# Greeks in Options

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## Delta

Delta measures the rate of change of the option price with respect to changes in the price of the underlying asset:

$$\Delta = \frac{\partial f}{\partial S}$$

- If delta is 0.6, then for small changes in the underlying asset's price, the option price will change by approximately 60% of that amount.
- **Example:** Suppose the stock price is \$100 and we sell 20 call options. Each option corresponds to 100 shares, so the position involves:

  $$20 \cdot 100 = 2000 \text{ shares of the underlying.}$$

  Let $\Delta = 0.6$. To hedge, we buy:

  $$0.6 \cdot 2000 = 1200 \text{ shares of the underlying.}$$

  If the stock price increases by \$1:

  - Gain from the underlying position: \$1200
  - Loss on the option position: Since $\Delta f = 0.6 \cdot \Delta S$, the value of the short option position increases by:

    $$0.6 \cdot 2000 = 1200$$

    Net result: \$0 (locally delta-neutral)

  If the stock price decreases by \$1:

  - Loss on the underlying: \$1200
  - Gain on the short option position: \$1200
- For hedging, we need to rebalance dynamically. As the price rises, the call option's delta increases, and more shares need to be purchased to maintain neutrality.

- **Black-Scholes closed-form expression for delta:**

$$c(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$d_1 = \frac{\ln(S/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Using the fact that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we obtain:

$$\Delta_c = N(d_1)$$
$$\Delta_p = N(d_1) - 1$$

Since $N(d_1) \in [0,1]$, $\Delta_p$ is negative. Therefore:

  - For a long put, buy $-\Delta_p$ shares of the underlying.
  - For a short put, short $-\Delta_p$ shares.

- **Delta as a function of the stock price:**

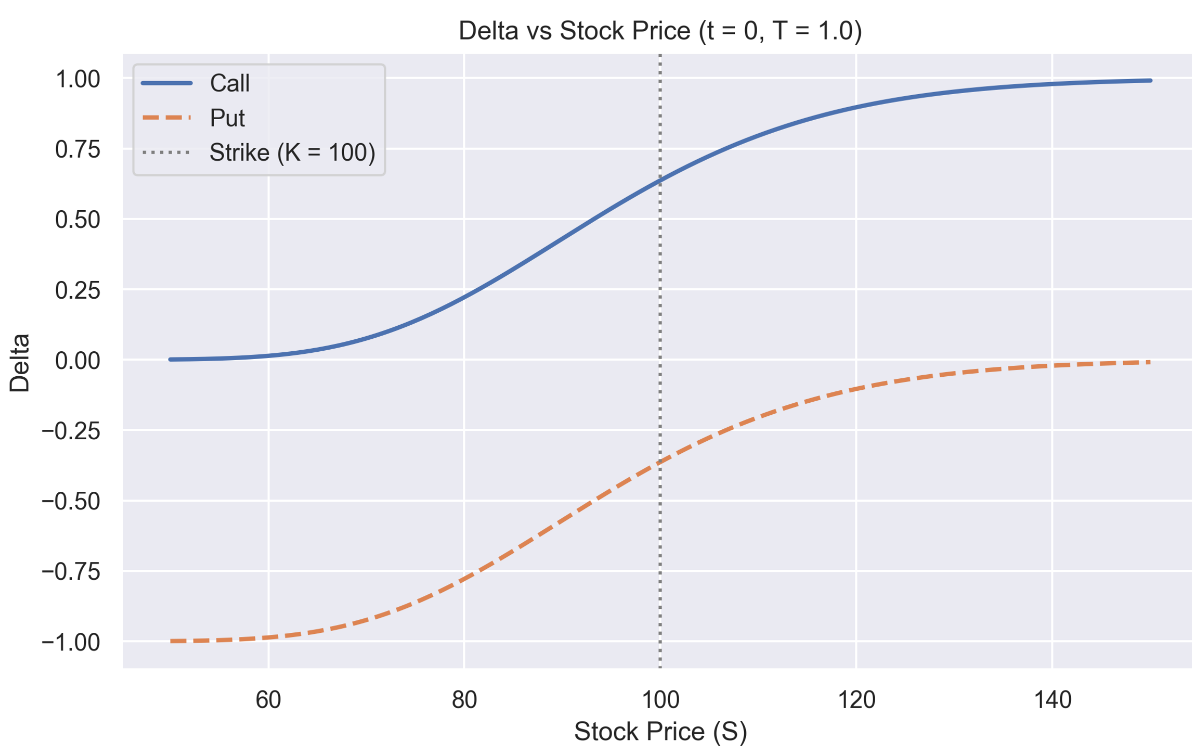


Figure 1: Delta vs. stock price

Intuitively:

