Symbolic Model of the Kinematics of the ABB IRb-7600 Manipulator

Direct and Inverse Kinematic Models

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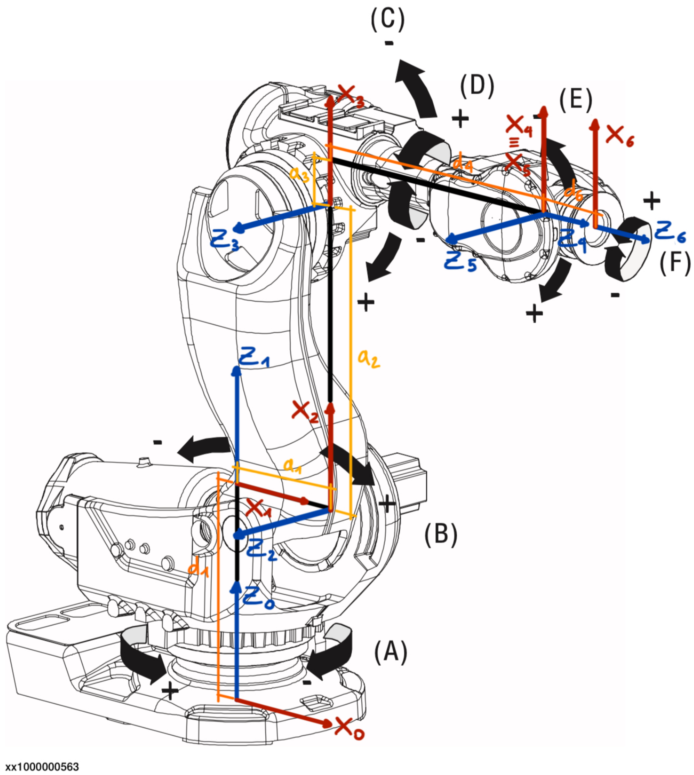
1 Introduction

In this document, I am going to explain my work which is about the definition of the mathematical model, the choice of the links-joints structure and frames placement, the derivation of the Direct and Inverse Kinematic models, and giving some numerical and visual examples of the ABB IRb-7600 Manipulator.
All my design choices will be discussed and motivated, when necessary.

An important assumption is made when modeling this manipulator, in order to simplify the solution of DK and IK. It is assumed that origins of Frames 1,2,3,4,5 lay in a plane, perpendicular to the base plane created by x_0-y_0.

2 Modeling of the Manipulator



Pos	Description	Pos	Description
A	Axis 1	D	Axis 4
B	Axis 2	E	Axis 5
C	Axis 3	F	Axis 6

Figure 1: Assignment of frames following Denavit-Hartenberg convention.

2.1 Frame Assignment

First of all, it is necessary to identify the axis of rotation (there are only revolute joints) and assign the z_i axes to the i-th axis of rotation, in our case from Frame 1 to Frame 6.

Then, find the common perpendicular between z_{i-1} and z_i axes and assign x-axes in order to have x_{i-1} aligned to the just computed common normal. For example, axes z_1 and z_2 intersect, and the common perpendicular is found along $z_1 \times z_2$; another particular example is between axes z_2 and z_3, which are parallel: in this case, there are an infinite number of common normals, so a simple choice is to fix it in such a way that it is aligned to the previous x-axes, x_2.

Finally, determine y_i axes to be $y_i = z_i \times x_i$ (right-handed frame).
The base coordinate Frame 0 should be placed in such way that a_0, α_0, and d_1 are equal to 0, meaning that $z_0 \equiv z_1$ and $O_0 \equiv O_1$. This is just used to simplify and speed up the computations, as 0-parameters means zero-ing out some matrix coefficients.
In this case, I decided to keep Frame 0 at the base of the manipulator to make the drawing and transformation matrices clearer, as this would just be for learning purposes. In this way, though, the benefits of 0-parameters as described before will not be exploited.

2.2 Design Choices of Axes

When assigning frames to links and joints, a bit of freedom is available in some of the choices which have to be made.

The first is that the direction of some of the z-axes can be freely decided, al long as they lay along the corresponding axis of rotation. For example when determining z_2 axes, it could be drawn in the current direction or the exact opposite, the difference would just be a "-" in the α_1 D-H parameter.

