Norms, condition numbers

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Example 1

Compare the solutions of the linear systems below:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Observation: `small` change in the right-hand side - `big` change in the solution. Explanation?

Suppose that $A \in \mathbb{R}^{n \times n}$ is invertible, $b \in \mathbb{R}^n$, $b \neq 0$. We are searching for the solution of $Ax = b$. In practice the right hand side (observations, measurements) is not exactly known, it contains errors (δb), so we *have* to solve $Ay = b + \delta b$.

The question: how big can be the vector $y - x$? We want to measure vectors, difference of vectors, so we introduce the notion of `norm`.

Norm

Let X a linear vector space over $\mathbb{R}$. The map $\|.\| : X \to \mathbb{R}$ is a norm on X if:

1. $\|x\| \geq 0$ for all $x \in X$
2. $\|x\| = 0 \iff x = 0$
3. $\|\lambda x\| = |\lambda| \|x\|$, for all $\lambda \in \mathbb{R}$ and $x \in X$
4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ triangle inequality

Norms on $\mathbb{R}^n$

Let $X = \mathbb{R}^n$

1. **1-norm** (or octahedron, or Manhattan)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

2. **2-norm** (or euclidean):

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

3. **∞-norm** (or maximum):

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

Example 2

For
$$x = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$
we have:

$\|x\|_1 = |-3| + |0| + |1| = 4$

$\|x\|_2 = \left(|-3|^2 + |0|^2 + |1|^2\right)^{1/2} = \sqrt{10}$

$\|x\|_\infty = \max\{|-3|, |0|, |1|\} = 3$

Matrix-norm

Let $\|\cdot\|$ be a vector-norm on $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a matrix. Then the quantity

$$\boxed{\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}}$$

defines a norm on $\mathbb{R}^{n\times n}$. It is the matrix norm induced by the given vector norm.

Remark 1

It can be verified that $\|.\|$ is a norm.

Properties of the matrix-norm

1. $\|E\| = 1$
2. $\|Ax\| \leq \|A\| \cdot \|x\|$ minden $x \in \mathbb{R}^n$ esetén
3. $\|AB\| \leq \|A\| \cdot \|B\|$ minden $A, B \in \mathbb{R}^{n \times n}$ esetén

Remark 2

The (induced) matrix norm can be defined as the smallest number M, for which

$$\|Ax\| \leq M \cdot \|x\|$$

holds, for all $x \in \mathbb{R}^n$

Computational rules for $1, 2, \infty$ matrix norms

1. 1-norm (column norm):

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

2. ∞ norm (row norm):

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

3. 2-norm (spectral norm):

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

where $\lambda_{\max}(A^T A)$ is the largest eigenvalue of $A^T A$.

Example 3

$$A = \begin{pmatrix} -3 & 0 & 4 \\ 1 & -1 & 2 \\ -2 & 1 & -2 \end{pmatrix}, \qquad \|A\|_1 =? \quad \|A\|_\infty =?$$

$$\begin{array}{cccc} \left(\begin{array}{ccc} -3 & 0 & 4 \\ 1 & -1 & 2 \\ -2 & 1 & -2 \end{array}\right) & \begin{array}{l} \leftarrow 7 \\ \leftarrow 4 \\ \leftarrow 5 \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 6 & 2 & 8 \end{array} & \end{array}$$

$\|A\|_1 = 8$ and $\|A\|_\infty = 7$

Example 4

$$A = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}, \qquad \|A\|_2 = ?$$

$$A^T A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 13 & -7 \\ -7 & 5 \end{pmatrix}$$

the eigenvalues of $A^T A$:

$$\begin{vmatrix} 13-\lambda & -7 \\ -7 & 5-\lambda \end{vmatrix} = (13-\lambda)(5-\lambda) - 49 = \lambda^2 - 18\lambda + 16 = 0$$

$$\lambda_{1,2} = \frac{18 \pm \sqrt{18^2 - 64}}{2} = 9 \pm \sqrt{65}$$

$\|A\|_2 = \sqrt{9 + \sqrt{65}} \approx 4.13$

The condition number

Suppose that $A \in \mathbb{R}^{n\times n}$ is invertible, $b \in \mathbb{R}^n$, $b \neq 0$. We are searching for the solution of $Ax = b$. Suppose that we have error in the right hand side, so we *have* to solve the $A(x+\delta x) = b + \delta b$ system.

Then, on the one hand:

$$A(x+\delta x) = b+\delta b$$

$$\underline{Ax} + A\cdot\delta x = \underline{b} + \delta b$$

$$A\cdot\delta x = \delta b$$

$$\delta x = A^{-1}\delta b$$

$$\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\|\cdot\|\delta b\|$$

On the other hand: $Ax = b \implies$

$$\|b\| = \|Ax\| \leq \|A\| \cdot \|x\|$$

$$\frac{1}{\|x\|} \leq \|A\| \cdot \frac{1}{\|b\|}$$

Putting them together:

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\| \cdot \|A^{-1}\|}_{cond(A):=} \frac{\|\delta b\|}{\|b\|}$$

$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq cond(A) \frac{\|\delta b\|}{\|b\|}}$$

Definition

Let A be an invertible matrix. Then

$$cond(A) = \|A\| \cdot \|A^{-1}\|$$

is called the `condition number` of A.

Properties

1. its value *does* depend on the norm used
2. $cond(A) \geq 1$
3. if A is orthogonal ($A^T A = E$), then $cond_2(A) = 1$
4. $$\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| \leq cond(A)$$
   where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues in absolute value of A
5. for $c \neq 0$, $cond(cA) = cond(A)$

Remark 3

Let $C = cond(A)$. The condition number tells us, that the relative error in the right-hand side can get C-times larger in the solution. Note that it is the worst case scenario.

Example 5

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \end{pmatrix}, \quad cond_\infty(A) =?$$

Then $\det(A) = 10^{-4}$,

$$A^{-1} = \begin{pmatrix} 10001 & -10000 \\ -10000 & 10000 \end{pmatrix}$$

$\|A\|_\infty = 2.0001$ and $\|A^{-1}\|_\infty = 20001$,

$cond_\infty(A) = 2.0001 \cdot 20001 \approx 40000.$

The error in the solution can be as large as 40000 times the error in the right hand side.

Remark 4

The condition number does not depend on the determinant!

Review 1

Inverse of a 2×2 matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{det(A)}$

Hilbert matrix

$$H_n = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+1) \\ 1/3 & 1/4 & 1/5 & \cdots & 1/(n+2) \\ \vdots & & & & \\ 1/n & 1/(n+1) & 1/(n+2) & \cdots & 1/(2n-1) \end{pmatrix}$$

Remark 5

For computing the condition numbers of large matrices use the `condest` function! It computes an estimation of $cond_1(A)$, without the expensive A^{-1} computations.

Suppose for the relative error of b:

$$\boxed{\frac{\|\delta b\|}{\|b\|} \approx \varepsilon_1}$$

If, in addition we have:

$$cond(A) \geq \frac{1}{\varepsilon_1}$$

then

$$cond(A)\frac{\|\delta b\|}{\|b\|} \geq 1,$$

that is, the error in the solution can be as large as the solution itself. It is bad. Such matrices (or linear systems) are called ill-conditioned .

In order to get at least 1 exact digit in the solution, we need smaller condition number. For example, if

$$cond(A) \leq \frac{1}{a\varepsilon_1}$$

then

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{1}{a}$$