A mod p^3 analogue of a theorem of Gauss on binomial coefficients

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We begin with a table:

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p	$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}$	(mod p)
5	2	2
13	20	7
17	70	2
29	3432	10
37	48620	2
41	184756	10
53	10400600	39
61		10
73		67
89		10
97		18

$p \equiv 1 \pmod 4$

Add in sums of squares:

p	$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}$	$\pmod p$	a	b
5	2	2	1	2
13	20	7	3	2
17	70	2	1	4
29	3432	10	5	2
37	48620	2	1	6
41	184756	10	5	4
53	10400600	39	7	2
61		10	5	6
73		67	3	8
89		10	5	8
97		18	9	4

$p \equiv 1 \pmod 4, \quad p = a^2 + b^2.$

Reformulating the table:

p	$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}$	$\pmod p$	$\lvert\cdots\rvert < \frac{p}{2}$	a	b
5	2	2	2	1	2
13	20	7	-6	3	2
17	70	2	2	1	4
29	3432	10	10	5	2
37	48620	2	2	1	6
41	184756	10	10	5	4
53	10400600	39	-14	7	2
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$p \equiv 1 \pmod 4, \quad p = a^2 + b^2.$

1. Introduction

The table is an illustration of the following celebrated result:

Theorem 1 (Gauss, 1828)

Let $p \equiv 1 \pmod 4$ *be a prime and write*

$$p = a^2 + b^2, \qquad a \equiv 1 \pmod 4.$$

Then

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod p.$$

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Several different proofs are known, some using “Jacobsthal sums".

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Beukers (1984) conjectured, and Chowla, Dwork & Evans (1986) proved:

Theorem 2 (Chowla, Dwork, Evans)

Let p and a be as before. Then

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \left(2a - \tfrac{p}{2a}\right)\left(1 + \tfrac{1}{2}pq_p(2)\right) \pmod{p^2}.$$

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Application: Search for Wilson primes.

2. Gauss Factorials

Recall *Wilson's Theorem*: p is a prime if and only if

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Theorem 3 (Gauss)

For any integer $n \ge 2$,

$$(n-1)_n! \equiv \begin{cases} -1 \pmod{n} & \text{for } n = 2, 4, p^\alpha, \text{ or } 2p^\alpha, \\ 1 \pmod{n} & \text{otherwise,} \end{cases}$$

where p is an odd prime and α is a positive integer.

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Idea: Use the mod p^2 extension by Chowla et al.

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Main technical device: We can show that

$$\left(\frac{p^2-1}{2}\right)_p ! \equiv (p-1)!^{\frac{p-1}{2}} \left(\frac{p-1}{2}\right)! \left(1 + \frac{p-1}{2} p \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j}\right) \pmod{p^2}.$$

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Altogether we have, after simplifying,

$$\frac{\left(\frac{p^2-1}{2}\right)_p!}{\left(\left(\frac{p^2-1}{4}\right)_p!\right)^2} \equiv \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \frac{1}{1+\frac{1}{2}pq_p(2)} \pmod{p^2}.$$

Combining this with the theorem of Chowla, Dwork & Evans:

Theorem 4

Let p and a be as before. Then

$$\frac{\left(\frac{p^2-1}{2}\right)_p!}{\left(\left(\frac{p^2-1}{4}\right)_p!\right)^2} \equiv 2a - \frac{p}{2a} \pmod{p^2}.$$

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- Hopeless to conjecture an extension of the theorem of Chowla et al.,

- but easily possible for the theorem above.

3. Extensions modulo p^3

By numerical experimentation we first conjectured

Theorem 5

Let p and a be as before. Then

$$\frac{\left(\frac{p^3-1}{2}\right)_p!}{\left(\left(\frac{p^3-1}{4}\right)_p!\right)^2} \equiv 2a - \frac{p}{2a} - \frac{p^2}{8a^3} \pmod{p^3}.$$

(Proof later).

