW_t + \frac{1}{2}\cdot 4\cdot 3W_t^2 d\langle W\rangle_t = 4W_t^3 dW_t + 6W_t^2 dt$. So $W_T^4 = 4\int_0^T W_t^3 dW_t + 6\int_0^T W_t^2 dt$. □

(ii)

Proof. $E[W_T^4] = 6\int_0^T t dt = 3T^2$. □

(iii)

Proof. $dW_t^6 = 6W_t^5 dW_t + \frac{1}{2}\cdot 6\cdot 5W_t^4 dt$. So $W_T^6 = 6\int_0^T W_t^5 dW_t + 15\int_0^T W_t^4 dt$. Hence $E[W_T^6] = 15\int_0^T 3t^2 dt = 15T^3$. □

4.8.

Proof. $d(e^{\beta t}R_t) = \beta e^{\beta t}R_t dt + e^{\beta t}dR_t = e^{\beta t(\alpha dt + \sigma dW_t)}$. Hence

$$e^{\beta t}R_t = R_0 + \int_0^t e^{\beta s}(\alpha ds + \sigma dW_s) = R_0 + \frac{\alpha}{\beta}(e^{\beta t}-1) + \sigma\int_0^t e^{\beta s}dW_s,$$

and $R_t = R_0 e^{-\beta t} + \frac{\alpha}{\beta}(1-e^{-\beta t}) + \sigma\int_0^t e^{-\beta(t-s)}dW_s$. □

4.9. (i)

Proof.

$$\begin{aligned} Ke^{-r(T-t)}N'(d_-) &= Ke^{-r(T-t)}\frac{e^{-\frac{d_-^2}{2}}}{\sqrt{2\pi}} \\ &= Ke^{-r(T-t)}\frac{e^{-\frac{(d_+ - \sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}} \\ &= Ke^{-r(T-t)}e^{\sigma\sqrt{T-t}d_+}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_+) \\ &= Ke^{-r(T-t)}\frac{x}{K}e^{(r+\frac{\sigma^2}{2})(T-t)}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_+) \\ &= xN'(d_+). \end{aligned}$$

□

(ii)

Proof.

$$\begin{aligned}
c_x &= N(d_+) + xN'(d_+)\frac{\partial}{\partial x}d_+(T-t,x) - Ke^{-r(T-t)}N'(d_-)\frac{\partial}{\partial x}d_-(T-t,x) \\
&= N(d_+) + xN'(d_+)\frac{\partial}{\partial x}d'_+(T-t,x) - xN'(d_+)\frac{\partial}{\partial x}d_+(T-t,x) \\
&= N(d_+).
\end{aligned}$$

□

(iii)

Proof.

$$\begin{aligned}
c_t &= xN'(d_+)\frac{\partial}{\partial x}d_+(T-t,x) - rKe^{-r(T-t)}N(d_-) - Ke^{-r(T-t)}N'(d_-)\frac{\partial}{\partial t}d_-(T-t,x) \\
&= xN'(d_+)\frac{\partial}{\partial t}d_+(T-t,x) - rKe^{-r(T-t)}N(d_-) - xN'(d_+)\left[\frac{\partial}{\partial t}d_+(T-t,x) + \frac{\sigma}{2\sqrt{T-t}}\right] \\
&= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+).
\end{aligned}$$

□

(iv)

Proof.

$$\begin{aligned}
&c_t + rxc_x + \frac{1}{2}\sigma^2x^2c_{xx} \\
=\ & -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) + \frac{1}{2}\sigma^2x^2N'(d_+)\frac{\partial}{\partial x}d_+(T-t,x) \\
=\ & rc - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + \frac{1}{2}\sigma^2x^2N'(d_+)\frac{1}{\sigma\sqrt{T-t}x} \\
=\ & rc.
\end{aligned}$$

□

(v)

Proof. For $x > K$, $d_+(T-t,x) > 0$ and $\lim_{t\uparrow T} d_+(T-t,x) = \lim_{\tau\downarrow 0} d_+(\tau,x) = \infty$. $\lim_{t\uparrow T} d_-(T-t,x) = \lim_{\tau\downarrow 0} d_-(\tau,x) = \lim_{\tau\downarrow 0}\left(\frac{1}{\sigma\sqrt{\tau}}\ln\frac{x}{K} + \frac{1}{\sigma}(r+\frac{1}{2}\sigma^2)\sqrt{\tau} - \sigma\sqrt{\tau}\right) = \infty$. Similarly, $\lim_{t\uparrow T} d_\pm = -\infty$ for $x \in (0,K)$. Also it's clear that $\lim_{t\uparrow T} d_\pm = 0$ for $x = K$. So

$$\lim_{t\uparrow T} c(t,x) = xN(\lim_{t\uparrow T} d_+) - KN(\lim_{t\uparrow T} d_-) = \begin{cases} x-K, & \text{if } x > K \\ 0, & \text{if } x \le K \end{cases} = (x-K)^+.$$

□

(vi)

Proof. It is easy to see $\lim_{x\downarrow 0} d_\pm = -\infty$. So for $t \in [0,T]$, $\lim_{x\downarrow 0} c(t,x) = \lim_{x\downarrow 0} xN(\lim_{x\downarrow} d_+(T-t,x)) - Ke^{-r(T-t)}N(\lim_{x\downarrow 0} d_-(T-t,x)) = 0$. □

(vii)

Proof. For $t \in [0,T]$, it is clear $\lim_{x\to\infty} d_\pm = \infty$. Note

$$\lim_{x\to\infty} x(N(d_+)-1) = \lim_{x\to\infty} \frac{N'(d_+)\frac{\partial}{\partial x}d_+}{-x^{-2}} = \lim_{x\to\infty} \frac{N'(d_+)\frac{1}{\sigma\sqrt{T-t}}}{-x^{-1}}.$$

By the expression of d_+, we get $x = K\exp\{\sigma\sqrt{T-t}d_+ - (T-t)(r+\frac{1}{2}\sigma^2)\}$. So we have

$$\lim_{x\to\infty} x(N(d_+)-1) = \lim_{x\to\infty} N'(d_+)\frac{-x}{\sigma\sqrt{T-t}} = \lim_{d_+\to\infty} \frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}} \frac{-Ke^{\sigma\sqrt{T-t}d_+-(T-t)(r+\frac{1}{2}\sigma^2)}}{\sigma\sqrt{T-t}} = 0.$$

Therefore

$$\begin{aligned}
& \lim_{x\to\infty}[c(t,x) - (x - e^{-r(T-t)}K)] \\
= \; & \lim_{x\to\infty}[xN(d_+) - Ke^{-r(T-t)N(d_-)} - x + Ke^{-r(T-t)}] \\
= \; & \lim_{x\to\infty}[x(N(d_+)-1) + Ke^{-r(T-t)}(1-N(d_-))] \\
= \; & \lim_{x\to\infty} x(N(d_+)-1) + Ke^{-r(T-t)}(1-N(\lim_{x\to\infty} d_-)) \\
= \; & 0.
\end{aligned}$$

□

4.10. (i)

Proof. We show (4.10.16) + (4.10.9) $\Longleftrightarrow$ (4.10.16) + (4.10.15), i.e. assuming X has the representation $X_t = \Delta_t S_t + \Gamma_t M_t$, "continuous-time self-financing condition" has two equivalent formulations, (4.10.9) or (4.10.15). Indeed, $dX_t = \Delta_t dS_t + \Gamma_t dM_t + (S_t d\Delta_t + dS_t d\Delta_t + M_t d\Gamma_t + dM_t d\Gamma_t)$. So $dX_t = \Delta_t dS_t + \Gamma_t dM_t \Longleftrightarrow S_t d\Delta_t + dS_t d\Delta_t + M_t d\Gamma_t + dM_t d\Gamma_t = 0$, i.e. (4.10.9) $\Longleftrightarrow$ (4.10.15). □

(ii)

Proof. First, we clarify the problems by stating explicitly the given conditions and the result to be proved. We assume we have a portfolio $X_t = \Delta_t S_t + \Gamma_t M_t$. We let $c(t,S_t)$ denote the price of call option at time t and set $\Delta_t = c_x(t,S_t)$. Finally, we assume the portfolio is self-financing. The problem is to show

$$rN_t dt = \left[c_t(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t,S_t)\right]dt,$$

where $N_t = c(t,S_t) - \Delta_t S_t$.

Indeed, by the self-financing property and $\Delta_t = c_x(t,S_t)$, we have $c(t,S_t) = X_t$ (by the calculations in Subsection 4.5.1-4.5.3). This uniquely determines Γ_t as

$$\Gamma_t = \frac{X_t - \Delta_t S_t}{M_t} = \frac{c(t,S_t) - c_x(t,S_t)S_t}{M_t} = \frac{N_t}{M_t}.$$

Moreover,

$$\begin{aligned}
dN_t &= \left[c_t(t,S_t)dt + c_x(t,S_t)dS_t + \frac{1}{2}c_{xx}(t,S_t)d\langle S_t\rangle_t\right] - d(\Delta_t S_t) \\
&= \left[c_t(t,S_t) + \frac{1}{2}c_{xx}(t,S_t)\sigma^2 S_t^2\right]dt + [c_x(t,S_t)dS_t - d(X_t - \Gamma_t M_t)] \\
&= \left[c_t(t,S_t) + \frac{1}{2}c_{xx}(t,S_t)\sigma^2 S_t^2\right]dt + M_t d\Gamma_t + dM_t d\Gamma_t + [c_x(t,S_t)dS_t + \Gamma_t dM_t - dX_t].
\end{aligned}$$

By self-financing property, $c_x(t,S_t)dt + \Gamma_t dM_t = \Delta_t dS_t + \Gamma_t dM_t = dX_t$, so

$$\left[c_t(t,S_t) + \frac{1}{2}c_{xx}(t,S_t)\sigma^2 S_t^2\right]dt = dN_t - M_t d\Gamma_t - dM_t d\Gamma_t = \Gamma_t dM_t = \Gamma_t r M_t dt = rN_t dt.$$

□

4.11.

Proof. First, we note $c(t,x)$ solves the Black-Scholes-Merton PDE with volatility σ_1:

$$\left(\frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}x^2\sigma_1^2\frac{\partial^2}{\partial x^2} - r\right)c(t,x) = 0.$$

So

$$c_t(t,S_t) + rS_t c_x(t,S_t) + \frac{1}{2}\sigma_1^2 S_t^2 c_{xx}(t,S_t) - rc(t,S_t) = 0,$$

and

$$\begin{aligned}
dc(t,S_t) &= c_t(t,S_t)dt + c_x(t,S_t)(\alpha S_t dt + \sigma_2 S_t dW_t) + \frac{1}{2}c_{xx}(t,S_t)\sigma_2^2 S_t^2 dt \\
&= \left[c_t(t,S_t) + \alpha c_x(t,S_t)S_t + \frac{1}{2}\sigma_2^2 S_t^2 c_{xx}(t,S_t)\right]dt + \sigma_2 S_t c_x(t,S_t)dW_t \\
&= \left[rc(t,S_t) + (\alpha - r)c_x(t,S_t)S_t + \frac{1}{2}S_t^2(\sigma_2^2 - \sigma_1^2)c_{xx}(t,S_t)\right]dt + \sigma_2 S_t c_x(t,S_t)dW_t.
\end{aligned}$$

Therefore

$$\begin{aligned}
dX_t &= \Bigg[rc(t,S_t) + (\alpha - r)c_x(t,S_t)S_t + \frac{1}{2}S_t^2(\sigma_2^2 - \sigma_1^2)\sigma_{xx}(t,S_t) + rX_t - rc(t,S_t) + rS_t c_x(t,S_t) \\
&\quad -\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S_t^2 c_{xx}(t,S_t) - c_x(t,S_t)\alpha S_t\Bigg]dt + [\sigma_2 S_t c_x(t,S_t) - c_x(t,S_t)\sigma_2 S_t]dW_t \\
&= rX_t dt.
\end{aligned}$$

This implies $X_t = X_0 e^{rt}$. By X_0, we conclude $X_t = 0$ for all $t \in [0,T]$.

□

4.12. (i)

Proof. By (4.5.29), $c(t,x) - p(t,x) = x - e^{-r(T-t)}K$. So $p_x(t,x) = c_x(t,x) - 1 = N(d_+(T-t,x)) - 1$, $p_{xx}(t,x) = c_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x))$ and

$$\begin{aligned}
p_t(t,x) &= c_t(t,x) + re^{-r(T-t)}K \\
&= -rKe^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) + rKe^{-r(T-t)} \\
&= rKe^{-r(T-t)}N(-d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)).
\end{aligned}$$

□

(ii)

Proof. For an agent hedging a short position in the put, since $\Delta_t = p_x(t,x) < 0$, he should short the underlying stock and put $p(t,S_t) - p_x(t,S_t)S_t (> 0)$ cash in the money market account. □

(iii)

Proof. By the put-call parity, it suffices to show $f(t,x) = x - Ke^{-r(T-t)}$ satisfies the Black-Scholes-Merton partial differential equation. Indeed,

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r\right)f(t,x) = -rKe^{-r(T-t)} + \frac{1}{2}\sigma^2 x^2 \cdot 0 + rx \cdot 1 - r(x - Ke^{-r(T-t)}) = 0.$$

Remark: The Black-Scholes-Merton PDE has many solutions. Proper boundary conditions are the key to uniqueness. For more details, see Wilmott [8]. □

4.13.

Proof. We suppose (W_1, W_2) is a pair of local martingale defined by SDE

$$\begin{cases} dW_1(t) = dB_1(t) \\ dW_2(t) = \alpha(t)dB_1(t) + \beta(t)dB_2(t). \end{cases} \tag{2.1}$$

We want to find $\alpha(t)$ and $\beta(t)$ such that

$$\begin{cases} (dW_2(t))^2 = [\alpha^2(t) + \beta^2(t) + 2\rho(t)\alpha(t)\beta(t)]dt = dt \\ dW_1(t)dW_2(t) = [\alpha(t) + \beta(t)\rho(t)]dt = 0. \end{cases} \tag{2.2}$$

Solve the equation for $\alpha(t)$ and $\beta(t)$, we have $\beta(t) = \frac{1}{\sqrt{1-\rho^2(t)}}$ and $\alpha(t) = -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}$. So

$$\begin{cases} W_1(t) = B_1(t) \\ W_2(t) = \int_0^t \frac{-\rho(s)}{\sqrt{1-\rho^2(s)}} dB_1(s) + \int_0^t \frac{1}{\sqrt{1-\rho^2(s)}} dB_2(s) \end{cases} \tag{2.3}$$

is a pair of independent BM's. Equivalently, we have

$$\begin{cases} B_1(t) = W_1(t) \\ B_2(t) = \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s). \end{cases} \tag{2.4}$$

□

4.14. (i)

Proof. Clearly $Z_j \in \mathcal{F}_{t_{j+1}}$. Moreover

$$E[Z_j|\mathcal{F}_{t_j}] = f''(W_{t_j})E[(W_{t_{j+1}} - W_{t_j})^2 - (t_{j+1} - t_j)|\mathcal{F}_{t_j}] = f''(W_{t_j})(E[W^2_{t_{j+1}-t_j}] - (t_{j+1} - t_j)) = 0,$$

since $W_{t_{j+1}} - W_{t_j}$ is independent of $\mathcal{F}_{t_j}$ and $W_t \sim N(0, t)$. Finally, we have

$$\begin{aligned} E[Z_j^2|\mathcal{F}_{t_j}] &= [f''(W_{t_j})]^2 E[(W_{t_{j+1}} - W_{t_j})^4 - 2(t_{j+1} - t_j)(W_{t_{j+1}} - W_{t_j})^2 + (t_{j+1} - t_j)^2|\mathcal{F}_{t_j}] \\ &= [f''(W_{t_j})]^2 (E[W^4_{t_{j+1}-t_j}] - 2(t_{j+1} - t_j)E[W^2_{t_{j+1}-t_j}] + (t_{j+1} - t_j)^2) \\ &= [f''(W_{t_j})]^2 [3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2] \\ &= 2[f''(W_{t_j})]^2 (t_{j+1} - t_j)^2, \end{aligned}$$

where we used the independence of Browian motion increment and the fact that $E[X^4] = 3E[X^2]^2$ if X is Gaussian with mean 0. □

(ii)

Proof. $E[\sum_{j=0}^{n-1} Z_j] = E[\sum_{j=0}^{n-1} E[Z_j|\mathcal{F}_{t_j}]] = 0$ by part (i). □

(iii)

Proof.

$$
\begin{aligned}
Var[\sum_{j=0}^{n-1} Z_j] &= E[(\sum_{j=0}^{n-1} Z_j)^2] \\
&= E[\sum_{j=0}^{n-1} Z_j^2 + 2 \sum_{0\le i<j\le n-1} Z_i Z_j] \\
&= \sum_{j=0}^{n-1} E[E[Z_j^2|\mathcal{F}_{t_j}]] + 2 \sum_{0\le i<j\le n-1} E[Z_i E[Z_j|\mathcal{F}_{t_j}]] \\
&= \sum_{j=0}^{n-1} E[2[f''(W_{t_j})]^2 (t_{j+1}-t_j)^2] \\
&= \sum_{j=0}^{n-1} 2E[(f''(W_{t_j}))^2](t_{j+1}-t_j)^2 \\
&\le 2 \max_{0\le j\le n-1} |t_{j+1}-t_j| \cdot \sum_{j=0}^{n-1} E[(f''(W_{t_j}))^2](t_{j+1}-t_j) \\
&\to 0,
\end{aligned}
$$

since $\sum_{j=0}^{n-1} E[(f''(W_{t_j}))^2](t_{j+1}-t_j) \to \int_0^T E[(f''(W_t))^2]dt < \infty$. □

4.15. (i)

Proof. B_i is a local martingale with

$$
(dB_i(t))^2 = \left(\sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t)\right)^2 = \sum_{j=1}^d \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)} dt = dt.
$$

So B_i is a Brownian motion. □

(ii)

Proof.

$$
\begin{aligned}
dB_i(t)dB_k(t) &= \left[\sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t)\right]\left[\sum_{l=1}^d \frac{\sigma_{kl}(t)}{\sigma_k(t)} dW_l(t)\right] \\
&= \sum_{1\le j,\, l\le d} \frac{\sigma_{ij}(t)\sigma_{kl}(t)}{\sigma_i(t)\sigma_k(t)} dW_j(t) dW_l(t) \\
&= \sum_{j=1}^d \frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)} dt \\
&= \rho_{ik}(t) dt.
\end{aligned}
$$

□

4.16.

Proof. To find the m independent Brownion motion $W_1(t), \cdots, W_m(t)$, we need to find $A(t) = (a_{ij}(t))$ so that

$$
(dB_1(t), \cdots, dB_m(t))^{tr} = A(t)(dW_1(t), \cdots, dW_m(t))^{tr},
$$

or equivalently

$$(dW_1(t),\cdots,dW_m(t))^{tr}=A(t)^{-1}(dB_1(t),\cdots,dB_m(t))^{tr},$$

and

$$\begin{aligned}
& (dW_1(t),\cdots,dW_m(t))^{tr}(dW_1(t),\cdots,dW_m(t))\\
= & A(t)^{-1}(dB_1(t),\cdots,dB_m(t))^{tr}(dB_1(t),\cdots,dB_m(t))(A(t)^{-1})^{tr}dt\\
= & I_{m\times m}dt,
\end{aligned}$$

where $I_{m\times m}$ is the $m\times m$ unit matrix. By the condition $dB_i(t)dB_k(t)=\rho_{ik}(t)dt$, we get

$$(dB_1(t),\cdots,dB_m(t))^{tr}(dB_1(t),\cdots,dB_m(t))=C(t).$$

So $A(t)^{-1}C(t)(A(t)^{-1})^{tr}=I_{m\times m}$, which gives $C(t)=A(t)A(t)^{tr}$. This motivates us to define A as the square root of C. Reverse the above analysis, we obtain a formal proof. □

4.17.

Proof. We will try to solve all the sub-problems in a single, long solution. We start with the general X_i:

$$X_i(t)=X_i(0)+\int_0^t\theta_i(u)du+\int_0^t\sigma_i(u)dB_i(u),\ i=1,2.$$

The goal is to show

$$\lim_{\epsilon\downarrow 0}\frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}}=\rho(t_0).$$

First, for $i=1,2$, we have

$$\begin{aligned}
M_i(\epsilon) &= E[X_i(t_0+\epsilon)-X_i(t_0)|\mathcal{F}_{t_0}]\\
&= E\left[\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du+\int_{t_0}^{t_0+\epsilon}\sigma_i(u)dB_i(u)|\mathcal{F}_{t_0}\right]\\
&= \Theta_i(t_0)\epsilon+E\left[\int_{t_0}^{t_0+\epsilon}(\Theta_i(u)-\Theta_i(t_0))du|\mathcal{F}_{t_0}\right].
\end{aligned}$$

By Conditional Jensen's Inequality,

$$\left|E\left[\int_{t_0}^{t_0+\epsilon}(\Theta_i(u)-\Theta_i(t_0))du|\mathcal{F}_{t_0}\right]\right|\le E\left[\int_{t_0}^{t_0+\epsilon}|\Theta_i(u)-\Theta_i(t_0)|du|\mathcal{F}_{t_0}\right]$$

Since $\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}|\Theta_i(u)-\Theta_i(t_0)|du\le 2M$ and $\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}|\Theta_i(u)-\Theta_i(t_0)|du=0$ by the continuity of Θ_i, the Dominated Convergence Theorem under Conditional Expectation implies

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}E\left[\int_{t_0}^{t_0+\epsilon}|\Theta_i(u)-\Theta_i(t_0)|du|\mathcal{F}_{t_0}\right]=E\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}|\Theta_i(u)-\Theta_i(t_0)|du|\mathcal{F}_{t_0}\right]=0.$$

So $M_i(\epsilon)=\Theta_i(t_0)\epsilon+o(\epsilon)$. This proves (iii).

To calculate the variance and covariance, we note $Y_i(t)=\int_0^t\sigma_i(u)dB_i(u)$ is a martingale and by Itô's formula $Y_i(t)Y_j(t)-\int_0^t\sigma_i(u)\sigma_j(u)du$ is a martingale ($i=1,2$). So

$$\begin{aligned}
& E[(X_i(t_0+\epsilon)-X_i(t_0))(X_j(t_0+\epsilon)-X_j(t_0))|\mathcal{F}_{t_0}]\\
= & E\left[\left(Y_i(t_0+\epsilon)-Y_i(t_0)+\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du\right)\left(Y_j(t_0+\epsilon)-Y_j(t_0)+\int_{t_0}^{t_0+\epsilon}\Theta_j(u)du\right)|\mathcal{F}_{t_0}\right]\\
= & E\left[(Y_i(t_0+\epsilon)-Y_i(t_0))\left(Y_j(t_0+\epsilon)-Y_j(t_0)\right)|\mathcal{F}_{t_0}\right]+E\left[\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du\int_{t_0}^{t_0+\epsilon}\Theta_j(u)du|\mathcal{F}_{t_0}\right]\\
& +E\left[(Y_i(t_0+\epsilon)-Y_i(t_0))\int_{t_0}^{t_0+\epsilon}\Theta_j(u)du|\mathcal{F}_{t_0}\right]+E\left[(Y_j(t_0+\epsilon)-Y_j(t_0))\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du|\mathcal{F}_{t_0}\right]\\
= & I+II+III+IV.
\end{aligned}$$

$$I = E[Y_i(t_0+\epsilon)Y_j(t_0+\epsilon) - Y_i(t_0)Y_j(t_0)|\mathcal{F}_{t_0}] = E\left[\int_{t_0}^{t_0+\epsilon} \sigma_i(u)\sigma_j(u)\rho_{ij}(t)dt|\mathcal{F}_{t_0}\right].$$

By an argument similar to that involved in the proof of part (iii), we conclude $I = \sigma_i(t_0)\sigma_j(t_0)\rho_{ij}(t_0)\epsilon + o(\epsilon)$ and

$$\begin{aligned} II &= E\left[\int_{t_0}^{t_0+\epsilon}(\Theta_i(u)-\Theta_i(t_0))du\int_{t_0}^{t_0+\epsilon}\Theta_j(u)du|\mathcal{F}_{t_0}\right] + \Theta_i(t_0)\epsilon E\left[\int_{t_0}^{t_0+\epsilon}\Theta_j(u)du|\mathcal{F}_{t_0}\right] \\ &= o(\epsilon) + (M_i(\epsilon)-o(\epsilon))M_j(\epsilon) \\ &= M_i(\epsilon)M_j(\epsilon) + o(\epsilon). \end{aligned}$$

By Cauchy's inequality under conditional expectation (note $E[XY|\mathcal{F}]$ defines an inner product on $L^2(\Omega)$),

$$\begin{aligned} III &\le E\left[|Y_i(t_0+\epsilon)-Y_i(t_0)|\int_{t_0}^{t_0+\epsilon}|\Theta_j(u)|du|\mathcal{F}_{t_0}\right] \\ &\le M\epsilon\sqrt{E[(Y_i(t_0+\epsilon)-Y_i(t_0))^2|\mathcal{F}_{t_0}]} \\ &\le M\epsilon\sqrt{E[Y_i(t_0+\epsilon)^2-Y_i(t_0)^2|\mathcal{F}_{t_0}]} \\ &\le M\epsilon\sqrt{E[\int_{t_0}^{t_0+\epsilon}\Theta_i(u)^2du|\mathcal{F}_{t_0}]} \\ &\le M\epsilon\cdot M\sqrt{\epsilon} \\ &= o(\epsilon) \end{aligned}$$

Similarly, $IV = o(\epsilon)$. In summary, we have

$$E[(X_i(t_0+\epsilon)-X_i(t_0))(X_j(t_0+\epsilon)-X_j(t_0))|\mathcal{F}_{t_0}] = M_i(\epsilon)M_j(\epsilon) + \sigma_i(t_0)\sigma_j(t_0)\rho_{ij}(t_0)\epsilon + o(\epsilon) + o(\epsilon).$$

This proves part (iv) and (v). Finally,

$$\lim_{\epsilon\downarrow 0}\frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}} = \lim_{\epsilon\downarrow 0}\frac{\rho(t_0)\sigma_1(t_0)\sigma_2(t_0)\epsilon + o(\epsilon)}{\sqrt{(\sigma_1^2(t_0)\epsilon+o(\epsilon))(\sigma_2^2(t_0)\epsilon+o(\epsilon))}} = \rho(t_0).$$

This proves part (vi). Part (i) and (ii) are consequences of general cases.

