Greeks in Options

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In this document f will denote the option price, c a call option and p a put option.

Delta

Delta measures the rate of change of the option price with respect to changes in the price of the underlying asset:

$$\Delta = \frac{\partial f}{\partial S}$$

- If delta is 0.6, then for small changes in the underlying asset's price, the option price will change by approximately 60% of that amount.
- **Example:** Suppose the stock price is \$100 and we sell 20 call options. Each option corresponds to 100 shares, so the position involves:

  $$20 \cdot 100 = 2000 \text{ shares of the underlying.}$$

  Let $\Delta = 0.6$. To hedge, we buy:

  $$0.6 \cdot 2000 = 1200 \text{ shares of the underlying.}$$

  If the stock price increases by \$1:

  - Gain from the underlying position: \$1200
  - Loss on the option position: Since $\Delta f = 0.6 \cdot \Delta S$, the value of the short option position increases by:

    $$0.6 \cdot 2000 = 1200$$

    Net result: \$0 (locally delta-neutral)

  If the stock price decreases by \$1:

  - Loss on the underlying: \$1200
  - Gain on the short option position: \$1200
- For hedging, we need to rebalance dynamically. As the price rises, the call option's delta increases, and more shares need to be purchased to maintain neutrality.

- **Black-Scholes closed-form expression for delta:** The closed form solutions for Black-Scholes PDE for put and call options are:

$$c(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$p(S,t) = Ke^{-r(T-t)}N(-d_2) - S_0N(-d_1)$$
$$d_1 = \frac{\ln(S/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Using the fact that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we obtain:

$$\Delta_c = N(d_1) \text{ Delta of call option}$$
$$\Delta_p = N(d_1) - 1 \text{ Delta of put option}$$

Since $N(d_1) \in [0,1]$, Δ_p is negative. Therefore:

  - For a long put, buy $-\Delta_p$ shares of the underlying.
  - For a short put, short $-\Delta_p$ shares.

