Gauss Factorials: Properties and Applications

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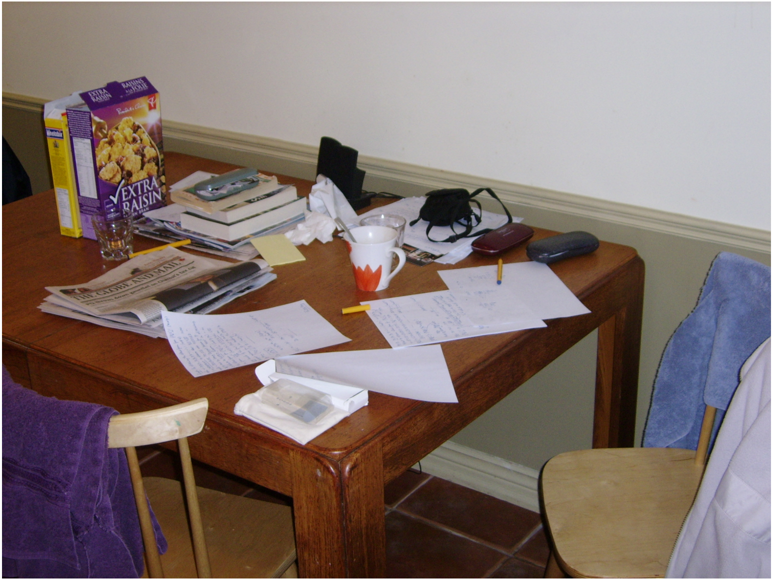
EXTRA RAISIN

1. Introduction

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(Converse is due to Lagrange). A proof depends on the fact that any integer a with $1 < a < p-1$ has its inverse $a^{-1} \not\equiv a \pmod{p}$.

If we write out the factorial $(p-1)!$ and exploit symmetry modulo p, we obtain

$$1\cdot 2\cdot\ldots\cdot\tfrac{p-1}{2}\,\tfrac{p+1}{2}\cdot\ldots\cdot(p-1)\equiv\left(\tfrac{p-1}{2}\right)!(-1)^{\frac{p-1}{2}}\left(\tfrac{p-1}{2}\right)!\pmod{p},$$

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and thus, with Wilson's Theorem,

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For $p \equiv 1 \pmod 4$ the RHS is -1, so

$$\mathrm{ord}_p \left(\left(\tfrac{p-1}{2}\right)! \right) = 4 \quad \text{for} \quad p \equiv 1 \pmod 4.$$

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$$\left(\frac{p-1}{2}\right)! \equiv \pm 1 \pmod{p}.$$

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Theorem 1 (Mordell, 1961)

For a prime $p \equiv 3 \pmod 4$,

$$\left(\frac{p-1}{2}\right)! \equiv -1 \pmod p \quad \Leftrightarrow \quad h(-p) \equiv 1 \pmod 4,$$

where $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$.

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Discovered independently by Chowla.

This completely determines the order $\pmod p$ of $\left(\frac{p-1}{2}\right)!$

Now consider the two halves of the product

$$1 \cdot 2 \cdots \frac{p-1}{2} \frac{p+1}{2} \cdots (p-1)$$

and denote them, respectively, by

$$\Pi_1^{(2)}, \qquad \Pi_2^{(2)}.$$

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By Wilson's theorem:

$$\Pi_1^{(2)} \Pi_2^{(2)} \equiv -1 \pmod{p},$$

and by symmetry:

$$\Pi_2^{(2)} \equiv (-1)^{\frac{p-1}{2}} \Pi_1^{(2)} \pmod{p}.$$

What can we say about the three partial products

$$\Pi_1^{(3)}, \quad \Pi_2^{(3)}, \quad \Pi_3^{(3)}$$

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$$\Pi_1^{(3)} = 1{\cdot}2 \cdots \tfrac{p-1}{3}, \quad \Pi_2^{(3)} = \tfrac{p+2}{3} \cdots \tfrac{2p-2}{3}, \quad \Pi_3^{(3)} = \tfrac{2p+1}{3} \cdots (p-1).$$

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(without a power of -1 since $\frac{p-1}{3}$ is always even.)

No obvious relation between $\Pi_1^{(3)}$ and the "middle third" $\Pi_2^{(3)}$.

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Example:

p	$\Pi_1^{(3)}$	$\Pi_2^{(3)}$	$\Pi_3^{(3)}$	p	$\Pi_1^{(4)}$	$\Pi_2^{(4)}$	$\Pi_3^{(4)}$	$\Pi_4^{(4)}$
7	**2**	$-$**2**	**2**	5	1	2	-2	-1
13	-2	3	-2	13	6	3	-3	-6
19	-2	-5	-2	17	7	-3	-3	7
31	2	-8	2	29	-6	-2	2	6
37	7	3	7	37	-16	5	-5	16
43	-3	19	-3	41	13	7	7	13
61	$-$**14**	**14**	$-$**14**	53	26	7	-7	-26
67	-20	-33	-20	61	19	7	-7	-19
73	33	-12	33	73	18	-35	-35	18
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19	-2	-5	-2	17	7	-3	-3	7
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37	7	3	7	37	-16	5	-5	16
43	-3	19	-3	41	13	7	7	13
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- The obvious symmetries.