Another choice is about the setting of the zeros of the axes/joint variables, namely the zero of z-axes/θ-variables, and consequently, the zero-position of the respective x-axes. An example is the zero of joint 2, or θ_2. A possibility would be to follow the configuration depicted in Figure 1, for example, and choose this vertical position of x_2 axes to be the zero for θ_2. In my case, this does not represent zero-position of the frame, instead in this configuration would be θ_2=90°.
A clearer schema of my choices of the zero-position for all the links of the manipulator can be seen in the following (Figure 2).

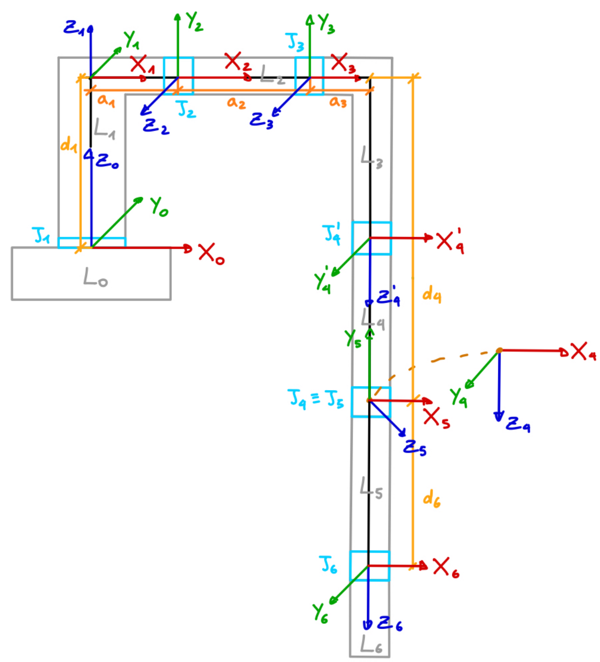


Figure 2: Frame and joint zero-positions.

When looking at this schema, it is possible to observe the introduction of the fictitious Frame 4'. It has been drawn just for clarity and visual reasons, but the real Frame 4 is placed with the origin coincident with the origin of Frame 5.

2.3 Denavit-Hartenberg Parameters

Once the frames are assigned and placed, it is possible to continue and define effectively the D-H parameters values. Denavit-Hartenberg parameters consist of 4 elements:

- a_{i-1}: the distance between axes z_{i-1} and z_i measured along axes x_{i-1}.
- α_{i-1}: the angle between axes z_{i-1} and z_i measured around axes x_{i-1}.
- d_i: the distance between axes x_{i-1} and x_i measured along axes z_i.
- θ_i: the angle between axes x_{i-1} and x_i measured around axes z_i.

The employment of D-H conventions for assigning the frames and the subsequent definition of the D-H parameters let us easily define the homogeneous transformation between two consecutive frames, by using just 4 variables instead of 6 (3 for position and 3 for orientation). Thus, the computation of direct kinematics becomes trivial.

In the following table are shown the D-H parameters which I chose for the modelling of this manipulator.

i	0	1	2	3	4	5	6
α	0	$\pi/2$	0	$\pi/2$	$-\pi/2$	$\pi/2$	-
a	0	0.41	1.075	0.165	0	0	-
d	-	0.78	0	0	1.056	0	0.25
θ	-	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6

3 Direct Kinematic Model

When the D-H parameters has been fixed, the homogeneous transformation between two consecutive frames, Frame $i-1$ and Frame i, is a simple composition of elementary transformations, namely rotation and translation of α_{i-1} and a_{i-1} around x_{i-1}, followed by a rotation and translation of θ_i and d_i around z_i:

$$ {}^{i-1}_{i}T = \mathrm{Rot}(x_{i-1}, \alpha_{i-1}) \cdot \mathrm{Trasl}(x_{i-1}, a_{i-1}) \cdot \mathrm{Rot}(z_i, \theta_i) \cdot \mathrm{Trasl}(z_i, d_i) $$

