Homework 2 - BIOS 6643 - Analysis of Longitudinal Data

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Question 1 Let **B** be defined as follows:

$$\begin{bmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}$$

Part A Are the column vectors of **B** linearly dependent? Explain or show.

The column vectors can be separated from the matrix, and the following system of equations can be constructed. If each scalar, represented by a, b, and c are equal to 0, then the column vectors are considered independent.

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this out into a system of equations:

$$a + 5b = 0$$
$$a + 5c = 0$$
$$a + 5c = 0$$

Where the second and third row are identical. After solving the system of equations we end up with the following equalities:

$$c = -\frac{1}{5}a$$
$$b = -\frac{1}{5}a$$

This can be summed up as follow:

$$-\frac{1}{5}a = b = c$$

If we take $a = -5$ we get the following operation of the column vectors:

$$-5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because $a = b = c = 0$ is not necessary for this operation to be true, the column vectors a linearly dependent.

Part B Are the row vectors of **B** linearly dependent? Explain or show?

Taking the transpose of the matrix returns the following matrix:

$$\begin{bmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 5 \end{bmatrix}$$

Separating the transposed matrix **B** into column vectors multiplied by scalars a, b, and c:

$$a \begin{bmatrix}1\\5\\0\end{bmatrix} + b \begin{bmatrix}1\\0\\5\end{bmatrix} + c \begin{bmatrix}1\\0\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Where the second and third column vectors are the same. Expanding this into a system of equations:

$$\begin{aligned} a+b+c &= 0 \\ 5a &= 0 \\ 5b+5c &= 0 \end{aligned}$$

From this we automatically know that $a = 0$, and $b = -c$. If we take $b = 1$ and $c = -1$ we get the following operation:

$$0 \begin{bmatrix}1\\5\\0\end{bmatrix} + 1 \begin{bmatrix}1\\0\\5\end{bmatrix} - 1 \begin{bmatrix}1\\0\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Because this operation does not depend on $a = b = c = 0$ this means the row vectors of **B** are linearly dependent.

Part C What is the rank of **B**?

Because the three columns of **B** are linearly dependent the rank cannot be 3. To find the rank we need to transform matrix **B** into row echelon form, where 1R indicates the first row, 2R the second row, and 3R the third row.

$$\begin{bmatrix}1&5&0\\1&0&5\\1&0&5\end{bmatrix} \xrightarrow{3R=3R-2R} \begin{bmatrix}1&5&0\\1&0&5\\0&0&0\end{bmatrix} \xrightarrow{2R=2R-1R} \begin{bmatrix}1&5&0\\0&-5&5\\0&0&0\end{bmatrix}$$

After transforming **B** into row echelon form the number of non-zero rows is 2, and thus the rank of **B** is 2.

Part D Will this matrix be invertible? Explain.

Matrix **B** is not invertible because the rank is 2, while the number of columns is 3. In other words, because not all columns are linearly independent it is not invertible.

Question 2 Let $\mathbf{A}$ be defined as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 8 \\ 0 & 3 & 1 \\ 0 & 5 & 5 \end{bmatrix}$$

Part A Are the column vectors of $\mathbf{A}$ linearly dependent? Explain or show.

Separating out the column vectors and multiplying each by either scalar a, b, or c:

$$a \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + c \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because the first column is all 0 a can be any scalar. Thus the column vectors of $\mathbf{A}$ are linearly dependent.

Part B Are the row vectors of $\mathbf{A}$ linearly dependent? Explain or show.

Taking the transpose of $\mathbf{A}$ results in the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 8 \\ 0 & 3 & 1 \\ 0 & 5 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 5 \\ 8 & 1 & 5 \end{bmatrix}$$

Separating out these column vectors and multiplying by scalars a, b, and c:

$$a \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this out into a system of equations:

$$a + 3b + 5c = 0$$
$$8a + b + 5c = 0$$

Because we have three unknown (a, b, and c) and only two equations (due to the top row of 0s) we cannot solve this system of equations. Therefore we cannot find a unique solution where all scalars are equal to 0, meaning the row vectors are linearly dependent.

Part C What is the rank of $\mathbf{A}$?

