1. Suppose that you wish to estimate a regression in deviations-from-means form. The deviations-from-means model is given by $y_i = \beta x_i + \varepsilon_i$ and all standard assumptions hold (i.e, x_i is nonrandom, $\mathbb{E}(\varepsilon_i) = 0$, $var(\varepsilon_i) = \sigma^2$ for all i, and $cov(\varepsilon_i, \varepsilon_j) = 0$ for all observations).

   (a) Derive the Ordinary Least Squares (OLS) estimator $\hat{\beta}^{OLS}$. Show all work and simplify completely to receive full credit.

$$\widehat{\beta}^{OLS} \equiv \underset{\widehat{\beta}}{\operatorname{argmin}} \sum_{i=1}^{n} \left(y_i - \widehat{\beta} x_i \right)^2 \qquad \text{(recalling that } e_i = y_i - \widehat{\beta} x_i\text{)}$$

Differentiate with respect to $\widehat{\beta}$, set it equal to zero, and solve for $\widehat{\beta}$.

$$\text{F.O.C.:} \qquad \sum_{i=1}^{n} -2(y_i - \widehat{\beta} x_i) x_i = 0$$

$$2 \sum_{i=1}^{n} y_i x_i - 2\widehat{\beta} \sum_{i=1}^{n} x_i^2 = 0$$

$$\sum_{i=1}^{n} y_i x_i = \widehat{\beta} \sum_{i=1}^{n} x_i^2$$

$$\boxed{\widehat{\beta}^{OLS} = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}}$$

(b) Determine if $\hat{\beta}^{OLS}$ is an unbiased estimator of β. Make sure to be clear about when you use an assumption, show all work, and simplify completely to receive full credit.

$$\begin{aligned}
\underline{A1}: &\quad y_i = \beta x_i + \varepsilon_i \text{ is the true DGP} \\
\underline{A2}: &\quad x_i \text{ is nonrandom} \\
\underline{A3}: &\quad \mathbb{E}[\varepsilon_i] = 0 \;\; \forall i \\
\underline{A4}: &\quad Var(\varepsilon_i) = \sigma^2 \;\; i = 1, \ldots, n \\
&\quad Cov(\varepsilon_i, \varepsilon_j) = 0 \;\; \forall i \neq j
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\widehat{\beta}^{OLS}\right] &= \mathbb{E}\left[\frac{\sum\limits_i y_i x_i}{\sum\limits_i x_i^2}\right] \\
&= \mathbb{E}\left[\frac{\sum\limits_i (\beta x_i + \varepsilon_i) x_i}{\sum\limits_i x_i^2}\right] \qquad \text{(A1)} \\
&= \mathbb{E}\left[\frac{\sum\limits_i (\beta x_i^2 + \varepsilon_i x_i)}{\sum\limits_i x_i^2}\right] \\
&= \mathbb{E}\left[\frac{\sum\limits_i \beta x_i^2}{\sum\limits_i x_i^2} + \frac{\sum\limits_i \varepsilon_i x_i}{\sum\limits_i x_i^2}\right] \\
&= \mathbb{E}\left[\beta + \frac{\sum\limits_i x_i \varepsilon_i}{\sum\limits_i x_i^2}\right] \\
&= \beta + \mathbb{E}\left[\frac{\sum\limits_i x_i \varepsilon_i}{\sum\limits_i x_i^2}\right] \\
&= \beta + \frac{\sum\limits_i x_i \mathbb{E}[\varepsilon_i]}{\sum\limits_i x_i^2} \qquad \text{(A2)} \\
&\boxed{\mathbb{E}\left[\widehat{\beta}^{OLS}\right] = \beta} \qquad \text{(A3)} \\
&\implies \widehat{\beta}^{OLS} \text{ is an unbiased estimator of } \beta
\end{aligned}$$

(c) Find the variance of $\hat{\beta}^{OLS}$. Make sure to be clear about when you use an assumption, show all work and simplify completely to receive full credit.

$$
\begin{aligned}
Var\left(\widehat{\beta}^{OLS}\right) &= Var\left(\frac{\sum_i x_i y_i}{\sum_i x_i^2}\right) \\
&= Var\left(\frac{\sum_i x_i(\beta x_i + \varepsilon_i)}{\sum_i x_i^2}\right) && \text{(A1)} \\
&= Var\left(\beta + \frac{\sum_i x_i \varepsilon_i}{\sum_i x_i^2}\right) \\
&= Var\left(\frac{\sum_i x_i \varepsilon_i}{\sum_i x_i^2}\right) && \text{(the variance of a constant is zero)} \\
&= \frac{1}{\left(\sum_i x_i^2\right)^2} Var\left(\sum_i x_i \varepsilon_i\right) && \text{(pull constant out and square)} \\
&= \frac{1}{\left(\sum_i x_i^2\right)^2}\left[\sum_i Var(x_i\varepsilon_i) + 2\sum_{i\neq j} Cov(x_i\varepsilon_i, x_j\varepsilon_j)\right] && \text{(property of variance)} \\
&= \frac{1}{\left(\sum_i x_i^2\right)^2}\left[\sum_i x_i^2 Var(\varepsilon_i) + 2\sum_{i\neq j} x_i x_j Cov(\varepsilon_i, \varepsilon_j)\right] && \text{(A2)} \\
&= \frac{1}{\left(\sum_i x_i^2\right)^2}\left[\sigma^2 \sum_i x_i^2\right] && \text{(A4)}
\end{aligned}
$$

$$\boxed{Var\left(\widehat{\beta}^{OLS}\right) = \frac{\sigma^2}{\sum_i x_i^2}}$$