where

$$ \mathrm{Rot}(x_{i-1}, \alpha_{i-1}) \cdot \mathrm{Trasl}(x_{i-1}, a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & cos(\alpha_{i-1}) & -sin(\alpha_{i-1}) & 0 \\ 0 & sin(\alpha_{i-1}) & cos(\alpha_{i-1}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \mathrm{Rot}(z_i, \theta_i) \cdot \mathrm{Trasl}(z_i, d_i) = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i) & 0 & 0 \\ sin(\theta_i) & cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Given the joint variables $\boldsymbol{\theta} = [\theta_1, \theta_2, ..., \theta_6]$, it is possible to use the formula recursively, efficiently looping through the variables and accumulate the result in order to get the overall transformation from base Frame 0 and Frame 6, ${}^{0}_{6}T$ in the following way:

$$ {}^{0}_{6}T = \prod_{i=1}^{6} {}^{i-1}_{i}T = {}^{0}_{1}T \cdot {}^{1}_{2}T \cdot {}^{2}_{3}T \cdot {}^{3}_{4}T \cdot {}^{4}_{5}T \cdot {}^{5}_{6}T $$

3.1 Numerical Example

Let's plug some numerical values in the just computed Direct Kinematics equation:

$$ q = [0.33, 2.476, -1.189, 2.127, 0.563, -2.138] $$

$$ {}^{0}_{6}T(\mathbf{q}) = \begin{bmatrix} 0.5330 & -0.0995 & 0.8403 & 0.8008 \\ 0.1197 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} $$

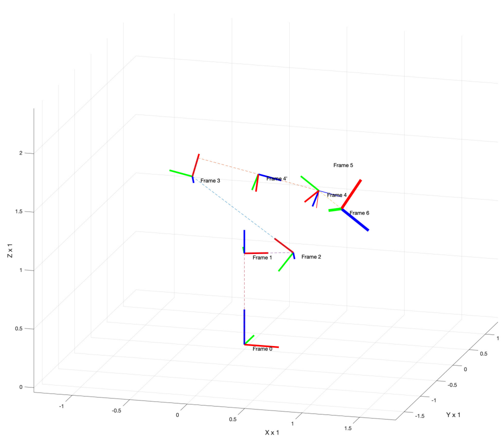


Figure 3: Pose example with given joint variables q.

4 Inverse Kinematic Model

The inverse problem is a bit more difficult and requires some inventive, tricks, and imagination.
My solution consists of a first geometric approach in order to find θ_1, θ_2, and θ_3, exploiting some simplifications in the modelling of the manipulator.
Once these are known, I opted for an analytical solution, by using transformation matrices and their inverse in order to find the last three angles, θ_4, θ_5, and θ_6.

4.1 Finding θ_1, θ_2, and θ_3

Finding θ_1:

For the first angle, θ_1, the solution is quite simple, after knowing that Frames 1,2,3,4,5 are in the same plane, as explained in the introduction, and the following trick.
The only knowns, at this beginning point, are Frame 6 position and orientation, namely its pose. This implies also the knowledge about z_6 direction, which can be found in the third column of orientation matrix of Frame 6, which is, again, given.
At this point, we can simply find the position of the origin of Frame 4/5, as they coincide, "going back" along z_6 by d_6 units. From there, exploit the fact that all those frames, up to frame 1, lay on the same plane, which passes through z_1-z_0 axes and perpendicular to x_0-y_0 plane.
In order to find θ_1, once ${}^0_5P$ is known, we can simply use its x and z components as follows:

$$\text{extract } z_6 \text{ from } {}^0_6T \text{ (third column, at [1:3,3])}$$

$$\begin{aligned} {}^0_5P &= {}^0_6P - d_6 \cdot z_6 \\ \theta_{1_1} &= atan2({}^0_5P_z, {}^0_5P_x) \\ \theta_{1_2} &= \theta_{1_1} - \pi \end{aligned}$$

Finding θ_3 and θ_2:

The trick that allows to compute θ_3 and θ_2 is, again, to exploit the fact that O_2, O_3, and O_4 lay in the same plane. The problem then transforms into finding the angles of a 2D planar manipulator, a way easier task.
The configuration at this point is represented in the following Figure 4.