- When the stock price is close to zero, the call option is deeply out-of-the-money. The chance it will be exercised is very low, so the option price is insensitive to changes in the underlying — delta is near zero.
- As the stock price approaches the strike price, the probability of exercise increases, and the option becomes more sensitive — delta increases rapidly.
- Once the stock price is far above the strike, the option is almost certainly going to be exercised. The call behaves similarly to the stock itself — delta approaches 1.

The result is the characteristic "S-shaped" delta curve. A similar argument applies for puts, but the curve is mirrored and shifted due to the negative slope.

## Theta

Theta represents the rate of change of the option price with respect to time:

$$\Theta = \frac{\partial f}{\partial t}$$

It captures the "time decay" of the option — how much value is lost as expiry approaches, holding all else constant.

- **Black-Scholes expressions:**

$$\Theta(c) = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$
$$\Theta(p) = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$$

- For long options, theta is typically negative — the option loses value over time. For short positions, theta is positive.

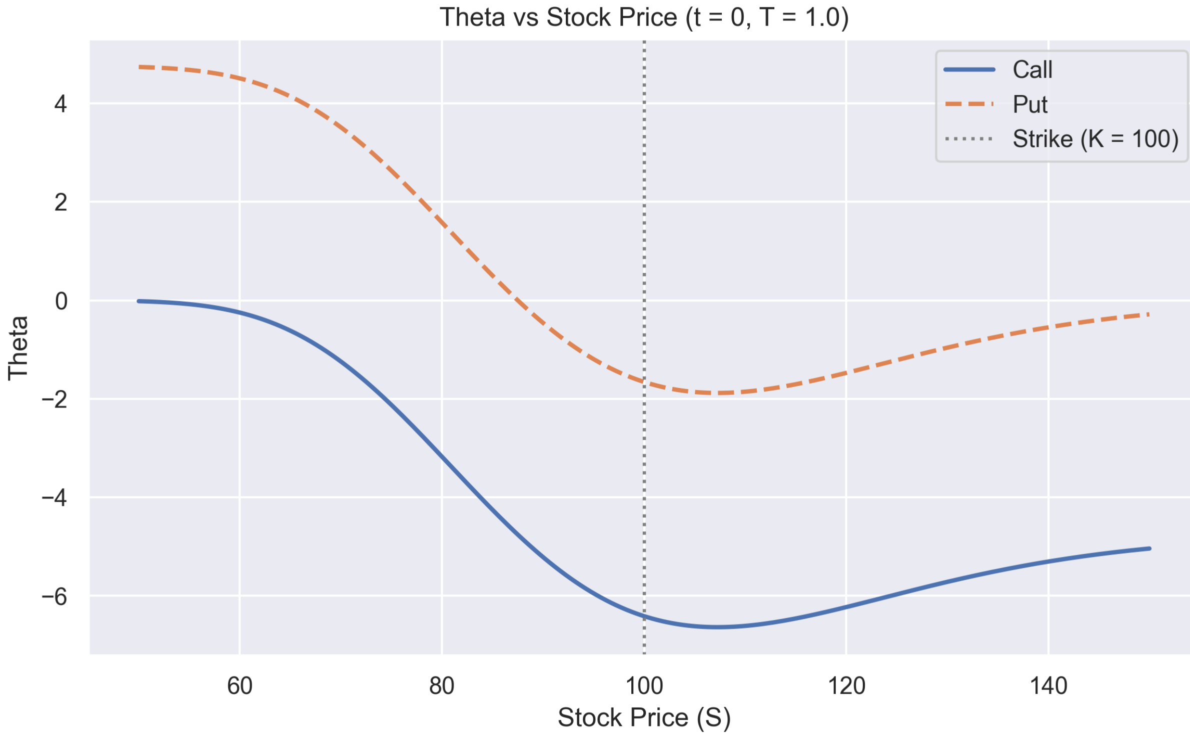


Figure 2: Tehta vs. Stock Price

## Gamma

Gamma is the rate of change of delta with respect to changes in the underlying price:

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

This is the same for both calls and puts.