□

4.18. (i)

Proof.

$$d(e^{rt}\zeta_t) = (de^{-\theta W_t - \frac{1}{2}\theta^2 t}) = -e^{-\theta W_t - \frac{1}{2}\theta^2 t}\theta dW_t = -\theta(e^{rt}\zeta_t)dW_t,$$

where for the second "=", we used the fact that $e^{-\theta W_t - \frac{1}{2}\theta^2 t}$ solves $dX_t = -\theta X_t dW_t$. Since $d(e^{rt}\zeta_t) = re^{rt}\zeta_t dt + e^{rt}d\zeta_t$, we get $d\zeta_t = -\theta\zeta_t dW_t - r\zeta_t dt$. □

(ii)

Proof.

$$\begin{aligned} d(\zeta_t X_t) &= \zeta_t dX_t + X_t d\zeta_t + dX_t d\zeta_t \\ &= \zeta_t(rX_t dt + \Delta_t(\alpha - r)S_t dt + \Delta_t\sigma S_t dW_t) + X_t(-\theta\zeta_t dW_t - r\zeta_t dt) \\ &\quad +(rX_t dt + \Delta_t(\alpha-r)S_t dt + \Delta_t\sigma S_t dW_t)(-\theta\zeta_t dW_t - r\zeta_t dt) \\ &= \zeta_t(\Delta_t(\alpha-r)S_t dt + \Delta_t\sigma S_t dW_t) - \theta X_t\zeta_t dW_t - \theta\Delta_t\sigma S_t\zeta_t dt \\ &= \zeta_t\Delta_t\sigma S_t dW_t - \theta X_t\zeta_t dW_t. \end{aligned}$$

So $\zeta_t X_t$ is a martingale. □

(iii)

Proof. By part (ii), $X_0 = \zeta_0 X_0 = E[\zeta_T X_t] = E[\zeta_T V_T]$. (This can be seen as a version of risk-neutral pricing, only that the pricing is carried out under the actual probability measure.) $\square$

4.19. (i)

Proof. B_t is a local martingale with $[B]_t = \int_0^t sign(W_s)^2 ds = t$. So by Lévy's theorem, B_t is a Brownian motion. $\square$

(ii)

Proof. $d(B_t W_t) = B_t dW_t + sign(W_t) W_t dW_t + sign(W_t)dt$. Integrate both sides of the resulting equation and the expectation, we get

$$E[B_t W_t] = \int_0^t E[sign(W_s)]ds = \int_0^t E[1_{\{W_s \geq 0\}} - 1_{\{W_s < 0\}}]ds = \frac{1}{2}t - \frac{1}{2}t = 0.$$

$\square$

(iii)

Proof. By Itô's formula, $dW_t^2 = 2W_t dW_t + dt$. $\square$

(iv)

Proof. By Itô's formula,

$$\begin{aligned}
d(B_t W_t^2) &= B_t dW_t^2 + W_t^2 dB_t + dB_t dW_t^2 \\
&= B_t(2W_t dW_t + dt) + W_t^2 sign(W_t) dW_t + sign(W_t) dW_t (2W_t dW_t + dt) \\
&= 2B_t W_t dW_t + B_t dt + sign(W_t) W_t^2 dW_t + 2 sign(W_t) W_t dt.
\end{aligned}$$

So

$$\begin{aligned}
E[B_t W_t^2] &= E[\int_0^t B_s ds] + 2E[\int_0^t sign(W_s) W_s ds] \\
&= \int_0^t E[B_s]ds + 2\int_0^t E[sign(W_s)W_s]ds \\
&= 2\int_0^t (E[W_s 1_{\{W_s \geq 0\}}] - E[W_s 1_{\{W_s < 0\}}])ds \\
&= 4\int_0^t \int_0^\infty x \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} dx ds \\
&= 4\int_0^t \sqrt{\frac{s}{2\pi}} ds \\
&\neq 0 = E[B_t] \cdot E[W_t^2].
\end{aligned}$$

Since $E[B_t W_t^2] \neq E[B_t] \cdot E[W_t^2]$, B_t and W_t are not independent. $\square$

4.20. (i)

Proof. $f(x) = \begin{cases} x - K, & \text{if } x \geq K \\ 0, & \text{if } x < K. \end{cases}$ So $f'(x) = \begin{cases} 1, & \text{if } x > K \\ \text{undefined}, & \text{if } x = K \\ 0, & \text{if } x < K \end{cases}$ and $f''(x) = \begin{cases} 0, & \text{if } x \neq K \\ \text{undefined}, & \text{if } x = K. \end{cases}$

$\square$

(ii)

Proof. $E[f(W_T)] = \int_K^\infty (x-K)\frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}}dx = \sqrt{\frac{T}{2\pi}}e^{-\frac{K^2}{2T}} - K\Phi(-\frac{K}{\sqrt{T}})$ where Φ is the distribution function of standard normal random variable. If we suppose $\int_0^T f''(W_t)dt = 0$, the expectation of RHS of (4.10.42) is equal to 0. So (4.10.42) cannot hold. □

(iii)

Proof. This is trivial to check. □

(iv)

Proof. If $x = K$, $\lim_{n\to\infty} f_n(x) = \frac{1}{8n} = 0$; if $x > K$, for n large enough, $x \geq K + \frac{1}{2n}$, so $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}(x-K) = x-K$; if $x < K$, for n large enough, $x \leq K - \frac{1}{2n}$, so $lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$. In summary, $lim_{n\to\infty} f_n(x) = (x-K)^+$. Similarly, we can show

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0, & \text{if } x < K \\ \frac{1}{2}, & \text{if } x = K \\ 1, & \text{if } x > K. \end{cases} \tag{2.5}$$

□

(v)

Proof. Fix ω, so that $W_t(\omega) < K$ for any $t \in [0,T]$. Since $W_t(\omega)$ can obtain its maximum on $[0,T]$, there exists n_0, so that for any $n \geq n_0$, $\max_{0\leq t\leq T} W_t(\omega) < K - \frac{1}{2n}$. So

$$L_K(T)(\omega) = \lim_{n\to\infty} n\int_0^T 1_{(K-\frac{1}{2n},K+\frac{1}{2n})}(W_t(\omega))dt = 0.$$

□

(vi)

Proof. Take expectation on both sides of the formula (4.10.45), we have

$$E[L_K(T)] = E[(W_T - K)^+] > 0.$$

So we cannot have $L_K(T) = 0$ a.s.. □

4.21. (i)

Proof. There are two problems. First, the transaction cost could be big due to active trading; second, the purchases and sales cannot be made at exactly the same price K. For more details, see Hull [2]. □

(ii)

Proof. No. The RHS of (4.10.26) is a martingale, so its expectation is 0. But $E[(S_T - K)^+] > 0$. So $X_T \neq (S_T - K)^+$. □

2.5 Risk-Neutral Pricing

5.1. (i)

Proof.

$$\begin{aligned}
df(X_t) &= f'(X_t)dt + \frac{1}{2}f''(X_t)d\langle X\rangle_t \\
&= f(X_t)(dX_t + \frac{1}{2}d\langle X\rangle_t) \\
&= f(X_t)\left[\sigma_t dW_t + (\alpha_t - R_t - \frac{1}{2}\sigma_t^2)dt + \frac{1}{2}\sigma_t^2 dt\right] \\
&= f(X_t)(\alpha_t - R_t)dt + f(X_t)\sigma_t dW_t.
\end{aligned}$$

This is formula (5.2.20). □

(ii)

Proof. $d(D_tS_t) = S_t dD_t + D_t dS_t + dD_t dS_t = -S_t R_t D_t dt + D_t\alpha_t S_t dt + D_t\sigma_t S_t dW_t = D_tS_t(\alpha_t - R_t)dt + D_tS_t\sigma_t dW_t$. This is formula (5.2.20). □

5.2.

Proof. By Lemma 5.2.2., $\widetilde{E}[D_TV_T|\mathcal{F}_t] = E\left[\frac{D_TV_TZ_T}{Z_t}|\mathcal{F}_t\right]$. Therefore (5.2.30) is equivalent to $D_tV_tZ_t = E[D_TV_TZ_T|\mathcal{F}_t]$. □

5.3. (i)

Proof.

$$\begin{aligned}
c_x(0,x) &= \frac{d}{dx}\widetilde{E}[e^{-rT}(xe^{\sigma\widetilde{W}_T + (r-\frac{1}{2}\sigma^2)T} - K)^+] \\
&= \widetilde{E}\left[e^{-rT}\frac{d}{dx}h(xe^{\sigma\widetilde{W}_T + (r-\frac{1}{2}\sigma^2)T})\right] \\
&= \widetilde{E}\left[e^{-rT}e^{\sigma\widetilde{W}_T + (r-\frac{1}{2}\sigma^2)T}1_{\{xe^{\sigma\widetilde{W}_T+(r-\frac{1}{2}\sigma^2)T}>K\}}\right] \\
&= e^{-\frac{1}{2}\sigma^2T}\widetilde{E}\left[e^{\sigma\widetilde{W}_T}1_{\{\widetilde{W}_T > \frac{1}{\sigma}(\ln\frac{K}{x} - (r-\frac{1}{2}\sigma^2)T)\}}\right] \\
&= e^{-\frac{1}{2}\sigma^2T}\widetilde{E}\left[e^{\sigma\sqrt{T}\frac{\widetilde{W}_T}{\sqrt{T}}}1_{\{\frac{\widetilde{W}_T}{\sqrt{T}} - \sigma\sqrt{T} > \frac{1}{\sigma\sqrt{T}}(\ln\frac{K}{x} - (r-\frac{1}{2}\sigma^2)T) - \sigma\sqrt{T}\}}\right] \\
&= e^{-\frac{1}{2}\sigma^2T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}e^{\sigma\sqrt{T}z}1_{\{z-\sigma\sqrt{T} > -d_+(T,x)\}}dz \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\sigma\sqrt{T})^2}{2}}1_{\{z-\sigma\sqrt{T} > -d_+(T,x)\}}dz \\
&= N(d_+(T,x)).
\end{aligned}$$

□

(ii)

Proof. If we set $\widehat{Z}_T = e^{\sigma\widetilde{W}_T - \frac{1}{2}\sigma^2T}$ and $\widehat{Z}_t = \widetilde{E}[\widehat{Z}_T|\mathcal{F}_t]$, then $\widehat{Z}$ is a $\widetilde{P}$-martingale, $\widehat{Z}_t > 0$ and $E[\widehat{Z}_T] = \widetilde{E}[e^{\sigma\widetilde{W}_T - \frac{1}{2}\sigma^2T}] = 1$. So if we define $\widehat{P}$ by $d\widehat{P} = Z_T d\widetilde{P}$ on $\mathcal{F}_T$, then $\widehat{P}$ is a probability measure equivalent to $\widetilde{P}$, and

$$c_x(0,x) = \widetilde{E}[\widehat{Z}_T 1_{\{S_T>K\}}] = \widehat{P}(S_T > K).$$

Moreover, by Girsanov's Theorem, $\widehat{W}_t = \widetilde{W}_t + \int_0^t(-\sigma)du = \widetilde{W}_t - \sigma t$ is a $\widehat{P}$-Brownian motion (set $\Theta = -\sigma$ in Theorem 5.4.1.) □

(iii)

Proof. $S_T = xe^{\sigma\widetilde{W}_T+(r-\frac{1}{2}\sigma^2)T} = xe^{\sigma\widehat{W}_T+(r+\frac{1}{2}\sigma^2)T}$. So

$$\widehat{P}(S_T > K) = \widehat{P}(xe^{\sigma\widehat{W}_T+(r+\frac{1}{2}\sigma^2)T} > K) = \widehat{P}\left(\frac{\widehat{W}_T}{\sqrt{T}} > -d_+(T,x)\right) = N(d_+(T,x)).$$

□

5.4. First, a few typos. In the SDE for S, "$\sigma(t)d\widetilde{W}(t)$" $\to$ "$\sigma(t)S(t)d\widetilde{W}(t)$". In the first equation for $c(0,S(0))$, $E \to \widetilde{E}$. In the second equation for $c(0,S(0))$, the variable for BSM should be

$$BSM\left(T, S(0); K, \frac{1}{T}\int_0^T r(t)dt, \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}\right).$$

(i)

Proof. $d\ln S_t = \frac{dS_t}{S_t} - \frac{1}{2S_t^2}d\langle S\rangle_t = r_t dt + \sigma_t d\widetilde{W}_t - \frac{1}{2}\sigma_t^2 dt$. So $S_T = S_0\exp\{\int_0^T(r_t - \frac{1}{2}\sigma_t^2)dt + \int_0^T \sigma_t d\widetilde{W}_t\}$. Let $X = \int_0^T(r_t - \frac{1}{2}\sigma_t^2)dt + \int_0^T \sigma_t d\widetilde{W}_t$. The first term in the expression of X is a number and the second term is a Gaussian random variable $N(0, \int_0^T \sigma_t^2 dt)$, since both r and σ ar deterministic. Therefore, $S_T = S_0 e^X$, with $X \sim N(\int_0^T(r_t - \frac{1}{2}\sigma_t^2)dt, \int_0^T \sigma_t^2 dt)$,. □

(ii)

Proof. For the standard BSM model with constant volatility Σ and interest rate R, under the risk-neutral measure, we have $S_T = S_0 e^Y$, where $Y = (R - \frac{1}{2}\Sigma^2)T + \Sigma\widetilde{W}_T \sim N((R-\frac{1}{2}\Sigma^2)T, \Sigma^2 T)$, and $\widetilde{E}[(S_0e^Y - K)^+] = e^{RT}BSM(T, S_0; K, R, \Sigma)$. Note $R = \frac{1}{T}(E[Y] + \frac{1}{2}Var(Y))$ and $\Sigma = \sqrt{\frac{1}{T}Var(Y)}$, we can get

$$\widetilde{E}[(S_0e^Y - K)^+] = e^{E[Y]+\frac{1}{2}Var(Y)}BSM\left(T, S_0; K, \frac{1}{T}\left(E[Y] + \frac{1}{2}Var(Y)\right), \sqrt{\frac{1}{T}Var(Y)}\right).$$

So for the model in this problem,

$$\begin{aligned}
c(0,S_0) &= e^{-\int_0^T r_t dt}\widetilde{E}[(S_0e^X - K)^+] \\
&= e^{-\int_0^T r_t dt}e^{E[X]+\frac{1}{2}Var(X)}BSM\left(T, S_0; K, \frac{1}{T}\left(E[X] + \frac{1}{2}Var(X)\right), \sqrt{\frac{1}{T}Var(X)}\right) \\
&= BSM\left(T, S_0; K, \frac{1}{T}\int_0^T r_t dt, \sqrt{\frac{1}{T}\int_0^T \sigma_t^2 dt}\right).
\end{aligned}$$

□

5.5. (i)

Proof. Let $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$. Note $dZ_t = -Z_t\Theta_t dW_t$, so

$$d\left(\frac{1}{Z_t}\right) = f'(Z_t)dZ_t + \frac{1}{2}f''(Z_t)dZ_t dZ_t = -\frac{1}{Z_t^2}(-Z_t)\Theta_t dW_t + \frac{1}{2}\frac{2}{Z_t^3}Z_t^2\Theta_t^2 dt = \frac{\Theta_t}{Z_t}dW_t + \frac{\Theta_t^2}{Z_t}dt.$$

□

(ii)

Proof. By Lemma 5.2.2., for $s, t \geq 0$ with $s < t$, $\widetilde{M}_s = \widetilde{E}[\widetilde{M}_t|\mathcal{F}_s] = E\left[\frac{Z_t\widetilde{M}_t}{Z_s}|\mathcal{F}_s\right]$. That is, $E[Z_t\widetilde{M}_t|\mathcal{F}_s] = Z_s\widetilde{M}_s$. So $M = Z\widetilde{M}$ is a P-martingale. □

(iii)

Proof.

$$d\widetilde{M}_t = d\left(M_t \cdot \frac{1}{Z_t}\right) = \frac{1}{Z_t}dM_t + M_t d\frac{1}{Z_t} + dM_t d\frac{1}{Z_t} = \frac{\Gamma_t}{Z_t}dW_t + \frac{M_t\Theta_t}{Z_t}dW_t + \frac{M_t\Theta_t^2}{Z_t}dt + \frac{\Gamma_t\Theta_t}{Z_t}dt.$$

□

(iv)

Proof. In part (iii), we have

$$d\widetilde{M}_t = \frac{\Gamma_t}{Z_t}dW_t + \frac{M_t\Theta_t}{Z_t}dW_t + \frac{M_t\Theta_t^2}{Z_t}dt + \frac{\Gamma_t\Theta_t}{Z_t}dt = \frac{\Gamma_t}{Z_t}(dW_t + \Theta_t dt) + \frac{M_t\Theta_t}{Z_t}(dW_t + \Theta_t dt).$$

Let $\widetilde{\Gamma}_t = \frac{\Gamma_t + M_t\Theta_t}{Z_t}$, then $d\widetilde{M}_t = \widetilde{\Gamma}_t d\widetilde{W}_t$. This proves Corollary 5.3.2. □

5.6.

Proof. By Theorem 4.6.5, it suffices to show $\widetilde{W}_i(t)$ is an $\mathcal{F}_t$-martingale under $\widetilde{P}$ and $[\widetilde{W}_i, \widetilde{W}_j](t) = t\delta_{ij}$ $(i, j = 1, 2)$. Indeed, for $i = 1, 2$, $\widetilde{W}_i(t)$ is an $\mathcal{F}_t$-martingale under $\widetilde{P}$ if and only if $\widetilde{W}_i(t)Z_t$ is an $\mathcal{F}_t$-martingale under P, since

$$\widetilde{E}[\widetilde{W}_i(t)|\mathcal{F}_s] = E\left[\frac{\widetilde{W}_i(t)Z_t}{Z_s}|\mathcal{F}_s\right].$$

By Itô's product formula, we have

$$\begin{aligned}
d(\widetilde{W}_i(t)Z_t) &= \widetilde{W}_i(t)dZ_t + Z_t d\widetilde{W}_i(t) + dZ_t d\widetilde{W}_i(t) \\
&= \widetilde{W}_i(t)(-Z_t)\Theta(t) \cdot dW_t + Z_t(dW_i(t) + \Theta_i(t)dt) + (-Z_t\Theta_t \cdot dW_t)(dW_i(t) + \Theta_i(t)dt) \\
&= \widetilde{W}_i(t)(-Z_t)\sum_{j=1}^d \Theta_j(t)dW_j(t) + Z_t(dW_i(t) + \Theta_i(t)dt) - Z_t\Theta_i(t)dt \\
&= \widetilde{W}_i(t)(-Z_t)\sum_{j=1}^d \Theta_j(t)dW_j(t) + Z_t dW_i(t)
\end{aligned}$$

This shows $\widetilde{W}_i(t)Z_t$ is an $\mathcal{F}_t$-martingale under P. So $\widetilde{W}_i(t)$ is an $\mathcal{F}_t$-martingale under $\widetilde{P}$. Moreover,

$$[\widetilde{W}_i, \widetilde{W}_j](t) = \left[W_i + \int_0^\cdot \Theta_i(s)ds, W_j + \int_0^\cdot \Theta_j(s)ds\right](t) = [W_i, W_j](t) = t\delta_{ij}.$$

Combined, this proves the two-dimensional Girsanov's Theorem. □

5.7. (i)

Proof. Let a be any strictly positive number. We define $X_2(t) = (a + X_1(t))D(t)^{-1}$. Then

$$P\left(X_2(T) \geq \frac{X_2(0)}{D(T)}\right) = P(a + X_1(T) \geq a) = P(X_1(T) \geq 0) = 1,$$

and $P\left(X_2(T) > \frac{X_2(0)}{D(T)}\right) = P(X_1(T) > 0) > 0$, since a is arbitrary, we have proved the claim of this problem.

Remark 4. *The intuition is that we invest the positive starting fund a into the money market account, and construct portfolio X_1 from zero cost. Their sum should be able to beat the return of money market account.*

$\square$

(ii)

Proof. We define $X_1(t) = X_2(t)D(t) - X_2(0)$. Then $X_1(0) = 0$,

$$P(X_1(T) \geq 0) = P\left(X_2(T) \geq \frac{X_2(0)}{D(T)}\right) = 1,\ P(X_1(T) > 0) = P\left(X_2(T) > \frac{X_2(0)}{D(T)}\right) > 0.$$

$\square$

5.8. The basic idea is that for any positive $\widetilde{P}$-martingale M, $dM_t = M_t \cdot \frac{1}{M_t} dM_t$. By Martingale Representation Theorem, $dM_t = \widetilde{\Gamma}_t d\widetilde{W}_t$ for some adapted process $\widetilde{\Gamma}_t$. So $dM_t = M_t(\frac{\widetilde{\Gamma}_t}{M_t})d\widetilde{W}_t$, i.e. any positive martingale must be the exponential of an integral w.r.t. Brownian motion. Taking into account discounting factor and apply Itô's product rule, we can show every strictly positive asset is a generalized geometric Brownian motion.

(i)

Proof. $V_t D_t = \widetilde{E}[e^{-\int_0^T R_u du} V_T | \mathcal{F}_t] = \widetilde{E}[D_T V_T | \mathcal{F}_t]$. So $(D_t V_t)_{t\geq 0}$ is a $\widetilde{P}$-martingale. By Martingale Representation Theorem, there exists an adapted process $\widetilde{\Gamma}_t$, $0 \leq t \leq T$, such that $D_t V_t = \int_0^t \widetilde{\Gamma}_s d\widetilde{W}_s$, or equivalently, $V_t = D_t^{-1} \int_0^t \widetilde{\Gamma}_s d\widetilde{W}_s$. Differentiate both sides of the equation, we get $dV_t = R_t D_t^{-1} \int_0^t \widetilde{\Gamma}_s d\widetilde{W}_s dt + D_t^{-1} \widetilde{\Gamma}_t d\widetilde{W}_t$, i.e. $dV_t = R_t V_t dt + \frac{\widetilde{\Gamma}_t}{D_t} dW_t$. $\square$

(ii)

Proof. We prove the following more general lemma.

Lemma 1. *Let X be an almost surely positive random variable (i.e. $X > 0$ a.s.) defined on the probability space $(\Omega, \mathcal{G}, P)$. Let $\mathcal{F}$ be a sub σ-algebra of $\mathcal{G}$, then $Y = E[X|\mathcal{F}] > 0$ a.s.*

Proof. By the property of conditional expectation $Y_t \geq 0$ a.s. Let $A = \{Y = 0\}$, we shall show $P(A) = 0$. Indeed, note $A \in \mathcal{F}$, $0 = E[YI_A] = E[E[X|\mathcal{F}]I_A] = E[XI_A] = E[X1_{A\cap\{X\geq 1\}}] + \sum_{n=1}^{\infty} E[X1_{A\cap\{\frac{1}{n} > X \geq \frac{1}{n+1}\}}] \geq P(A\cap\{X \geq 1\}) + \sum_{n=1}^{\infty} \frac{1}{n+1} P(A\cap\{\frac{1}{n} > X \geq \frac{1}{n+1}\})$. So $P(A\cap\{X \geq 1\}) = 0$ and $P(A\cap\{\frac{1}{n} > X \geq \frac{1}{n+1}\}) = 0$, $\forall n \geq 1$. This in turn implies $P(A) = P(A \cap \{X > 0\}) = P(A \cap \{X \geq 1\}) + \sum_{n=1}^{\infty} P(A \cap \{\frac{1}{n} > X \geq \frac{1}{n+1}\}) = 0$. $\square$

By the above lemma, it is clear that for each $t \in [0,T]$, $V_t = \widetilde{E}[e^{-\int_t^T R_u du} V_T | \mathcal{F}_t] > 0$ a.s.. Moreover, by a classical result of martingale theory (Revuz and Yor [4], Chapter II, Proposition (3.4)), we have the following stronger result: for a.s. ω, $V_t(\omega) > 0$ for any $t \in [0,T]$. $\square$

(iii)

Proof. By (ii), $V > 0$ a.s., so $dV_t = V_t \frac{1}{V_t} dV_t = V_t \frac{1}{V_t}\left(R_t V_t dt + \frac{\widetilde{\Gamma}_t}{D_t} d\widetilde{W}_t\right) = V_t R_t dt + V_t \frac{\widetilde{\Gamma}_t}{V_t D_t} d\widetilde{W}_t = R_t V_t dt + \sigma_t V_t d\widetilde{W}_t$, where $\sigma_t = \frac{\widetilde{\Gamma}_t}{V_t D_t}$. This shows V follows a generalized geometric Brownian motion. $\square$

5.9.

Proof. $c(0,T,x,K) = xN(d_+) - Ke^{-rT}N(d_-)$ with $d_\pm = \frac{1}{\sigma\sqrt{T}}(\ln\frac{x}{K} + (r \pm \frac{1}{2}\sigma^2)T)$. Let $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, then $f'(y) = -yf(y)$,

$$\begin{aligned} c_K(0,T,x,K) &= xf(d_+)\frac{\partial d_+}{\partial y} - e^{-rT}N(d_-) - Ke^{-rT}f(d_-)\frac{\partial d_-}{\partial y} \\ &= xf(d_+)\frac{-1}{\sigma\sqrt{T}K} - e^{-rT}N(d_-) + e^{-rT}f(d_-)\frac{1}{\sigma\sqrt{T}}, \end{aligned}$$

and

$$
\begin{aligned}
& c_{KK}(0,T,x,K) \\
= \quad & xf(d_+)\frac{1}{\sigma\sqrt{T}K^2} - \frac{x}{\sigma\sqrt{T}K}f(d_+)(-d_+)\frac{\partial d_+}{\partial y} - e^{-rT}f(d_-)\frac{\partial d_-}{\partial y} + \frac{e^{-rT}}{\sigma\sqrt{T}}(-d_-)f(d_-)\frac{d_-}{\partial y} \\
= \quad & \frac{x}{\sigma\sqrt{T}K^2}f(d_+) + \frac{xd_+}{\sigma\sqrt{T}K}f(d_+)\frac{-1}{K\sigma\sqrt{T}} - e^{-rT}f(d_-)\frac{-1}{K\sigma\sqrt{T}} - \frac{e^{-rT}d_-}{\sigma\sqrt{T}}f(d_-)\frac{-1}{K\sigma\sqrt{T}} \\
= \quad & f(d_+)\frac{x}{K^2\sigma\sqrt{T}}[1-\frac{d_+}{\sigma\sqrt{T}}] + \frac{e^{-rT}f(d_-)}{K\sigma\sqrt{T}}[1+\frac{d_-}{\sigma\sqrt{T}}] \\
= \quad & \frac{e^{-rT}}{K\sigma^2 T}f(d_-)d_+ - \frac{x}{K^2\sigma^2 T}f(d_+)d_-.
\end{aligned}
$$

□

5.10. (i)

Proof. At time t_0, the value of the chooser option is $V(t_0) = \max\{C(t_0), P(t_0)\} = \max\{C(t_0), C(t_0) - F(t_0)\} = C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + (e^{-r(T-t_0)}K - S(t_0))^+$. □

(ii)

Proof. By the risk-neutral pricing formula, $V(0) = \widetilde{E}[e^{-rt_0}V(t_0)] = \widetilde{E}[e^{-rt_0}C(t_0) + (e^{-rT}K - e^{-rt_0}S(t_0)^+] = C(0) + \widetilde{E}[e^{-rt_0}(e^{-r(T-t_0)}K - S(t_0))^+]$. The first term is the value of a call expiring at time T with strike price K and the second term is the value of a put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$.

