Tipsy Cops & Robbers on Infinite Trees

By David Wu, Supervised by Florian Lehner

1 Introduction

Cops and robbers is a classic game played on connected graphs. There are any number of cops and one robber, and the players take turns moving to adjacent vertices. If at any point a cop lands on the same vertex as a robber, the cop has captured the robber, and the game is over. A player being tipsy is defined as having some proportion of moves made randomly, as opposed to deterministically.

In this paper I look at the circumstances under which a game is cop-win, and if so, the expected duration of the game, given a single tipsy cop and tipsy robber on an infinite Δ-regular tree. Given such a tree, our deterministic movements for the robber will be to move to any neighbour such that it increases their distance to the cop, and for the cop, to move to the neighbour that decreases their distance to the robber.

2 Setup

The game will function as follows: The cop and robber will be placed at vertices k distance away from each other, where $k \geq 2$. The cop will move first, and the cop and robber will take turns moving to neighbour vertices, where vertices are not reflexive. The game is a cop-win if the probability that the cop lands on the same vertex as a robber is 1, and robber-win otherwise. The length of the game is the number of moves that the cop makes. As such, each time step starts with a cop move then ends with a robber move.

Let x_c be the probability that the cop moves tipsily, x_r the probability the robber moves tipsily, and Δ the number of neighbours each vertex in the tree has. Then we have the following probabilities:

$$\mathit{robber} \to \mathit{cop} : \frac{x_r}{\Delta}$$

$$\mathit{robber} \not\to \mathit{cop} : 1 - \frac{x_r}{\Delta}$$

$$\mathit{cop} \to \mathit{robber} : 1 - x_c + \frac{x_c}{\Delta}$$

$$\textit{cop} \rightharpoonup \textit{robber} : x_c - \frac{x_c}{\Delta}$$

Let $d(t)$ be the smallest distance between the cop and robber at time t. Since each time step involves the movement of both players, we have three scenarios for $d(t+1)$: the cop and robber move away from each other; the cop and robber move towards each other; either the cop or robber moves towards the other whilst the other moves away from them. Using the above probabilities, we can calculate the following:

$$d(t+1) = \begin{cases} d(t)+2 & \text{with probability } \left(1 - \frac{x_r}{\Delta}\right)\left(x_c - \frac{x_c}{\Delta}\right) \\ d(t)-2 & \text{with probability } \frac{x_r}{\Delta}\left(1 - x_c + \frac{x_c}{\Delta}\right) \\ d(t) & \text{otherwise} \end{cases}$$

3 Calculating Cop-Win

Let X be a random variable, where

$$X = \{(c_t, r_t),\ (c_s, r_t),\ (c_s, r_s),\ (c_t, r_s)\},$$

with probability distribution

$$Y = \{(x_c x_r),\ ((1-x_c)x_r),\ ((1-x_c)(1-x_r)),\ (x_c(1-x_r))\}.$$

X is a random variable of the different outcomes for each time step, where c, r are the cop and robber, and t, s are tipsy and sober moves. Let γ_c, γ_r be uniformly distributed over the unit interval $[0, 1)$.

Define an auxiliary random process by

$$H_t = \begin{cases} 2 & \text{if } X = (c_t, r_s) \text{ and } \gamma_c > \frac{1}{\Delta} \text{ or } X = (c_t, r_t) \text{ and } \gamma_c > \frac{1}{\Delta},\ \gamma_r > \frac{1}{\Delta} \\ -2 & \text{if } X = (c_t, r_t \text{ and } \gamma_c < \frac{1}{\Delta}, \gamma_r < \frac{1}{\Delta} \text{ or } x = (c_s, r_t) \text{ and } \gamma_c < \frac{1}{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