- **Delta as a function of the stock price:**

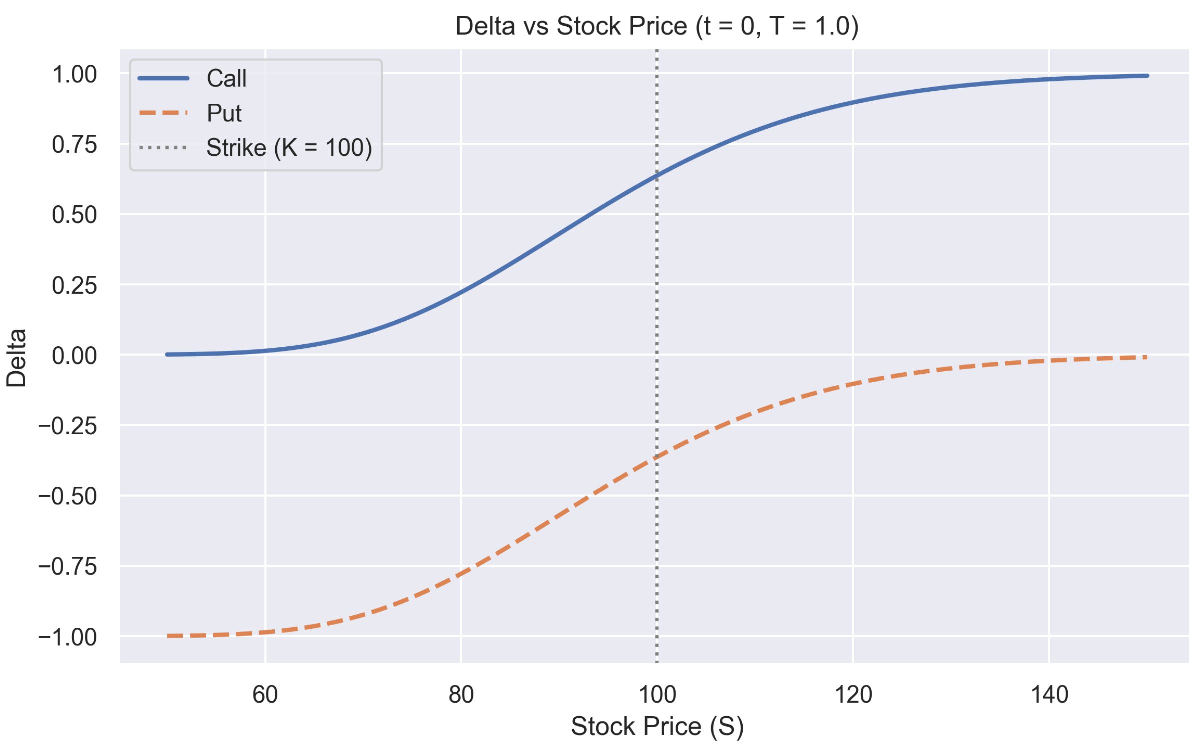

Figure 1: Delta vs. stock price

Intuitively:

- When the stock price is close to zero, the call option is deeply out-of-the-money. The chance it will be exercised is very low, so the option price is insensitive to changes in the underlying — delta is near zero.
- As the stock price approaches the strike price, the probability of exercise increases, and the option becomes more sensitive — delta increases rapidly.
- Once the stock price is far above the strike, the option is almost certainly going to be exercised. The call behaves similarly to the stock itself — delta approaches 1.

The result is the characteristic "S-shaped" delta curve. A similar argument applies for puts, but the curve is mirrored and shifted due to the negative slope.

Theta

Theta represents the rate of change of the option price with respect to time:

$$\Theta = \frac{\partial f}{\partial t}$$

It captures the "time decay" of the option — how much value is lost as expiry approaches, holding all else constant.

- **Black-Scholes expressions:**

$$\Theta(c) = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$
$$\Theta(p) = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$$

- For long options, theta is typically negative — the option loses value over time. For short positions, theta is positive.

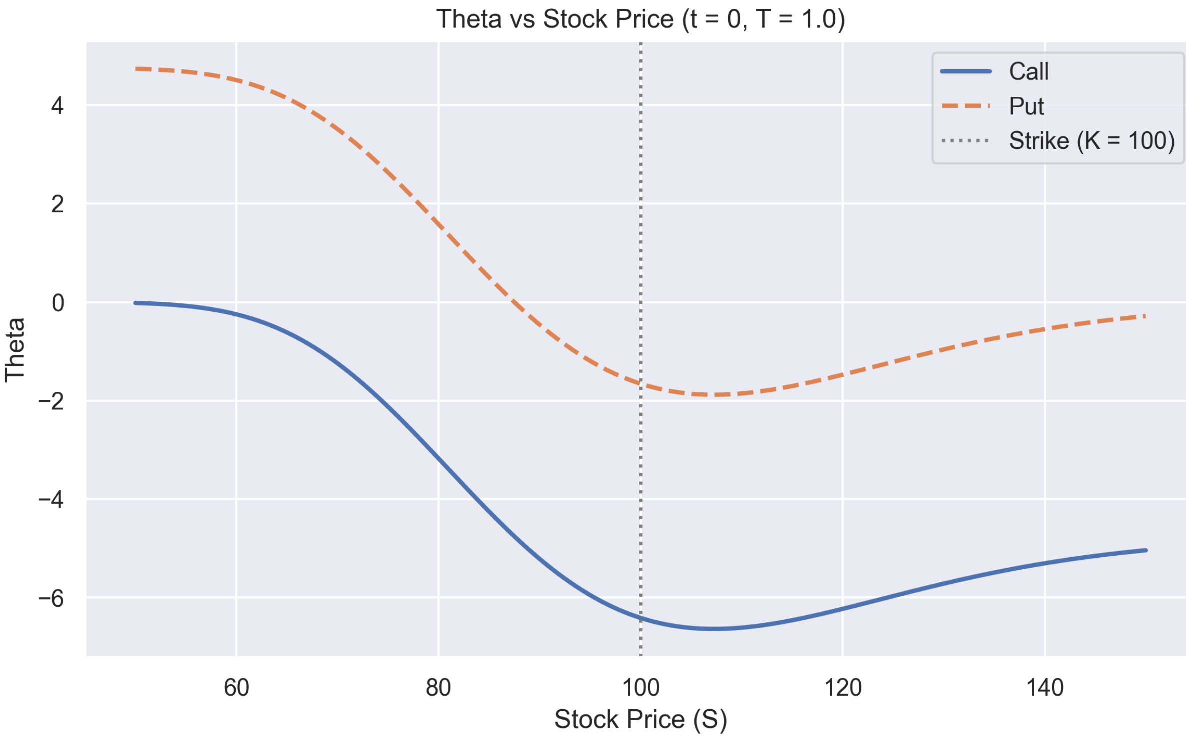

Figure 2: Tehta vs. Stock Price

Gamma

Gamma is the rate of change of delta with respect to changes in the underlying price:

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

This is the same for both calls and puts.