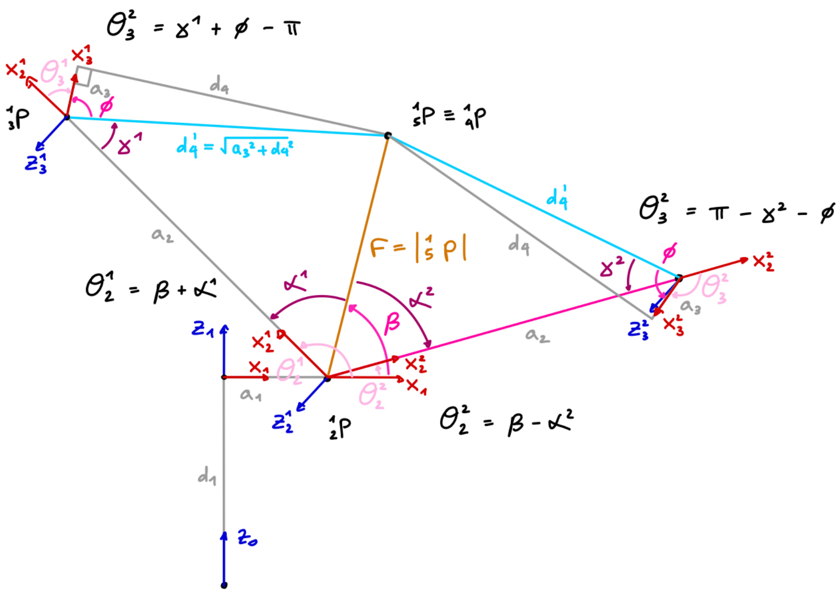


Figure 4: Planar view of the manipulator w.r.t. Frame 1.

There are two solutions for both θ_1 values found at the previous step. They correspond to elbow-up and elbow-down configurations. In theory, they are computable, but in reality one of the two could not be reached as the links would collide with themselves. At this step, it would be possible to check a predetermined set of physical limits for the axes in order to decide if the solution is feasible.

To find the two solutions for θ_3 it is sufficient to apply the cosine rule to the triangle formed by link a_2, the vector that connects O_2 and O_4, that is ${}^1_5P$, and the hypotenuse given by the right triangle with sides a_3 and d_4, which I called, in Figure 4, d'_4 (see Figure 4).
It is easy to check that ${}^1_5P = {}^0_1T^{-1} \cdot {}^0_5P = {}^1_0T \cdot {}^0_5P$, where ${}^0_1T$ is computed using one of the values of previously computed θ_1.

$$
\begin{aligned}
|{}^1_5P|^2 &= a_2^2 + d_4'^2 - 2a_2d'_4cos(\gamma) \\
cos(\gamma) &= \frac{|{}^1_5P|^2 - a_2^2 - d_4'^2}{-2a_2d'_4} \\
\gamma &= atan2(\sqrt{1 - cos(\gamma)^2}, cos(\gamma)) \\
\theta_{3_1} &= \gamma + \phi - \pi \\
\theta_{3_2} &= \pi - \gamma - \phi
\end{aligned}
$$

Another problem could now rise, when computing θ_3 and θ_2, as one of the two positions of Frame 1, i.e. the value of θ_{1_1} or θ_{1_2}, could make it impossible to reach the desired position. This can happen when the arm is too far extended outwards, and link length a_1 gets in the opposite direction with respect to the desired one.
It is quite easy to intercept this impossible solution, it is enough to check $cos(\gamma)$ given by the equation above, and if $|cos(\gamma)| > 1$ then it is sure that the configuration with that specific θ_1 is impossible to reach, while the other one should be reachable.

To find the two solutions for θ_2, the angle between link a_2 and vector ${}^1_5P$ is computed, α, and then the sine law is employed to compute its sine. Then, the angle between vector ${}^1_5P$ and x_1 axes, β, is computed.
Once these two angles are known, the two solutions for θ_2 can be computed.

$$
\begin{aligned}
\frac{d'_4}{sin(\alpha)} &= \frac{|{}^1_5P|}{sin(\gamma)} \\
sin(\alpha) &= \frac{d'_4}{|{}^1_5P|}sin(\gamma) \\
\alpha &= atan2(sin(\alpha), \sqrt{1 - sin(\alpha)^2} \\
\beta &= atan2({}^1_5P_x, {}^1_5P_z) \\
\theta_{2_1} &= \beta + \alpha \\
\theta_{2_2} &= \beta - \alpha
\end{aligned}
$$