Note that the probability that $H_t = 2$ is the same as the probability that $d(t+1) = d(t) + 2$, and the probability that $H_t = -2$ is the same as the probability that $d(t+1) = d(t) - 2$. Thus, if $H_t = 2$, $d(t+1) = d(t) = 2$, and if $H_t = -2$, $d(t+1) = d(t) - 2$, otherwise $d(t+1) = d(t)$. Therefore:

$$d(0) + \sum_{i=1}^{t} H_i \leq d(t)\ \forall t$$

H_t are i.i.d and $d(0) = k$, so $\mathbb{E}[d(0) + \sum_{i=1}^{t} H_i] = k + t\,\mathbb{E}\,H_t$, and calculating $\mathbb{E}\,H_t$ gives:

$$\begin{aligned}
\mathbb{E}\,H_t &= 2\mathbb{P}[H_t = 2] - 2\mathbb{P}[H_t = -2] && (1)\\
&= 2\left(1 - \frac{x_r}{\Delta}\right)\left(x_c - \frac{x_c}{\Delta}\right) - 2\frac{x_r}{\Delta}\left(1 - x_c + \frac{x_c}{\Delta}\right) && (2)\\
&= 2x_c - \frac{2x_c + 2x_r}{\Delta} && (3)
\end{aligned}$$

When $\mathbb{E}\,H_t < 0$, by the strong law of large numbers

$$d(t) = d(0) + \sum_{i=1}^{t} H_i \xrightarrow{a.s.} -\infty \text{ as } t \to \infty,$$

and so we have that $\mathbb{P}[\text{cop-win}] = 1$ when $2x_c - \frac{2x_c+2x_r}{1-\Delta}$, $x_c < \frac{x_r}{1-\Delta}$.

When $x_c > \frac{x_r}{1-\Delta}$, $\mathbb{E}\,H_t > 0$, and so the probability that $\sum_{i=1}^{t} H_i$ is non-negative for all values is positive. Therefore, the probability that $d(t) > d(0) \geq 2\ \forall t$ is positive. Thus, since there is a positive probability that the distance between the cop and robber is always greater than 0, the robber may win with positive probability.

4 Expected Capture Time

We can now calculate the expected capture time for cop-win games, which is when $\mathbb{E}\,H_t < 0$, or $x_c < \frac{x_r}{1-\Delta}$. Since the length of a game is the number of moves the cop makes and the cop goes first, it must be split into the even and odd cases. In the even case, at the end of the last time step the robber moves onto the cop's vertex. In the odd case, the cop must make an extra accurate move onto the robber's vertex. In this paper, I'll only deal with the even case.

Define $\tau_0 = \inf\{t : d(t) = 0\}$, and $c_k = \mathbb{E}[\tau_0 \mid d(0) = k]$, $k \geq 0$. Then we get the recursion

$$c_k = \begin{cases} 1 + pc_{k+2} + qc_{k-2} + rc_k & \text{when } k > 0 \\ 0 & \text{when } k = 0 \end{cases}$$

where p is the probability of the cop and robber both moving away from each other, q the probability of both moving towards each other, and $r = 1 - p - q$. This recursion is solved by $c_k = \frac{k}{2(q-p)}$. We know from before that $p = (1 - \frac{x_r}{\Delta})(x_c - \frac{x_c}{\Delta})$ and $q = \frac{x_c}{\Delta}(1 - x_c + \frac{x_c}{\Delta})$, and so

$$c_k = \frac{k}{2(\frac{x_c+x_r}{\Delta} - x_c)}\ ,$$

giving our expected capture time.

5 Future Work

The result of the game when $x_c = \frac{x_c}{1-\Delta}$, $\mathbb{E}\, H_t = 0$ could be formalised. It is presumed that the game will be cop-win but with an expected capture time of infinity.

An analysis of the expected capture time in the odd case should also be made, potentially with some recursion looking similar to:

$$c_k = \begin{cases} 1 + pc_{k+2} + qc_{k-2} + rc_k & \text{when } k > 1 \\ 1 + (1 - x_c + \frac{x_c}{\Delta})c_0 + (1 - \frac{x_r}{\Delta})(x_c - \frac{x_c}{\Delta})c_3 + (x_c - \frac{x_c}{\Delta})(\frac{x_r}{\Delta})c_1 & \text{when } k = 1 \\ 0 & \text{when } k = 0 \end{cases}$$