□

5.11.

Proof. We first make an analysis which leads to the hint, then we give a formal proof.

(Analysis) If we want to construct a portfolio X that exactly replicates the cash flow, we must find a solution to the backward SDE

$$
\begin{cases} dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt - C_t dt \\ X_T = 0. \end{cases}
$$

Multiply D_t on both sides of the first equation and apply Itô's product rule, we get $d(D_tX_t) = \Delta_t d(D_tS_t) - C_tD_tdt$. Integrate from 0 to T, we have $D_TX_T - D_0X_0 = \int_0^T \Delta_t d(D_tS_t) - \int_0^T C_tD_tdt$. By the terminal condition, we get $X_0 = D_0^{-1}(\int_0^T C_tD_tdt - \int_0^T \Delta_t d(D_tS_t))$. X_0 is the theoretical, no-arbitrage price of the cash flow, provided we can find a trading strategy Δ that solves the BSDE. Note the SDE for S gives $d(D_tS_t) = (D_tS_t)\sigma_t(\theta_t dt + dW_t)$, where $\theta_t = \frac{\alpha_t - R_t}{\sigma_t}$. Take the proper change of measure so that $\widetilde{W}_t = \int_0^t \theta_s ds + W_t$ is a Brownian motion under the new measure $\widetilde{P}$, we get

$$
\int_0^T C_tD_tdt = D_0X_0 + \int_0^T \Delta_t d(D_tS_t) = D_0X_0 + \int_0^T \Delta_t(D_tS_t)\sigma_t d\widetilde{W}_t.
$$

This says the random variable $\int_0^T C_tD_tdt$ has a stochastic integral representation $D_0X_0 + \int_0^T \Delta_tD_tS_t\sigma_t d\widetilde{W}_t$. This inspires us to consider the martingale generated by $\int_0^T C_tD_tdt$, so that we can apply Martingale Representation Theorem and get a formula for Δ by comparison of the integrands.

(Formal proof) Let $M_T = \int_0^T C_tD_tdt$, and $M_t = \widetilde{E}[M_T|\mathcal{F}_t]$. Then by Martingale Representation Theorem, we can find an adapted process $\widetilde{\Gamma}_t$, so that $M_t = M_0 + \int_0^t \widetilde{\Gamma}_t d\widetilde{W}_t$. If we set $\Delta_t = \frac{\widetilde{\Gamma}_t}{D_tS_t\sigma_t}$, we can check $X_t = D_t^{-1}(D_0X_0 + \int_0^t \Delta_u d(D_uS_u) - \int_0^t C_uD_udu)$, with $X_0 = M_0 = \widetilde{E}[\int_0^T C_tD_tdt]$ solves the SDE

$$
\begin{cases} dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt - C_t dt \\ X_T = 0. \end{cases}
$$

Indeed, it is easy to see that X satisfies the first equation. To check the terminal condition, we note $X_T D_T = D_0 X_0 + \int_0^T \Delta_t D_t S_t \sigma_t d\widetilde{W}_t - \int_0^T C_t D_t dt = M_0 + \int_0^T \widetilde{\Gamma}_t d\widetilde{W}_t - M_T = 0$. So $X_T = 0$. Thus, we have found a trading strategy Δ, so that the corresponding portfolio X replicates the cash flow and has zero terminal value. So $X_0 = \widetilde{E}[\int_0^T C_t D_t dt]$ is the no-arbitrage price of the cash flow at time zero.

Remark 5. *As shown in the analysis, $d(D_t X_t) = \Delta_t d(D_t S_t) - C_t D_t dt$. Integrate from t to T, we get $0 - D_t X_t = \int_t^T \Delta_u d(D_u S_u) - \int_t^T C_u D_u du$. Take conditional expectation w.r.t. $\mathcal{F}_t$ on both sides, we get $-D_t X_t = -\widetilde{E}[\int_t^T C_u D_u du|\mathcal{F}_t]$. So $X_t = D_t^{-1}\widetilde{E}[\int_t^T C_u D_u du|\mathcal{F}_t]$. This is the no-arbitrage price of the cash flow at time t, and we have justified formula (5.6.10) in the textbook.*

□

5.12. (i)

Proof. $d\widetilde{B}_i(t) = dB_i(t) + \gamma_i(t)dt = \sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t) + \sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)}\Theta_j(t)dt = \sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)} d\widetilde{W}_j(t)$. So B_i is a martingale. Since $d\widetilde{B}_i(t)d\widetilde{B}_i(t) = \sum_{j=1}^d \frac{\sigma_{ij}(t)^2}{\sigma_i(t)^2}dt = dt$, by Lévy's Theorem, $\widetilde{B}_i$ is a Brownian motion under $\widetilde{P}$. □

(ii)

Proof.

$$\begin{aligned} dS_i(t) &= R(t)S_i(t)dt + \sigma_i(t)S_i(t)d\widetilde{B}_i(t) + (\alpha_i(t) - R(t))S_i(t)dt - \sigma_i(t)S_i(t)\gamma_i(t)dt \\ &= R(t)S_i(t)dt + \sigma_i(t)S_i(t)d\widetilde{B}_i(t) + \sum_{j=1}^d \sigma_{ij}(t)\Theta_j(t)S_i(t)dt - S_i(t)\sum_{j=1}^d \sigma_{ij}(t)\Theta_j(t)dt \\ &= R(t)S_i(t)dt + \sigma_i(t)S_i(t)d\widetilde{B}_i(t). \end{aligned}$$

□

(iii)

Proof. $d\widetilde{B}_i(t)d\widetilde{B}_k(t) = (dB_i(t) + \gamma_i(t)dt)(dB_j(t) + \gamma_j(t)dt) = dB_i(t)dB_j(t) = \rho_{ik}(t)dt$. □

(iv)

Proof. By Itô's product rule and martingale property,

$$\begin{aligned} E[B_i(t)B_k(t)] &= E[\int_0^t B_i(s)dB_k(s)] + E[\int_0^t B_k(s)dB_i(s)] + E[\int_0^t dB_i(s)dB_k(s)] \\ &= E[\int_0^t \rho_{ik}(s)ds] = \int_0^t \rho_{ik}(s)ds. \end{aligned}$$

Similarly, by part (iii), we can show $\widetilde{E}[\widetilde{B}_i(t)\widetilde{B}_k(t)] = \int_0^t \rho_{ik}(s)ds$. □

(v)

Proof. By Itô's product formula,

$$E[B_1(t)B_2(t)] = E[\int_0^t sign(W_1(u))du] = \int_0^t [P(W_1(u) \geq 0) - P(W_1(u) < 0)]du = 0.$$

Meanwhile,

$$\begin{aligned}
\widetilde{E}[\widetilde{B}_1(t)\widetilde{B}_2(t)] &= \widetilde{E}[\int_0^t sign(W_1(u))du \\
&= \int_0^t [\widetilde{P}(W_1(u) \geq 0) - \widetilde{P}(W_1(u) < 0)]du \\
&= \int_0^t [\widetilde{P}(\widetilde{W}_1(u) \geq u) - \widetilde{P}(\widetilde{W}_1(u) < u)]du \\
&= \int_0^t 2\left(\frac{1}{2} - \widetilde{P}(\widetilde{W}_1(u) < u)\right) du \\
&< 0,
\end{aligned}$$

for any $t > 0$. So $E[B_1(t)B_2(t)] = \widetilde{E}[\widetilde{B}_1(t)\widetilde{B}_2(t)]$ for all $t > 0$. □

5.13. (i)

Proof. $\widetilde{E}[W_1(t)] = \widetilde{E}[\widetilde{W}_1(t)] = 0$ and $\widetilde{E}[W_2(t)] = \widetilde{E}[\widetilde{W}_2(t) - \int_0^t \widetilde{W}_1(u)du] = 0$, for all $t \in [0, T]$. □

(ii)

Proof.

$$\begin{aligned}
\widetilde{Cov}[W_1(T), W_2(T)] &= \widetilde{E}[W_1(T)W_2(T)] \\
&= \widetilde{E}\left[\int_0^T W_1(t)dW_2(t) + \int_0^T W_2(t)dW_1(t)\right] \\
&= \widetilde{E}\left[\int_0^T \widetilde{W}_1(t)(d\widetilde{W}_2(t) - \widetilde{W}_1(t)dt)\right] + \widetilde{E}\left[\int_0^T W_2(t)d\widetilde{W}_1(t)\right] \\
&= -\widetilde{E}\left[\int_0^T \widetilde{W}_1(t)^2 dt\right] \\
&= -\int_0^T tdt \\
&= -\frac{1}{2}T^2.
\end{aligned}$$

□

5.14. Equation (5.9.6) can be transformed into $d(e^{-rt}X_t) = \Delta_t[d(e^{-rt}S_t) - ae^{-rt}dt] = \Delta_t e^{-rt}[dS_t - rS_t dt - adt]$. So, to make the discounted portfolio value $e^{-rt}X_t$ a martingale, we are motivated to change the measure in such a way that $S_t - r\int_0^t S_u du - at$ is a martingale under the new measure. To do this, we note the SDE for S is $dS_t = \alpha_t S_t dt + \sigma S_t dW_t$. Hence $dS_t - rS_t dt - adt = [(\alpha_t - r)S_t - a]dt + \sigma S_t dW_t = \sigma S_t \left[\frac{(\alpha_t - r)S_t - a}{\sigma S_t} dt + dW_t\right]$. Set $\theta_t = \frac{(\alpha_t - r)S_t - a}{\sigma S_t}$ and $\widetilde{W}_t = \int_0^t \theta_s ds + W_t$, we can find an equivalent probability measure $\widetilde{P}$, under which S satisfies the SDE $dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t + adt$ and $\widetilde{W}_t$ is a BM. This is the rational for formula (5.9.7).

This is a good place to pause and think about the meaning of "martingale measure." *What is to be a martingale?* The new measure $\widetilde{P}$ should be such that *the discounted value process of the replicating portfolio* is a martingale, *not the discounted price process of the underlying*. First, we want $D_t X_t$ to be a martingale under $\widetilde{P}$ because we suppose that X is able to replicate the derivative payoff at terminal time, $X_T = V_T$. In order to avoid arbitrage, we must have $X_t = V_t$ for any $t \in [0, T]$. The difficulty is how to calculate X_t and the magic is brought by the martingale measure in the following line of reasoning: $V_t = X_t = D_t^{-1}\widetilde{E}[D_T X_T | \mathcal{F}_t] = D_t^{-1}\widetilde{E}[D_T V_T | \mathcal{F}_t]$. You can think of martingale measure as a calculational convenience. That is *all* about martingale measure! *Risk neutral* is just a *perception*, referring to the actual effect of constructing a hedging portfolio! Second, we note when the portfolio is self-financing, the discounted price process of the underlying is a martingale under $\widetilde{P}$, as in the classical Black-Scholes-Merton model without dividends or cost of carry. This is not a coincidence. Indeed, we have in this case the relation $d(D_tX_t) = \Delta_t d(D_tS_t)$. So D_tX_t being a martingale under $\widetilde{P}$ is more or less equivalent to D_tS_t being a martingale under $\widetilde{P}$. However, when the underlying pays dividends, or there is cost of carry, $d(D_tX_t) = \Delta_t d(D_tS_t)$ no longer holds, as shown in formula (5.9.6). The portfolio is no longer *self-financing*, but *self-financing with consumption*. What we still want to retain is the martingale property of D_tX_t, not that of D_tS_t. This is how we choose martingale measure in the above paragraph.

Let V_T be a payoff at time T, then for the martingale $M_t = \widetilde{E}[e^{-rT}V_T|\mathcal{F}_t]$, by Martingale Representation Theorem, we can find an adapted process $\widetilde{\Gamma}_t$, so that $M_t = M_0 + \int_0^t \widetilde{\Gamma}_s d\widetilde{W}_s$. If we let $\Delta_t = \frac{\widetilde{\Gamma}_t e^{rt}}{\sigma S_t}$, then the value of the corresponding portfolio X satisfies $d(e^{-rt}X_t) = \widetilde{\Gamma}_t d\widetilde{W}_t$. So by setting $X_0 = M_0 = \widetilde{E}[e^{-rT}V_T]$, we must have $e^{-rt}X_t = M_t$, for all $t \in [0,T]$. In particular, $X_T = V_T$. Thus the portfolio perfectly hedges V_T. This justifies the risk-neutral pricing of European-type contingent claims in the model where cost of carry exists. Also note the risk-neutral measure is different from the one in case of no cost of carry.

Another perspective for perfect replication is the following. We need to solve the backward SDE

$$\begin{cases} dX_t = \Delta_t dS_t - a\Delta_t dt + r(X_t - \Delta_t S_t)dt \\ X_T = V_T \end{cases}$$

for two unknowns, X and Δ. To do so, we find a probability measure $\widetilde{P}$, under which $e^{-rt}X_t$ is a martingale, then $e^{-rt}X_t = \widetilde{E}[e^{-rT}V_T|\mathcal{F}_t] := M_t$. Martingale Representation Theorem gives $M_t = M_0 + \int_0^t \widetilde{\Gamma}_u d\widetilde{W}_u$ for some adapted process $\widetilde{\Gamma}$. This would give us a theoretical representation of Δ by comparison of integrands, hence a perfect replication of V_T.

(i)

Proof. As indicated in the above analysis, if we have (5.9.7) under $\widetilde{P}$, then $d(e^{-rt}X_t) = \Delta_t[d(e^{-rt}S_t) - ae^{-rt}dt] = \Delta_t e^{-rt}\sigma S_t d\widetilde{W}_t$. So $(e^{-rt}X_t)_{t\geq 0}$, where X is given by (5.9.6), is a $\widetilde{P}$-martingale. □

(ii)

Proof. By Itô's formula, $dY_t = Y_t[\sigma d\widetilde{W}_t + (r - \frac{1}{2}\sigma^2)dt] + \frac{1}{2}Y_t\sigma^2 dt = Y_t(\sigma d\widetilde{W}_t + rdt)$. So $d(e^{-rt}Y_t) = \sigma e^{-rt}Y_t d\widetilde{W}_t$ and $e^{-rt}Y_t$ is a $\widetilde{P}$-martingale. Moreover, if $S_t = S_0Y_t + Y_t\int_0^t \frac{a}{Y_s}ds$, then

$$dS_t = S_0 dY_t + \int_0^t \frac{a}{Y_s}ds dY_t + adt = \left(S_0 + \int_0^t \frac{a}{Y_s}ds\right)Y_t(\sigma d\widetilde{W}_t + rdt) + adt = S_t(\sigma d\widetilde{W}_t + rdt) + adt.$$

This shows S satisfies (5.9.7).

Remark 6. *To obtain this formula for S, we first set $U_t = e^{-rt}S_t$ to remove the rS_tdt term. The SDE for U is $dU_t = \sigma U_t d\widetilde{W}_t + ae^{-rt}dt$. Just like solving linear ODE, to remove U in the $d\widetilde{W}_t$ term, we consider $V_t = U_t e^{-\sigma\widetilde{W}_t}$. Itô's product formula yields*

$$\begin{aligned} dV_t &= e^{-\sigma\widetilde{W}_t}dU_t + U_t e^{-\sigma\widetilde{W}_t}\left((-\sigma)d\widetilde{W}_t + \frac{1}{2}\sigma^2 dt\right) + dU_t \cdot e^{-\sigma\widetilde{W}_t}\left((-\sigma)d\widetilde{W}_t + \frac{1}{2}\sigma^2 dt\right) \\ &= e^{-\sigma\widetilde{W}_t}ae^{-rt}dt - \frac{1}{2}\sigma^2 V_t dt. \end{aligned}$$

Note V appears only in the dt term, so multiply the integration factor $e^{\frac{1}{2}\sigma^2 t}$ on both sides of the equation, we get

$$d(e^{\frac{1}{2}\sigma^2 t}V_t) = ae^{-rt-\sigma\widetilde{W}_t+\frac{1}{2}\sigma^2 t}dt.$$

Set $Y_t = e^{\sigma\widetilde{W}_t+(r-\frac{1}{2}\sigma^2)t}$, we have $d(S_t/Y_t) = adt/Y_t$. So $S_t = Y_t(S_0 + \int_0^t \frac{ads}{Y_s})$.

□

(iii)

Proof.

$$
\begin{aligned}
\widetilde{E}[S_T|\mathcal{F}_t] &= S_0\widetilde{E}[Y_T|\mathcal{F}_t] + \widetilde{E}\left[Y_T\int_0^t \frac{a}{Y_s}ds + Y_T\int_t^T \frac{a}{Y_s}ds|\mathcal{F}_t\right] \\
&= S_0\widetilde{E}[Y_T|\mathcal{F}_t] + \int_0^t \frac{a}{Y_s}ds\widetilde{E}[Y_T|\mathcal{F}_t] + a\int_t^T \widetilde{E}\left[\frac{Y_T}{Y_s}|\mathcal{F}_t\right]ds \\
&= S_0Y_t\widetilde{E}[Y_{T-t}] + \int_0^t \frac{a}{Y_s}dsY_t\widetilde{E}[Y_{T-t}] + a\int_t^T \widetilde{E}[Y_{T-s}]ds \\
&= \left(S_0 + \int_0^t \frac{a}{Y_s}ds\right)Y_te^{r(T-t)} + a\int_t^T e^{r(T-s)}ds \\
&= \left(S_0 + \int_0^t \frac{ads}{Y_s}\right)Y_te^{r(T-t)} - \frac{a}{r}(1-e^{r(T-t)}).
\end{aligned}
$$

In particular, $\widetilde{E}[S_T] = S_0e^{rT} - \frac{a}{r}(1-e^{rT})$. $\square$

(iv)

Proof.

$$
\begin{aligned}
d\widetilde{E}[S_T|\mathcal{F}_t] &= ae^{r(T-t)}dt + \left(S_0 + \int_0^t \frac{ads}{Y_s}\right)(e^{r(T-t)}dY_t - rY_te^{r(T-t)}dt) + \frac{a}{r}e^{r(T-t)}(-r)dt \\
&= \left(S_0 + \int_0^t \frac{ads}{Y_s}\right)e^{r(T-t)}\sigma Y_t d\widetilde{W}_t.
\end{aligned}
$$

So $\widetilde{E}[S_T|\mathcal{F}_t]$ is a $\widetilde{P}$-martingale. As we have argued at the beginning of the solution, risk-neutral pricing is valid even in the presence of cost of carry. So by an argument similar to that of §5.6.2, the process $\widetilde{E}[S_T|\mathcal{F}_t]$ is the futures price process for the commodity. $\square$

(v)

Proof. We solve the equation $\widetilde{E}[e^{-r(T-t)}(S_T-K)|\mathcal{F}_t]=0$ for K, and get $K=\widetilde{E}[S_T|\mathcal{F}_t]$. So $For_S(t,T) = Fut_S(t,T)$. $\square$

(vi)

Proof. We follow the hint. First, we solve the SDE

$$
\begin{cases}
dX_t = dS_t - adt + r(X_t - S_t)dt \\
X_0 = 0.
\end{cases}
$$

By our analysis in part (i), $d(e^{-rt}X_t) = d(e^{-rt}S_t) - ae^{-rt}dt$. Integrate from 0 to t on both sides, we get $X_t = S_t - S_0e^{rt} + \frac{a}{r}(1-e^{rt}) = S_t - S_0e^{rt} - \frac{a}{r}(e^{rt}-1)$. In particular, $X_T = S_T - S_0e^{rT} - \frac{a}{r}(e^{rT}-1)$. Meanwhile, $For_S(t,T) = Fut_s(t,T) = \widetilde{E}[S_T|\mathcal{F}_t] = \left(S_0 + \int_0^t \frac{ads}{Y_s}\right)Y_te^{r(T-t)} - \frac{a}{r}(1-e^{r(T-t)})$. So $For_S(0,T) = S_0e^{rT} - \frac{a}{r}(1-e^{rT})$ and hence $X_T = S_T - For_S(0,T)$. After the agent delivers the commodity, whose value is S_T, and receives the forward price $For_S(0,T)$, the portfolio has exactly zero value. $\square$

2.6 Connections with Partial Differential Equations

6.1. (i)

Proof. $Z_t = 1$ is obvious. Note the form of Z is similar to that of a geometric Brownian motion. So by Itô's formula, it is easy to obtain $dZ_u = b_u Z_u du + \sigma_u Z_u dW_u$, $u \geq t$. $\square$

(ii)

Proof. If $X_u = Y_u Z_u$ $(u \geq t)$, then $X_t = Y_t Z_t = x \cdot 1 = x$ and

$$\begin{aligned} dX_u &= Y_u dZ_u + Z_u dY_u + dY_u Z_u \\ &= Y_u(b_u Z_u du + \sigma_u Z_u dW_u) + Z_u \left(\frac{a_u - \sigma_u \gamma_u}{Z_u} du + \frac{\gamma_u}{Z_u} dW_u \right) + \sigma_u Z_u \frac{\gamma_u}{Z_u} du \\ &= [Y_u b_u Z_u + (a_u - \sigma_u \gamma_u) + \sigma_u \gamma_u] du + (\sigma_u Z_u Y_u + \gamma_u) dW_u \\ &= (b_u X_u + a_u) du + (\sigma_u X_u + \gamma_u) dW_u. \end{aligned}$$

Remark 7. *To see how to find the above solution, we manipulate the equation (6.2.4) as follows. First, to remove the term $b_u X_u du$, we multiply on both sides of (6.2.4) the integrating factor $e^{-\int_t^u b_v dv}$. Then*

$$d(X_u e^{-\int_t^u b_v dv}) = e^{-\int_t^u b_v dv}(a_u du + (\gamma_u + \sigma_u X_u) dW_u).$$

Let $\bar{X}_u = e^{-\int_t^u b_v dv} X_u$, $\bar{a}_u = e^{-\int_t^u b_v dv} a_u$ and $\bar{\gamma}_u = e^{-\int_t^u b_v dv} \gamma_u$, then $\bar{X}$ satisfies the SDE

$$d\bar{X}_u = \bar{a}_u du + (\bar{\gamma}_u + \sigma_u \bar{X}_u) dW_u = (\bar{a}_u du + \bar{\gamma}_u dW_u) + \sigma_u \bar{X}_u dW_u.$$

To deal with the term $\sigma_u \bar{X}_u dW_u$, we consider $\hat{X}_u = \bar{X}_u e^{-\int_t^u \sigma_v dW_v}$. Then

$$\begin{aligned} d\hat{X}_u &= e^{-\int_t^u \sigma_v dW_v}[(\bar{a}_u du + \bar{\gamma}_u dW_u) + \sigma_u \bar{X}_u dW_u] + \bar{X}_u \left(e^{-\int_t^u \sigma_v dW_v}(-\sigma_u) dW_u + \frac{1}{2} e^{-\int_t^u \sigma_v dW_v} \sigma_u^2 du \right) \\ &\quad + (\bar{\gamma}_u + \sigma_u \bar{X}_u)(-\sigma_u) e^{-\int_t^u \sigma_v dW_v} du \\ &= \hat{a}_u du + \hat{\gamma}_u dW_u + \sigma_u \hat{X}_u dW_u - \sigma_u \hat{X}_u dW_u + \frac{1}{2} \hat{X}_u \sigma_u^2 du - \sigma_u(\hat{\gamma}_u + \sigma_u \hat{X}_u) du \\ &= (\hat{a}_u - \sigma_u \hat{\gamma}_u - \frac{1}{2} \hat{X}_u \sigma_u^2) du + \hat{\gamma}_u dW_u, \end{aligned}$$

where $\hat{a}_u = \bar{a}_u e^{-\int_t^u \sigma_v dW_v}$ and $\hat{\gamma}_u = \bar{\gamma}_u e^{-\int_t^u \sigma_v dW_v}$. Finally, use the integrating factor $e^{\int_t^u \frac{1}{2} \sigma_v^2 dv}$, we have

$$d\left(\hat{X}_u e^{\frac{1}{2} \int_t^u \sigma_v^2 dv} \right) = e^{\frac{1}{2} \int_t^u \sigma_v^2 dv}(d\hat{X}_u + \hat{X}_u \cdot \frac{1}{2} \sigma_u^2 du) = e^{\frac{1}{2} \int_t^u \sigma_v^2 dv}[(\hat{a}_u - \sigma_u \hat{\gamma}_u) du + \hat{\gamma}_u dW_u].$$

Write everything back into the original X, a and γ, we get

$$d\left(X_u e^{-\int_t^u b_v dv - \int_t^u \sigma_v dW_v + \frac{1}{2} \int_t^u \sigma_v^2 dv} \right) = e^{\frac{1}{2} \int_t^u \sigma_v^2 dv - \int_t^u \sigma_v dW_v - \int_t^u b_v dv}[(a_u - \sigma_u \gamma_u) du + \gamma_u dW_u],$$

i.e.

$$d\left(\frac{X_u}{Z_u} \right) = \frac{1}{Z_u}[(a_u - \sigma_u \gamma_u) du + \gamma_u dW_u] = dY_u.$$

This inspired us to try $X_u = Y_u Z_u$.