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Using more complicated congruences than the ones leading to Theorem 4 (but the same ideas), and going *backwards*, we obtain

Theorem 6 (Main result)

Let p and a be as before. Then

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \left(2a - \frac{p}{2a} - \frac{p^2}{8a^3}\right) \times \left(1 + \tfrac{1}{2}pq_p(2) + \tfrac{1}{8}p^2\left(2E_{p-3} - q_p(2)^2\right)\right) \pmod{p^3}.$$

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Here E_{p-3} is the Euler number defined by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n \qquad (|t| < \pi).$$

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How can we prove Theorem 5?
By further experimentation we first conjectured, and then proved the following generalization.

Theorem 7

Let p and a be as before and let $\alpha \geq 2$ be an integer. Then

$$\frac{\left(\frac{p^\alpha-1}{2}\right)_p!}{\left(\left(\frac{p^\alpha-1}{4}\right)_p!\right)^2} \equiv 2a-1\cdot\frac{p}{2a}-1\cdot\frac{p^2}{8a^3}$$

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Here $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number which is always an integer.

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Theorem 5 is obviously a special case of Theorem 7.

4. Main Ingredients in the Proof

- The Jacobi sum

$$J(\chi,\psi)=\sum_{j \bmod p}\chi(j)\psi(1-j),$$

where χ and ψ are characters modulo p.

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$$p=a'^2+b'^2,\quad a'\equiv\left(\frac{2}{p}\right)\pmod 4,\quad b'\equiv a'g^{(p-1)/4}\pmod p.$$

These are uniquely defined, differ from a and b of Gauss' theorem only (possibly) in sign.

- Then

$$J(\chi,\chi) = (-1)^{\frac{p-1}{4}}(a' + ib'),$$
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- On the other hand,

$$J(\chi,\chi) \equiv 0 \pmod{p},$$
$$J(\chi^3,\chi^3) = \frac{\Gamma_p(1-\frac{1}{2})}{\Gamma_p(1-\frac{1}{4})^2}.$$

These are deep results, related to the “Gross-Koblitz formula" (see, e.g., *Gauss and Jacobi Sums* by B. Berndt, R. Evans and K. Williams).

- $\Gamma_p(z)$ is the p-adic gamma function defined by

$$F(n) := (-1)^n \prod_{\substack{0<j<n \\ p\nmid j}} j,$$

$$\Gamma_p(z) = \lim_{n\to z} F(n) \qquad (z \in \mathbb{Z}_p),$$

where n runs through any sequence of positive integers p-adically approaching z.

- In particular,

$$
\begin{aligned}
(-1)^{\frac{p-1}{4}}(a' - ib') = J(\chi^3, \chi^3) &= \frac{\Gamma_p(1-\frac{1}{2})}{\Gamma_p(1-\frac{1}{4})^2} \\
&\equiv \frac{\Gamma_p(1+\frac{p^\alpha-1}{2})}{\Gamma_p(1+\frac{p^\alpha-1}{4})^2} \pmod{p^\alpha} \\
&= \frac{F(1+\frac{p^\alpha-1}{2})}{F(1+\frac{p^\alpha-1}{4})^2} \\
&= -\frac{\left(\frac{p^\alpha-1}{2}\right)_p!}{\left(\left(\frac{p^\alpha-1}{4}\right)_p!\right)^2}.
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- use the combinatorial identity ($k = 0, 1, \ldots, n-1$)

$$\sum_{j=0}^{k} \frac{(-1)^j}{j+1} \binom{2j}{j} \binom{n+j-k}{k-j} = \binom{n-1-k}{k};$$

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- putting everything together, we obtain Theorem 7.