- Delta hedging relies on a first-order (linear) approximation of option price movements. But option prices are convex functions of the underlying — particularly around the strike price. Gamma measures the curvature.
- **Why Gamma Matters:** When price changes are small, delta hedging works well. For larger moves, however, ignoring convexity introduces error. Gamma improves hedging accuracy by accounting for this second-order effect.
- **Gamma Hedging Example:** Suppose a portfolio is delta neutral and has gamma of -3000. A traded call option has delta 0.63 and gamma 1.5. To make the portfolio gamma-neutral:

$$\frac{3000}{1.5} = 2000 \text{ call options (long)}$$

This changes the delta of the portfolio by:

$$2000 \cdot 0.63 = 1260$$

To preserve delta neutrality, we must sell 1260 units of the underlying.

- **Gamma Hedging and the Underlying:** Since the underlying asset has zero gamma (it is a linear instrument), we cannot use it to hedge gamma risk. Instead, we use other options on the same underlying to construct gamma-neutral portfolios.

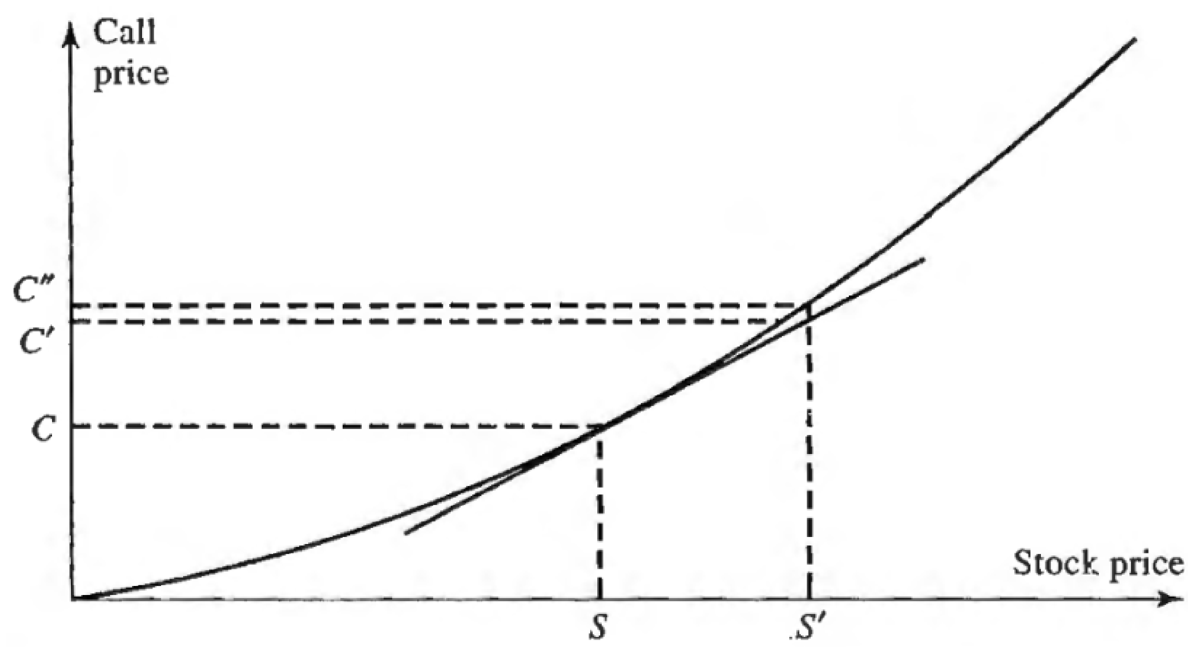

Figure 3: Gamma and curvature of option price
From J. Hull - options, futures, and other derivatives)