$\square$

6.2. (i)

Proof. The portfolio is self-financing, so for any $t \leq T_1$, we have

$$dX_t = \Delta_1(t)df(t, R_t, T_1) + \Delta_2(t)df(t, R_t, T_2) + R_t(X_t - \Delta_1(t)f(t, R_t, T_1) - \Delta_2(t)f(t, R_t, T_2))dt,$$

and

$$\begin{aligned}
& d(D_tX_t) \\
= \quad & -R_tD_tX_tdt + D_tdX_t \\
= \quad & D_t[\Delta_1(t)df(t, R_t, T_1) + \Delta_2(t)df(t, R_t, T_2) - R_t(\Delta_1(t)f(t, R_t, T_1) + \Delta_2(t)f(t, R_t, T_2))dt] \\
= \quad & D_t[\Delta_1(t)\left(f_t(t, R_t, T_1)dt + f_r(t, R_t, T_1)dR_t + \frac{1}{2}f_{rr}(t, R_t, T_1)\gamma^2(t, R_t)dt\right) \\
& +\Delta_2(t)\left(f_t(t, R_t, T_2)dt + f_r(t, R_t, T_2)dR_t + \frac{1}{2}f_{rr}(t, R_t, T_2)\gamma^2(t, R_t)dt\right) \\
& -R_t(\Delta_1(t)f(t, R_t, T_1) + \Delta_2(t)f(t, R_t, T_2))dt] \\
= \quad & \Delta_1(t)D_t[-R_tf(t, R_t, T_1) + f_t(t, R_t, T_1) + \alpha(t, R_t)f_r(t, R_t, T_1) + \frac{1}{2}\gamma^2(t, R_t)f_{rr}(t, R_t, T_1)]dt \\
& +\Delta_2(t)D_t[-R_tf(t, R_t, T_2) + f_t(t, R_t, T_2) + \alpha(t, R_t)f_r(t, R_t, T_2) + \frac{1}{2}\gamma^2(t, R_t)f_{rr}(t, R_t, T_2)]dt \\
& +D_t\gamma(t, R_t)[D_t\gamma(t, R_t)[\Delta_1(t)f_r(t, R_t, T_1) + \Delta_2(t)f_r(t, R_t, T_2)]]dW_t \\
= \quad & \Delta_1(t)D_t[\alpha(t, R_t) - \beta(t, R_t, T_1)]f_r(t, R_t, T_1)dt + \Delta_2(t)D_t[\alpha(t, R_t) - \beta(t, R_t, T_2)]f_r(t, R_t, T_2)dt \\
& +D_t\gamma(t, R_t)[\Delta_1(t)f_r(t, R_t, T_1) + \Delta_2(t)f_r(t, R_t, T_2)]dW_t.
\end{aligned}$$

□

(ii)

Proof. Let $\Delta_1(t) = S_tf_r(t, R_t, T_2)$ and $\Delta_2(t) = -S_tf_r(t, R_t, T_1)$, then

$$\begin{aligned}
d(D_tX_t) &= D_tS_t[\beta(t, R_t, T_2) - \beta(t, R_t, T_1)]f_r(t, R_t, T_1)f_r(t, R_t, T_2)dt \\
&= D_t|[\beta(t, R_t, T_1) - \beta(t, R_t, T_2)]f_r(t, R_t, T_1)f_r(t, R_t, T_2)|dt.
\end{aligned}$$

Integrate from 0 to T on both sides of the above equation, we get

$$D_TX_T - D_0X_0 = \int_0^T D_t|[\beta(t, R_t, T_1) - \beta(t, R_t, T_2)]f_r(t, R_t, T_1)f_r(t, R_t, T_2)|dt.$$

If $\beta(t, R_t, T_1) \neq \beta(t, R_t, T_2)$ for some $t \in [0, T]$, under the assumption that $f_r(t, r, T) \neq 0$ for all values of r and $0 \leq t \leq T$, $D_TX_T - D_0X_0 > 0$. To avoid arbitrage (see, for example, Exercise 5.7), we must have for a.s. ω, $\beta(t, R_t, T_1) = \beta(t, R_t, T_2)$, $\forall t \in [0, T]$. This implies $\beta(t, r, T)$ does not depend on T. □

(iii)

Proof. In (6.9.4), let $\Delta_1(t) = \Delta(t)$, $T_1 = T$ and $\Delta_2(t) = 0$, we get

$$\begin{aligned}
d(D_tX_t) &= \Delta(t)D_t\left[-R_tf(t, R_t, T) + f_t(t, R_t, T) + \alpha(t, R_t)f_r(t, R_t, T) + \frac{1}{2}\gamma^2(t, R_t)f_{rr}(t, R_t, T)\right]dt \\
&\quad +D_t\gamma(t, R_t)\Delta(t)f_r(t, R_t, T)dW_t.
\end{aligned}$$

This is formula (6.9.5).

If $f_r(t, r, T) = 0$, then $d(D_tX_t) = \Delta(t)D_t\left[-R_tf(t, R_t, T) + f_t(t, R_t, T) + \frac{1}{2}\gamma^2(t, R_t)f_{rr}(t, R_t, T)\right]dt$. We choose $\Delta(t) = \text{sign}\left\{\left[-R_tf(t, R_t, T) + f_t(t, R_t, T) + \frac{1}{2}\gamma^2(t, R_t)f_{rr}(t, R_t, T)\right]\right\}$. To avoid arbitrage in this case, we must have $f_t(t, R_t, T) + \frac{1}{2}\gamma^2(t, R_t)f_{rr}(t, R_t, T) = R_tf(t, R_t, T)$, or equivalently, for any r in the range of R_t, $f_t(t, r, T) + \frac{1}{2}\gamma^2(t, r)f_{rr}(t, r, T) = rf(t, r, T)$. □

6.3.

Proof. We note

$$\frac{d}{ds}\left[e^{-\int_0^s b_v dv}C(s,T)\right] = e^{-\int_0^s b_v dv}[C(s,T)(-b_s) + b_s C(s,T) - 1] = -e^{-\int_0^s b_v dv}.$$

So integrate on both sides of the equation from t to T, we obtain

$$e^{-\int_0^T b_v dv}C(T,T) - e^{-\int_0^t b_v dv}C(t,T) = -\int_t^T e^{-\int_0^s b_v dv}ds.$$

Since $C(T,T) = 0$, we have $C(t,T) = e^{\int_0^t b_v dv}\int_t^T e^{-\int_0^s b_v dv}ds = \int_t^T e^{\int_s^t b_v dv}ds$. Finally, by $A'(s,T) = -a(s)C(s,T) + \frac{1}{2}\sigma^2(s)C^2(s,T)$, we get

$$A(T,T) - A(t,T) = -\int_t^T a(s)C(s,T)ds + \frac{1}{2}\int_t^T \sigma^2(s)C^2(s,T)ds.$$

Since $A(T,T) = 0$, we have $A(t,T) = \int_t^T (a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T))ds$. $\square$

6.4. (i)

Proof. By the definition of φ, we have

$$\varphi'(t) = e^{\frac{1}{2}\sigma^2\int_t^T C(u,T)du}\frac{1}{2}\sigma^2(-1)C(t,T) = -\frac{1}{2}\varphi(t)\sigma^2 C(t,T).$$

So $C(t,T) = -\frac{2\varphi'(t)}{\phi(t)\sigma^2}$. Differentiate both sides of the equation $\varphi'(t) = -\frac{1}{2}\varphi(t)\sigma^2 C(t,T)$, we get

$$\begin{aligned}
\varphi''(t) &= -\frac{1}{2}\sigma^2[\varphi'(t)C(t,T) + \varphi(t)C'(t,T)] \\
&= -\frac{1}{2}\sigma^2[-\frac{1}{2}\varphi(t)\sigma^2 C^2(t,T) + \varphi(t)C'(t,T)] \\
&= \frac{1}{4}\sigma^4\varphi(t)C^2(t,T) - \frac{1}{2}\sigma^2\varphi(t)C'(t,T).
\end{aligned}$$

So $C'(t,T) = \left[\frac{1}{4}\sigma^4\varphi(t)C^2(t,T) - \varphi''(t)\right]/\frac{1}{2}\varphi(t)\sigma^2 = \frac{1}{2}\sigma^2C^2(t,T) - \frac{2\varphi''(t)}{\sigma^2\varphi(t)}$. $\square$

(ii)

Proof. Plug formulas (6.9.8) and (6.9.9) into (6.5.14), we get

$$-\frac{2\varphi''(t)}{\sigma^2\varphi(t)} + \frac{1}{2}\sigma^2C^2(t,T) = b(-1)\frac{2\varphi'(t)}{\sigma^2\varphi(t)} + \frac{1}{2}\sigma^2C^2(t,T) - 1,$$

i.e. $\varphi''(t) - b\varphi'(t) - \frac{1}{2}\sigma^2\varphi(t) = 0$. $\square$

(iii)

Proof. The characteristic equation of $\varphi''(t) - b\varphi'(t) - \frac{1}{2}\sigma^2\varphi(t) = 0$ is $\lambda^2 - b\lambda - \frac{1}{2}\sigma^2 = 0$, which gives two roots $\frac{1}{2}(b \pm \sqrt{b^2 + 2\sigma^2}) = \frac{1}{2}b \pm \gamma$ with $\gamma = \frac{1}{2}\sqrt{b^2+2\sigma^2}$. Therefore by standard theory of ordinary differential equations, a general solution of φ is $\varphi(t) = e^{\frac{1}{2}bt}(a_1e^{\gamma t} + a_2e^{-\gamma t})$ for some constants a_1 and a_2. It is then easy to see that we can choose appropriate constants c_1 and c_2 so that

$$\varphi(t) = \frac{c_1}{\frac{1}{2}b+\gamma}e^{-(\frac{1}{2}b+\gamma)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma}e^{-(\frac{1}{2}b-\gamma)(T-t)}.$$

$\square$

(iv)

Proof. From part (iii), it is easy to see $\varphi'(t) = c_1 e^{-(\frac{1}{2}b+\gamma)(T-t)} - c_2 e^{-(\frac{1}{2}b-\gamma)(T-t)}$. In particular,

$$0 = C(T,T) = -\frac{2\varphi'(T)}{\sigma^2\varphi(T)} = -\frac{2(c_1-c_2)}{\sigma^2\varphi(T)}.$$

So $c_1 = c_2$. □

(v)

Proof. We first recall the definitions and properties of sinh and cosh:

$$\sinh z = \frac{e^z - e^{-z}}{2},\ \cosh z = \frac{e^z + e^{-z}}{2},\ (\sinh z)' = \cosh z, \text{ and } (\cosh z)' = \sinh z.$$

Therefore

$$\begin{aligned}
\varphi(t) &= c_1 e^{-\frac{1}{2}b(T-t)}\left[\frac{e^{-\gamma(T-t)}}{\frac{1}{2}b+\gamma} - \frac{e^{\gamma(T-t)}}{\frac{1}{2}b-\gamma}\right] \\
&= c_1 e^{-\frac{1}{2}b(T-t)}\left[\frac{\frac{1}{2}b-\gamma}{\frac{1}{4}b^2-\gamma^2}e^{-\gamma(T-t)} - \frac{\frac{1}{2}b+\gamma}{\frac{1}{4}b^2-\gamma^2}e^{\gamma(T-t)}\right] \\
&= \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}\left[-(\frac{1}{2}b-\gamma)e^{-\gamma(T-t)} + (\frac{1}{2}b+\gamma)e^{\gamma(T-t)}\right] \\
&= \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}[b\sinh(\gamma(T-t)) + 2\gamma\cosh(\gamma(T-t))].
\end{aligned}$$

and

$$\begin{aligned}
\varphi'(t) &= \frac{1}{2}b\cdot\frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}[b\sinh(\gamma(T-t)) + 2\gamma\cosh(\gamma(T-t))] \\
&\quad + \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}[-\gamma b\cosh(\gamma(T-t)) - 2\gamma^2\sinh(\gamma(T-t))] \\
&= 2c_1 e^{-\frac{1}{2}b(T-t)}\left[\frac{b^2}{2\sigma^2}\sinh(\gamma(T-t)) + \frac{b\gamma}{\sigma^2}\cosh(\gamma(T-t)) - \frac{b\gamma}{\sigma^2}\cosh(\gamma(T-t)) - \frac{2\gamma^2}{\sigma^2}\sinh(\gamma(T-t))\right] \\
&= 2c_1 e^{-\frac{1}{2}b(T-t)}\frac{b^2-4\gamma^2}{2\sigma^2}\sinh(\gamma(T-t)) \\
&= -2c_1 e^{-\frac{1}{2}b(T-t)}\sinh(\gamma(T-t)).
\end{aligned}$$

This implies

$$C(t,T) = -\frac{2\varphi'(t)}{\sigma^2\varphi(t)} = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}b\sinh(\gamma(T-t))}.$$

□

(vi)

Proof. By (6.5.15) and (6.9.8), $A'(t,T) = \frac{2a\varphi'(t)}{\sigma^2\varphi(t)}$. Hence

$$A(T,T) - A(t,T) = \int_t^T \frac{2a\varphi'(s)}{\sigma^2\varphi(s)}ds = \frac{2a}{\sigma^2}\ln\frac{\varphi(T)}{\varphi(t)},$$

and

$$A(t,T) = -\frac{2a}{\sigma^2}\ln\frac{\varphi(T)}{\varphi(t)} = -\frac{2a}{\sigma^2}\ln\left[\frac{\gamma e^{\frac{1}{2}b(T-t)}}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}b\sinh(\gamma(T-t))}\right].$$

□

6.5. (i)

Proof. Since $g(t, X_1(t), X_2(t)) = E[h(X_1(T), X_2(T))|\mathcal{F}_t]$ and $e^{-rt}f(t, X_1(t), X_2(t)) = E[e^{-rT}h(X_1(T), X_2(T))|\mathcal{F}_t]$, iterated conditioning argument shows $g(t, X_1(t), X_2(t))$ and $e^{-rt}f(t, X_1(t), X_2(t))$ ar both martingales. □

(ii) and (iii)

Proof. We note

$$\begin{aligned}
& dg(t, X_1(t), X_2(t)) \\
= \; & g_t dt + g_{x_1}dX_1(t) + g_{x_2}dX_2(t) + \frac{1}{2}g_{x_1x_2}dX_1(t)dX_1(t) + \frac{1}{2}g_{x_2x_2}dX_2(t)dX_2(t) + g_{x_1x_2}dX_1(t)dX_2(t) \\
= \; & \Big[g_t + g_{x_1}\beta_1 + g_{x_2}\beta_2 + \frac{1}{2}g_{x_1x_1}(\gamma_{11}^2 + \gamma_{12}^2 + 2\rho\gamma_{11}\gamma_{12}) + g_{x_1x_2}(\gamma_{11}\gamma_{21} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21} + \gamma_{12}\gamma_{22}) \\
& + \frac{1}{2}g_{x_2x_2}(\gamma_{21}^2 + \gamma_{22}^2 + 2\rho\gamma_{21}\gamma_{22})\Big] dt + \text{martingale part.}
\end{aligned}$$

So we must have

$$\begin{aligned}
& g_t + g_{x_1}\beta_1 + g_{x_2}\beta_2 + \frac{1}{2}g_{x_1x_1}(\gamma_{11}^2 + \gamma_{12}^2 + 2\rho\gamma_{11}\gamma_{12}) + g_{x_1x_2}(\gamma_{11}\gamma_{21} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21} + \gamma_{12}\gamma_{22}) \\
& + \frac{1}{2}g_{x_2x_2}(\gamma_{21}^2 + \gamma_{22}^2 + 2\rho\gamma_{21}\gamma_{22}) = 0.
\end{aligned}$$

Taking $\rho = 0$ will give part (ii) as a special case. The PDE for f can be similarly obtained. □

6.6. (i)

Proof. Multiply $e^{\frac{1}{2}bt}$ on both sides of (6.9.15), we get

$$d(e^{\frac{1}{2}bt}X_j(t)) = e^{\frac{1}{2}bt}\left(X_j(t)\frac{1}{2}bdt + (-\frac{b}{2}X_j(t)dt + \frac{1}{2}\sigma dW_j(t)\right) = e^{\frac{1}{2}bt}\frac{1}{2}\sigma dW_j(t).$$

So $e^{\frac{1}{2}bt}X_j(t) - X_j(0) = \frac{1}{2}\sigma\int_0^t e^{\frac{1}{2}bu}dW_j(u)$ and $X_j(t) = e^{-\frac{1}{2}bt}\left(X_j(0) + \frac{1}{2}\sigma\int_0^t e^{\frac{1}{2}bu}dW_j(u)\right)$. By Theorem 4.4.9, $X_j(t)$ is normally distributed with mean $X_j(0)e^{-\frac{1}{2}bt}$ and variance $\frac{e^{-bt}}{4}\sigma^2\int_0^t e^{bu}du = \frac{\sigma^2}{4b}(1 - e^{-bt})$. □

(ii)

Proof. Suppose $R(t) = \sum_{j=1}^d X_j^2(t)$, then

$$\begin{aligned}
dR(t) &= \sum_{j=1}^d (2X_j(t)dX_j(t) + dX_j(t)dX_j(t)) \\
&= \sum_{j=1}^d \left(2X_j(t)dX_j(t) + \frac{1}{4}\sigma^2 dt\right) \\
&= \sum_{j=1}^d \left(-bX_j^2(t)dt + \sigma X_j(t)dW_j(t) + \frac{1}{4}\sigma^2 dt\right) \\
&= \left(\frac{d}{4}\sigma^2 - bR(t)\right)dt + \sigma\sqrt{R(t)}\sum_{j=1}^d \frac{X_j(t)}{\sqrt{R(t)}}dW_j(t).
\end{aligned}$$

Let $B(t) = \sum_{j=1}^d \int_0^t \frac{X_j(s)}{\sqrt{R(s)}}dW_j(s)$, then B is a local martingale with $dB(t)dB(t) = \sum_{j=1}^d \frac{X_j^2(t)}{R(t)}dt = dt$. So by Lévy's Theorem, B is a Brownian motion. Therefore $dR(t) = (a - bR(t))dt + \sigma\sqrt{R(t)}dB(t)$ ($a := \frac{d}{4}\sigma^2$) and R is a CIR interest rate process. □

(iii)

Proof. By (6.9.16), $X_j(t)$ is dependent on W_j only and is normally distributed with mean $e^{-\frac{1}{2}bt}X_j(0)$ and variance $\frac{\sigma^2}{4b}[1-e^{-bt}]$. So $X_1(t)$, $\cdots$, $X_d(t)$ are i.i.d. normal with the same mean $\mu(t)$ and variance $v(t)$. □

(iv)

Proof.

$$\begin{aligned}
E\left[e^{uX_j^2(t)}\right] &= \int_{-\infty}^{\infty} e^{ux^2}\frac{e^{-\frac{(x-\mu(t))^2}{2v(t)}}dx}{\sqrt{2\pi v(t)}} \\
&= \int_{-\infty}^{\infty}\frac{e^{-\frac{(1-2uv(t))x^2-2\mu(t)x+\mu^2(t)}{2v(t)}}}{\sqrt{2\pi v(t)}}dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi v(t)}}e^{-\frac{\left(x-\frac{\mu(t)}{1-2uv(t)}\right)^2+\frac{\mu^2(t)}{1-2uv(t)}-\frac{\mu^2(t)}{(1-2uv(t))^2}}{2v(t)/(1-2uv(t))}}dx \\
&= \int_{-\infty}^{\infty}\frac{\sqrt{1-2uv(t)}}{\sqrt{2\pi v(t)}}e^{-\frac{\left(x-\frac{\mu(t)}{1-2uv(t)}\right)^2}{2v(t)/(1-2uv(t))}}dx\cdot\frac{e^{-\frac{\mu^2(t)(1-2uv(t))-\mu^2(t)}{2v(t)(1-2uv(t))}}}{\sqrt{1-2uv(t)}} \\
&= \frac{e^{-\frac{u\mu^2(t)}{1-2uv(t)}}}{\sqrt{1-2uv(t)}}.
\end{aligned}$$

□

(v)

Proof. By $R(t)=\sum_{j=1}^d X_j^2(t)$ and the fact $X_1(t)$, $\cdots$, $X_d(t)$ are i.i.d.,

$$E[e^{uR(t)}]=(E[e^{uX_1^2(t)}])^d=(1-2uv(t))^{-\frac{d}{2}}e^{\frac{ud\mu^2(t)}{1-2uv(t)}}=(1-2uv(t))^{-\frac{2a}{\sigma^2}}e^{-\frac{e^{-bt}uR(0)}{1-2uv(t)}}.$$

□

6.7. (i)

Proof. $e^{-rt}c(t,S_t,V_t)=\widetilde{E}[e^{-rT}(S_T-K)^+|\mathcal{F}_t]$ is a martingale by iterated conditioning argument. Since

$$\begin{aligned}
& d(e^{-rt}c(t,S_t,V_t)) \\
= \;& e^{-rt}\Big[c(t,S_t,V_t)(-r)+c_t(t,S_t,V_t)+c_s(t,S_t,V_t)rS_t+c_v(t,S_t,V_t)(a-bV_t)+\frac{1}{2}c_{ss}(t,S_t,V_t)V_tS_t^2+ \\
& \frac{1}{2}c_{vv}(t,S_t,V_t)\sigma^2V_t+c_{sv}(t,S_t,V_t)\sigma V_tS_t\rho\Big]dt+\text{martingale part},
\end{aligned}$$

we conclude $rc=c_t+rsc_s+c_v(a-bv)+\frac{1}{2}c_{ss}vs^2+\frac{1}{2}c_{vv}\sigma^2v+c_{sv}\sigma sv\rho$. This is equation (6.9.26). □

(ii)

Proof. Suppose $c(t,s,v)=sf(t,\log s,v)-e^{-r(T-t)}Kg(t,\log s,v)$, then

$$\begin{aligned}
c_t &= sf_t(t,\log s,v)-re^{-r(T-t)}Kg(t,\log s,v)-e^{-r(T-t)}Kg_t(t,\log s,v), \\
c_s &= f(t,\log s,v)+sf_s(t,\log s,v)\frac{1}{s}-e^{-r(T-t)}Kg_s(t,\log s,v)\frac{1}{s}, \\
c_v &= sf_v(t,\log s,v)-e^{-r(T-t)}Kg_v(t,\log s,v), \\
c_{ss} &= f_s(t,\log s,v)\frac{1}{s}+f_{ss}(t,\log s,v)\frac{1}{s}-e^{-r(T-t)}Kg_{ss}(t,\log s,v)\frac{1}{s^2}+e^{-r(T-t)}Kg_s(t,\log s,v)\frac{1}{s^2}, \\
c_{sv} &= f_v(t,\log s,v)+f_{sv}(t,\log s,v)-e^{-r(T-t)}\frac{K}{s}g_{sv}(t,\log s,v), \\
c_{vv} &= sf_{vv}(t,\log s,v)-e^{-r(T-t)}Kg_{vv}(t,\log s,v).
\end{aligned}$$

So

$$\begin{aligned}
& c_t + rsc_s + (a-bv)c_v + \frac{1}{2}s^2vc_{ss} + \rho\sigma svc_{sv} + \frac{1}{2}\sigma^2vc_{vv} \\
= \quad & sf_t - re^{-r(T-t)}Kg - e^{-r(T-t)}Kg_t + rsf + rsf_s - rKe^{-r(T-t)}g_s + (a-bv)(sf_v - e^{-r(T-t)}Kg_v) \\
& + \frac{1}{2}s^2v\left[-\frac{1}{s}f_s + \frac{1}{s}f_{ss} - e^{-r(T-t)}\frac{K}{s^2}g_{ss} + e^{-r(T-t)}K\frac{g_s}{s^2}\right] + \rho\sigma sv\left(f_v + f_{sv} - e^{-r(T-t)}\frac{K}{s}g_{sv}\right) \\
& + \frac{1}{2}\sigma^2v(sf_{vv} - e^{-r(T-t)}Kg_{vv}) \\
= \quad & s\left[f_t + (r + \frac{1}{2}v)f_s + (a - bv + \rho\sigma v)f_v + \frac{1}{2}vf_{ss} + \rho\sigma vf_{sv} + \frac{1}{2}\sigma^2vf_{vv}\right] - Ke^{-r(T-t)}\left[g_t + (r - \frac{1}{2}v)g_s\right. \\
& \left. + (a-bv)g_v + \frac{1}{2}vg_{ss} + \rho\sigma vg_{sv} + \frac{1}{2}\sigma^2vg_{vv}\right] + rsf - re^{-r(T-t)}Kg \\
= \quad & rc.
\end{aligned}$$

That is, c satisfies the PDE (6.9.26). □

(iii)

Proof. First, by Markov property, $f(t, X_t, V_t) = E[1_{\{X_T \geq \log K\}}|\mathcal{F}_t]$. So $f(T, X_t, V_t) = 1_{\{X_T \geq \log K\}}$, which implies $f(T, x, v) = 1_{\{x \geq \log K\}}$ for all $x \in \mathbb{R}$, $v \geq 0$. Second, $f(t, X_t, V_t)$ is a martingale, so by differentiating f and setting the dt term as zero, we have the PDE (6.9.32) for f. Indeed,

$$\begin{aligned}
df(t, X_t, V_t) \quad = \quad & \left[f_t(t, X_t, V_t) + f_x(t, X_t, V_t)(r + \frac{1}{2}V_t) + f_v(t, X_t, V_t)(a - bv_t + \rho\sigma V_t) + \frac{1}{2}f_{xx}(t, X_t, V_t)V_t\right. \\
& \left. + \frac{1}{2}f_{vv}(t, X_t, V_t)\sigma^2V_t + f_{xv}(t, X_t, V_t)\sigma V_t\rho\right] dt + \text{martingale part.}
\end{aligned}$$

So we must have $f_t + (r + \frac{1}{2}v)f_x + (a - bv + \rho\sigma v)f_v + \frac{1}{2}f_{xx}v + \frac{1}{2}f_{vv}\sigma^2v + \sigma v\rho f_{xv} = 0$. This is (6.9.32). □

(iv)

Proof. Similar to (iii). □

(v)

Proof. $c(T, s, v) = sf(T, \log s, v) - e^{-r(T-t)}Kg(T, \log s, v) = s1_{\{\log s \geq \log K\}} - K1_{\{\log s \geq \log K\}} = 1_{\{s \geq K\}}(s - K) = (s - K)^+$. □

6.8.

Proof. We follow the hint. Suppose h is smooth and compactly supported, then it is legitimate to exchange integration and differentiation:

$$\begin{aligned}
g_t(t, x) &= \frac{\partial}{\partial t}\int_0^\infty h(y)p(t, T, x, y)dy = \int_0^\infty h(y)p_t(t, T, x, y)dy, \\
g_x(t, x) &= \int_0^\infty h(y)p_x(t, T, x, y)dy, \\
g_{xx}(t, x) &= \int_0^\infty h(y)p_{xx}(t, T, x, y)dy.
\end{aligned}$$

So (6.9.45) implies $\int_0^\infty h(y)\left[p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y)\right]dy = 0$. By the arbitrariness of h and assuming β, p_t, p_x, v, p_{xx} are all continuous, we have

$$p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y) = 0.$$

This is (6.9.43). □

6.9.