- Delta hedging relies on a first-order (linear) approximation of option price movements. But option prices are convex functions of the underlying — particularly around the strike price. Gamma measures the curvature.
- **Why Gamma Matters:** When price changes are small, delta hedging works well. For larger moves, however, ignoring convexity introduces error. Gamma improves hedging accuracy by accounting for this second-order effect.
- **Gamma Hedging Example:** Suppose a portfolio is delta neutral and has gamma of $-3000$. A traded call option has delta 0.63 and gamma 1.5. To make the portfolio gamma-neutral:

$$\frac{3000}{1.5} = 2000 \text{ call options (long)}$$

This changes the delta of the portfolio by:

$$2000 \cdot 0.63 = 1260$$

To preserve delta neutrality, we must sell 1260 units of the underlying.

- **Gamma Hedging and the Underlying:** Since the underlying asset has zero gamma (it is a linear instrument), we cannot use it to hedge gamma risk. Instead, we use other options on the same underlying to construct gamma-neutral portfolios.

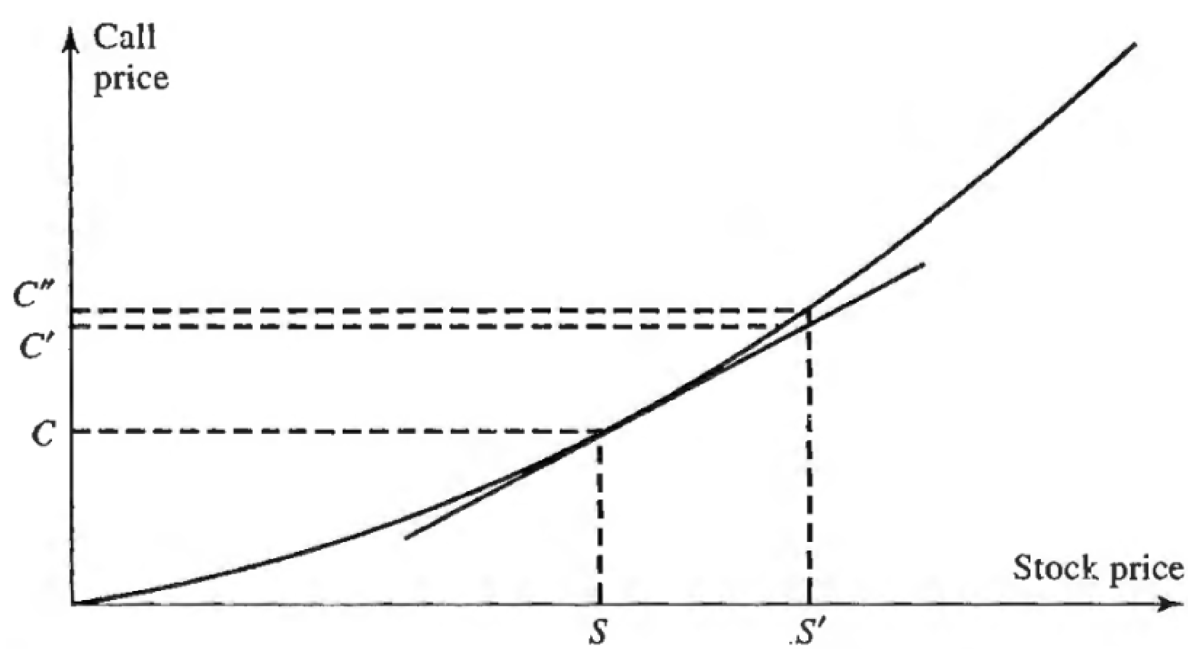


Figure 3: Gamma and curvature of option price
From J. Hull - options, futures, and other derivatives)

