Cauchy meets Raabe - again

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Abstract

A mixed type test for convergence is proposed.

Lemma.
Consider a positive real series $\sum_n a_n$ and a real sequence λ_n for which:

$$\frac{\lambda_n}{\log n} \to \infty$$

Then:

$$\underline{\lim}\ A_n > 1 \quad \Longrightarrow \quad \sum_n a_n < \infty$$

$$A_n \leq 1 \quad \text{for all sufficiently large } n \quad \Longrightarrow \quad \sum_n a_n = \infty$$

where

$$A_n = \frac{\lambda_n}{\log(n)} \left(\left(\frac{a_n}{a_{n+1}} \right)^{\frac{1}{\lambda_n}} - 1 \right)$$

Proof:
Set $L = \underline{\lim}\ A_n$.
If $L > 1$ then by the properties of $\underline{\lim}$ there is a N_q such that if $n \geq N_q$ we have:

$$A_n > q \quad \Longrightarrow$$

$$\frac{a_n}{a_{n+1}} > \left(1 + \frac{1}{\frac{\lambda_n}{q \log(n)}} \right)^{\frac{\lambda_n}{q \log(n)} q \log(n)} = (*)$$

with $q = \min(\frac{1+L}{2}, \frac{3}{2})$. For the sequence $b_n = \left(1 + \frac{1}{\frac{\lambda_n}{q \log(n)}} \right)^{\frac{\lambda_n}{q \log(n)}}$, it is clear that $b_n \to e$, so $\log(b_n) \to 1$, hence there is an M_q, for which

$$q \log(b_n) > \frac{1+q}{2} = r > 1$$

if $n > M_q$. Putting these observations together, if $n > max(N_q, M_q)$:

$$(*) = e^{q \log(b_n) \log(n)} > e^{\log(n^r)} = n^r$$

which means, that $\frac{1}{n^r}$ (essentially) majorizes our a_n.

In the other case we have:

$$\frac{1}{a_n} \leq e^{\log(b_n) \log(n)} \overset{b_n < e}{\leq} n$$

if $n > N$, that is a_n is minorized by $\frac{1}{n}$.

Example.
Later. Examples are welcome :-)

Summary.
Later.