Proof. We first note $dh_b(X_u) = h_b'(X_u)dX_u + \frac{1}{2}h_b''(X_u)dX_udX_u = \left[h_b'(X_u)\beta(u,X_u) + \frac{1}{2}\gamma^2(u,X_u)h_b''(X_u)\right]du + h_b'(X_u)\gamma(u,X_u)dW_u$. Integrate on both sides of the equation, we have

$$h_b(X_T) - h_b(X_t) = \int_t^T \left[h_b'(X_u)\beta(u,X_u) + \frac{1}{2}\gamma^2(u,X_u)h_b''(X_u)\right]du + \text{martingale part}.$$

Take expectation on both sides, we get

$$\begin{aligned}
E^{t,x}[h_b(X_T) - h_b(X_t)] &= \int_{-\infty}^{\infty} h_b(y)p(t,T,x,y)dy - h(x) \\
&= \int_t^T E^{t,x}[h_b'(X_u)\beta(u,X_u) + \frac{1}{2}\gamma^2(u,X_u)h_b''(X_u)]du \\
&= \int_t^T \int_{-\infty}^{\infty} \left[h_b'(y)\beta(u,y) + \frac{1}{2}\gamma^2(u,y)h_b''(y)\right] p(t,u,x,y)dydu.
\end{aligned}$$

Since h_b vanishes outside $(0,b)$, the integration range can be changed from $(-\infty,\infty)$ to $(0,b)$, which gives (6.9.48).

By integration-by-parts formula, we have

$$\begin{aligned}
\int_0^b \beta(u,y)p(t,u,x,y)h_b'(y)dy &= h_b(y)\beta(u,y)p(t,u,x,y)|_0^b - \int_0^b h_b(y)\frac{\partial}{\partial y}(\beta(u,y)p(t,u,x,y))dy \\
&= -\int_0^b h_b(y)\frac{\partial}{\partial y}(\beta(u,y)p(t,u,x,y))dy,
\end{aligned}$$

and

$$\int_0^b \gamma^2(u,y)p(t,u,x,y)h_b''(y)dy = -\int_0^b \frac{\partial}{\partial y}(\gamma^2(u,y)p(t,u,x,y))h_b'(y)dy = \int_0^b \frac{\partial^2}{\partial y}(\gamma^2(u,y)p(t,u,x,y))h_b(y)dy.$$

Plug these formulas into (6.9.48), we get (6.9.49).

Differentiate w.r.t. T on both sides of (6.9.49), we have

$$\int_0^b h_b(y)\frac{\partial}{\partial T}p(t,T,x,y)dy = -\int_0^b \frac{\partial}{\partial y}[\beta(T,y)p(t,T,x,y)]h_b(y)dy + \frac{1}{2}\int_0^b \frac{\partial^2}{\partial y^2}[\gamma^2(T,y)p(t,T,x,y)]h_b(y)dy,$$

that is,

$$\int_0^b h_b(y)\left[\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}(\beta(T,y)p(t,T,x,y)) - \frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T,y)p(t,T,x,y))\right]dy = 0.$$

This is (6.9.50).

By (6.9.50) and the arbitrariness of h_b, we conclude for any $y \in (0,\infty)$,

$$\frac{\partial}{\partial T}p(t,T,x,y) + \frac{\partial}{\partial y}(\beta(T,y)p(t,T,x,y)) - \frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T,y)p(t,T,x,y)) = 0.$$

□

6.10.

Proof. Under the assumption that $\lim_{y\to\infty}(y-K)ry\widetilde{p}(0,T,x,y) = 0$, we have

$$-\int_K^\infty (y-K)\frac{\partial}{\partial y}(ry\widetilde{p}(0,T,x,y))dy = -(y-K)ry\widetilde{p}(0,T,x,y)|_K^\infty + \int_K^\infty ry\widetilde{p}(0,T,x,y)dy = \int_K^\infty ry\widetilde{p}(0,T,x,y)dy.$$

If we further assume (6.9.57) and (6.9.58), then use integration-by-parts formula twice, we have

$$
\begin{aligned}
& \frac{1}{2}\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}(\sigma^2(T,y)y^2\widetilde{p}(0,T,x,y))dy \\
= & \frac{1}{2}\left[(y-K)\frac{\partial}{\partial y}(\sigma^2(T,y)y^2\widetilde{p}(0,T,x,y))|_K^\infty - \int_K^\infty \frac{\partial}{\partial y}(\sigma^2(T,y)y^2\widetilde{p}(0,T,x,y))dy\right] \\
= & -\frac{1}{2}(\sigma^2(T,y)y^2\widetilde{p}(0,T,x,y)|_K^\infty) \\
= & \frac{1}{2}\sigma^2(T,K)K^2\widetilde{p}(0,T,x,K).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c_T(0,T,x,K) &= -rc(0,T,x,K) + e^{-rT}\int_K^\infty (y-K)\widetilde{p}_T(0,T,x,y)dy \\
&= -re^{-rT}\int_K^\infty (y-K)\widetilde{p}(0,T,x,y)dy + e^{-rT}\int_K^\infty (y-K)\widetilde{p}_T(0,T,x,y)dy \\
&= -re^{-rT}\int_K^\infty (y-K)\widetilde{p}(0,T,x,y)dy - e^{-rT}\int_K^\infty (y-K)\frac{\partial}{\partial y}(ry\widetilde{p}(t,T,x,y))dy \\
&\quad + e^{-rT}\int_K^\infty (y-K)\frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(T,y)y^2\widetilde{p}(t,T,x,y))dy \\
&= -re^{-rT}\int_K^\infty (y-K)\widetilde{p}(0,T,x,y)dy + e^{-rT}\int_K^\infty ry\widetilde{p}(0,T,x,y)dy \\
&\quad + e^{-rT}\frac{1}{2}\sigma^2(T,K)K^2\widetilde{p}(0,T,x,K) \\
&= re^{-rT}K\int_K^\infty \widetilde{p}(0,T,x,y)dy + \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\widetilde{p}(0,T,x,K) \\
&= -rKc_K(0,T,x,K) + \frac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K).
\end{aligned}
$$

□

2.7 Exotic Options

7.1. (i)

Proof. Since $\delta_\pm(\tau,s) = \frac{1}{\sigma\sqrt{\tau}}[\log s + (r\pm\frac{1}{2}\sigma^2)\tau] = \frac{\log s}{\sigma}\tau^{-\frac{1}{2}} + \frac{r\pm\frac{1}{2}\sigma^2}{\sigma}\sqrt{\tau}$,

$$
\begin{aligned}
\frac{\partial}{\partial t}\delta_\pm(\tau,s) &= \frac{\log s}{\sigma}(-\frac{1}{2})\tau^{-\frac{3}{2}}\frac{\partial\tau}{\partial t} + \frac{r\pm\frac{1}{2}\sigma^2}{\sigma}\frac{1}{2}\tau^{-\frac{1}{2}}\frac{\partial\tau}{\partial t} \\
&= -\frac{1}{2\tau}\left[\frac{\log s}{\sigma}\frac{1}{\sqrt{\tau}}(-1) - \frac{r\pm\frac{1}{2}\sigma^2}{\sigma}\sqrt{\tau}(-1)\right] \\
&= -\frac{1}{2\tau}\cdot\frac{1}{\sigma\sqrt{\tau}}\left[-\log ss + (r\pm\frac{1}{2}\sigma^2)\tau)\right] \\
&= -\frac{1}{2\tau}\delta_\pm(\tau,\frac{1}{s}).
\end{aligned}
$$

□

(ii)

Proof.

$$\frac{\partial}{\partial x}\delta_\pm(\tau,\frac{x}{c}) = \frac{\partial}{\partial x}\left(\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{c}+(r\pm\frac{1}{2}\sigma^2)\tau\right]\right) = \frac{1}{x\sigma\sqrt{\tau}},$$
$$\frac{\partial}{\partial x}\delta_\pm(\tau,\frac{c}{x}) = \frac{\partial}{\partial x}\left(\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{c}{x}+(r\pm\frac{1}{2}\sigma^2)\tau\right]\right) = -\frac{1}{x\sigma\sqrt{\tau}}.$$

□

(iii)

Proof.

$$N'(\delta_\pm(\tau,s)) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\delta_\pm(\tau,s)}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(\log s+r\tau)^2\pm\sigma^2\tau(\log s+r\tau)+\frac{1}{4}\sigma^4\tau^2}{2\sigma^2\tau}}.$$

Therefore

$$\frac{N'(\delta_+(\tau,s))}{N'(\delta_-(\tau,s))} = e^{-\frac{2\sigma^2\tau(\log s+r\tau)}{2\sigma^2\tau}} = \frac{e^{-r\tau}}{s}$$

and $e^{-r\tau}N'(\delta_-(\tau,s)) = sN'(\delta_+(\tau,s))$. □

(iv)

Proof.

$$\frac{N'(\delta_\pm(\tau,s))}{N'(\delta_\pm(\tau,s^{-1}))} = e^{-\frac{\left[(\log s+r\tau)^2-(\log\frac{1}{s}+r\tau)^2\right]\pm\sigma^2\tau(\log s-\log\frac{1}{s})}{2\sigma^2\tau}} = e^{-\frac{4r\tau\log s\pm 2\sigma^2\tau\log s}{2\sigma^2\tau}} = e^{-(\frac{2r}{\sigma^2}\pm1)\log s} = s^{-(\frac{2r}{\sigma^2}\pm1)}.$$

So $N'(\delta_\pm(\tau,s^{-1})) = s^{(\frac{2r}{\sigma^2}\pm1)}N'(\delta_\pm(\tau,s))$. □

(v)

Proof. $\delta_+(\tau,s)-\delta_-(\tau,s) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s+(r+\frac{1}{2}\sigma^2)\tau\right] - \frac{1}{\sigma\sqrt{\tau}}\left[\log s+(r-\frac{1}{2}\sigma^2)\tau\right] = \frac{1}{\sigma\sqrt{\tau}}\sigma^2\tau = \sigma\sqrt{\tau}$. □

(vi)

Proof. $\delta_\pm(\tau,s)-\delta_\pm(\tau,s^{-1}) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s+(r\pm\frac{1}{2}\sigma^2)\tau\right] - \frac{1}{\sigma\sqrt{\tau}}\left[\log s^{-1}+(r\pm\frac{1}{2}\sigma^2)\tau\right] = \frac{2\log s}{\sigma\sqrt{\tau}}$. □

(vii)

Proof. $N'(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$, so $N''(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}(-\frac{y^2}{2})' = -yN'(y)$. □

To be continued ...

7.3.

Proof. We note $S_T = S_0e^{\sigma\widehat{W}_T} = S_te^{\sigma(\widehat{W}_T-\widehat{W}_t)}$, $\widehat{W}_T-\widehat{W}_t = (\widetilde{W}_T-\widetilde{W}_t)+\alpha(T-t)$ is independent of $\mathcal{F}_t$, $\sup_{t\le u\le T}(\widehat{W}_u-\widehat{W}_t)$ is independent of $\mathcal{F}_t$, and

$$\begin{aligned}
Y_T &= S_0e^{\sigma\widehat{M}_T}\\
&= S_0e^{\sigma\sup_{t\le u\le T}\widehat{W}_u}1_{\{\widehat{M}_t\le\sup_{t\le u\le T}\widehat{W}_t\}} + S_0e^{\sigma\widehat{M}_t}1_{\{\widehat{M}_t>\sup_{t\le u\le T}\widehat{W}_u\}}\\
&= S_te^{\sigma\sup_{t\le u\le T}(\widehat{W}_u-\widehat{W}_t)}1_{\{\frac{Y_t}{S_t}\le e^{\sigma\sup_{t\le u\le T}(\widehat{W}_u-\widehat{W}_t)}\}} + Y_t1_{\{\frac{Y_t}{S_t}\le e^{\sigma\sup_{t\le u\le T}(\widehat{W}_u-\widehat{W}_t)}\}}.
\end{aligned}$$

So $E[f(S_T,Y_T)|\mathcal{F}_t] = E[f(x\frac{S_{T-t}}{S_0}, x\frac{Y_{T-t}}{S_0}1_{\{\frac{y}{x}\le\frac{Y_{T-t}}{S_0}\}} + y1_{\{\frac{y}{x}\le\frac{Y_{T-t}}{S_0}\}})]$, where $x=S_t$, $y=Y_t$. Therefore $E[f(S_T,Y_T)|\mathcal{F}_t]$ is a Borel function of (S_t,Y_t).

□

7.4.

Proof. By Cauchy's inequality and the monotonicity of Y, we have

$$\begin{aligned}
|\sum_{j=1}^m (Y_{t_j} - Y_{t_{j-1}})(S_{t_j} - S_{t_{j-1}})| &\leq \sum_{j=1}^m |Y_{t_j} - Y_{t_{j-1}}||S_{t_j} - S_{t_{j-1}}| \\
&\leq \sqrt{\sum_{j=1}^m (Y_{t_j} - Y_{t_{j-1}})^2}\sqrt{\sum_{j=1}^m (S_{t_j} - S_{t_{j-1}})^2} \\
&\leq \sqrt{\max_{1\leq j\leq m} |Y_{t_j} - Y_{t_{j-1}}|(Y_T - Y_0)}\sqrt{\sum_{j=1}^m (S_{t_j} - S_{t_{j-1}})^2}.
\end{aligned}$$

If we increase the number of partition points to infinity and let the length of the longest subinterval $\max_{1\leq j\leq m}|t_j - t_{j-1}|$ approach zero, then $\sqrt{\sum_{j=1}^m (S_{t_j} - S_{t_{j-1}})^2} \to \sqrt{[S]_T - [S]_0} < \infty$ and $\max_{1\leq j\leq m}|Y_{t_j} - Y_{t_{j-1}}| \to 0$ a.s. by the continuity of Y. This implies $\sum_{j=1}^m (Y_{t_j} - Y_{t_{j-1}})(S_{t_j} - S_{t_{j-1}}) \to 0$. □

2.8 American Derivative Securities

8.1.

Proof. $v_L'(L+) = (K-L)(-\frac{2r}{\sigma^2})(\frac{x}{L})^{-\frac{2r}{\sigma^2}-1}\frac{1}{L}\Big|_{x=L} = -\frac{2r}{\sigma^2 L}(K-L)$. So $v_L'(L+) = v_L'(L-)$ if and only if $-\frac{2r}{\sigma^2 L}(K-L) = -1$. Solve for L, we get $L = \frac{2rK}{2r+\sigma^2}$. □

8.2.

Proof. By the calculation in Section 8.3.3, we can see $v_2(x) \geq (K_2 - x)^+ \geq (K_1 - x)^+$, $rv_2(x) - rxv_2'(x) - \frac{1}{2}\sigma^2x^2v_2''(x) \geq 0$ for all $x \geq 0$, and for $0 \leq x < L_{1*} < L_{2*}$,

$$rv_2(x) - rxv_2'(x) - \frac{1}{2}\sigma^2x^2v_2''(x) = rK_2 > rK_1 > 0.$$

So the linear complementarity conditions for v_2 imply $v_2(x) = (K_2 - x)^+ = K_2 - x > K_1 - x = (K_1 - x)^+$ on $[0, L_{1*}]$. Hence $v_2(x)$ does not satisfy the third linear complementarity condition for v_1: for each $x \geq 0$, equality holds in either (8.8.1) or (8.8.2) or both. □

8.3. (i)

Proof. Suppose x takes its values in a domain bounded away from 0. By the general theory of linear differential equations, if we can find two linearly independent solutions $v_1(x)$, $v_2(x)$ of (8.8.4), then any solution of (8.8.4) can be represented in the form of $C_1v_1 + C_2v_2$ where C_1 and C_2 are constants. So it suffices to find two linearly independent special solutions of (8.8.4). Assume $v(x) = x^p$ for some constant p to be determined, (8.8.4) yields $x^p(r - pr - \frac{1}{2}\sigma^2p(p-1)) = 0$. Solve the quadratic equation $0 = r - pr - \frac{1}{2}\sigma^2p(p-1) = (-\frac{1}{2}\sigma^2p - r)(p-1)$, we get $p = 1$ or $-\frac{2r}{\sigma^2}$. So a general solution of (8.8.4) has the form $C_1x + C_2x^{-\frac{2r}{\sigma^2}}$. □

(ii)

Proof. Assume there is an interval $[x_1, x_2]$ where $0 < x_1 < x_2 < \infty$, such that $v(x) \not\equiv 0$ satisfies (8.3.19) with equality on $[x_1, x_2]$ and satisfies (8.3.18) with equality for x at and immediately to the left of x_1 and for x at and immediately to the right of x_2, then we can find some C_1 and C_2, so that $v(x) = C_1x + C_2x^{-\frac{2r}{\sigma^2}}$ on $[x_1, x_2]$. If for some $x_0 \in [x_1, x_2]$, $v(x_0) = v'(x_0) = 0$, by the uniqueness of the solution of (8.8.4), we would conclude $v \equiv 0$. This is a contradiction. So such an x_0 cannot exist. This implies $0 < x_1 < x_2 < K$

(if $K \leq x_2$, $v(x_2) = (K - x_2)^+ = 0$ and $v'(x_2)$=the right derivative of $(K-x)^+$ at x_2, which is 0). [1] Thus we have four equations for C_1 and C_2:

$$\begin{cases} C_1 x_1 + C_2 x_1^{-\frac{2r}{\sigma^2}} = K - x_1 \\ C_1 x_2 + C_2 x_2^{-\frac{2r}{\sigma^2}} = K - x_2 \\ C_1 - \frac{2r}{\sigma^2} C_2 x_1^{-\frac{2r}{\sigma^2}-1} = -1 \\ C_1 - \frac{2r}{\sigma^2} C_2 x_2^{-\frac{2r}{\sigma^2}-1} = -1. \end{cases}$$

Since $x_1 \neq x_2$, the last two equations imply $C_2 = 0$. Plug $C_2 = 0$ into the first two equations, we have $C_1 = \frac{K-x_1}{x_1} = \frac{K-x_2}{x_2}$; plug $C_2 = 0$ into the last two equations, we have $C_1 = -1$. Combined, we would have $x_1 = x_2$. Contradiction. Therefore our initial assumption is incorrect, and the only solution v that satisfies the specified conditions in the problem is the zero solution. □

(iii)

Proof. If in a right neighborhood of 0, v satisfies (8.3.19) with equality, then part (i) implies $v(x) = C_1 x + C_2 x^{-\frac{2r}{\sigma^2}}$ for some constants C_1 and C_2. Then $v(0) = \lim_{x \downarrow 0} v(x) = 0 < (K-0)^+$, i.e. (8.3.18) will be violated. So we must have $rv - rxv' - \frac{1}{2}\sigma^2 x^2 v'' > 0$ in a right neighborhood of 0. According to (8.3.20), $v(x) = (K-x)^+$ near o. So $v(0) = K$. We have thus concluded simultaneously that v cannot satisfy (8.3.19) with equality near 0 and $v(0) = K$, starting from first principles (8.3.18)-(8.3.20). □

(iv)

Proof. This is already shown in our solution of part (iii): near 0, v cannot satisfy (8.3.19) with equality. □

(v)

Proof. If v satisfy $(K-x)^+$ with equality for all $x \geq 0$, then v cannot have a continuous derivative as stated in the problem. This is a contradiction. □

(vi)

Proof. By the result of part (i), we can start with $v(x) = (K-x)^+$ on $[0, x_1]$ and $v(x) = C_1 x + C_2 x^{-\frac{2r}{\sigma^2}}$ on $[x_1, \infty)$. By the assumption of the problem, both v and v' are continuous. Since $(K-x)^+$ is not differentiable at K, we must have $x_1 \leq K$.This gives us the equations

$$\begin{cases} K - x_1 = (K-x_1)^+ = C_1 x_1 + C_2 x_1^{-\frac{2r}{\sigma^2}} \\ -1 = C_1 - \frac{2r}{\sigma^2} C_2 x_1^{-\frac{2r}{\sigma^2}-1}. \end{cases}$$

Because v is assumed to be bounded, we must have $C_1 = 0$ and the above equations only have two unknowns: C_2 and x_1. Solve them for C_2 and x_1, we are done. □

8.4. (i)

Proof. This is already shown in part (i) of Exercise 8.3. □

(ii)

[1] Note we have interpreted the condition "$v(x)$ satisfies (8.3.18) with equality for x at and immediately to the right of x_2" as "$v(x_2) = (K-x_2)^+$ and $v'(x_2)$ =the right derivative of $(K-x)^+$ at x_2." This is weaker than "$v(x) = (K-x)$ in a right neighborhood of x_2."

Proof. We solve for A, B the equations

$$\begin{cases} AL^{-\frac{2r}{\sigma^2}} + BL = K - L \\ -\frac{2r}{\sigma^2}AL^{-\frac{2r}{\sigma^2}-1} + B = -1, \end{cases}$$

and we obtain $A = \frac{\sigma^2 KL^{\frac{2r}{\sigma^2}}}{\sigma^2+2r}$, $B = \frac{2rK}{L(\sigma^2+2r)} - 1$. □

(iii)

Proof. By (8.8.5), $B > 0$. So for $x \geq K$, $f(x) \geq BK > 0 = (K-x)^+$. If $L \leq x < K$,

$$f(x) - (K-x)^+ = \frac{\sigma^2 KL^{\frac{2r}{\sigma^2}}}{\sigma^2+2r}x^{-\frac{2r}{\sigma^2}} + \frac{2rKx}{L(\sigma^2+2r)} - K = x^{-\frac{2r}{\sigma^2}}\frac{KL^{\frac{2r}{\sigma^2}}\left[\sigma^2 + 2r(\frac{x}{L})^{\frac{2r}{\sigma^2}+1} - (\sigma^2+2r)(\frac{x}{L})^{\frac{2r}{\sigma^2}}\right]}{(\sigma^2+2r)L}.$$

Let $g(\theta) = \sigma^2 + 2r\theta^{\frac{2r}{\sigma^2}+1} - (\sigma^2+2r)\theta^{\frac{2r}{\sigma^2}}$ with $\theta \geq 1$. Then $g(1) = 0$ and $g'(\theta) = 2r(\frac{2r}{\sigma^2}+1)\theta^{\frac{2r}{\sigma^2}} - (\sigma^2+2r)\frac{2r}{\sigma^2}\theta^{\frac{2r}{\sigma^2}-1} = \frac{2r}{\sigma^2}(\sigma^2+2r)\theta^{\frac{2r}{\sigma^2}-1}(\theta-1) \geq 0$. So $g(\theta) \geq 0$ for any $\theta \geq 1$. This shows $f(x) \geq (K-x)^+$ for $L \leq x < K$. Combined, we get $f(x) \geq (K-x)^+$ for all $x \geq L$. □

(iv)

Proof. Since $\lim_{x\to\infty} v(x) = \lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} v_{L_*}(x) = \lim_{x\to\infty}(K-L_*)(\frac{x}{L_*})^{-\frac{2r}{\sigma^2}} = 0$, $v(x)$ and $v_{L_*}(x)$ are different. By part (iii), $v(x) \geq (K-x)^+$. So v satisfies (8.3.18). For $x \geq L$, $rv - rxv' - \frac{1}{2}\sigma^2x^2v'' = rf - rxf - \frac{1}{2}\sigma^2x^2f'' = 0$. For $0 \leq x \leq L$, $rv - rxv' - \frac{1}{2}\sigma^2x^2v'' = r(K-x) + rx = rK$. Combined, $rv - rxv' - \frac{1}{2}\sigma^2x^2v'' \geq 0$ for $x \geq 0$. So v satisfies (8.3.19). Along the way, we also showed v satisfies (8.3.20). In summary, v satisfies the linear complementarity condition (8.3.18)-(8.3.20), but v is not the function v_{L_*} given by (8.3.13). □

(v)

Proof. By part (ii), $B = 0$ if and only if $\frac{2rK}{L(\sigma^2+2r)} - 1 = 0$, i.e. $L = \frac{2rK}{2r+\sigma^2}$. In this case, $v(x) = Ax^{-\frac{2r}{\sigma^2}} = \frac{\sigma^2K}{\sigma^2+2r}(\frac{x}{L})^{-\frac{2r}{\sigma^2}} = (K-L)(\frac{x}{L})^{-\frac{2r}{\sigma^2}} = v_{L_*}(x)$, on the interval $[L,\infty)$. □

8.5. The difficulty of the dividend-paying case is that from Lemma 8.3.4, we can only obtain $\widetilde{E}[e^{-(r-a)\tau_L}]$, not $\widetilde{E}[e^{-r\tau_L}]$. So we have to start from Theorem 8.3.2.