4.2 Finding θ_4, θ_5, and θ_6

Now, as anticipated, an analytical approach is used to compute the remaining three angles by means of the homogeneous transformation matrices used in the direct kinematics.
By symbolically multiplying the last three transformation matrices, it results:

$$
{}^3_6T = {}^3_4T \cdot {}^4_5T \cdot {}^5_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -s_4c_6 - c_4c_5s_6 & c_4s_5 & a_3 + c_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 - d_4 \\ s_4c_5c_6 + c_4s_6 & c_4c_6 - s_4c_5s_6 & s_4s_5 & s_4s_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Finding θ_5:

Looking at the computed matrix, it is now possible to directly extract θ_5 from ${}^3_4T$, at[2,3], it is then just enough to compute the corresponding value to equate to. For this reason, for the right-hand side we compute:

$$ {}^3_6T = {}^0_3T^{-1} \cdot {}^0_6T = {}^2_3T^{-1} \cdot {}^1_2T^{-1} \cdot {}^0_1T^{-1} \cdot {}^0_6T = {}^3_2T \cdot {}^2_1T \cdot {}^1_0T \cdot {}^0_6T $$

which is a matrix of coefficients, as we know all the variables, θ_1, θ_2, and θ_3. For each value of θ_2 and θ_3 there are two solutions for θ_5. From here:

$$ \theta_{5_{1-2}} = atan2(\pm\sqrt{1 - cos(\theta_5)^2}, cos(\theta_5)) $$

If $\theta_5 = 0$ the manipulator is in a singular configuration, called wrist singularity, which happens when z_4 and z_6 axes are aligned. For this configuration no correct analytical solution is possible, mathematically there are infinite solutions. I decided that in this case, the algorithm should return no solutions for the angles which bring to this singularity.

If $\theta_5 \neq 0$, it is possible to continue with the computation of θ_4 and θ_6.

Finding θ_4 and θ_6:

Again, by looking at the symbolic transformation matrix ${}^3_6T$, it is easy to extract $cos(\theta_6)$, at [2,2], and $sin(\theta_6)$, at [2,1], dividing by $sin(\theta_5)$, which is for sure different from zero as θ_5 is different from zero.
In the same way, $cos(\theta_4)$, at [1,3], and $sin(\theta_4)$, at [3,3], are extracted. Then, for each value of θ_5, the values for θ_4 and θ_6 are:

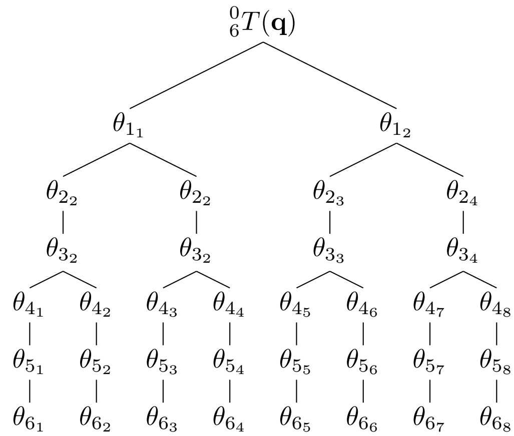
$$ \theta_4 = atan2(sin(\theta_4), cos(\theta_4)) $$
$$ \theta_6 = atan2(sin(\theta_6), cos(\theta_6)) $$

4.3 Final Solution

Finally, it is just a matter of arranging the solutions correctly together. This can be done in the following manner:

$\mathbf{q}_1$	θ_{1_1}	θ_{2_1}	θ_{3_1}	θ_{4_1}	θ_{5_1}	θ_{6_1}
$\mathbf{q}_2$	θ_{1_1}	θ_{2_1}	θ_{3_1}	θ_{4_2}	θ_{5_2}	θ_{6_2}
$\mathbf{q}_3$	θ_{1_1}	θ_{2_2}	θ_{3_2}	θ_{4_3}	θ_{5_3}	θ_{6_3}
$\mathbf{q}_4$	θ_{1_1}	θ_{2_2}	θ_{3_2}	θ_{4_4}	θ_{5_4}	θ_{6_4}
$\mathbf{q}_5$	θ_{1_2}	θ_{2_3}	θ_{3_3}	θ_{4_5}	θ_{5_5}	θ_{6_5}
$\mathbf{q}_6$	θ_{1_2}	θ_{2_3}	θ_{3_3}	θ_{4_6}	θ_{5_6}	θ_{6_6}
$\mathbf{q}_7$	θ_{1_2}	θ_{2_4}	θ_{3_4}	θ_{4_7}	θ_{5_7}	θ_{6_7}
$\mathbf{q}_8$	θ_{1_2}	θ_{2_4}	θ_{3_4}	θ_{4_8}	θ_{5_8}	θ_{6_8}

A tree view of how the various multiple angles found during the algorithm influence each other and create the eight possible solutions of Inverse Kinematics.



4.4 Numerical Example

In order to check if the solutions found by the Inverse Kinematics are correct, let's plug the pose used in the Direct Kinematics numeric example, and for each of the eight results given by the IK, recompute the DK and compare the results with the original pose. If everything is exact, the eight poses have to match the original one.

$$ {}^0_6T = \begin{bmatrix} 0.5330 & -0.0995 & 0.8403 & 0.8008 \\ 0.1197 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} $$

When plugged into the Inverse Kinematics algorithm, the resulting eight joint variables configurations are:

$\mathbf{q}_1$	0.3300	2.4760	-1.1890	2.1270	0.5630	-2.1380
$\mathbf{q}_2$	0.3300	2.4760	-1.1890	-1.0146	-0.5630	-5.2796
$\mathbf{q}_3$	0.3300	-0.1078	-2.2626	1.9334	2.6355	-4.2364
$\mathbf{q}_4$	0.3300	-0.1078	-2.2626	-1.2082	-2.6355	-1.0948
$\mathbf{q}_5$	-2.8116	3.6649	-0.5815	-0.5732	2.1520	-3.4155
$\mathbf{q}_6$	-2.8116	3.6649	-0.5815	2.5684	-2.1520	-0.2739
$\mathbf{q}_7$	-2.8116	1.6766	-2.8701	-1.1523	0.5191	-1.9775
$\mathbf{q}_8$	-2.8116	1.6766	-2.8701	1.9893	-0.5191	-5.1191

The first row, corresponding to solution $\mathbf{q}_1$, is consistent with the initial joint angle vector which was used to derive the pose of Frame 6, ${}^0_6T(\mathbf{q})$ in the DK numerical example.
Now, let's compute the associated Frame 6 pose using the Direct Kinematic equation for each of the given joint variables set.

$$ {}^0_6T(\mathbf{q}_1) = \begin{bmatrix} 0.5330 & -0.0995 & 0.8403 & 0.8008 \\ 0.1197 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \quad {}^0_6T(\mathbf{q}_2) = \begin{bmatrix} 0.5330 & -0.0995 & 0.8403 & 0.8008 \\ 0.1197 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} $$

$$ {}^0_6T(\mathbf{q}_3) = \begin{bmatrix} 0.5330 & -0.0995 & 0.8402 & 0.8008 \\ 0.1197 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \quad {}^0_6T(\mathbf{q}_4) = \begin{bmatrix} 0.5330 & -0.0995 & 0.8402 & 0.8008 \\ 0.1197 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} $$

$$ {}^0_6T(\mathbf{q}_5) = \begin{bmatrix} 0.5329 & -0.0995 & 0.8403 & 0.8008 \\ 0.1198 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1798 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \quad {}^0_6T(\mathbf{q}_6) = \begin{bmatrix} 0.5329 & -0.0995 & 0.8403 & 0.8008 \\ 0.1197 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1798 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} $$

$$ {}^0_6T(\mathbf{q}_7) = \begin{bmatrix} 0.5330 & -0.0995 & 0.8403 & 0.8008 \\ 0.1198 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \quad {}^0_6T(\mathbf{q}_8) = \begin{bmatrix} 0.5330 & -0.0995 & 0.8403 & 0.8008 \\ 0.1198 & -0.9742 & -0.1913 & 0.1545 \\ 0.8376 & 0.2026 & -0.5073 & 1.1797 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} $$

Here are shown the eight solutions from the Inverse Kinematics: each vector, $\mathbf{q}_i \mid i \in [1, 6]$, have been plugged into the manipulator 3D model and plotted.
The important thing to notice is that the pose of Frame 6, among the plots, is constant and consistent with the original value.