Because matrix $\mathbf{A}$ is linearly dependent the rank cannot be 3. To find the rank we need to transform the matrix into row echelon form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 8 \\ 0 & 3 & 1 \\ 0 & 5 & 5 \end{bmatrix} \xrightarrow{3R=3R-2R} \begin{bmatrix} 0 & 1 & 8 \\ 0 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{2R=2(2R)-3(3R)} \begin{bmatrix} 0 & 1 & 8 \\ 0 & 0 & -10 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{3R=3R-2(1R)}$$

$$\xrightarrow{3R=3R-2(1R)} \begin{bmatrix} 0 & 1 & 8 \\ 0 & 0 & -10 \\ 0 & 0 & -12 \end{bmatrix} \xrightarrow{3R=3R-2R} \begin{bmatrix} 0 & 1 & 8 \\ 0 & 0 & -10 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{2R=-0.20(2R)} \begin{bmatrix} 0 & 1 & 8 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{3R=3R+2R}$$

$$\xrightarrow{3R=3R+2R} \begin{bmatrix} 0 & 1 & 8 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

After transforming $\mathbf{A}$ into row echelon form the number of non-zero rows is 2. Therefore the rank of $\mathbf{A}$ is 2.

Part D Will this matrix be invertable? Explain.

Matrix $\mathbf{A}$ is not invertible because the rank is 2, while the number of columns is 3. In other words, because not all columns are linearly independent it is not invertible.

Question 3 Consider the following functions of the random variables Y_1, Y_2, and Y_3:

$$W_1 = Y_1 + Y_2 + Y_3$$
$$W_2 = Y_1 - Y_2$$
$$W_3 = Y_1 - Y_2 - Y_3$$

Part A State the above in matrix notation.

We can write this in matrix notation as follows:

$$\mathbf{W} = \mathbf{XY}$$

Where the matrices are equal to the following:

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

Part B Find the expectation of the random vector $\mathbf{W}$.

$$E[\mathbf{W}] = E[\mathbf{XY}] = E[\mathbf{X}]E[\mathbf{Y}]$$

$\mathbf{X}$ is a constant, meaning taking the expectation does not change the matrix. For $\mathbf{W}$ and $\mathbf{Y}$ the expectation can be brought in for every element of the matrices.

$$\begin{bmatrix} E[W_1] \\ E[W_2] \\ E[W_3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} E[Y_1] \\ E[Y_2] \\ E[Y_3] \end{bmatrix}$$

Multiplying the right hand side together:

$$\begin{bmatrix} E[W_1] \\ E[W_2] \\ E[W_3] \end{bmatrix} = \begin{bmatrix} E[Y_1] + E[Y_2] + E[Y_3] \\ E[Y_1] - E[Y_2] \\ E[Y_1] - E[Y_2] - E[Y_3] \end{bmatrix}$$

Part C Find the variance-covariance matrix of $\mathbf{W}$.

$$Var(\mathbf{W}) = \begin{bmatrix} Var(Y_1) & Cov(Y_1, Y_2) & Cov(Y_1, Y_3) \\ Cov(Y_2, Y_1) & Var(Y_2) & Cov(Y_2, Y_3) \\ Cov(Y_3, Y_1) & Cov(Y_3, Y_2) & Var(Y_3) \end{bmatrix}$$

Question 4 Consider the following functions of the random variables Y_1,Y_2,Y_3,and Y_4:

$$W_1 = \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4) W_2 \qquad = \frac{1}{2}(Y_1 + Y_2) - \frac{1}{2}(Y_3 + Y_4)$$

Part A State the above in matrix notation.

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

This can be simplified with the following notation:

$$\mathbf{W} = \mathbf{XY}$$

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Homework 2 Question 3

Consider the following functions of the random variables Y_1, Y_2, and Y_3.

$$W_1 = Y_1 + Y_2 + Y_3$$
$$W_2 = Y_1 - Y_2$$
$$W_3 = Y_1 - Y_2 - Y_3$$

A) State the above in matrix notation

We can write this in matrix notation as follows:

$$W = XY$$

where the matrices are equal to the following:

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

B) Find the expectation of the random vector W

$$E[W] = E[XY] = E[X]\,E[Y]$$

X is a constant, meaning taking the expectation does not change the matrix. For X and Y the expectation can be brought in for every element of the matrices.

$$\begin{bmatrix} E[W_1] \\ E[W_2] \\ E[W_3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} E[Y_1] \\ E[Y_2] \\ E[Y_3] \end{bmatrix}$$

multiplying the right hand side:

$$\begin{bmatrix} E[W_1] \\ E[W_2] \\ E[W_3] \end{bmatrix} = \begin{bmatrix} E[Y_1] + E[Y_2] + E[Y_3] \\ E[Y_1] - E[Y_2] \\ E[Y_1] - E[Y_2] - E[Y_3] \end{bmatrix}$$