(i)

Proof. By (8.8.9), $S_t = S_0e^{\sigma\widetilde{W}_t+(r-a-\frac{1}{2}\sigma^2)t}$. Assume $S_0 = x$, then $S_t = L$ if and only if $-\widetilde{W}_t - \frac{1}{\sigma}(r-a-\frac{1}{2}\sigma^2)t = \frac{1}{\sigma}\log\frac{x}{L}$. By Theorem 8.3.2,

$$\widetilde{E}[e^{-r\tau_L}] = e^{-\frac{1}{\sigma}\log\frac{x}{L}\left[\frac{1}{\sigma}(r-a-\frac{1}{2}\sigma^2)+\sqrt{\frac{1}{\sigma^2}(r-a-\frac{1}{2}\sigma^2)^2+2r}\right]}.$$

If we set $\gamma = \frac{1}{\sigma^2}(r-a-\frac{1}{2}\sigma^2) + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}(r-a-\frac{1}{\sigma^2})^2+2r}$, we can write $\widetilde{E}[e^{-r\tau_L}]$ as $e^{-\gamma\log\frac{x}{L}} = (\frac{x}{L})^{-\gamma}$. So the risk-neutral expected discounted pay off of this strategy is

$$v_L(x) = \begin{cases} K-x, & 0 \leq x \leq L \\ (K-L)(\frac{x}{L})^{-\gamma}, & x > L. \end{cases}$$

□

(ii)

Proof. $\frac{\partial}{\partial L}v_L(x) = -(\frac{x}{L})^{-\gamma}(1 - \frac{\gamma(K-L)}{L})$. Set $\frac{\partial}{\partial L}v_L(x) = 0$ and solve for L_*, we have $L_* = \frac{\gamma K}{\gamma+1}$. □

(iii)

Proof. By Itô's formula, we have

$$d\left[e^{-rt}v_{L_*}(S_t)\right] = e^{-rt}\left[-rv_{L_*}(S_t) + v'_{L_*}(S_t)(r-a)S_t + \frac{1}{2}v''_{L_*}(S_t)\sigma^2 S_t^2\right]dt + e^{-rt}v'_{L_*}(S_t)\sigma S_t d\widetilde{W}_t.$$

If $x > L_*$,

$$\begin{aligned}
& -rv_{L_*}(x) + v'_{L_*}(x)(r-a)x + \frac{1}{2}v''_{L_*}(x)\sigma^2 x^2 \\
= \ & -r(K-L_*)\left(\frac{x}{L_*}\right)^{-\gamma} + (r-a)x(K-L_*)(-\gamma)\frac{x^{-\gamma-1}}{L_*^{-\gamma}} + \frac{1}{2}\sigma^2 x^2(-\gamma)(-\gamma-1)(K-L_*)\frac{x^{-\gamma-2}}{L_*^{-\gamma}} \\
= \ & (K-L_*)\left(\frac{x}{L_*}\right)^{-\gamma}\left[-r-(r-a)\gamma+\frac{1}{2}\sigma^2\gamma(\gamma+1)\right].
\end{aligned}$$

By the definition of γ, if we define $u = r - a - \frac{1}{2}\sigma^2$, we have

$$\begin{aligned}
& r + (r-a)\gamma - \frac{1}{2}\sigma^2\gamma(\gamma+1) \\
= \ & r - \frac{1}{2}\sigma^2\gamma^2 + \gamma(r-a-\frac{1}{2}\sigma^2) \\
= \ & r - \frac{1}{2}\sigma^2\left(\frac{u}{\sigma^2}+\frac{1}{\sigma}\sqrt{\frac{u^2}{\sigma^2}+2r}\right)^2 + \left(\frac{u}{\sigma^2}+\frac{1}{\sigma}\sqrt{\frac{u^2}{\sigma^2}+2r}\right)u \\
= \ & r - \frac{1}{2}\sigma^2\left(\frac{u^2}{\sigma^4}+\frac{2u}{\sigma^3}\sqrt{\frac{u^2}{\sigma^2}+2r}+\frac{1}{\sigma^2}\left(\frac{u^2}{\sigma^2}+2r\right)\right) + \frac{u^2}{\sigma^2} + \frac{u}{\sigma}\sqrt{\frac{u^2}{\sigma^2}+2r} \\
= \ & r - \frac{u^2}{2\sigma^2} - \frac{u}{\sigma}\sqrt{\frac{u^2}{\sigma^2}+2r} - \frac{1}{2}\left(\frac{u^2}{\sigma^2}+2r\right) + \frac{u^2}{\sigma^2} + \frac{u}{\sigma}\sqrt{\frac{u^2}{\sigma^2}+2r} \\
= \ & 0.
\end{aligned}$$

If $x < L_*$, $-rv_{L_*}(x) + v'_{L_*}(x)(r-a)x + \frac{1}{2}v''_{L_*}(x)\sigma^2x^2 = -r(K-x) + (-1)(r-a)x = -rK + ax$. Combined, we get

$$d\left[e^{-rt}v_{L_*}(S_t)\right] = -e^{-rt}1_{\{S_t<L_*\}}(rK - aS_t)dt + e^{-rt}v'_{L_*}(S_t)\sigma S_t d\widetilde{W}_t.$$

Following the reasoning in the proof of Theorem 8.3.5, we only need to show $1_{\{x<L_*\}}(rK-ax) \geq 0$ to finish the solution. This is further equivalent to proving $rK - aL_* \geq 0$. Plug $L_* = \frac{\gamma K}{\gamma+1}$ into the expression and note $\gamma \geq \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}(r-a-\frac{1}{2}\sigma^2)^2} + \frac{1}{\sigma^2}(r-a-\frac{1}{2}\sigma^2) \geq 0$, the inequality is further reduced to $r(\gamma+1) - a\gamma \geq 0$. We prove this inequality as follows.

Assume for some K, r, a and σ (K and σ are assumed to be strictly positive, r and a are assumed to be non-negative), $rK - aL_* < 0$, then necessarily $r < a$, since $L_* = \frac{\gamma K}{\gamma+1} \leq K$. As shown before, this means $r(\gamma+1) - a\gamma < 0$. Define $\theta = \frac{r-a}{\sigma}$, then $\theta < 0$ and $\gamma = \frac{1}{\sigma^2}(r-a-\frac{1}{2}\sigma^2) + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}(r-a-\frac{1}{2}\sigma^2)^2 + 2r} =$

$\frac{1}{\sigma}(\theta - \frac{1}{2}\sigma) + \frac{1}{\sigma}\sqrt{(\theta - \frac{1}{2}\sigma)^2 + 2r}$. We have

$$
\begin{aligned}
r(\gamma+1) - a\gamma < 0 &\iff (r-a)\gamma + r < 0 \\
&\iff (r-a)\left[\frac{1}{\sigma}(\theta - \frac{1}{2}\sigma) + \frac{1}{\sigma}\sqrt{(\theta - \frac{1}{2}\sigma)^2 + 2r}\right] + r < 0 \\
&\iff \theta(\theta - \frac{1}{2}\sigma) + \theta\sqrt{(\theta - \frac{1}{2}\sigma)^2 + 2r} + r < 0 \\
&\iff \theta\sqrt{(\theta - \frac{1}{2}\sigma)^2 + 2r} < -r - \theta(\theta - \frac{1}{2}\sigma)(< 0) \\
&\iff \theta^2[(\theta - \frac{1}{2}\sigma)^2 + 2r] > r^2 + \theta^2(\theta - \frac{1}{2}\sigma)^2 + 2\theta r(\theta - \frac{1}{2}\sigma^2) \\
&\iff 0 > r^2 - \theta r\sigma^2 \\
&\iff 0 > r - \theta\sigma^2.
\end{aligned}
$$

Since $\theta\sigma^2 < 0$, we have obtained a contradiction. So our initial assumption is incorrect, and $rK - aL_* \geq 0$ must be true. □

(iv)

Proof. The proof is similar to that of Corollary 8.3.6. Note the only properties used in the proof of Corollary 8.3.6 are that $e^{-rt}v_{L_*}(S_t)$ is a supermartingale, $e^{-rt\wedge\tau_{L_*}}v_{L_*}(S_t\wedge\tau_{L_*})$ is a martingale, and $v_{L_*}(x) \geq (K-x)^+$. Part (iii) already proved the supermartingale-martingale property, so it suffices to show $v_{L_*}(x) \geq (K-x)^+$ in our problem. Indeed, by $\gamma \geq 0$, $L_* = \frac{\gamma K}{\gamma+1} < K$. For $x \geq K > L_*$, $v_{L_*}(x) > 0 = (K-x)^+$; for $0 \leq x < L_*$, $v_{L_*}(x) = K - x = (K-x)^+$; finally, for $L_* \leq x \leq K$,

$$
\frac{d}{dx}(v_{L_*}(x) - (K-x)) = -\gamma(K - L_*)\frac{x^{-\gamma-1}}{L_*^{-\gamma}} + 1 \geq -\gamma(K - L_*)\frac{L_*^{-\gamma-1}}{L_*^{-\gamma}} + 1 = -\gamma(K - \frac{\gamma K}{\gamma+1})\frac{1}{\frac{\gamma K}{\gamma+1}} + 1 = 0.
$$

and $(v_{L_*}(x) - (K-x))|_{x=L_*} = 0$. So for $L_* \leq x \leq K$, $v_{L_*}(x) - (K-x)^+ \geq 0$. Combined, we have $v_{L_*}(x) \geq (K-x)^+ \geq 0$ for all $x \geq 0$. □

8.6.

Proof. By Lemma 8.5.1, $X_t = e^{-rt}(S_t - K)^+$ is a submartingale. For any $\tau \in \Gamma_{0,T}$, Theorem 8.8.1 implies

$$
\widetilde{E}[e^{-rT}(S_T - K)^+] \geq \widetilde{E}[e^{-r\tau\wedge T}(S_{\tau\wedge T} - K)^+] \geq E[e^{-r\tau}(S_\tau - K)^+ 1_{\{\tau<\infty\}}] = E[e^{-r\tau}(S_\tau - K)^+],
$$

where we take the convention that $e^{-r\tau}(S_\tau - K)^+ = 0$ when $\tau = \infty$. Since τ is arbitrarily chosen, $\widetilde{E}[e^{-rT}(S_T - K)^+] \geq \max_{\tau\in\Gamma_{0,T}} \widetilde{E}[e^{-r\tau}(S_\tau - K)^+]$. The other direction "$\leq$" is trivial since $T \in \Gamma_{0,T}$. □

8.7.

Proof. Suppose $\lambda \in [0,1]$ and $0 \leq x_1 \leq x_2$, we have $f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2) \leq (1-\lambda)h(x_1) + \lambda h(x_2)$. Similarly, $g((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)h(x_1) + \lambda h(x_2)$. So

$$
h((1-\lambda)x_1 + \lambda x_2) = \max\{f((1-\lambda)x_1 + \lambda x_2), g((1-\lambda)x_1 + \lambda x_2)\} \leq (1-\lambda)h(x_1) + \lambda h(x_2).
$$

That is, h is also convex. □

2.9 Change of Numéraire

To provide an intuition for change of numéraire, we give **a summary of results for change of numéraire in discrete case**. This summary is based on Shiryaev [5].

Consider a model of financial market $(\widetilde{B}, \bar{B}, S)$ as in [1] Definition 2.1.1 or [5] page 383. Here $\widetilde{B}$ and $\bar{B}$ are both one-dimensional while S could be a vector price process. Suppose $\widetilde{B}$ and $\bar{B}$ are both strictly positive, then both of them can be chosen as numéaire.

Several results hold under this model. First, no-arbitrage and completeness properties of market are independent of the choice of numéraire (see, for example, Shiryaev [5] page 413 Remark and page 481). Second, if the market is arbitrage-free, then corresponding to $\widetilde{B}$ (resp. $\bar{B}$), there is an equivalent probability $\widetilde{P}$ (resp. $\bar{P}$), such that $\left(\frac{\bar{B}}{\widetilde{B}}, \frac{S}{\widetilde{B}}\right)$ (resp. $\left(\frac{\widetilde{B}}{\bar{B}}, \frac{S}{\bar{B}}\right)$) is a martingale under $\widetilde{P}$ (resp. $\bar{P}$). Third, if the market is both arbitrage-free and complete, we have the relation

$$d\bar{P} = \frac{\bar{B}_T}{\widetilde{B}_T} \frac{1}{E\left[\frac{\bar{B}_0}{\widetilde{B}_0}\right]} d\widetilde{P}.$$

See Shiryaev [5], page 510, formula (12). Finally, if f_T is a European contingent claim with maturity N and the market is both arbitrage-free and complete, then [2]

$$\bar{B}_t \bar{E}\left[\frac{f_T}{\bar{B}_T}|\mathcal{F}_t\right] = \widetilde{B}_t \widetilde{E}\left[\frac{f_T}{\widetilde{B}_T}|\mathcal{F}_t\right].$$

That is, the price of f_T is independent of the choice of numéraire. See Shiryaev [5], Chapter VI, §1b.2.

The above theoretical results can be applied to market involving foreign money market account. We consider the following market: a domestic money market account M ($M_0 = 1$), a foreign money market account M^f ($M_0^f = 1$), a (vector) asset price process S denominated in domestic currency called stock. Suppose the domestic vs. foreign currency exchange rate is Q. Note Q is not a traded asset. Denominated by domestic currency, the traded assets are $(M, M^f Q, S)$, where $M^f Q$ can be seen as the price process of one unit foreign currency. Domestic risk-neutral measure $\widetilde{P}$ is such that $\left(\frac{M^f Q}{M}, \frac{S}{M}\right)$ is a $\widetilde{P}$-martingale. Denominated by foreign currency, the traded assets are $\left(M^f, \frac{M}{Q}, \frac{S}{Q}\right)$. Foreign risk-neutral measure $\widetilde{P}^f$ is such that $\left(\frac{M}{QM^f}, \frac{S}{QM^f}\right)$ is a $\widetilde{P}^f$-martingale. This is a change of numéraire in the market denominated by domestic currency, from M to $M^f Q$. If we assume the market is arbitrage-free and complete, the foreign risk-neutral measure is

$$d\widetilde{P}^f = \frac{Q_T M_T^f}{M_T E\left[\frac{Q_0 M_0^f}{M_0}\right]} d\widetilde{P} = \frac{Q_T D_T M_T^f}{Q_0} d\widetilde{P}$$

on $\mathcal{F}_T$. Under the above set-up, for a European contingent claim f_T, denominated in domestic currency, its payoff in foreign currency is f_T/Q_T. Therefore its foreign price is $\widetilde{E}^f\left[\frac{D_T^f f_T}{D_t^f Q_T}|\mathcal{F}_t\right]$. Convert this price into domestic currency, we have $Q_t \widetilde{E}^f\left[\frac{D_T^f f_T}{D_t^f Q_T}|\mathcal{F}_t\right]$. Use the relation between $\widetilde{P}^f$ and $\widetilde{P}$ on $\mathcal{F}_T$ and the Bayes formula, we get

$$Q_t \widetilde{E}^f\left[\frac{D_T^f f_T}{D_t^f Q_T}|\mathcal{F}_t\right] = \widetilde{E}\left[\frac{D_T f_T}{D_t}|\mathcal{F}_t\right].$$

[2] If the market is incomplete but the contingent claim is still replicable, this result still holds for $t = 0$. Indeed, in Shiryaev [5], Chapter V §1c.2, it is easy to generalize formula (12) to the case of $N \geq 1$ with $x^* = \sup_{\widetilde{P} \in \mathcal{P}(P)} \widetilde{E}\left[\frac{f_N}{B_N}\right] B_0$, $x_* = \inf_{\widetilde{P} \in \mathcal{P}(P)} \widetilde{E}\left[\frac{f_N}{B_N}\right] B_0$, $C^*(P) = \inf\{x \geq 0 : \exists \pi \in SF, x_0^\pi = x, x_N^\pi \geq f_N\}$, and $C_*(P) = \sup\{x \geq 0 : \exists \pi \in SF, x_0^\pi = x, x_N^\pi \leq f_N\}$. No-arbitrage implies $C_* \leq C^*$. Since

$$\frac{X_N^\pi}{B_N} = \frac{X_0^\pi}{B_0} + \sum_{k=1}^N \gamma_k \Delta\left(\frac{S_k}{B_k}\right),$$

we must have $C_* \leq x_* \leq x^* \leq C^*$. The replicability of f_N implies $C^* \leq C_*$. So, if the market is incomplete but the contingent claim is still replicable, we still have $C_* = x_* = x^* = C^*$, i.e. the pricing formula still holds for $t = 0$.

The RHS is exactly the price of f_T in domestic market if we apply risk-neutral pricing.

9.1. (i)

Proof. For any $0 \leq t \leq T$, by Lemma 5.5.2,

$$E^{(M_2)}\left[\frac{M_1(T)}{M_2(T)}\bigg|\mathcal{F}_t\right] = E\left[\frac{M_2(T)}{M_2(t)}\frac{M_1(T)}{M_2(T)}\bigg|\mathcal{F}_t\right] = \frac{E[M_1(T)|\mathcal{F}_t]}{M_2(t)} = \frac{M_1(t)}{M_2(t)}.$$

So $\frac{M_1(t)}{M_2(t)}$ is a martingale under P^{M_2}. □

(ii)

Proof. Let $M_1(t) = D_tS_t$ and $M_2(t) = D_tN_t/N_0$. Then $\widetilde{P}^{(N)}$ as defined in (9.2.6) is $P^{(M_2)}$ as defined in Remark 9.2.5. Hence $\frac{M_1(t)}{M_2(t)} = \frac{S_t}{N_t}N_0$ is a martingale under $\widetilde{P}^{(N)}$, which implies $S_t^{(N)} = \frac{S_t}{N_t}$ is a martingale under $\widetilde{P}^{(N)}$. □

9.2. (i)

Proof. Since $N_t^{-1} = N_0^{-1}e^{-\nu\widetilde{W}_t-(r-\frac{1}{2}\nu^2)t}$, we have

$$d(N_t^{-1}) = N_0^{-1}e^{-\nu\widetilde{W}_t-(r-\frac{1}{2}\nu^2)t}[-\nu d\widetilde{W}_t - (r-\frac{1}{2}\nu^2)dt + \frac{1}{2}\nu^2dt] = N_t^{-1}(-\nu d\widehat{W}_t - rdt).$$

□

(ii)

Proof.

$$d\widehat{M}_t = M_td\left(\frac{1}{N_t}\right) + \frac{1}{N_t}dM_t + d\left(\frac{1}{N_t}\right)dM_t = \widehat{M}_t(-\nu d\widehat{W}_t - rdt) + r\widehat{M}_tdt = -\nu\widehat{M}_td\widehat{W}_t.$$

Remark: This can also be obtained directly from Theorem 9.2.2. □

(iii)

Proof.

$$\begin{aligned}
d\widehat{X}_t &= d\left(\frac{X_t}{N_t}\right) = X_td\left(\frac{1}{N_t}\right) + \frac{1}{N_t}dX_t + d\left(\frac{1}{N_t}\right)dX_t \\
&= (\Delta_tS_t + \Gamma_tM_t)d\left(\frac{1}{N_t}\right) + \frac{1}{N_t}(\Delta_tdS_t + \Gamma_tdM_t) + d\left(\frac{1}{N_t}\right)(\Delta_tdS_t + \Gamma_tdM_t) \\
&= \Delta_t\left[S_td\left(\frac{1}{N_t}\right) + \frac{1}{N_t}dS_t + d\left(\frac{1}{N_t}\right)dS_t\right] + \Gamma_t\left[M_td\left(\frac{1}{N_t}\right) + \frac{1}{N_t}dM_t + d\left(\frac{1}{N_t}\right)dM_t\right] \\
&= \Delta_td\widehat{S}_t + \Gamma_td\widehat{M}_t.
\end{aligned}$$

□

9.3. To avoid singular cases, we need to assume $-1 < \rho < 1$.

(i)

Proof. $N_t = N_0 e^{\nu \widetilde{W}_3(t)+(r-\frac{1}{2}\nu^2)t}$. So

$$\begin{aligned} dN_t^{-1} &= d(N_0^{-1}e^{-\nu\widetilde{W}_3(t)-(r-\frac{1}{2}\nu^2)t}) \\ &= N_0^{-1}e^{-\nu\widetilde{W}_3(t)-(r-\frac{1}{2}\nu^2)t}\left[-\nu d\widetilde{W}_3(t)-(r-\frac{1}{2}\nu^2)dt+\frac{1}{2}\nu^2dt\right] \\ &= N_t^{-1}[-\nu d\widetilde{W}_3(t)-(r-\nu^2)dt], \end{aligned}$$

and

$$\begin{aligned} dS_t^{(N)} &= N_t^{-1}dS_t + S_t dN_t^{-1} + dS_t dN_t^{-1} \\ &= N_t^{-1}(rS_tdt+\sigma S_t d\widetilde{W}_1(t)) + S_t N_t^{-1}[-\nu d\widetilde{W}_3(t)-(r-\nu^2)dt] \\ &= S_t^{(N)}(rdt+\sigma d\widetilde{W}_1(t)) + S_t^{(N)}[-\nu d\widetilde{W}_3(t)-(r-\nu^2)dt] - \sigma S_t^{(N)}\rho dt \\ &= S_t^{(N)}(\nu^2-\sigma\rho)dt + S_t^{(N)}(\sigma d\widetilde{W}_1(t)-\nu d\widetilde{W}_3(t)). \end{aligned}$$

Define $\gamma = \sqrt{\sigma^2-2\rho\sigma\nu+\nu^2}$ and $\widetilde{W}_4(t) = \frac{\sigma}{\gamma}\widetilde{W}_1(t) - \frac{\nu}{\gamma}\widetilde{W}_3(t)$, then $\widetilde{W}_4$ is a martingale with quadratic variation

$$[\widetilde{W}_4]_t = \frac{\sigma^2}{\gamma^2}t - 2\frac{\sigma\nu}{\gamma^2}\rho t + \frac{\nu^2}{r^2}t = t.$$

By Lévy's Theorem, $\widetilde{W}_4$ is a BM and therefore, $S_t^{(N)}$ has volatility $\gamma = \sqrt{\sigma^2-2\rho\sigma\nu+\nu^2}$. □

(ii)

Proof. This problem is the same as Exercise 4.13, we define $\widetilde{W}_2(t) = \frac{-\rho}{\sqrt{1-\rho^2}}\widetilde{W}_1(t) + \frac{1}{\sqrt{1-\rho^2}}\widetilde{W}_3(t)$, then $\widetilde{W}_2$ is a martingale, with

$$(d\widetilde{W}_2(t))^2 = \left(-\frac{\rho}{\sqrt{1-\rho^2}}d\widetilde{W}_1(t) + \frac{1}{\sqrt{1-\rho^2}}d\widetilde{W}_3(t)\right)^2 = \left(\frac{\rho^2}{1-\rho^2}+\frac{1}{1-\rho^2}-\frac{2\rho^2}{1-\rho^2}\right)dt = dt,$$

and $d\widetilde{W}_2(t)d\widetilde{W}_1(t) = -\frac{\rho}{\sqrt{1-\rho^2}}dt + \frac{\rho}{\sqrt{1-\rho^2}}dt = 0$. So $\widetilde{W}_2$ is a BM independent of $\widetilde{W}_1$, and $dN_t = rN_tdt + \nu N_t d\widetilde{W}_3(t) = rN_tdt + \nu N_t[\rho d\widetilde{W}_1(t) + \sqrt{1-\rho^2}d\widetilde{W}_2(t)]$. □

(iii)

Proof. Under $\widetilde{P}$, $(\widetilde{W}_1, \widetilde{W}_2)$ is a two-dimensional BM, and

$$\begin{cases} dS_t = rS_tdt + \sigma S_t d\widetilde{W}_1(t) = rS_tdt + S_t(\sigma, 0)\cdot\begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t)\end{pmatrix} \\ dN_t = rN_tdt + \nu N_t d\widetilde{W}_3(t) = rN_tdt + N_t(\nu\rho, \nu\sqrt{1-\rho^2})\cdot\begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t)\end{pmatrix}. \end{cases}$$

So under $\widetilde{P}$, the volatility vector for S is $(\sigma, 0)$, and the volatility vector for N is $(\nu\rho, \nu\sqrt{1-\rho^2})$. By Theorem 9.2.2, under the measure $\widetilde{P}^{(N)}$, the volatility vector for $S^{(N)}$ is $(v_1, v_2) = (\sigma-\nu\rho, -\nu\sqrt{1-\rho^2}$. In particular, the volatility of $S^{(N)}$ is

$$\sqrt{v_1^2+v_2^2} = \sqrt{(\sigma-\nu\rho)^2+(-\nu\sqrt{1-\rho^2})^2} = \sqrt{\sigma^2-2\nu\rho\sigma+\nu^2},$$

consistent with the result of part (i). □

9.4.

Proof. From (9.3.15), we have $M_t^f Q_t = M_0^f Q_0 e^{\int_0^t \sigma_2(s)d\widetilde{W}_3(s)+\int_0^t (R_s-\frac{1}{2}\sigma_2^2(s))ds}$. So

$$\frac{D_t^f}{Q_t} = D_0^f Q_0^{-1} e^{-\int_0^t \sigma_2(s)d\widetilde{W}_3(s)-\int_0^t (R_s-\frac{1}{2}\sigma_2^2(s))ds}$$

and

$$d\left(\frac{D_t^f}{Q_t}\right) = \frac{D_t^f}{Q_t}[-\sigma_2(t)d\widetilde{W}_3(t) - (R_t - \frac{1}{2}\sigma_2^2(t))dt + \frac{1}{2}\sigma_2^2(t)dt] = \frac{D_t^f}{Q_t}[-\sigma_2(t)d\widetilde{W}_3(t) - (R_t - \sigma_2^2(t))dt].$$

To get (9.3.22), we note

$$\begin{aligned} d\left(\frac{M_t D_t^f}{Q_t}\right) &= M_t d\left(\frac{D_t^f}{Q_t}\right) + \frac{D_t^f}{Q_t} dM_t + dM_t d\left(\frac{D_t^f}{Q_t}\right) \\ &= \frac{M_t D_t^f}{Q_t}[-\sigma_2(t)d\widetilde{W}_3(t) - (R_t - \sigma_2^2(t))dt] + \frac{R_t M_t D_t^f}{Q_t} dt \\ &= -\frac{M_t D_t^f}{Q_t}(\sigma_2(t)d\widetilde{W}_3(t) - \sigma_2^2(t)dt) \\ &= -\frac{M_t D_t^f}{Q_t}\sigma_2(t)d\widetilde{W}_3^f(t). \end{aligned}$$

To get (9.3.23), we note

$$\begin{aligned} d\left(\frac{D_t^f S_t}{Q_t}\right) &= \frac{D_t^f}{Q_t} dS_t + S_t d\left(\frac{D_t^f}{Q_t}\right) + dS_t d\left(\frac{D_t^f}{Q_t}\right) \\ &= \frac{D_t^f}{Q_t} S_t (R_t dt + \sigma_1(t)d\widetilde{W}_1(t)) + \frac{S_t D_t^f}{Q_t}[-\sigma_2(t)d\widetilde{W}_3(t) - (R_t - \sigma_2^2(t))dt] \\ &\quad + S_t\sigma_1(t)d\widetilde{W}_1(t)\frac{D_t^f}{Q_t}(-\sigma_2(t))d\widetilde{W}_3(t) \\ &= \frac{D_t^f S_t}{Q_t}[\sigma_1(t)d\widetilde{W}_1(t) - \sigma_2(t)d\widetilde{W}_3(t) + \sigma_2^2(t)dt - \sigma_1(t)\sigma_2(t)\rho_t dt] \\ &= \frac{D_t^f S_t}{Q_t}[\sigma_1(t)d\widetilde{W}_1^f(t) - \sigma_2 d\widetilde{W}_3^f(t)]. \end{aligned}$$

□

9.5.