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- The obvious symmetries.
- $\Pi_1^{(3)} \equiv -\Pi_2^{(3)} \pmod{p}$ for $p = 7$ and $p = 61$.

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It turns out:

$\Pi_1^{(3)} \equiv -\Pi_2^{(3)} \pmod{p}$ also for $p = 331$, $p = 547$, $p = 1951$, and further relatively rare primes (explained later).

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In contrast: No primes p for which

$\Pi_1^{(3)} \equiv \Pi_2^{(3)} \pmod{p}, \qquad p \equiv 1 \pmod{6}$, or

$\Pi_1^{(4)} \equiv \pm\Pi_2^{(4)} \pmod{p}, \qquad p \equiv 1 \pmod{4}$.

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This will be explained later.

2. Composite Moduli

Define the *Gauss factorial* by

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Analogue of Wilson's theorem for composite moduli:

Theorem 2 (Gauss)

For any integer $n \geq 2$ we have

$$(n-1)_n! \equiv \begin{cases} -1 \pmod{n} & \textit{for} \quad n = 2, 4, p^\alpha, \textit{ or } 2p^\alpha, \\ 1 \pmod{n} & \textit{otherwise}, \end{cases}$$

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The first case indicates exactly those n that have primitive roots.

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Next, divide the product $(n-1)_n!$ into $M \geq 2$ partial products: For $n \equiv 1 \pmod{M}$, set

$$\Pi_j^{(M)} := \prod_{i \in I_j^{(M)}} i, \qquad (j = 1, 2, \ldots, M),$$

where, for $j = 1, 2, \ldots, M$,

$$I_j^{(M)} := \left\{ i \mid (j-1)\tfrac{n-1}{M} + 1 \leq i \leq j\tfrac{n-1}{M},\ \gcd(i, n) = 1 \right\}.$$

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- Dependence on n is implied in the notation;
- When $n = p$: reduces to previous case.

Example:

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We see: In contrast to prime case, it *can* happen that all partial products are congruent to each other.

3. The Distribution of Totatives

We pause to consider the *number* of elements in our subintervals:

$$\phi_{M,j}(n) := \# I_j^{(M)}.$$

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In general, the situation is less straightforward.
E.g., for $n = 4$:
$\phi_{3,1}(n) = \phi_{3,3}(n) = 1$, but $\phi_{3,2}(n) = 0$.

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Theorem 3 (Lehmer, 1955)

Let $M \geq 2$ and $n \equiv 1 \pmod{M}$.
If n has at least one prime factor $p \equiv 1 \pmod{M}$, then

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Note: Condition is sufficient, but not necessary.

E.g., $M = 8$ and $n = 105 = 3 \cdot 5 \cdot 7$.
None of the prime factors are $\equiv 1 \pmod{8}$, but
$\phi_{M,j}(n) = \frac{1}{8}\phi(105) = 6$ for $j = 1, \ldots, 8$.

4. When Are the Partial Products Congruent?

Return to our table:

n	$\Pi_1^{(3)}$	$\Pi_2^{(3)}$	$\Pi_3^{(3)}$	n	$\Pi_1^{(4)}$	$\Pi_2^{(4)}$	$\Pi_3^{(4)}$	$\Pi_4^{(4)}$
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Note:

$91 = 7 \cdot 13, \quad 65 = 5 \cdot 13, \quad 85 = 5 \cdot 17.$

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Theorem 4

Let $M \geq 2$ and $n \equiv 1 \pmod{M}$.
If n has at least two distinct prime factors $\equiv 1 \pmod{M}$, then

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Result is best possible:
E.g., $M = 3$ and $n = 70 = 2 \cdot 5 \cdot 7$.
- Only one factor $\equiv 1 \pmod{3}$,
- $\Pi_1^{(3)} \equiv 29 \pmod{70}, \Pi_2^{(3)} \equiv 1 \pmod{70}$.

This observation holds in general:

Theorem 4

Let $M \geq 2$ and $n \equiv 1 \pmod{M}$.
If n has at least two distinct prime factors $\equiv 1 \pmod{M}$, then

$$\Pi_j^{(M)} \equiv \left(\tfrac{n-1}{M}\right)_n! \pmod{n}, \qquad j = 1, 2, \ldots, M.$$

Result is best possible:
E.g., $M = 3$ and $n = 70 = 2 \cdot 5 \cdot 7$.
• Only one factor $\equiv 1 \pmod{3}$,
• $\Pi_1^{(3)} \equiv 29 \pmod{70}, \Pi_2^{(3)} \equiv 1 \pmod{70}$.

On the other hand, condition is sufficient but not necessary.
E.g., $M = 3$ and $n = 2^2 \cdot 61$; statement still holds.