5. A Jacobi Analogue

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$$4p = r^2 + 3s^2, \qquad r \equiv 1 \pmod 3, \qquad 3 \mid s,$$

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In analogy to Gauss' Theorem 1 we have

Theorem 8 (Jacobi, 1837)

Let p and r be as above. Then

$$\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}} \equiv -r \pmod p.$$

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This was generalized to mod p^2 independently by Evans (unpublished, 1985) and Yeung (1989):

Theorem 9 (Evans; Yeung)

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With methods similar to those in the first part of this talk, we proved

Theorem 10

Let p and r be as above. Then

$$\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}} \equiv \left(-r + \frac{p}{r} + \frac{p^2}{r^3}\right)\left(1 + \frac{1}{6}p^2 B_{p-2}(\tfrac{1}{3})\right) \pmod{p^3}.$$

Here $B_n(x)$ is the nth Bernoulli polynomial.

6. More on Gauss Factorials

Write out the factorial $(p-1)!$, exploit symmetry mod p:

$$1\cdot 2\cdot \ldots \cdot \tfrac{p-1}{2}\,\tfrac{p+1}{2}\cdot \ldots \cdot (p-1) \equiv \left(\tfrac{p-1}{2}\right)!(-1)^{\frac{p-1}{2}}\left(\tfrac{p-1}{2}\right)! \pmod{p}.$$

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Thus, with Wilson's Theorem,

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For $p \equiv 1 \pmod 4$ the RHS is -1, so

$$\operatorname{ord}_p\left(\left(\tfrac{p-1}{2}\right)!\right) = 4 \quad \text{for} \quad p \equiv 1 \pmod 4.$$

In the case $p \equiv 3 \pmod 4$ we get

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Theorem 11 (Mordell, 1961)

For a prime $p \equiv 3 \pmod 4$,

$$\left(\tfrac{p-1}{2}\right)! \equiv -1 \pmod p \quad \Leftrightarrow \quad h(-p) \equiv 1 \pmod 4,$$

where $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$.

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Discovered independently by Chowla.

This completely determines the order mod p of $\left(\tfrac{p-1}{2}\right)!$.

Now consider the two halves of the product

$$1 \cdot 2 \cdots \tfrac{p-1}{2} \, \tfrac{p+1}{2} \cdots (p-1)$$

and denote them, respectively, by

$$\Pi_1^{(2)}, \qquad \Pi_2^{(2)}.$$

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By Wilson's theorem:

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and by symmetry:

$$\Pi_2^{(2)} \equiv (-1)^{\frac{p-1}{2}} \Pi_1^{(2)} \pmod{p}.$$

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No obvious relation between $\Pi_1^{(3)}$ and the “middle third" $\Pi_2^{(3)}$.

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For any $M \geq 2$ and for a prime $p \equiv 1 \pmod{M}$, divide $(p-1)!$ into the products

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Example:

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7	**2**	$-$**2**	**2**	5	1	2	-2	-1
13	-2	3	-2	13	6	3	-3	-6
19	-2	-5	-2	17	7	-3	-3	7
31	2	-8	2	29	-6	-2	2	6
37	7	3	7	37	-16	5	-5	16
43	-3	19	-3	41	13	7	7	13
61	$-$**14**	**14**	$-$**14**	53	26	7	-7	-26
67	-20	-33	-20	61	19	7	-7	-19
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- The obvious symmetries.
- $\Pi_1^{(3)} \equiv -\Pi_2^{(3)} \pmod{p}$ for $p = 7$ and $p = 61$.

Are these last cases just coincidences?

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It turns out:

$\Pi_1^{(3)} \equiv -\Pi_2^{(3)} \pmod{p}$ also for $p = 331$, $p = 547$, $p = 1951$, and further relatively rare primes (explained below).

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In contrast: No primes p for which

$$\Pi_1^{(3)} \equiv \Pi_2^{(3)} \pmod{p}, \qquad p \equiv 1 \pmod{6}, \text{ or}$$

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The theorems of Gauss and Jacobi can be used to explain this:

Consider the quotient

$$Q_4(p) := \frac{\Pi_2^{(4)}}{\Pi_1^{(4)}}$$

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Recall Gauss' theorem:

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod{p},$$

where $p \equiv 1 \pmod 4$, $p = a^2 + b^2$, $a \equiv 1 \pmod 4$.