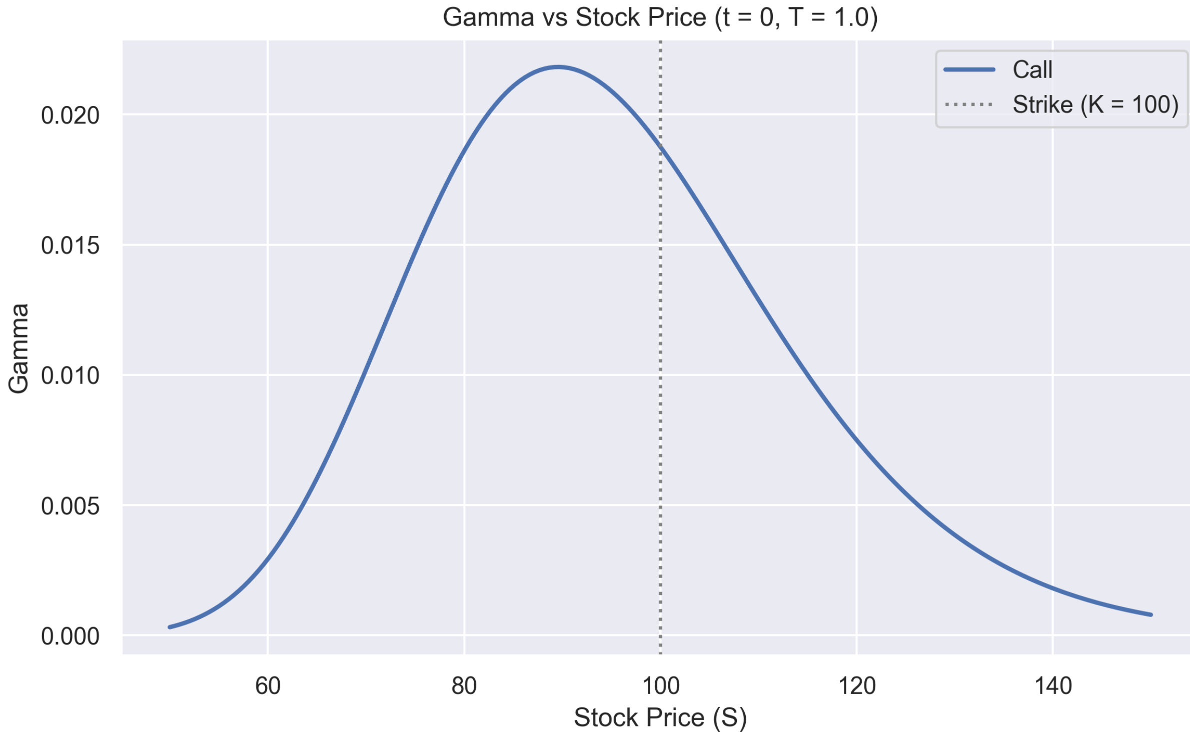


Figure 4: Gamma vs. Stock Price

## Vega

Vega measures the sensitivity of the option price to changes in the volatility of the underlying asset:

$$\nu = \frac{\partial f}{\partial \sigma} = S\sqrt{T-t}N'(d_1)$$

- **Vega Hedging:** Vega exposure can be neutralized by adding a position in a traded option. If a portfolio has vega $\nu$ and a traded option has vega $\nu_T$, then adding:

  $$-\frac{\nu}{\nu_T}$$

  units of the traded option will make the portfolio vega-neutral (locally).

- However, a portfolio that is gamma neutral is not necessarily vega neutral, and vice versa. Achieving both typically requires two or more traded options.

- **Example (from J. Hull - options, futures and other derivatives) - Vega and Gamma Neutrality:** Consider a delta-neutral portfolio with:

  $$\Gamma = -5000, \quad \nu = -8000$$

  We can trade the following options:

  - Option 1: $\Delta = 0.6$, $\Gamma = 0.5$, $\nu = 2$
  - Option 2: $\Delta = 0.5$, $\Gamma = 0.8$, $\nu = 1.2$

  Vega neutrality using Option 1 alone requires:

$$\frac{8000}{2} = 4000 \text{ long calls}$$

  This increases delta by $4000 \cdot 0.6 = 2400$, and gamma by $4000 \cdot 0.5 = 2000$. Gamma changes to $-3000$.