Proof is based on an observation:

$$\Pi_j^{(M)} = \frac{\left(j\frac{n-1}{M}\right)_n!}{\left((j-1)\frac{n-1}{M}\right)_n!}, \qquad j = 1, 2, \ldots, M,$$

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and a lemma:

Lemma 5

Let $M \geq 2$ and $n \equiv 1 \pmod{M}$, $n = p^\alpha q^\beta w$ for distinct prime $p, q \equiv 1 \pmod{M}$, $\alpha, \beta \geq 1$, and $\gcd(pq, w) = 1$. Then for $j = 1, 2, \ldots, M$,

$$\left(j\frac{n-1}{M}\right)_n! \equiv \frac{\varepsilon^{j\frac{p-1}{M}}}{p^{jA}} \pmod{q^\beta w}, \qquad A = \frac{p^{\alpha-1}}{M}\phi(q^\beta w),$$

where $\varepsilon = -1$ if $w = 1$, and $\varepsilon = 1$ if $w > 1$.

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To prove the Theorem, use this and the Chinese Remainder Theorem; dependence on j disappears.

Idea of proof:

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Break the range of the product in $\left(j\frac{n-1}{M}\right)_n!$ into
• a number of products of approximately equal length,
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Carefully count which elements to include/exclude.

5. Some Consequences

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This implies:

Corollary 6

Let $M \geq 2$ and $n \equiv 1 \pmod{M}$.
If n has at least two distinct prime factors $\equiv 1 \pmod{M}$, then the multiplicative order of $(\frac{n-1}{M})_n!$ modulo n is a divisor of M.

2. In the case of at least *three* distinct prime factors $\equiv 1 \pmod{M}$ we can say more:

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Let $M \geq 2$ and $n \equiv 1 \pmod{M}$.
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Method of proof is similar to that of the previous lemma.

Summary:

# of prime factors $\equiv 1 \pmod{M}$	All $\Pi_1^{(M)}, \ldots, \Pi_M^{(M)}$:
1	have the same number of factors
2	are congruent to each other $\pmod{M}$
3	are congruent to $1 \pmod{M}$

6. The Gauss and Jacobi Theorems

Return to the question of how $\Pi_1^{(4)}$ and $\Pi_2^{(4)}$ are related, for prime moduli $p \equiv 1 \pmod 4$.

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There exists a celebrated congruence for this binomial coefficient:

Fix p, a, and b such that

$$p \equiv 1 \pmod 4, \qquad p = a^2 + b^2, \qquad a \equiv 1 \pmod 4.$$

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Let p and a be as above. Then

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A similar theorem, due to Jacobi (1837), implies that

$$\Pi_2^{(3)} \not\equiv \Pi_1^{(3)} \pmod p \quad \text{for all} \quad p \equiv 1 \pmod 6.$$

A number of further consequences can be derived from these classical theorems.

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Corollary 9

For a prime $p \equiv 1 \pmod 6$ we have

$$\Pi_2^{(3)} \equiv -\Pi_1^{(3)} \pmod p \quad \Leftrightarrow \quad p = 27x^2 + 27x + 7, x \in \mathbb{Z}.$$

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The first such primes are 7, 61 (seen earlier), 331, 547, 1951.

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(a) $\frac{p-1}{4}! \equiv 1 \pmod p$ *only if* $p = 5$.

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(c) $\left(\frac{p-1}{4}!\right)^8 \equiv -1 \pmod p$ *holds for* $p = 17, 241, 3361, 46817, 652081, \ldots$

Part (c) is related to the solution of a certain Pell equation.

7. Extensions of Gauss' Theorem

Recall:

Theorem 11 (Gauss, 1828)

If p and a are such that $p = a^2 + b^2,\ a \equiv 1 \pmod 4$, then

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If p and a are such that $p = a^2 + b^2,\ a \equiv 1 \pmod 4$, then

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod{p}.$$

Theorem 12 (Chowla, Dwork, Evans, 1986)

With p and a as above,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \left(1 + \frac{1}{2} p q_p(2)\right)\left(2a - \frac{p}{2a}\right) \pmod{p^2},$$

where $q_p(2) := (2^{p-1} - 1)/p$ is the Fermat quotient to base 2.

The concept of Gauss factorial was essential in the proof of the following further extension:

Theorem 13

With p and a as above,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \left(2a - \frac{p}{2a} - \frac{p^2}{8a^3}\right) \times \left(1 + \tfrac{1}{2}pq_p(2) + \tfrac{1}{8}p^2\left(2E_{p-3} - q_p(2)^2\right)\right) \pmod{p^3},$$

where E_n is the nth Euler number.

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Similar extensions were also obtained for Jacobi's theorem, concerning the binomial coefficient

$$\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}} \qquad (p \equiv 1 \pmod 6).)$$

Thank you


Welcome to
The IRMACS Centre
Friday, September 21
9:30 am - Presentation Studio
9:30 am - ASB 10908
10:00 am - ASB 10940
10:45 am - ASB 10908
FAB Research Project
11:30 am - Presentation Studio
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