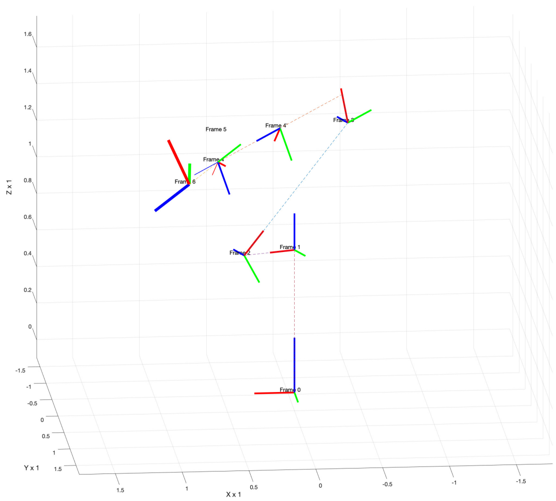


Figure 5: ${}^0_6T(\mathbf{q}_1)$

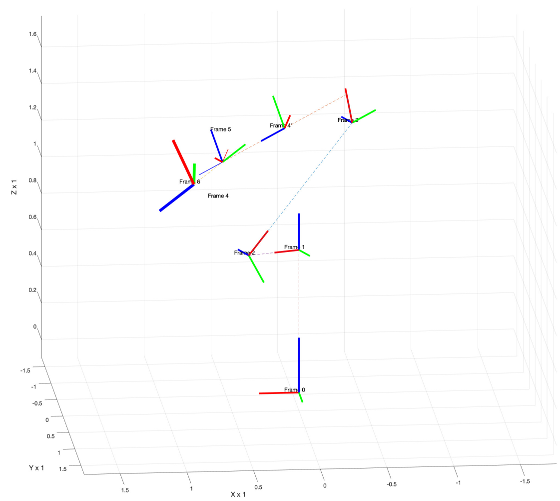


Figure 6: ${}^0_6T(\mathbf{q}_2)$

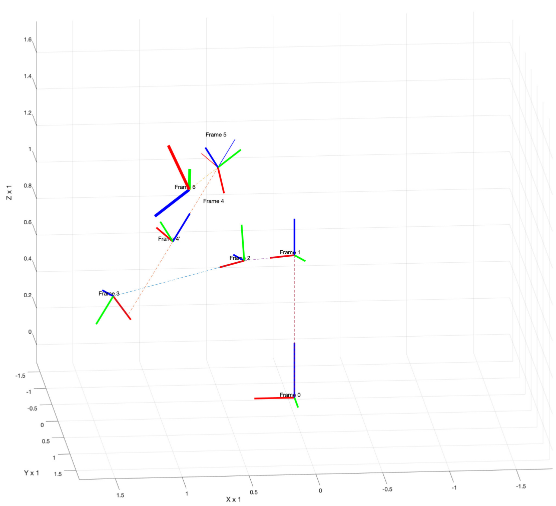


Figure 7: ${}^0_6T(\mathbf{q}_3)$

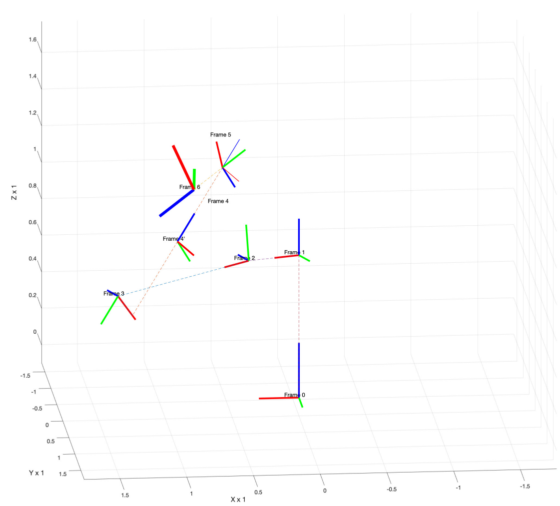


Figure 8: ${}^0_6T(\mathbf{q}_4)$