  To achieve both gamma and vega neutrality, we solve:

$$\begin{aligned} -5000 + 0.5w_1 + 0.8w_2 &= 0 \\ -8000 + 2w_1 + 1.2w_2 &= 0 \end{aligned}$$

  Solution:

$$w_1 = 400, \quad w_2 = 6000$$

  The resulting delta is:

$$400 \cdot 0.6 + 6000 \cdot 0.5 = 3240$$

  So, to remain delta neutral, we sell 3240 units of the underlying.

- **Vega Shape Intuition:**
  - When $S \ll K$, the option is deep out-of-the-money. Volatility has little effect — vega is near zero.
  - When $S \approx K$, the option is at-the-money. Its value is highly sensitive to volatility — vega peaks.
  - When $S \gg K$, the option is deep in-the-money. Again, volatility has little effect — vega decreases.

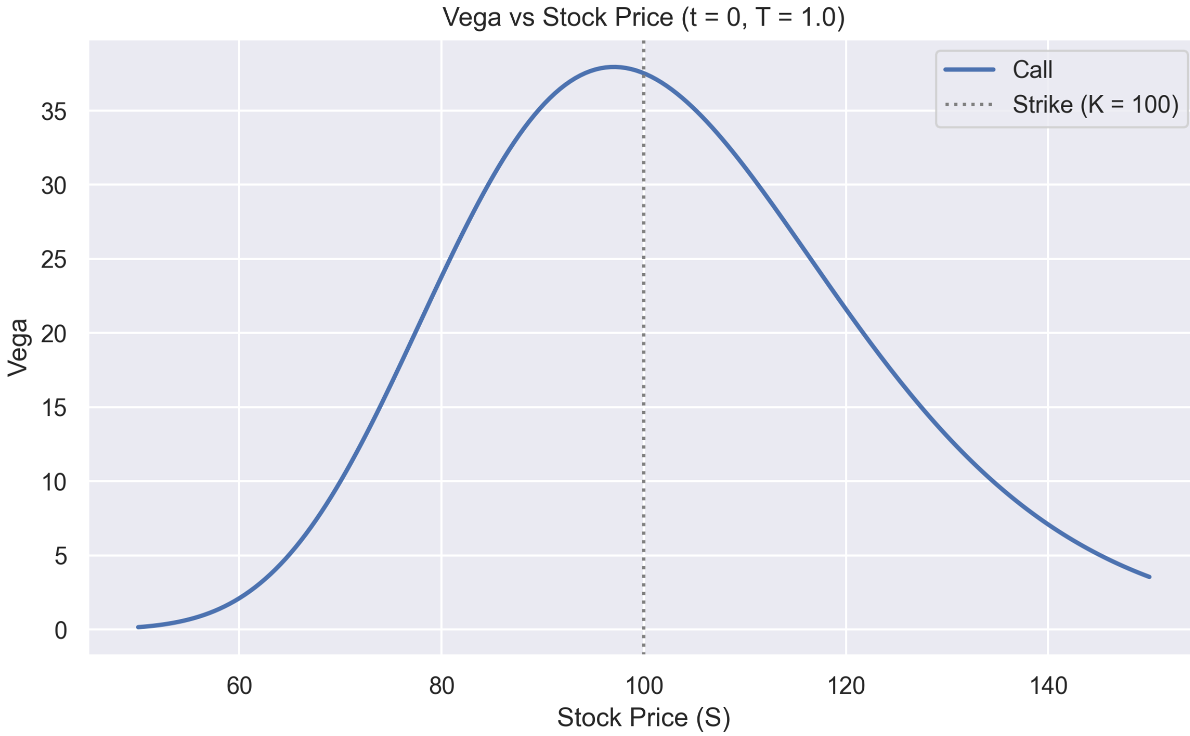


Figure 5: Vega vs. Stock Price

In summary, vega is highest when the option is at-the-money and decreases as the option becomes either deep in-the-money or out-of-the-money.

# References

[1] John C. Hull, *Options, Futures, and Other Derivatives*, 10th Edition, Pearson, 2017.