Proof. We combine the solutions of all the sub-problems into a single solution as follows. The payoff of a quanto call is $(\frac{S_T}{Q_T} - K)^+$ units of domestic currency at time T. By risk-neutral pricing formula, its price at time t is $\widetilde{E}[e^{-r(T-t)}(\frac{S_T}{Q_T} - K)^+|\mathcal{F}_t]$. So we need to find the SDE for $\frac{S_t}{Q_t}$ under risk-neutral measure $\widetilde{P}$. By formula (9.3.14) and (9.3.16), we have $S_t = S_0 e^{\sigma_1\widetilde{W}_1(t)+(r-\frac{1}{2}\sigma_1^2)t}$ and

$$Q_t = Q_0 e^{\sigma_2\widetilde{W}_3(t)+(r-r^f-\frac{1}{2}\sigma_2^2)t} = Q_0 e^{\sigma_2\rho\widetilde{W}_1(t)+\sigma_2\sqrt{1-\rho^2}\widetilde{W}_2(t)+(r-r^f-\frac{1}{2}\sigma_2^2)t}.$$

So $\frac{S_t}{Q_t} = \frac{S_0}{Q_0} e^{(\sigma_1-\sigma_2\rho)\widetilde{W}_1(t)-\sigma_2\sqrt{1-\rho^2}\widetilde{W}_2(t)+(r^f+\frac{1}{2}\sigma_2^2-\frac{1}{2}\sigma_1^2)t}$. Define

$$\sigma_4 = \sqrt{(\sigma_1-\sigma_2\rho)^2 + \sigma_2^2(1-\rho^2)} = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \text{ and } \widetilde{W}_4(t) = \frac{\sigma_1-\sigma_2\rho}{\sigma_4}\widetilde{W}_1(t) - \frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\widetilde{W}_2(t).$$

Then $\widetilde{W}_4$ is a martingale with $[\widetilde{W}_4]_t = \frac{(\sigma_1-\sigma_2\rho)^2}{\sigma_4^2}t + \frac{\sigma_2(1-\rho^2)}{\sigma_4^2}t + t$. So $\widetilde{W}_4$ is a Brownian motion under $\widetilde{P}$. So if we set $a = r - r^f + \rho\sigma_1\sigma_2 - \sigma_2^2$, we have

$$\frac{S_t}{Q_t} = \frac{S_0}{Q_0}e^{\sigma_4\widetilde{W}_4(t)+(r-a-\frac{1}{2}\sigma_4^2)t} \text{ and } d\left(\frac{S_t}{Q_t}\right) = \frac{S_t}{Q_t}[\sigma_4 d\widetilde{W}_4(t) + (r-a)dt].$$

Therefore, under $\widetilde{P}$, $\frac{S_t}{Q_t}$ behaves like dividend-paying stock and the price of the quanto call option is like the price of a call option on a dividend-paying stock. Thus formula (5.5.12) gives us the desired price formula for quanto call option. ☐

9.6. (i)

Proof. $d_+(t) - d_-(t) = \frac{1}{\sigma\sqrt{T-t}}\sigma^2(T-t) = \sigma\sqrt{T-t}$. So $d_-(t) = d_+(t) - \sigma\sqrt{T-t}$. ☐

(ii)

Proof. $d_+(t)+d_-(t) = \frac{2}{\sigma\sqrt{T-t}}\log\frac{\text{For}_S(t,T)}{K}$. So $d_+^2(t)-d_-^2(t) = (d_+(t)+d_-(t))(d_+(t)-d_-(t)) = 2\log\frac{\text{For}_S(t,T)}{K}$. ☐

(iii)

Proof.

$$\begin{aligned}
\text{For}_S(t,T)e^{-d_+^2(t)/2} - Ke^{-d_-^2(t)} &= e^{-d_+^2(t)/2}[\text{For}_S(t,T) - Ke^{d_+^2(t)/2-d_-^2(t)/2}] \\
&= e^{-d_+^2(t)/2}[\text{For}_S(t,T) - Ke^{\log\frac{\text{For}_S(t,T)}{K}}] \\
&= 0.
\end{aligned}$$

☐

(iv)

Proof.

$$\begin{aligned}
& dd_+(t) \\
= \;& \frac{1}{2}\sqrt{1}\sigma\sqrt{(T-t)^3}[\log\frac{\text{For}_S(t,T)}{K} + \frac{1}{2}\sigma^2(T-t)]dt + \frac{1}{\sigma\sqrt{T-t}}\left[\frac{d\text{For}_S(t,T)}{\text{For}_S(t,T)} - \frac{(d\text{For}_S(t,T))^2}{2\text{For}_S(t,T)^2} - \frac{1}{2}\sigma dt\right] \\
= \;& \frac{1}{2\sigma\sqrt{(T-t)^3}}\log\frac{\text{For}_S(t,T)}{K}dt + \frac{\sigma}{4\sqrt{T-t}}dt + \frac{1}{\sigma\sqrt{T-t}}(\sigma d\widetilde{W}^T(t) - \frac{1}{2}\sigma^2 dt - \frac{1}{2}\sigma^2 dt) \\
= \;& \frac{1}{2\sigma(T-t)^{3/2}}\log\frac{\text{For}_S(t,T)}{K}dt - \frac{3\sigma}{4\sqrt{T-t}}dt + \frac{d\widetilde{W}^T(t)}{\sqrt{T-t}}.
\end{aligned}$$

☐

(v)

Proof. $dd_-(t) = dd_+(t) - d(\sigma\sqrt{T-t}) = dd_+(t) + \frac{\sigma dt}{2\sqrt{T-t}}$. ☐

(vi)

Proof. By (iv) and (v), $(dd_-(t))^2 = (dd_+(t))^2 = \frac{dt}{T-t}$. ☐

(vii)

Proof. $dN(d_+(t)) = N'(d_+(t))dd_+(t) + \frac{1}{2}N''(d_+(t))(dd_+(t))^2 = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2(t)}{2}}dd_+(t) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2(t)}{2}}(-d_+(t))\frac{dt}{T-t}$. ☐

(viii)

Proof.

$$
\begin{aligned}
dN(d_-(t)) &= N'(d_-(t))dd_-(t) + \frac{1}{2}N''(d_-(t))(dd_-(t))^2 \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_-^2(t)}{2}}\left(dd_+(t) + \frac{\sigma dt}{2\sqrt{T-t}}\right) + \frac{1}{2}\frac{e^{-\frac{d_-^2(t)}{2}}}{\sqrt{2\pi}}(-d_-(t))\frac{dt}{T-t} \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_-^2(t)/2}dd_+(t) + \frac{\sigma e^{-d_-^2(t)/2}}{2\sqrt{2\pi(T-t)}}dt + \frac{e^{-\frac{d_-^2(t)(\sigma\sqrt{T-t}-d_+(t))}{2}}}{2(T-t)\sqrt{2\pi}}dt \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_-^2(t)/2}dd_+(t) + \frac{\sigma e^{-d_-^2(t)/2}}{\sqrt{2\pi(T-t)}}dt - \frac{d_+(t)e^{-\frac{d_-^2(t)}{2}}}{2(T-t)\sqrt{2\pi}}dt.
\end{aligned}
$$

□

(ix)

Proof.

$$
d\text{For}_S(t,T)dN(d_+(t)) = \sigma\text{For}_S(t,T)d\widetilde{W}^T(t)\frac{e^{-d_+^2(t)/2}}{\sqrt{2\pi}}\frac{1}{\sqrt{T-t}}d\widehat{W}^T(t) = \frac{\sigma\text{For}_S(t,T)e^{-d_+^2(t)/2}}{\sqrt{2\pi(T-t)}}dt.
$$

□

(x)

Proof.

$$
\begin{aligned}
& \text{For}_S(t,T)dN(d_+(t)) + d\text{For}_S(t,T)dN(d_+(t)) - KdN(d_-(t)) \\
= \;& \text{For}_S(t,T)\left[\frac{1}{\sqrt{2\pi}}e^{-d_+^2(t)/2}dd_+(t) - \frac{d_+(t)}{2(T-t)\sqrt{2\pi}}e^{-d_+^2(t)/2}dt\right] + \frac{\sigma\text{For}_S(t,T)e^{-d_+^2(t)/2}}{\sqrt{2\pi(T-t)}}dt \\
& -K\left[\frac{e^{-d_-^2(t)/2}}{\sqrt{2\pi}}dd_+(t) + \frac{\sigma}{\sqrt{2\pi(T-t)}}e^{-d_-^2(t)/2}dt - \frac{d_+(t)}{2(T-t)\sqrt{2\pi}}e^{-d_-^2(t)/2}dt\right] \\
= \;& \left[\frac{\text{For}_S(t,T)d_+(t)}{2(T-t)\sqrt{2\pi}}e^{-d_+^2(t)/2} + \frac{\sigma\text{For}_S(t,T)e^{-d_+^2(t)/2}}{\sqrt{2\pi(T-t)}} - \frac{K\sigma e^{-d_-^2(t)/2}}{\sqrt{2\pi(T-t)}} - \frac{Kd_+(t)}{2(T-t)\sqrt{2\pi}}e^{-d_-^2(t)/2}\right]dt \\
& +\frac{1}{\sqrt{2\pi}}\left(\text{For}_S(t,T)e^{-d_+^2(t)/2} - Ke^{-d_-^2(t)/2}\right)dd_+(t) \\
= \;& 0.
\end{aligned}
$$

The last "=" comes from (iii), which implies $e^{-d_-^2(t)/2} = \frac{\text{For}_S(t,T)}{K}e^{-d_+^2(t)/2}$. □

2.10 Term-Structure Models

10.1. (i)

Proof. Using the notation $I_1(t)$, $I_2(t)$, $I_3(t)$ and $I_4(t)$ introduced in the problem, we can write $Y_1(t)$ and $Y_2(t)$ as $Y_1(t) = e^{-\lambda_1 t}Y_1(0) + e^{-\lambda_1 t}I_1(t)$ and

$$Y_2(t) = \begin{cases} \frac{\lambda_{21}}{\lambda_1-\lambda_2}(e^{-\lambda_1 t} - e^{-\lambda_2 t})Y_1(0) + e^{-\lambda_2 t}Y_2(0) + \frac{\lambda_{21}}{\lambda_1-\lambda_2}\left[e^{-\lambda_1 t}I_1(t) - e^{-\lambda_2 t}I_2(t)\right] - e^{-\lambda_2 t}I_3(t), & \text{if } \lambda_1 \neq \lambda_2; \\ -\lambda_{21}te^{-\lambda_1 t}Y_1(0) + e^{-\lambda_1 t}Y_2(0) - \lambda_{21}\left[te^{-\lambda_1 t}I_1(t) - e^{-\lambda_1 t}I_4(t)\right] + e^{-\lambda_1 t}I_3(t), & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Since all the $I_k(t)$'s ($k = 1, \cdots, 4$) are normally distributed with zero mean, we can conclude $\widetilde{E}[Y_1(t)] = e^{-\lambda_1 t}Y_1(0)$ and

$$\widetilde{E}[Y_2(t)] = \begin{cases} \frac{\lambda_{21}}{\lambda_1-\lambda_2}(e^{-\lambda_1 t} - e^{-\lambda_2 t})Y_1(0) + e^{-\lambda_2 t}Y_2(0), & \text{if } \lambda_1 \neq \lambda_2; \\ -\lambda_{21}te^{-\lambda_1 t}Y_1(0) + e^{-\lambda_1 t}Y_2(0), & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

□

(ii)

Proof. The calculation relies on the following fact: if X_t and Y_t are both martingales, then $X_tY_t - [X,Y]_t$ is also a martingale. In particular, $\widetilde{E}[X_tY_t] = \widetilde{E}\{[X,Y]_t\}$. Thus

$$\widetilde{E}[I_1^2(t)] = \int_0^t e^{2\lambda_1 u}du = \frac{e^{2\lambda_1 t} - 1}{2\lambda_1},\ \widetilde{E}[I_1(t)I_2(t)] = \int_0^t e^{(\lambda_1+\lambda_2)u}du = \frac{e^{(\lambda_1+\lambda_2)t} - 1}{\lambda_1 + \lambda_2},$$

$$\widetilde{E}[I_1(t)I_3(t)] = 0,\ \widetilde{E}[I_1(t)I_4(t)] = \int_0^t ue^{2\lambda_1 u}du = \frac{1}{2\lambda_1}\left[te^{2\lambda_1 t} - \frac{e^{2\lambda_1 t} - 1}{2\lambda_1}\right]$$

and

$$\widetilde{E}[I_4^2(t)] = \int_0^t u^2 e^{2\lambda_1 u}du = \frac{t^2 e^{2\lambda_1 t}}{2\lambda_1} - \frac{te^{2\lambda_1 t}}{2\lambda_1^2} + \frac{e^{2\lambda_1 t} - 1}{4\lambda_1^3}.$$

□

(iii)

Proof. Following the hint, we have

$$\widetilde{E}[I_1(s)I_2(t)] = \widetilde{E}[J_1(t)I_2(t)] = \int_0^t e^{(\lambda_1+\lambda_2)u}1_{\{u\leq s\}}du = \frac{e^{(\lambda_1+\lambda_2)s} - 1}{\lambda_1 + \lambda_2}.$$

□

10.2. (i)

Proof. Assume $B(t,T) = E[e^{-\int_t^T R_s ds}|\mathcal{F}_t] = f(t, Y_1(t), Y_2(t))$. Then $d(D_tB(t,T)) = D_t[-R_t f(t, Y_1(t), Y_2(t))dt + df(t, Y_1(t), Y_2(t))]$. By Itô's formula,

$$\begin{aligned} df(t, Y_1(t), Y_2(t)) &= [f_t(t, Y_1(t), Y_2(t)) + f_{y_1}(t, Y_1(t), Y_2(t))(\mu - \lambda_1 Y_1(t)) + f_{y_2}(t, Y_1(t), Y_2(t))(-\lambda_2)Y_2(t)] \\ &\quad + f_{y_1y_2}(t, Y_1(t), Y_2(t))\sigma_{21}Y_1(t) + \frac{1}{2}f_{y_1y_1}(t, Y_1(t), Y_2(t))Y_1(t) \\ &\quad + \frac{1}{2}f_{y_2y_2}(t, Y_1(t), Y_2(t))(\sigma_{21}^2 Y_1(t) + \alpha + \beta Y_1(t))]dt + \text{martingale part}. \end{aligned}$$

Since $D_tB(t,T)$ is a martingale, we must have

$$\left[-(\delta_0 + \delta_1 y_1 + \delta_2 y_2) + \frac{\partial}{\partial t} + (\mu - \lambda_1 y_1)\frac{\partial}{\partial y_1} - \lambda_2 y_2\frac{\partial}{\partial y_2} + \frac{1}{2}\left(2\sigma_{21}y_1\frac{\partial^2}{\partial y_1 \partial y_2} + y_1\frac{\partial^2}{\partial y_1^2} + (\sigma_{21}^2 y_1 + \alpha + \beta y_1)\frac{\partial^2}{\partial y_2^2}\right)\right] f = 0.$$

□

(ii)

Proof. If we suppose $f(t,y_1,y_2) = e^{-y_1C_1(T-t)y_2C_2(T-t)-A(T-t)}$, then $\frac{\partial}{\partial t}f = [y_1C_1'(T-t) + y_2C_2'(T-t) + A'(T-t)]f$, $\frac{\partial}{\partial y_1}f = -C_1(T-t)f$, $\frac{\partial f}{\partial y_2} = -C_2(T-t)f$, $\frac{\partial^2 f}{\partial y_1 \partial y_2} = C_1(T-t)C_2(T-t)f$, $\frac{\partial^2 f}{\partial y_1^2} = C_1^2(T-t)f$, and $\frac{\partial^2 f}{\partial y_2^2} = C_2^2(T-t)f$. So the PDE in part (i) becomes

$$-(\delta_0+\delta_1y_1+\delta_2y_2)+y_1C_1'+y_2C_2'+A'-(\mu-\lambda_1y_1)C_1+\lambda_2y_2C_2+\frac{1}{2}\left[2\sigma_{21}y_1C_1C_2+y_1C_1^2+(\sigma_{21}^2y_1+\alpha+\beta y_1)C_2^2\right]=0.$$

Sorting out the LHS according to the independent variables y_1 and y_2, we get

$$\begin{cases} -\delta_1 + C_1' + \lambda_1 C_1 + \sigma_{21}C_1C_2 + \frac{1}{2}C_1^2 + \frac{1}{2}(\sigma_{21}^2+\beta)C_2^2 = 0 \\ -\delta_2 + C_2' + \lambda_2 C_2 = 0 \\ -\delta_0 + A' - \mu C_1 + \frac{1}{2}\alpha C_2^2 = 0. \end{cases}$$

In other words, we can obtain the ODEs for C_1, C_2 and A as follows

$$\begin{cases} C_1' = -\lambda_1 C_1 - \sigma_{21}C_1C_2 - \frac{1}{2}C_1^2 - \frac{1}{2}(\sigma_{21}^2+\beta)C_2^2 + \delta_1 \quad \textbf{different from (10.7.4), check!} \\ C_2' = -\lambda_2 C_2 + \delta_2 \\ A' = \mu C_1 - \frac{1}{2}\alpha C_2^2 + \delta_0. \end{cases}$$

□

10.3. (i)

Proof. $d(D_tB(t,T)) = D_t[-R_tf(t,T,Y_1(t),Y_2(t))dt + df(t,T,Y_1(t),Y_2(t))]$ and

$$\begin{aligned} & df(t,T,Y_1(t),Y_2(t)) \\ = \ & [f_t(t,T,Y_1(t),Y_2(t)) + f_{y_1}(t,T,Y_1(t),Y_2(t))(-\lambda_1Y_1(t)) + f_{y_2}(t,T,Y_1(t),Y_2(t))(-\lambda_{21}Y_1(t)-\lambda_2Y_2(t)) \\ & +\frac{1}{2}f_{y_1y_1}(t,T,Y_1(t),Y_2(t)) + \frac{1}{2}f_{y_2y_2}(t,T,Y_1(t),Y_2(t))]dt + \text{martingale part.} \end{aligned}$$

Since $D_tB(t,T)$ is a martingale under risk-neutral measure, we have the following PDE:

$$\left[-(\delta_0(t)+\delta_1y_1+\delta_2y_2)+\frac{\partial}{\partial t}-\lambda_1y_1\frac{\partial}{\partial y_1}-(\lambda_{21}y_1+\lambda_2y_2)\frac{\partial}{\partial y_2}+\frac{1}{2}\frac{\partial^2}{\partial y_1^2}+\frac{1}{2}\frac{\partial}{\partial y_2^2}\right]f(t,T,y_1,y_2)=0.$$

Suppose $f(t,T,y_1,y_2) = e^{-y_1C_1(t,T)-y_2C_2(t,T)-A(t,T)}$, then

$$\begin{cases} f_t(t,T,y_1,y_2) = \left[-y_1\frac{d}{dt}C_1(t,T) - y_2\frac{d}{dt}C_2(t,T) - \frac{d}{dt}A(t,T)\right]f(t,T,y_1,y_2), \\ f_{y_1}(t,T,y_1,y_2) = -C_1(t,T)f(t,T,y_1,y_2), \\ f_{y_2}(t,T,y_1,y_2) = -C_2(t,T)f(t,T,y_1,y_2), \\ f_{y_1y_2}(t,T,y_1,y_2) = C_1(t,T)C_2(t,T)f(t,T,y_1,y_2), \\ f_{y_1y_1}(t,T,y_1,y_2) = C_1^2(t,T)f(t,T,y_1,y_2), \\ f_{y_2y_2}(t,T,y_1,y_2) = C_2^2(t,T)f(t,T,y_1,y_2). \end{cases}$$

So the PDE becomes

$$\begin{aligned} & -(\delta_0(t)+\delta_1y_1+\delta_2y_2)+\left(-y_1\frac{d}{dt}C_1(t,T)-y_2\frac{d}{dt}C_2(t,T)-\frac{d}{dt}A(t,T)\right)+\lambda_1y_1C_1(t,T) \\ & +(\lambda_{21}y_1+\lambda_2y_2)C_2(t,T)+\frac{1}{2}C_1^2(t,T)+\frac{1}{2}C_2^2(t,T)=0. \end{aligned}$$

Sorting out the terms according to independent variables y_1 and y_2, we get

$$\begin{cases} -\delta_0(t) - \frac{d}{dt}A(t,T) + \frac{1}{2}C_1^2(t,T) + \frac{1}{2}C_2^2(t,T) = 0 \\ -\delta_1 - \frac{d}{dt}C_1(t,T) + \lambda_1 C_1(t,T) + \lambda_{21}C_2(t,T) = 0 \\ -\delta_2 - \frac{d}{dt}C_2(t,T) + \lambda_2 C_2(t,T) = 0. \end{cases}$$

That is

$$\begin{cases} \frac{d}{dt}C_1(t,T) = \lambda_1 C_1(t,T) + \lambda_{21}C_2(t,T) - \delta_1 \\ \frac{d}{dt}C_2(t,T) = \lambda_2 C_2(t,T) - \delta_2 \\ \frac{d}{dt}A(t,T) = \frac{1}{2}C_1^2(t,T) + \frac{1}{2}C_2^2(t,T) - \delta_0(t). \end{cases}$$

□

(ii)

Proof. For C_2, we note $\frac{d}{dt}[e^{-\lambda_2 t}C_2(t,T)] = -e^{-\lambda_2 t}\delta_2$ from the ODE in (i). Integrate from t to T, we have $0 - e^{-\lambda_2 t}C_2(t,T) = -\delta_2\int_t^T e^{-\lambda_2 s}ds = \frac{\delta_2}{\lambda_2}(e^{-\lambda_2 T} - e^{-\lambda_2 t})$. So $C_2(t,T) = \frac{\delta_2}{\lambda_2}(1 - e^{-\lambda_2(T-t)})$. For C_1, we note

$$\frac{d}{dt}(e^{-\lambda_1 t}C_1(t,T)) = (\lambda_{21}C_2(t,T) - \delta_1)e^{-\lambda_1 t} = \frac{\lambda_{21}\delta_2}{\lambda_2}(e^{-\lambda_1 t} - e^{-\lambda_2 T + (\lambda_2-\lambda_1)t}) - \delta_1 e^{-\lambda_1 t}.$$

Integrate from t to T, we get

$$\begin{aligned} & -e^{-\lambda_1 t}C_1(t,T) \\ = & \begin{cases} -\frac{\lambda_{21}\delta_2}{\lambda_2\lambda_1}(e^{-\lambda_1 T} - e^{-\lambda_1 t}) - \frac{\lambda_{21}\delta_2}{\lambda_2}e^{-\lambda_2 T}\frac{e^{(\lambda_2-\lambda_1)T} - e^{(\lambda_2-\lambda_1)t}}{\lambda_2-\lambda_1} + \frac{\delta_1}{\lambda_1}(e^{-\lambda_1 T} - e^{-\lambda_1 T}) & \text{if } \lambda_1 \neq \lambda_2 \\ -\frac{\lambda_{21}\delta_2}{\lambda_2\lambda_1}(e^{-\lambda_1 T} - e^{-\lambda_1 t}) - \frac{\lambda_{21}\delta_2}{\lambda_2}e^{-\lambda_2 T}(T-t) + \frac{\delta_1}{\lambda_1}(e^{-\lambda_1 T} - e^{-\lambda_1 T}) & \text{if } \lambda_1 = \lambda_2. \end{cases} \end{aligned}$$

So

$$C_1(t,T) = \begin{cases} \frac{\lambda_{21}\delta_2}{\lambda_2\lambda_1}(e^{-\lambda_1(T-t)} - 1) + \frac{\lambda_{21}\delta_2}{\lambda_2}\frac{e^{-\lambda_1(T-t)} - e^{-\lambda_2(T-t)}}{\lambda_2-\lambda_1} - \frac{\delta_1}{\lambda_1}(e^{-\lambda_1(T-t)} - 1) & \text{if } \lambda_1 \neq \lambda_2 \\ \frac{\lambda_{21}\delta_2}{\lambda_2\lambda_1}(e^{-\lambda_1(T-t)} - 1) + \frac{\lambda_{21}\delta_2}{\lambda_2}e^{-\lambda_2 T + \lambda_1 t}(T-t) - \frac{\delta_1}{\lambda_1}(e^{-\lambda_1(T-t)} - 1) & \text{if } \lambda_1 = \lambda_2. \end{cases}.$$

□

(iii)

Proof. From the ODE $\frac{d}{dt}A(t,T) = \frac{1}{2}(C_1^2(t,T) + C_2^2(t,T)) - \delta_0(t)$, we get

$$A(t,T) = \int_t^T \left[\delta_0(s) - \frac{1}{2}(C_1^2(s,T) + C_2^2(s,T))\right] ds.$$

□

(iv)

Proof. We want to find δ_0 so that $f(0,T,Y_1(0),Y_2(0)) = e^{-Y_1(0)C_1(0,T) - Y_2(0)C_2(0,T) - A(0,T)} = B(0,T)$ for all $T > 0$. Take logarithm on both sides and plug in the expression of $A(t,T)$, we get

$$\log B(0,T) = -Y_1(0)C_1(0,T) - Y_2(0)C_2(0,T) + \int_0^T \left[\frac{1}{2}(C_1^2(s,T) + C_2^2(s,T)) - \delta_0(s)\right] ds.$$

Taking derivative w.r.t. T, we have

$$\frac{\partial}{\partial T}\log B(0,T) = -Y_1(0)\frac{\partial}{\partial T}C_1(0,T) - Y_2(0)\frac{\partial}{\partial T}C_2(0,T) + \frac{1}{2}C_1^2(T,T) + \frac{1}{2}C_2^2(T,T) - \delta_0(T).$$

□

So

$$\begin{aligned}\delta_0(T) &= -Y_1(0)\frac{\partial}{\partial T}C_1(0,T) - Y_2(0)\frac{\partial}{\partial T}C_2(0,T) - \frac{\partial}{\partial T}\log B(0,T)\\ &= \begin{cases}-Y_1(0)\left[\delta_1 e^{-\lambda_1 T} - \frac{\lambda_{21}\delta_2}{\lambda_2}e^{-\lambda_2 T}\right] - Y_2(0)\delta_2 e^{-\lambda_2 T} - \frac{\partial}{\partial T}\log B(0,T) & \text{if } \lambda_1 \neq \lambda_2\\ -Y_1(0)\left[\delta_1 e^{-\lambda_1 T} - \lambda_{21}\delta_2 e^{-\lambda_2 T}T\right] - Y_2(0)\delta_2 e^{-\lambda_2 T} - \frac{\partial}{\partial T}\log B(0,T) & \text{if } \lambda_1 = \lambda_2.\end{cases}\end{aligned}$$