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Similarly, Jacobi's theorem implies that

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A number of further consequences can be derived from these classical theorems.

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Corollary 12

For a prime $p \equiv 1 \pmod 6$ *we have*

$$\Pi_2^{(3)} \equiv -\Pi_1^{(3)} \pmod p \quad \Leftrightarrow \quad p = 27x^2 + 27x + 7, x \in \mathbb{Z}.$$

A number of further consequences can be derived from these classical theorems. Two samples:

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The first such primes are 7, 61 (seen earlier), 331, 547, 1951.

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Let $p \equiv 1 \pmod 4$. *Then*

(a) $\frac{p-1}{4}! \equiv 1 \pmod p$ *only if* $p = 5$.

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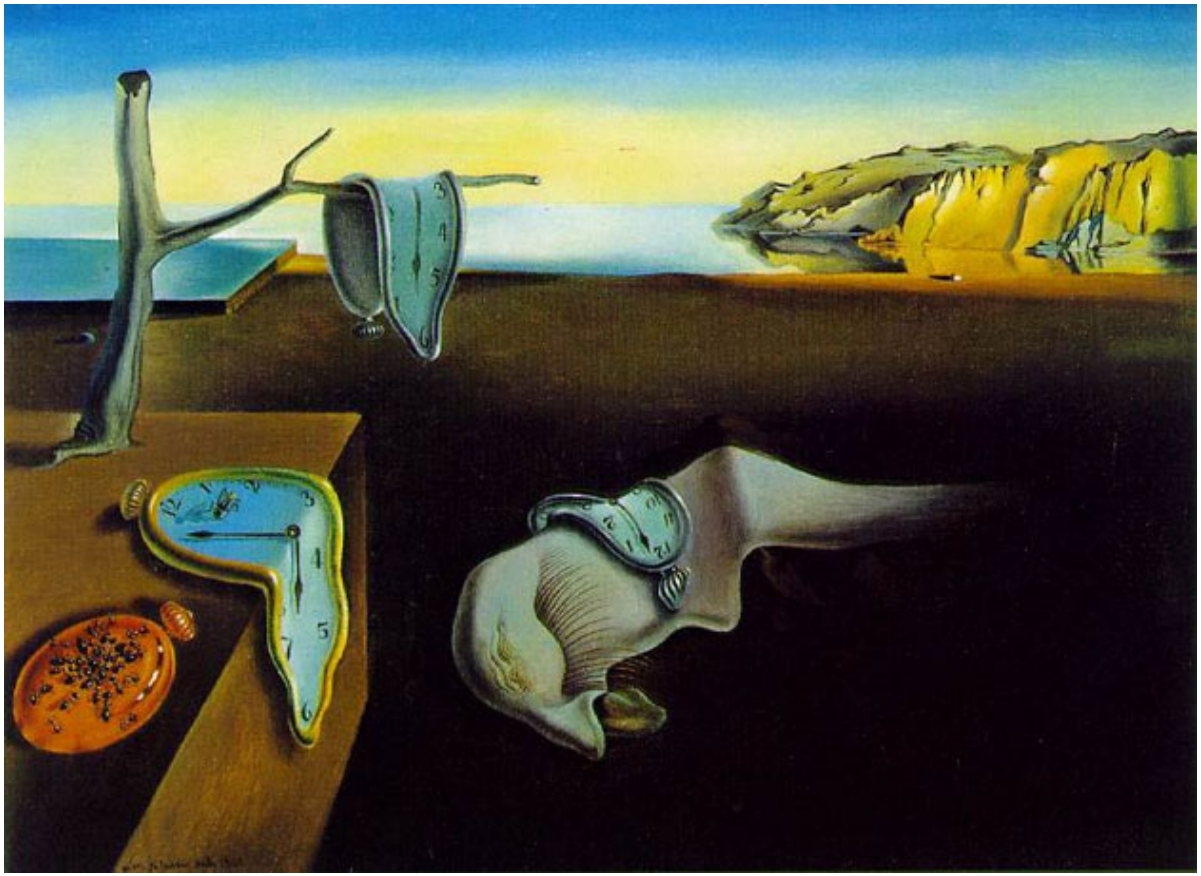
Part (c) is related to the solution of a certain Pell equation.

