Homework 5 - BIOS 6643 Part 2

Question 1

Derive the variance of the normal, binomial, and poisson using the exponential family framework. In other words, derive the variance of these distributions using the fact that $Var(Y) = \phi \frac{\partial^2 a(\theta)}{\partial \theta}$

Exponential family form in the generalized linear format:

$$f(y;\theta,\phi) = \exp\left[\frac{y_i\theta_i - a(\theta_i)}{\phi} + b(y_i,\phi)\right]$$

- Normal Distribution

$$f(y|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(y-\mu)^2\right] = \exp\left[-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y^2 - 2y\mu + \mu^2)\right]$$

$$= \exp\left[-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}y^2 + \frac{y\mu}{\sigma^2} - \frac{1}{2\sigma^2}\mu^2\right] = \exp\left[\frac{y\mu}{\sigma^2} - \frac{1}{2\sigma^2}\mu^2 - \frac{1}{2\sigma^2}y^2 - \frac{1}{2}\log(2\pi\sigma^2)\right]$$

$$= \exp\left[\frac{y\mu - \frac{\mu^2}{2}}{\sigma^2} + -\frac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right)\right]$$

$y = y$; $\theta = \mu$; $a(\theta) = \frac{\mu^2}{2}$; $\phi = \sigma^2$; $b(y,\phi) = -\frac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right)$

$$\frac{\partial^2 a(\theta)}{\partial^2\theta} = \frac{\partial^2}{\partial\theta}\left(\frac{\mu^2}{2}\right) = 1$$

$$Var(Y) = \sigma^2(1) = 1 \Rightarrow \boxed{Var(Y) = \sigma^2}$$

Homework 5 Part II Question 1 Continued

- Binomial Distribution

$$f(y,\mu) = \binom{n}{y} \pi^y (1-\pi)^{n-y} = \exp\left[\log\binom{n}{y} + y\log(\pi) + (n-y)\log(1-\pi)\right]$$

$$= \exp\left[y\log(\pi) - y\log(1-\pi) + n\log(1-\pi) + \log\binom{n}{y}\right] = \exp\left[y\log\left(\frac{\pi}{1-\pi}\right) + n\log(1-\pi) + \log\binom{n}{y}\right]$$

$$= \exp\left[\frac{y\log\left(\frac{\pi}{1-\pi}\right) + n\log(1-\pi)}{1} + \log\binom{n}{y}\right]$$

$$y = y;\ \theta = \log\left(\frac{\pi}{1-\pi}\right);\ a(\theta) = -n\log(1-\pi);\ \phi = 1;\ b(y,\phi) = \log\binom{n}{y}$$

Getting π in terms of θ:

$$\theta = \log\left(\frac{\pi}{1-\pi}\right) \Rightarrow \frac{\pi}{1-\pi} = e^\theta \Rightarrow \pi = e^\theta - \pi e^\theta \Rightarrow \pi + \pi e^\theta = e^\theta$$

$$\pi = \frac{e^\theta}{1+e^\theta}$$

Finding $a(\theta)$ with θ instead of π.

$$a(\theta) = -n\log(1-\pi) = -n\log\left(1 - \frac{e^\theta}{1+e^\theta}\right) = -n\log\left(\frac{1+e^\theta}{1+e^\theta} - \frac{e^\theta}{1+e^\theta}\right) = -n\log\left(\frac{1}{1+e^\theta}\right) = n\log(1+e^\theta)$$

$$\frac{\partial}{\partial\theta} a(\theta) = \frac{ne^\theta}{1+e^\theta} \Rightarrow \frac{\partial^2}{\partial\theta^2} a(\theta) = \frac{\partial^2}{\partial\theta^2}\left(\frac{ne^\theta}{1+e^\theta}\right) = \frac{ne^\theta(1+e^\theta) - ne^{2\theta}}{(1+e^\theta)^2} = \frac{ne^\theta}{(1+e^\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2} a(\theta)\cdot\phi = \frac{ne^\theta}{(1+e^\theta)^2} \Rightarrow \frac{n\exp\left[\log\left(\frac{\pi}{1-\pi}\right)\right]}{\left(1+\exp\left[\log\left(\frac{\pi}{1-\pi}\right)\right]\right)^2} = \frac{n\left(\frac{\pi}{1-\pi}\right)}{\left(1+\frac{\pi}{1-\pi}\right)^2} = \frac{n\left(\frac{\pi}{1-\pi}\right)}{\left(\frac{1-\pi}{1-\pi}+\frac{\pi}{1-\pi}\right)^2}$$

$$= \frac{n\pi(1-\pi)^2}{(1-\pi)} = n\pi(1-\pi) \Rightarrow \boxed{\mathrm{var}(y) = n\pi(1-\pi)}$$