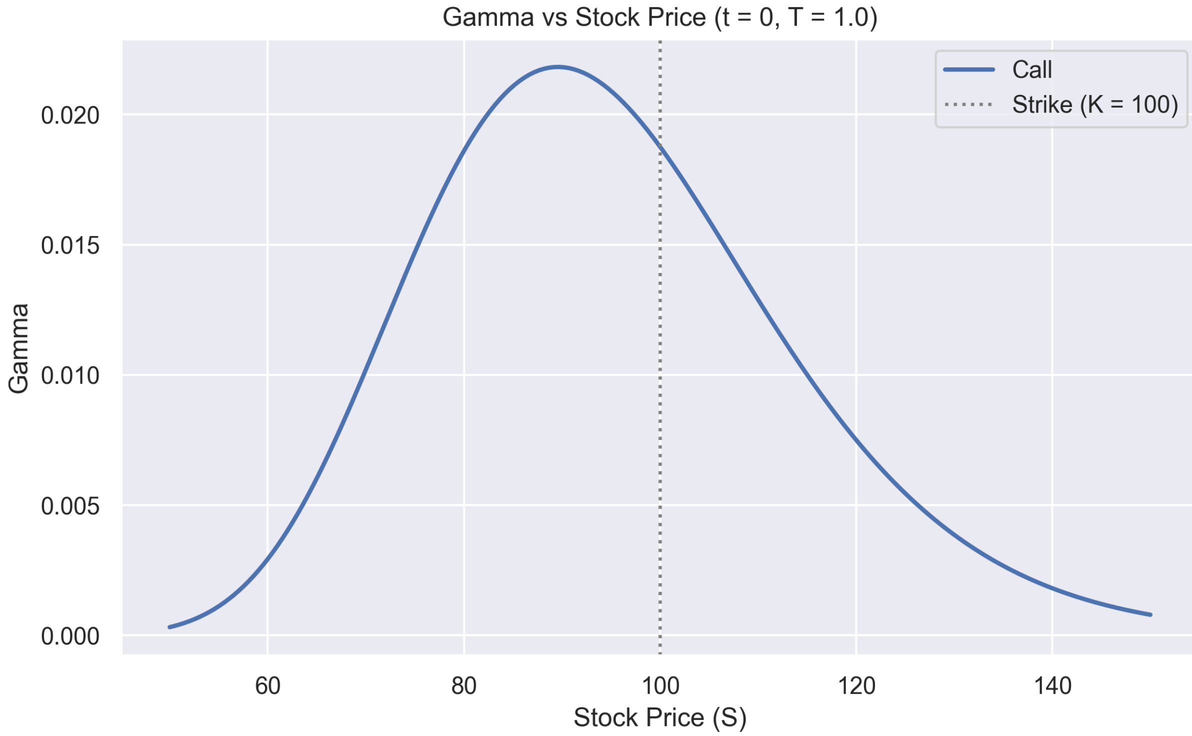

Figure 4: Gamma vs. Stock Price

Vega

Vega measures the sensitivity of the option price to changes in the volatility of the underlying asset:

$$\nu = \frac{\partial f}{\partial \sigma} = S\sqrt{T-t}N'(d_1)$$

- **Vega Hedging:** Vega exposure can be neutralized by adding a position in a traded option. If a portfolio has vega ν and a traded option has vega ν_T, then adding:

$$-\frac{\nu}{\nu_T}$$

  units of the traded option will make the portfolio vega-neutral (locally).

- However, a portfolio that is gamma neutral is not necessarily vega neutral, and vice versa. Achieving both typically requires two or more traded options.

- **Example (from J. Hull - options, futures and other derivatives) - Vega and Gamma Neutrality:** Consider a delta-neutral portfolio with:

$$\Gamma = -5000, \quad \nu = -8000$$

  We can trade the following options:

- Option 1: $\Delta = 0.6$, $\Gamma = 0.5$, $\nu = 2$
- Option 2: $\Delta = 0.5$, $\Gamma = 0.8$, $\nu = 1.2$

Vega neutrality using Option 1 alone requires:

$$\frac{8000}{2} = 4000 \text{ long calls}$$

This increases delta by $4000 \cdot 0.6 = 2400$, and gamma by $4000 \cdot 0.5 = 2000$. Gamma changes to -3000.