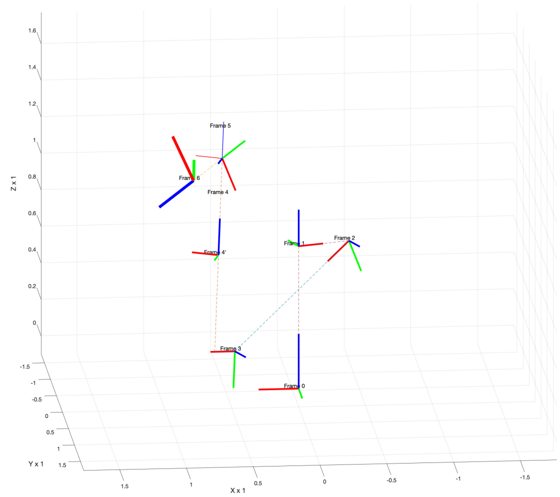


Figure 9: ${}^{0}_{6}T(\mathbf{q}_5)$

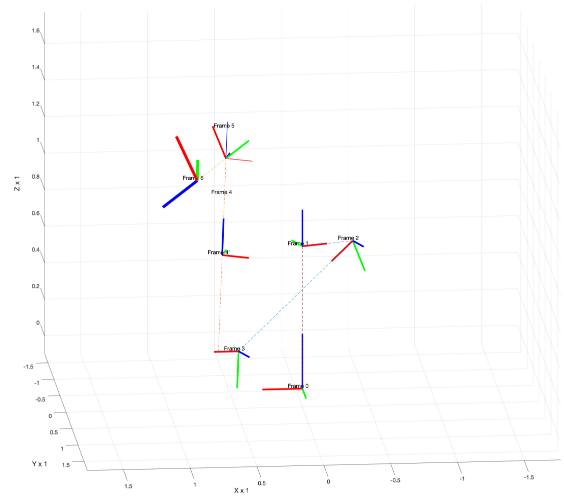


Figure 10: ${}^{0}_{6}T(\mathbf{q}_6)$

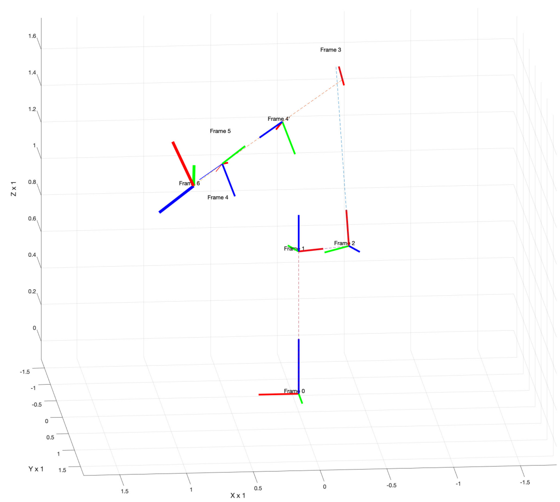


Figure 11: ${}^{0}_{6}T(\mathbf{q}_7)$

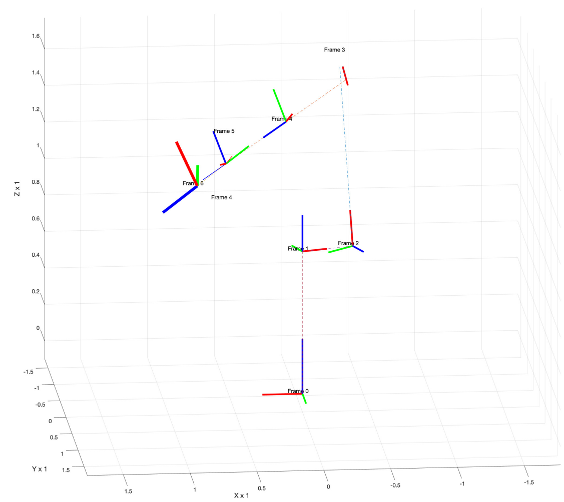


Figure 12: ${}^{0}_{6}T(\mathbf{q}_8)$