Homework 5 Part II Question 1 Continued

- Poisson Distribution

$$f(y;\lambda) = \frac{1}{y!}\lambda^y e^{-\lambda} = \exp\left[\log\left(\frac{1}{y!}\right) + y\log(\lambda) - \lambda\right]$$

$$= \exp\left[\frac{y\log(\lambda) - \lambda}{1} + \log\left(\frac{1}{y!}\right)\right]$$

$y = y$; $\theta = \log(\lambda)$; $a(\theta) = \lambda$; $\phi = 1$; $b(y,\theta) = \log\left(\frac{1}{y!}\right)$

Finding λ in terms of θ

$$\theta = \log(\lambda) \Rightarrow \lambda = e^{\theta} \Rightarrow a(\theta) = \lambda = e^{\theta}$$

$$\frac{\partial^2}{\partial\theta^2}a(\theta) = e^{\theta} \Rightarrow \frac{\partial}{\partial\theta}a(\theta)\,\phi = e^{\theta} = e^{\log(\lambda)} = \lambda \,;\ \boxed{Var(Y) = \lambda}$$

Homework 5 BIOSS 6643 Part II

Question 2

A) Find the canonical link for the Gamma distribution (which may be used for an outcome that is greater than 0) and write out the three components of the GLM for this model (the distribution of the outcome, the systematic component, and the link function for a canonical link). Use pdf with an α and β parameterization with a mean $= \frac{\alpha}{\beta}$ and a variance of $\frac{\alpha}{\beta^2}$.

Gamma function: $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) \, I_{(0,\infty)}(x) \quad \alpha > 0, \beta > 0$

$= \exp\left[\alpha \log(\beta) - \log(\Gamma(\alpha)) + (\alpha-1)\log(x) - \beta x \right]$
$= \exp\left[-\beta x + \alpha \log(\beta) + (\alpha-1)\log(x) - \log(\Gamma(\alpha)) \right]$

moving α into the denominator for the two terms on the left.

$$= \left[\frac{\frac{\beta}{\alpha}x - \log(\beta)}{-\frac{1}{\alpha}} + (\alpha-1)\log(x) - \log(\Gamma(\alpha)) \right]$$

$\theta = \frac{\beta}{\alpha}$; $\phi = \frac{1}{\alpha}$; $a(\theta) = \log(\beta)$; $b(x,\phi) = (\alpha-1)\log(x) - \log(\Gamma(\alpha))$

$\beta = \theta\alpha \Rightarrow \beta = \frac{\theta}{\phi}$
$a(\theta) = \log(\beta) = \log\left(\frac{\theta}{\phi}\right) = \log(\theta) - \log(\phi)$

Plugging these in:

$$= \left[\frac{\theta x - (\log(\theta) - \log(\phi))}{-\phi} + \left(\frac{1}{\phi} - 1\right)\log(x) - \log\left(\Gamma\left(\frac{1}{\phi}\right)\right) \right]$$

$$= \boxed{\left[\frac{\theta x - \log(\theta)}{-\phi} + \frac{\log(\phi)}{\phi} + \left(\frac{1}{\phi} - 1\right)\log(x) - \log\left(\Gamma\left(\frac{1}{\phi}\right)\right) \right]}$$

$a(\theta) = \log(\theta)$ $\quad b(y, \phi) = \frac{\log(\phi)}{\phi} + \left(\frac{1}{\phi} - 1\right)\log(x) - \log\left(\Gamma\left(\frac{1}{\phi}\right)\right)$

where $\theta = \frac{\beta}{\alpha}$ and $\phi = \frac{1}{\alpha}$

B) For the Gamma Distribution, show the $E[Y_i] = \frac{\partial a(\theta_i)}{\partial \theta_i}$ and $Var(Y_i) = \frac{\partial^2 a(\theta_i)}{\partial \theta_i^2}$

$$E[X] = \frac{\partial}{\partial \theta} a(\theta) = \frac{\partial}{\partial \theta} \log(\theta) = \frac{1}{\theta} = \frac{\alpha}{\beta}$$

$$V[X] = \frac{\partial^2 a(\theta)}{\partial \theta^2} \cdot -\phi = -\phi \frac{\partial}{\partial \theta}(\theta^{-1}) = \frac{\phi}{\theta^2} = \frac{1}{\alpha} \cdot \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

So,

$$\boxed{E[X_i] = \frac{\alpha}{\beta} \quad ; \quad V[X_i] = \frac{\alpha}{\beta^2}}$$