To achieve both gamma and vega neutrality, we solve:

$$-5000 + 0.5w_1 + 0.8w_2 = 0$$
$$-8000 + 2w_1 + 1.2w_2 = 0$$

Solution:

$$w_1 = 400, \quad w_2 = 6000$$

The resulting delta is:

$$400 \cdot 0.6 + 6000 \cdot 0.5 = 3240$$

So, to remain delta neutral, we sell 3240 units of the underlying.

- **Vega Shape Intuition:**
  - When $S \ll K$, the option is deep out-of-the-money. Volatility has little effect — vega is near zero.
  - When $S \approx K$, the option is at-the-money. Its value is highly sensitive to volatility — vega peaks.
  - When $S \gg K$, the option is deep in-the-money. Again, volatility has little effect — vega decreases.

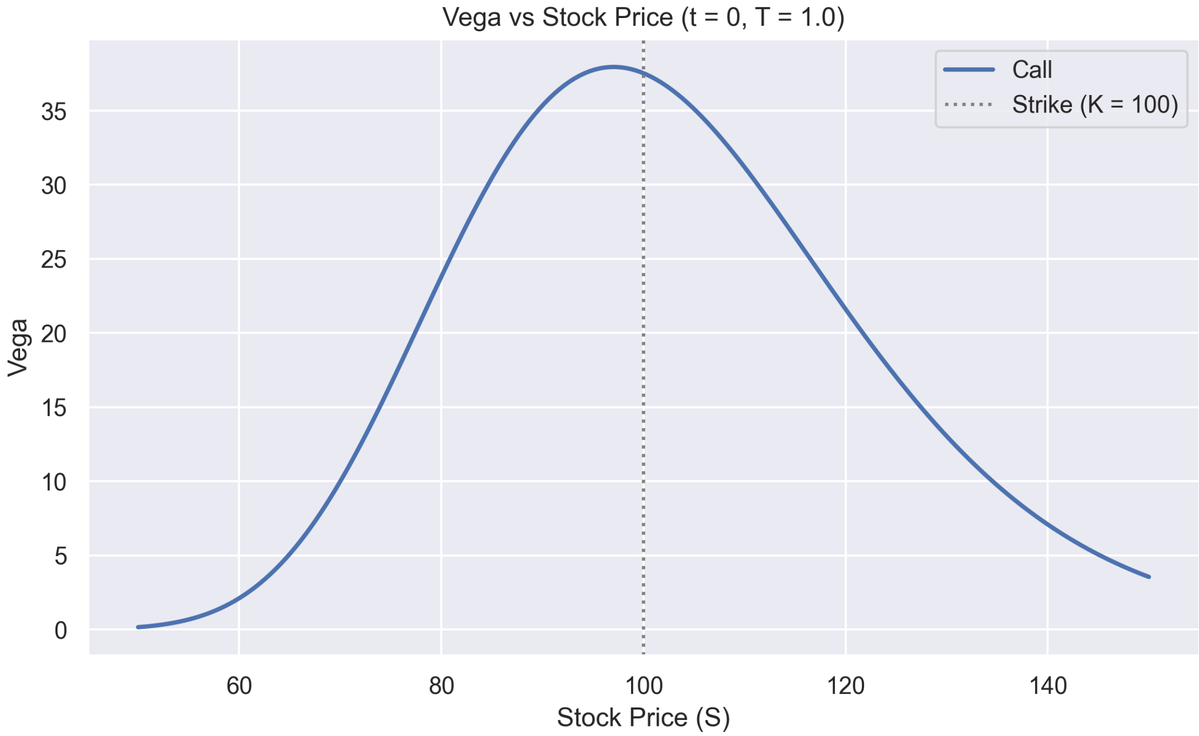

Figure 5: Vega vs. Stock Price

In summary, vega is highest when the option is at-the-money and decreases as the option becomes either deep in-the-money or out-of-the-money.

References

[1] John C. Hull, *Options, Futures, and Other Derivatives*, 10th Edition, Pearson, 2017.