10.4. (i)

Proof.

$$\begin{aligned}d\widehat{X}_t &= dX_t + Ke^{-Kt}\int_0^t e^{Ku}\Theta(u)dudt - \Theta(t)dt\\ &= -KX_tdt + \Sigma d\widehat{B}_t + Ke^{-Kt}\int_0^t e^{Ku}\Theta(u)dudt\\ &= -K\widehat{X}_tdt + \Sigma d\widehat{B}_t.\end{aligned}$$

□

(ii)

Proof.

$$\widetilde{W}_t = C\Sigma\widetilde{B}_t = \begin{pmatrix}\frac{1}{\sigma_1} & 0\\ -\frac{\rho}{\sigma_1\sqrt{1-\rho^2}} & \frac{1}{\sigma_2\sqrt{1-\rho^2}}\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\widetilde{B}_t = \begin{pmatrix}1 & 0\\ -\frac{\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}}\end{pmatrix}\widetilde{B}_t.$$

So $\widetilde{W}$ is a martingale with $\langle\widetilde{W}^1\rangle_t = \langle\widetilde{B}^1\rangle_t = t$, $\langle\widetilde{W}^2\rangle_t = \langle -\frac{\rho}{\sqrt{1-\rho^2}}\widetilde{B}^1 + \frac{1}{\sqrt{1-\rho^2}}\widetilde{B}^2\rangle_t = \frac{\rho^2 t}{1-\rho^2} + \frac{t}{1-\rho^2} - 2\frac{\rho}{1-\rho^2}\rho t = \frac{\rho^2+1-2\rho^2}{1-\rho^2}t = t$, and $\langle\widetilde{W}^1,\widetilde{W}^2\rangle_t = \langle\widetilde{B}^1, -\frac{\rho}{\sqrt{1-\rho^2}}\widetilde{B}^1 + \frac{1}{\sqrt{1-\rho^2}}\widetilde{B}^2\rangle_t = -\frac{\rho t}{\sqrt{1-\rho^2}} + \frac{\rho t}{\sqrt{1-\rho^2}} = 0$. Therefore $\widetilde{W}$ is a two-dimensional BM. Moreover, $dY_t = Cd\widehat{X}_t = -CK\widehat{X}_tdt + C\Sigma d\widetilde{B}_t = -CKC^{-1}Y_tdt + d\widetilde{W}_t = -\Lambda Y_tdt + d\widetilde{W}_t$, where

$$\begin{aligned}\Lambda &= CKC^{-1} = \begin{pmatrix}\frac{1}{\sigma_1} & 0\\ -\frac{\rho}{\sigma_1\sqrt{1-\rho^2}} & \frac{1}{\sigma_2\sqrt{1-\rho^2}}\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ -1 & \lambda_2\end{pmatrix}\cdot\frac{1}{|C|}\begin{pmatrix}\frac{1}{\sigma_2\sqrt{1-\rho^2}} & 0\\ \frac{\rho}{\sigma_1\sqrt{1-\rho^2}} & \frac{1}{\sigma_1}\end{pmatrix}\\ &= \begin{pmatrix}\frac{\lambda_1}{\sigma_1} & 0\\ -\frac{\rho\lambda_1}{\sigma_1\sqrt{1-\rho^2}} - \frac{1}{\sigma_2\sqrt{1-\rho^2}} & \frac{\lambda_2}{\sigma_2\sqrt{1-\rho^2}}\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2}\end{pmatrix}\\ &= \begin{pmatrix}\lambda_1 & 0\\ \frac{\rho\sigma_2(\lambda_2-\lambda_1)-\sigma_1}{\sigma_2\sqrt{1-\rho^2}} & \lambda_2\end{pmatrix}.\end{aligned}$$

□

(iii)

Proof.

$$\begin{aligned}X_t &= \widehat{X}_t + e^{-Kt}\int_0^t e^{Ku}\Theta(u)du = C^{-1}Y_t + e^{-Kt}\int_0^t e^{Ku}\Theta(u)du\\ &= \begin{pmatrix}\sigma_1 & 0\\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2}\end{pmatrix}\begin{pmatrix}Y_1(t)\\ Y_2(t)\end{pmatrix} + e^{-Kt}\int_0^t e^{Ku}\Theta(u)du\\ &= \begin{pmatrix}\sigma_1 Y_1(t)\\ \rho\sigma_2 Y_1(t) + \sigma_2\sqrt{1-\rho^2}Y_2(t)\end{pmatrix} + e^{-Kt}\int_0^t e^{Ku}\Theta(u)du.\end{aligned}$$

So $R_t = X_2(t) = \rho\sigma_2 Y_1(t) + \sigma_2\sqrt{1-\rho^2}Y_2(t) + \delta_0(t)$, where $\delta_0(t)$ is the second coordinate of $e^{-Kt}\int_0^t e^{Ku}\Theta(u)du$ and can be derived explicitly by Lemma 10.2.3. Then $\delta_1 = \rho\sigma_2$ and $\delta_2 = \sigma_2\sqrt{1-\rho^2}$. □

10.5.

Proof. We note $C(t,T)$ and $A(t,T)$ are dependent only on $T-t$. So $C(t,t+\bar{\tau})$ and $A(t,t+\bar{\tau})$ aare constants when $\bar{\tau}$ is fixed. So

$$\begin{aligned}
\frac{d}{dt}L_t &= -\frac{B(t,t+\bar{\tau})[-C(t,t+\bar{\tau})R'(t)-A(t,t+\bar{\tau})]}{\bar{\tau}B(t,t+\bar{\tau})} \\
&= \frac{1}{\bar{\tau}}[C(t,t+\bar{\tau})R'(t)+A(t,t+\bar{\tau})] \\
&= \frac{1}{\bar{\tau}}[C(0,\bar{\tau})R'(t)+A(0,\bar{\tau})].
\end{aligned}$$

Hence $L(t_2)-L(t_1)=\frac{1}{\bar{\tau}}C(0,\bar{\tau})[R(t_2)-R(t_1)]+\frac{1}{\bar{\tau}}A(0,\bar{\tau})(t_2-t_1)$. Since $L(t_2)-L(t_1)$ is a linear transformation, it is easy to verify that their correlation is 1. □

10.6. (i)

Proof. If $\delta_2=0$, then $dR_t=\delta_1 dY_1(t)=\delta_1(-\lambda_1 Y_1(t)dt+d\widetilde{W}_1(t))=\delta_1\left[(\frac{\delta_0}{\delta_1}-\frac{R_t}{\delta_1})\lambda_1 dt+d\widetilde{W}_1(t)\right]=(\delta_0\lambda_1-\lambda_1 R_t)dt+\delta_1 d\widetilde{W}_1(t)$. So $a=\delta_0\lambda_1$ and $b=\lambda_1$. □

(ii)

Proof.

$$\begin{aligned}
dR_t &= \delta_1 dY_1(t)+\delta_2 dY_2(t) \\
&= -\delta_1\lambda_1 Y_1(t)dt+\lambda_1 d\widetilde{W}_1(t)-\delta_2\lambda_{21}Y_1(t)dt-\delta_2\lambda_2 Y_2(t)dt+\delta_2 d\widetilde{W}_2(t) \\
&= -Y_1(t)(\delta_1\lambda_1+\delta_2\lambda_{21})dt-\delta_2\lambda_2 Y_2(t)dt+\delta_1 d\widetilde{W}_1(t)+\delta_2 d\widetilde{W}_2(t) \\
&= -Y_1(t)\lambda_2\delta_1 dt-\delta_2\lambda_2 Y_2(t)dt+\delta_1 d\widetilde{W}_1(t)+\delta_2 d\widetilde{W}_2(t) \\
&= -\lambda_2(Y_1(t)\delta_1+Y_2(t)\delta_2)dt+\delta_1 d\widetilde{W}_1(t)+\delta_2 d\widetilde{W}_2(t) \\
&= -\lambda_2(R_t-\delta_0)dt+\sqrt{\delta_1^2+\delta_2^2}\left[\frac{\delta_1}{\sqrt{\delta_1^2+\delta_2^2}}d\widetilde{W}_1(t)+\frac{\delta_2}{\sqrt{\delta_1^2+\delta_2^2}}d\widetilde{W}_2(t)\right].
\end{aligned}$$

So $a=\lambda_2\delta_0$, $b=\lambda_2$, $\sigma=\sqrt{\delta_1^2+\delta_2^2}$ and $\widetilde{B}_t=\frac{\delta_1}{\sqrt{\delta_1^2+\delta_2^2}}\widetilde{W}_1(t)+\frac{\delta_2}{\sqrt{\delta_1^2+\delta_2^2}}\widetilde{W}_2(t)$. □

10.7. (i)

Proof. We use the canonical form of the model as in formulas (10.2.4)-(10.2.6). By (10.2.20),

$$\begin{aligned}
dB(t,T) &= df(t,Y_1(t),Y_2(t)) \\
&= de^{-Y_1(t)C_1(T-t)-Y_2(t)C_2(T-t)-A(T-t)} \\
&= dt\text{ term}+B(t,T)[-C_1(T-t)d\widetilde{W}_1(t)-C_2(T-t)d\widetilde{W}_2(t)] \\
&= dt\text{ term}+B(t,T)(-C_1(T-t),-C_2(T-t))\begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t)\end{pmatrix}.
\end{aligned}$$

So the volatility vector of $B(t,T)$ under $\widetilde{P}$ is $(-C_1(T-t),-C_2(T-t))$. By (9.2.5), $\widetilde{W}_j^T(t)=\int_0^t C_j(T-u)du+\widetilde{W}_j(t)$ $(j=1,2)$ form a two-dimensional $\widetilde{P}^T-$BM. □

(ii)

Proof. Under the T-forward measure, the numeraire is $B(t,T)$. By risk-neutral pricing, at time zero the risk-neutral price V_0 of the option satisfies

$$\frac{V_0}{B(0,T)} = \widetilde{E}^T\left[\frac{1}{B(T,T)}(e^{-C_1(\bar{T}-T)Y_1(T)-C_2(\bar{T}-T)Y_2(T)-A(\bar{T}-T)} - K)^+\right].$$

Note $B(T,T)=1$, we get (10.7.19). $\square$

(iii)

Proof. We can rewrite (10.2.4) and (10.2.5) as

$$\begin{cases} dY_1(t) = -\lambda_1 Y_1(t)dt + d\widetilde{W}_1^T(t) - C_1(T-t)dt \\ dY_2(t) = -\lambda_{21}Y_1(t)dt - \lambda_2 Y_2(t)dt + d\widetilde{W}_2^T(t) - C_2(T-t)dt. \end{cases}$$

Then

$$\begin{cases} Y_1(t) = Y_1(0)e^{-\lambda_1 t} + \int_0^t e^{\lambda_1(s-t)}d\widetilde{W}_1^T(s) - \int_0^t C_1(T-s)e^{\lambda_1(s-t)}ds \\ Y_2(t) = Y_0 e^{-\lambda_2 t} - \lambda_{21}\int_0^t Y_1(s)e^{\lambda_2(s-t)}ds + \int_0^t e^{\lambda_2(s-t)}d\widetilde{W}_2(s) - \int_0^t C_2(T-s)e^{\lambda_2(s-t)}ds. \end{cases}$$

So (Y_1, Y_2) is jointly Gaussian and X is therefore Gaussian. $\square$

(iv)

Proof. First, we recall the Black-Scholes formula for call options: if $dS_t = \mu S_t dt + \sigma S_t d\widetilde{W}_t$, then

$$\widetilde{E}[e^{-\mu T}(S_0 e^{\sigma W_T + (\mu - \frac{1}{2}\sigma^2)T} - K)^+] = S_0 N(d_+) - Ke^{-\mu T}N(d_-)$$

with $d_\pm = \frac{1}{\sigma\sqrt{T}}(\log\frac{S_0}{K} + (\mu \pm \frac{1}{2}\sigma^2)T)$. Let $T=1$, $S_0=1$ and $\xi = \sigma W_1 + (\mu - \frac{1}{2}\sigma^2)$, then $\xi \stackrel{d}{=} N(\mu - \frac{1}{2}\sigma^2, \sigma^2)$ and

$$\widetilde{E}[(e^\xi - K)^+] = e^\mu N(d_+) - KN(d_-),$$

where $d_\pm = \frac{1}{\sigma}(-\log K + (\mu \pm \frac{1}{2}\sigma^2))$ (**different from the problem. Check!**). Since under $\widetilde{P}^T$, $X \stackrel{d}{=} N(\mu - \frac{1}{2}\sigma^2, \sigma^2)$, we have

$$B(0,T)\widetilde{E}^T[(e^X - K)^+] = B(0,T)(e^\mu N(d_+) - KN(d_-)).$$

$\square$

10.11.

Proof. On each payment date T_j, the payoff of this swap contract is $\delta(K - L(T_{j-1}, T_{j-1}))$. Its no-arbitrage price at time 0 is $\delta(KB(0,T_j) - B(0,T_j)L(0,T_{j-1}))$ by Theorem 10.4. So the value of the swap is

$$\sum_{j=1}^{n+1}\delta[KB(0,T_j) - B(0,T_j)L(0,T_{j-1})] = \delta K\sum_{j=1}^{n+1}B(0,T_j) - \delta\sum_{j=1}^{n+1}B(0,T_j)L(0,T_{j-1}).$$

$\square$

10.12.

Proof. Since $L(T,T) = \frac{1-B(T,T+\delta)}{\delta B(T,T+\delta)} \in \mathcal{F}_T$, we have

$$\begin{aligned}
\widetilde{E}[D(T+\delta)L(T,T)] &= \widetilde{E}[\widetilde{E}[D(T+\delta)L(T,T)|\mathcal{F}_T]] \\
&= \widetilde{E}\left[\frac{1-B(T,T+\delta)}{\delta B(T,T+\delta)}\widetilde{E}[D(T+\delta)|\mathcal{F}_T]\right] \\
&= \widetilde{E}\left[\frac{1-B(T,T+\delta)}{\delta B(T,T+\delta)}D(T)B(T,T+\delta)\right] \\
&= \widetilde{E}\left[\frac{D(T)-D(T)B(T,T+\delta)}{\delta}\right] \\
&= \frac{B(0,T)-B(0,T+\delta)}{\delta} \\
&= B(0,T+\delta)L(0,T).
\end{aligned}$$

□

2.11 Introduction to Jump Processes

11.1. (i)

Proof. First, $M_t^2 = N_t^2 - 2\lambda t N_t + \lambda^2 t^2$. So $E[M_t^2] < \infty$. $f(x) = x^2$ is a convex function. So by conditional Jensen's inequality,

$$E[f(M_t)|\mathcal{F}_s] \geq f(E[M_t|\mathcal{F}_s]) = f(M_s),\ \forall s \leq t.$$

So M_t^2 is a submartingale. □

(ii)

Proof. We note M_t has independent and stationary increment. So $\forall s \leq t$, $E[M_t^2 - M_s^2|\mathcal{F}_s] = E[(M_t - M_s)^2|\mathcal{F}_s] + E[(M_t - M_s)\cdot 2M_s|\mathcal{F}_s] = E[M_{t-s}^2] + 2M_s E[M_{t-s}] = Var(N_{t-s}) + 0 = \lambda(t-s)$. That is, $E[M_t^2 - \lambda t|\mathcal{F}_s] = M_s^2 - \lambda s$. □

11.2.

Proof. $P(N_{s+t} = k|N_s = k) = P(N_{s+t} - N_s = 0|N_s = k) = P(N_t = 0) = e^{-\lambda t} = 1 - \lambda t + O(t^2)$. Similarly, we have $P(N_{s+t} = k+1|N_s = k) = P(N_t = 1) = \frac{(\lambda t)^1}{1!}e^{-\lambda t} = \lambda t(1 - \lambda t + O(t^2)) = \lambda t + O(t^2)$, and $P(N_{s+t} \geq k+2|N_2 = k) = P(N_t \geq 2) = \sum_{k=2}^{\infty} \frac{(\lambda t)^k}{k!}e^{-\lambda t} = O(t^2)$. □

11.3.

Proof. For any $t \leq u$, we have

$$\begin{aligned}
E\left[\frac{S_u}{S_t}\middle|\mathcal{F}_t\right] &= E[(\sigma+1)^{N_t - N_u}e^{-\lambda\sigma(t-u)}|\mathcal{F}_t] \\
&= e^{-\lambda\sigma(t-u)}E[(\sigma+1)^{N_{t-u}}] \\
&= e^{-\lambda\sigma(t-u)}E[e^{N_{t-u}\log(\sigma+1)}] \\
&= e^{-\lambda\sigma(t-u)}e^{\lambda(t-u)(e^{\log(\sigma+1)}-1)} \quad \text{(by (11.3.4))} \\
&= e^{-\lambda\sigma(t-u)}e^{\lambda\sigma(t-u)} \\
&= 1.
\end{aligned}$$

So $S_t = E[S_u|\mathcal{F}_t]$ and S is a martingale. □

11.4.

Proof. The problem is ambiguous in that the relation between N_1 and N_2 is not clearly stated. According to page 524, paragraph 2, we would guess the condition should be that N_1 and N_2 are independent.

Suppose N_1 and N_2 are independent. Define $M_1(t) = N_1(t) - \lambda_1 t$ and $M_2(t) = N_2(t) - \lambda_2 t$. Then by independence $E[M_1(t)M_2(t)] = E[M_1(t)]E[M_2(t)] = 0$. Meanwhile, by Itô's product formula, $M_1(t)M_2(t) = \int_0^t M_1(s-)dM_2(s) + \int_0^t M_2(s-)dM_1(s) + [M_1, M_2]_t$. Both $\int_0^t M_1(s-)dM_2(s)$ and $\int_0^t M_2(s-)dM_1(s)$ are martingales. So taking expectation on both sides, we get $0 = 0 + E\{[M_1, M_2]_t\} = E[\sum_{0<s\le t} \Delta N_1(s)\Delta N_2(s)]$. Since $\sum_{0<s\le t} \Delta N_1(s)\Delta N_2(s) \ge 0$ a.s., we conclude $\sum_{0<s\le t} \Delta N_1(s)\Delta N_2(s) = 0$ a.s. By letting $t = 1, 2, \cdots$, we can find a set Ω_0 of probability 1, so that $\forall \omega \in \Omega_0$, $\sum_{0<s\le t} \Delta N_1(s)\Delta N_2(s) = 0$ for all $t > 0$. Therefore N_1 and N_2 can have no simultaneous jump. $\square$

11.5.

Proof. We shall prove the whole path of N_1 is independent of the whole path of N_2, following the scheme suggested by page 489, paragraph 1.

Fix $s \ge 0$, we consider $X_t = u_1(N_1(t) - N_1(s)) + u_2(N_2(t) - N_2(s)) - \lambda_1(e^{u_1} - 1)(t-s) - \lambda_2(e^{u_2} - 1)(t-s)$, $t > s$. Then by Itô's formula for jump process, we have

$$\begin{aligned} e^{X_t} - e^{X_s} &= \int_s^t e^{X_u} dX_u^c + \frac{1}{2}\int_s^t e^{X_u} dX_u^c dX_u^c + \sum_{s<u\le t} (e^{X_u} - e^{X_{u-}}) \\ &= \int_s^t e^{X_u}[-\lambda_1(e^{u_1} - 1) - \lambda_2(e^{u_2} - 1)]du + \sum_{0<u\le t} (e^{X_u} - e^{X_{u-}}). \end{aligned}$$

Since $\Delta X_t = u_1 \Delta N_1(t) + u_2 \Delta N_2(t)$ and N_1, N_2 have no simultaneous jump, $e^{X_u} - e^{X_{u-}} = e^{X_{u-}}(e^{\Delta X_u} - 1) = e^{X_{u-}}[(e^{u_1} - 1)\Delta N_1(u) + (e^{u_2} - 1)\Delta N_2(u)]$. So

$$\begin{aligned} & e^{X_t} - 1 \\ = & \int_s^t e^{X_{u-}}[-\lambda_1(e^{u_1} - 1) - \lambda_2(e^{u_2} - 1)]du + \sum_{s<u\le t} e^{X_{u-}}[(e^{u_1} - 1)\Delta N_1(u) + (e^{u_2} - 1)\Delta N_2(u)] \\ = & \int_s^t e^{X_{u-}}[(e^{u_1} - 1)d(N_1(u) - \lambda_1 u) - (e^{u_2} - 1)d(N_2(u) - \lambda_2 u)]. \end{aligned}$$

This shows $(e^{X_t})_{t\ge s}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t\ge s}$. So $E[e^{X_t}] \equiv 1$, i.e.

$$E[e^{u_1(N_1(t)-N_1(s))+u_2(N_2(t)-N_2(s))}] = e^{\lambda_1(e^{u_1}-1)(t-s)}e^{\lambda_2(e^{u_2}-1)(t-s)} = E[e^{u_1(N_1(t)-N_1(s))}]E[e^{u_2(N_2(t)-N_2(s))}].$$

This shows $N_1(t) - N_1(s)$ is independent of $N_2(t) - N_2(s)$.

Now, suppose we have $0 \le t_1 < t_2 < t_3 < \cdots < t_n$, then the vector $(N_1(t_1), \cdots, N_1(t_n))$ is independent of $(N_2(t_1), \cdots, N_2(t_n))$ if and only if $(N_1(t_1), N_1(t_2) - N_1(t_1), \cdots, N_1(t_n) - N_1(t_{n-1}))$ is independent of $(N_2(t_1), N_2(t_2) - N_2(t_1), \cdots, N_2(t_n) - N_2(t_{n-1}))$. Let $t_0 = 0$, then

$$\begin{aligned} & E[e^{\sum_{i=1}^n u_i(N_1(t_i)-N_1(t_{i-1}))+\sum_{j=1}^n v_j(N_2(t_j)-N_2(t_{j-1}))}] \\ = & E[e^{\sum_{i=1}^{n-1} u_i(N_1(t_i)-N_1(t_{i-1}))+\sum_{j=1}^{n-1} v_j(N_2(t_j)-N_2(t_{j-1}))}E[e^{u_n(N_1(t_n)-N_1(t_{n-1}))+v_n(N_2(t_n)-N_2(t_{n-1}))}|\mathcal{F}_{t_{n-1}}]] \\ = & E[e^{\sum_{i=1}^{n-1} u_i(N_1(t_i)-N_1(t_{i-1}))+\sum_{j=1}^{n-1} v_j(N_2(t_j)-N_2(t_{j-1}))}]E[e^{u_n(N_1(t_n)-N_1(t_{n-1}))+v_n(N_2(t_n)-N_2(t_{n-1}))}] \\ = & E[e^{\sum_{i=1}^{n-1} u_i(N_1(t_i)-N_1(t_{i-1}))+\sum_{j=1}^{n-1} v_j(N_2(t_j)-N_2(t_{j-1}))}]E[e^{u_n(N_1(t_n)-N_1(t_{n-1}))}]E[e^{v_n(N_2(t_n)-N_2(t_{n-1}))}], \end{aligned}$$

where the second equality comes from the independence of $N_i(t_n) - N_i(t_{n-1})$ $(i = 1, 2)$ relative to $\mathcal{F}_{t_{n-1}}$ and the third equality comes from the result obtained in the above paragraph. Working by induction, we have

$$\begin{aligned} & E[e^{\sum_{i=1}^n u_i(N_1(t_i)-N_1(t_{i-1}))+\sum_{j=1}^n v_j(N_2(t_j)-N_2(t_{j-1}))}] \\ = & \prod_{i=1}^n E[e^{u_i(N_1(t_i)-N_1(t_{i-1}))}] \prod_{j=1}^n E[e^{v_j(N_2(t_j)-N_2(t_{j-1}))}] \\ = & E[e^{\sum_{i=1}^n u_i(N_1(t_i)-N_1(t_{i-1}))}]E[e^{\sum_{j=1}^n v_j(N_2(t_j)-N_2(t_{j-1}))}]. \end{aligned}$$

This shows the whole path of N_1 is independent of the whole path of N_2. $\square$

11.6.

Proof. Let $X_t = u_1 W_t - \frac{1}{2}u_1^2 t + u_2 Q_t - \lambda t(\varphi(u_2) - 1)$ where φ is the moment generating function of the jump size Y. Itô's formula for jump process yields

$$e^{X_t} - 1 = \int_0^t e^{X_s}(u_1 dW_s - \frac{1}{2}u_1^2 ds - \lambda(\varphi(u_2) - 1)ds) + \frac{1}{2}\int_0^t e^{X_s} u_1^2 ds + \sum_{0<s\leq t}(e^{X_s} - e^{X_{s-}}).$$

Note $\Delta X_t = u_2 \Delta Q_t = u_2 Y_{N_t} \Delta N_t$, where N_t is the Poisson process associated with Q_t. So $e^{X_t} - e^{X_{t-}} = e^{X_{t-}}(e^{\Delta X_t} - 1) = e^{X_{t-}}(e^{u_2 Y_{N_t}} - 1)\Delta N_t$. Consider the compound Poisson process $H_t = \sum_{i=1}^{N_t}(e^{u_2 Y_i} - 1)$, then $H_t - \lambda E[e^{u_2 Y_{N_t}} - 1]t = H_t - \lambda(\varphi(u_2) - 1)t$ is a martingale, $e^{X_t} - e^{X_{t-}} = e^{X_{t-}}\Delta H_t$ and

$$\begin{aligned} e^{X_t} - 1 &= \int_0^t e^{X_s}(u_1 dW_s - \frac{1}{2}u_1^2 ds - \lambda(\varphi(u_2) - 1)ds) + \frac{1}{2}\int_0^t e^{X_s} u_1^2 ds + \int_0^t e^{X_{s-}} dH_s \\ &= \int_0^t e^{X_s} u_1 dW_s + \int_0^t e^{X_{s-}} d(H_s - \lambda(\varphi(u_2) - 1)s). \end{aligned}$$

This shows e^{X_t} is a martingale and $E[e^{X_t}] \equiv 1$. So $E[e^{u_1 W_t + u_2 Q_t}] = e^{\frac{1}{2}u_1 t} e^{\lambda t(\varphi(u_2) - 1)t} = E[e^{u_1 W_t}]E[e^{u_2 Q_t}]$. This shows W_t and Q_t are independent. □

11.7.

Proof. $E[h(Q_T)|\mathcal{F}_t] = E[h(Q_T - Q_t + Q_t)|\mathcal{F}_t] = E[h(Q_{T-t} + x)]|_{x=Q_t} = g(t, Q_t)$, where $g(t, x) = E[h(Q_{T-t} + x)]$. □

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