C) Find the Variance-Covariance Matrix of W.

$$Var(W) = X\,Var(Y)\,X^T$$

$$X^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \quad Var(Y) = \begin{bmatrix} Var(Y_1) & Cov(Y_1,Y_2) & Cov(Y_1,Y_3) \\ Cov(Y_2,Y_1) & Var(Y_2) & Cov(Y_2,Y_3) \\ Cov(Y_3,Y_1) & Cov(Y_3,Y_2) & Var(Y_3) \end{bmatrix} \quad X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$X\,Var(Y) = \begin{bmatrix} Var(Y_1) + Cov(Y_2,Y_1) + Cov(Y_3,Y_1), & Cov(Y_1,Y_2) + Var(Y_2) + Cov(Y_3,Y_2), & \dots \\ Var(Y_1) - Cov(Y_2,Y_1), & Cov(Y_1,Y_2) - Var(Y_2), & \dots \\ \vdots & \vdots & \end{bmatrix}$$

$$X\,Var(Y)\,X^T = \begin{bmatrix} Var(Y_1) + Cov(Y_2,Y_1) + Cov(Y_3,Y_1) + Cov(Y_1,Y_2) + Var(Y_2) + Cov(Y_3,Y_2) + \dots \\ Var(Y_1) - Cov(Y_2,Y_1) + Cov(Y_1,Y_2) - Var(Y_2) + \dots \\ \vdots \quad\quad \vdots \end{bmatrix}$$

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Homework 2 Question 4

Consider the following functions of the random variables Y_1, Y_2, Y_3, and Y_4:

$$W_1 = \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4)$$
$$W_2 = \frac{1}{2}(Y_1 + Y_2) - \frac{1}{2}(Y_3 + Y_4)$$

A.) State the above in matrix notation

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 1/4 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1/2 & 1/2 & -\frac{1}{2} & -1/2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \quad ; \quad W = XY$$

B.) Find the expectation of the random vector W.

$E[W] = E[XY] = E[X]E[Y] \rightarrow X$ is a constant, so $E[X] = X$ and $E[W] = XE[Y]$

$$E[W] = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} E\left[\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}\right]$$ $\rightarrow$ Bring the expectation inside the matrix.

$$E[W] = \begin{bmatrix} 1/4 & 1/4 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} E[Y_1] \\ E[Y_2] \\ E[Y_3] \\ E[Y_4] \end{bmatrix}$$

$$E[W] = \begin{bmatrix} \frac{1}{4}E[Y_1] + \frac{1}{4}E[Y_2] + \frac{1}{4}E[Y_3] + \frac{1}{4}E[Y_4] \\ \frac{1}{2}E[Y_1] + \frac{1}{2}E[Y_2] - \frac{1}{2}E[Y_3] - 1/2E[Y_4] \end{bmatrix}$$

$$E[W] = \begin{bmatrix} \frac{1}{4}\left(E[Y_1] + E[Y_2] + E[Y_3] + E[Y_4]\right) \\ \frac{1}{2}\left(E[Y_1] + E[Y_2]\right) - \frac{1}{2}\left(E[Y_3] + E[Y_4]\right) \end{bmatrix}$$

c) Find the variance-covariance matrix of W.

$Var(W) = X\,Var(Y)X^T \rightarrow X$ is a matrix of constant values.

$$Var(Y) = \begin{bmatrix} Var(Y_1) & Cov(Y_1,Y_2) & Cov(Y_1,Y_3) & Cov(Y_1,Y_4) \\ Cov(Y_2,Y_1) & Var(Y_2) & Cov(Y_2,Y_3) & Cov(Y_2,Y_4) \\ Cov(Y_3,Y_1) & Cov(Y_3,Y_2) & Var(Y_3) & Cov(Y_3,Y_4) \\ Cov(Y_4,Y_1) & Cov(Y_4,Y_2) & Cov(Y_4,Y_3) & Var(Y_4) \end{bmatrix} \quad \text{dim: } 4\times4$$