How are we doing with time?



Salvador Dalí, *The Persistence of Memory*, 1931.

7. Composite Moduli

Recall Gauss' generalization of Wilson's theorem,
the prototype of all composite modulus extensions:

$$(n-1)_n! \equiv \begin{cases} -1 \pmod{n} & \text{for} \quad n = 2, 4, p^\alpha, \text{ or } 2p^\alpha, \\ 1 \pmod{n} & \text{otherwise,} \end{cases}$$

where p is an odd prime and α is a positive integer.

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Everything we did before can be extended to composite moduli.

E.g., the multiplicative orders of $(\frac{n-1}{2})_n!$ modulo n
were completely determined; only orders 1, 2, 4 occur.
(JBC & KD, 2008).

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where, for $j = 1, 2, \ldots, M$,

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- Dependence on n is implied in the notation;
- When $n = p$: reduces to previous case.

Example:

n	$\Pi_1^{(3)}$	$\Pi_2^{(3)}$	$\Pi_3^{(3)}$	n	$\Pi_1^{(4)}$	$\Pi_2^{(4)}$	$\Pi_3^{(4)}$	$\Pi_4^{(4)}$
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We see: In contrast to prime case, it *can* happen that all partial products are congruent to each other.

We pause to consider the *number* of elements in our subintervals:

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In general, the situation is less straightforward.
E.g., for $n = 4$:
$\phi_{3,1}(n) = \phi_{3,3}(n) = 1$, but $\phi_{3,2}(n) = 0$.

When do we have equal distribution?

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Theorem 14 (Lehmer, 1955)

Let $M \geq 2$ and $n \equiv 1 \pmod{M}$.
If n has at least one prime factor $p \equiv 1 \pmod{M}$, then

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Note: Condition is sufficient, but not necessary.

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82	-33	-25	-33	77	16	31	31	16
85	-28	9	-28	81	2	40	-40	2
88	5	-7	5	85	**13**	**13**	**13**	**13**
91	**29**	**29**	**29**	89	22	42	42	22
94	-23	43	-23	93	34	-10	-10	34
97	21	-11	21	97	20	-28	-28	20

Note:

$91 = 7 \cdot 13, \quad 65 = 5 \cdot 13, \quad 85 = 5 \cdot 17.$

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Let $M \geq 2$ and $n \equiv 1 \pmod{M}$.
If n has at least **two** *distinct prime factors $\equiv 1 \pmod{M}$, then*

$$\Pi_j^{(M)} \equiv \left(\tfrac{n-1}{M}\right)_n! \pmod{n}, \qquad j = 1, 2, \ldots, M.$$

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On the other hand, condition is sufficient but not necessary.

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6. Use Chinese Remainder Theorem modulo $q^\beta w$ and $q^\alpha w$.

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Method of proof is similar to that of previous result.

Summary:

# of prime factors $\equiv 1 \pmod{M}$	All $\Pi_1^{(M)}, \ldots, \Pi_M^{(M)}$:
1	have the same number of factors
2	are congruent to each other $\pmod{M}$
3	are congruent to 1 $\pmod{M}$

Outlook

Main task: Understand the multiplicative orders of

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- n has 0 or 1 prime factor $\equiv 1 \pmod{M}$:
  – The most interesting and difficult case.
  – Several results already published.
  – Further work in progress.

Thank you – Danke