$Var(W)$ dim: 2×2

$$X = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \qquad X^T = \begin{bmatrix} 1/4 & 1/2 \\ 1/4 & 1/2 \\ 1/4 & -1/2 \\ 1/4 & -1/2 \end{bmatrix} \quad \text{dim: } 4\times2$$

dim: 2×4

$$X\,Var(Y) = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} Var(Y_1) & Cov(Y_1,Y_2) & Cov(Y_1,Y_3) & Cov(Y_1,Y_4) \\ Cov(Y_2,Y_1) & Var(Y_2) & Cov(Y_2,Y_3) & Cov(Y_2,Y_4) \\ Cov(Y_3,Y_1) & Cov(Y_3,Y_2) & Var(Y_3) & Cov(Y_3,Y_4) \\ Cov(Y_4,Y_1) & Cov(Y_4,Y_2) & Cov(Y_4,Y_3) & Var(Y_4) \end{bmatrix}$$

$$X\,Var(X) = \begin{bmatrix} \frac{1}{4}\left(Var(Y_1) + Cov(Y_2,Y_1) + Cov(Y_3,Y_1) + Cov(Y_4,Y_1)\right), \frac{1}{4}\left(Cov(Y_1,Y_2) + Var(Y_2) + Cov(Y_3,Y_2) + Cov(Y_4,Y_2)\right)\ldots \\ \frac{1}{2}\left(Var(Y_1) + Cov(Y_2,Y_1) - Cov(Y_3,Y_4) - Cov(Y_4,Y_1)\right), \frac{1}{2}\left(Cov(Y_1,Y_2) + Var(Y_2) - Cov(Y_3,Y_2) - Cov(Y_4,Y_2)\right)\ldots \end{bmatrix}$$

$$X\,Var(X)X^T = \begin{bmatrix} \frac{1}{16}\left(Var(Y_1) + Cov(Y_2,Y_1) + Cov(Y_3,Y_1) + Cov(Y_4,Y_1)\right) + \frac{1}{16}\left(Cov(Y_1,Y_2) + Var(Y_2) + Cov(Y_3,Y_2) + Cov(Y_4,Y_2)\right)\ldots \\ \frac{1}{8}\left(Var(Y_1) + Cov(Y_2,Y_1) - Cov(Y_3,Y_4) - Cov(Y_4,Y_1)\right) + \frac{1}{8}\left(Cov(Y_1,Y_2) + Var(Y_2) - Cov(Y_3,Y_4) - Cov(Y_4,Y_2)\right)\ldots \end{bmatrix}$$

Following through for this will return a 2×2 matrix for $Var(W)$.

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Homework 2 Question 5

Consider the least squares estimator $b = (X^T X)^{-1} X^T Y$ in the reading or $\hat{\beta} = (X^T X)^{-1} X^T Y$ in more general notation. Using matrix methods, show that the estimator is an unbiased estimator of β from a multiple regression model where $Y = X\beta + \epsilon$ and $\epsilon_i \sim N(0, \sigma^2)$.

An unbiased estimator is defined as follows:

$$\text{Bias}[\hat{\theta}] = E[\hat{\theta}] - \theta_0 = 0$$

$\hat{\beta} = (X^T X)^{-1} X^T Y$ —> Plugging in for $Y = X\beta + \epsilon$

$\hat{\beta} = (X^T X)^{-1} X^T (X\beta + \epsilon)$ —> Expanding this out

$$\hat{\beta} = \underbrace{(X^T X)^{-1} X^T X}\beta + (X^T X)^{-1} X^T \epsilon$$

$(X^T X)^{-1} X^T X = I$, multiplying a matrix by its inverse returns the identity matrix.

$$\hat{\beta} = \underbrace{I\beta} + (X^T X)^{-1} X^T \epsilon$$

multiplying by identity matrix returns β: $I\beta = \beta$

$$\hat{\beta} = \beta + (X^T X)^{-1} X^T \epsilon$$

Taking the expected value

$$E[\hat{\beta}] = E[\beta] + \underbrace{E[(X^T X)^{-1} X^T \epsilon]}_{=0}$$

Constant, so $E[\beta] = \beta$

$= 0$, because X is constant and $\epsilon \sim N(0, R_i)$, so $E[(X^T X)^{-1} X^T \epsilon] = (X^T X)^{-1} X^T E[\epsilon] = 0$

So,

$$\boxed{E[\hat{\beta}] = \beta}$$

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Homework 2 Question 6

Obtain an expression for the variance-covariance matrix of the fitted values $\hat{Y}_i, i = 1, \ldots, n$, in terms of the hat matrix.

Hat Matrix: $H = X(X^TX)^{-1}X^T$

$$\hat{Y} = X(X^TX)^{-1}X^TY = HY = X(X^TX)^{-1}X^TX\beta = X\beta$$

X is a matrix of fixed values (not random), so the hat matrix is also made up of fixed values (not random).

$$\text{Var}(\hat{Y}) = X\,\text{Var}(\beta)X^T \longrightarrow \text{Var}(\beta) = \sigma^2(X^TX)^{-1}$$

$$\text{Var}(\hat{Y}) = X\sigma^2(X^TX)^{-1}X^T = \sigma^2 X(X^TX)^{-1}X^T = \sigma^2 H$$

In short

$$\boxed{\text{Var}(\hat{Y}) = \sigma^2 H}$$

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Homework 2 Question 7

Consider the multiple regression model:

$$Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i; \quad i = 1, \ldots, n$$

where the ε_i are uncorrelated, with $E\{\varepsilon_i\} = 0$ and $Var(\varepsilon_i) = \sigma^2$.

A.) State the least squares criterion and derive the least squares normal equations.

The least squares criterion for this model:

$$Q = \sum_{i=1}^{n} (Y_i - \beta_1 X_{i1} - \beta_2 X_{i2})^2$$

To get the least squares normal equations we need to take the derivative of Q relative to β_1 and β_2, and then set these derivatives to 0.

$$\frac{\partial Q}{\partial \beta_1} = -2 \sum_{i=1}^{n} X_{i1} (Y_i - \beta_1 X_{i1} - \beta_2 X_{i2}) = 0$$

$$\sum_{i=1}^{n} X_{i1} (Y_i - \beta_1 X_{i1} - \beta_2 X_{i2}) = 0$$

$$\sum_{i=1}^{n} X_{i1} Y_i - \beta_1 \sum_{i=1}^{n} X_{i1}^2 - \beta_2 \sum_{i=1}^{n} X_{i1} X_{i2} = 0$$

$$\Downarrow$$

$$\sum_{i=1}^{n} X_{i1} Y_i = \beta_1 \sum_{i=1}^{n} X_{i1}^2 + \beta_2 \sum_{i=1}^{n} X_{i1} X_{i2}$$

Doing this for β_2

$$\frac{\partial Q}{\partial \beta_2} = -2 \sum_{i=1}^{n} X_{i2} (Y_i - \beta_1 X_{i1} - \beta_2 X_{i2}) = 0$$

$$\sum_{i=1}^{n} X_{i2} (Y_i - \beta_1 X_{i1} - \beta_2 X_{i2}) = 0$$

$$\sum_{i=1}^{n} X_{i2} Y_i - \beta_1 \sum_{i=1}^{n} X_{i1} - \beta_2 \sum_{i=1}^{n} X_{i2}^2 = 0$$

$$\Downarrow$$

$$\sum_{i=1}^{n} X_{i2} Y_i = \beta_1 \sum_{i=1}^{n} X_{i1} - \beta_2 \sum_{i=1}^{n} X_{i2}^2$$

The least squares normal equations are:

$$\boxed{\begin{aligned} \sum_{i=1}^{n} X_{i1} Y_i &= \beta_1 \sum_{i=1}^{n} X_{i1}^2 + \beta_2 \sum_{i=1}^{n} X_{i1} X_{i2} \\ \sum_{i=1}^{n} X_{i2} Y_i &= \beta_1 \sum_{i=1}^{n} X_{i1} X_{i2} + \beta_2 \sum_{i=1}^{n} X_{i2}^2 \end{aligned}}$$

B.) State the likelihood function and explain why the maximum likelihood estimators will be the same as the least squares estimators.

The expected value of Y_i

$$E[Y_i] = E[\beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i] = \beta_1 X_{i1} + \beta_2 X_{i2}, \text{ because } E[\varepsilon_i] = 0$$

The variance of Y_i

$$Var[Y_i] = Var[\beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i] = \sigma^2, \text{ because } V[\varepsilon_i] = \sigma^2$$

Meaning

$$Y_i \sim N(\beta_1 X_{i1} + \beta_2 X_{i2}, \sigma^2)$$

The likelihood function of this:

$$L(\beta_1, \beta_2) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}\left(Y_i - (\beta_1 X_{i1} + \beta_2 X_{i2})\right)^2\right)\right]$$

$$L(\beta_1, \beta_2) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - \beta_1 X_{i1} - \beta_2 X_{i2})^2\right)$$

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Homework 2 Question 7 [Continued]

Taking the log to get the log likelihood function.

$$\text{Log}L(\beta_1, \beta_2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 x_{i1} - \beta_2 x_{i2})^2$$

The term with the summation is the least square criterion. Finding the maximum likelihood estimates is the same process here as finding the least squares estimators. The derivative for each β term is taken and set to 0.