Partitioning $SS[Tot]$ in $a \times b$ design (Two-way ANOVA)

$$\boxed{Y_{ijk} \quad = \quad \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + E_{ijk}}$$
$$(i = 1, 2, \ldots, a \text{ and } j = 1, 2, \ldots, b \text{ and } k = 1, 2, \ldots, n)$$

Deviations:
total : $y_{ijk} - \overline{y}_{\ldots}$
due to level i of factor A: ______________
due to level j of factor B: ______________
due to levels i of factor A and j of factor B after subtracting main effects:

$$\bar{y}_{ij.} - \bar{y}_{\ldots} -$$

$$SS[Tot] = \sum_i \sum_j \sum_k (y_{ijk} - \qquad)^2 = \sum_i \sum_j \sum_k (\overline{y}_{ij.} - \qquad)^2 + \sum_i \sum_j \sum_k (y_{ijk} - \qquad)^2$$

$$SS[A] = \sum_i \sum_j \sum_k (\qquad)^2, \quad SS[B] = \sum_i \sum_j \sum_k (\qquad)^2$$

$$SS[AB] = \sum_i \sum_j \sum_k (\qquad)^2, SS[E] = \sum_i \sum_j \sum_k (\qquad)^2$$

ANOVA for two-factor crossed design

$$y_{ijk} = \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + y_{ijk} - \bar{y}_{ij.}$$

$$y_{ijk} - \bar{y}_{...} = (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + y_{ijk} - \bar{y}_{ij.}$$

Square both sides, sum over i, j, k, and the $\times$-products vanish.

$$\begin{aligned} SS(Tot) &= SS(Trt) + SS(\qquad\quad) \\ SS(Trt) &= SS(A) + \qquad\quad + \end{aligned}$$

Analysis of replicated two (or more) factor designs often proceed according to the following steps:

1. Check for interaction
    1. If no interaction, analyze main effects
    2. If interaction, analyze simple effects

$a \times b$ example continued

Test for interaction effect in 2×2 generalizes to $a \times b$:

$$H_0 : (\alpha\beta)_{ij} \equiv 0 \text{ vs. } H_1 : (\alpha\beta)_{ij} \neq 0 \text{ for some } i, j$$

$$F = \frac{MS[AB]}{MS[E]}$$

on $(a-1)(b-1)$ and $N - ab$ numerator, denominator df.

$$SS[AB] = n \sum_{i=1}^{3} \sum_{j=1}^{3} (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 = 0.597$$

$$F = \frac{.597/2}{0.025} = 11.96$$

which is highly significant ($p = 0.0014$) on 2,12 df.

We could proceed to test for main effects, but we won't.

Q: Why not?

A: Because effect of one factor depends on the level of the other factor, it might not make sense to talk about main effects.

If one insists on main effects, the appropriate F-ratios are

$$F_A = \frac{SS[A]/(a-1)}{MS[E]} \text{ on } a-1, N-ab \ \ df$$

$$F_B = \frac{SS[B]/(b-1)}{MS[E]} \text{ on } b-1, N-ab \ \ df$$

but the significance of the interaction effect suggests that the effect of one factor, say A, differs across levels of the other factor. A test for the main effect of A is based on the effect of A after *averaging over levels of B*. (Draw a picture.)

$a \times b$ designs

Yields on 36 tomato crops from balanced, complete, crossed design with $a = 3$ varieties (A) at $b = 4$ planting densities (B) :

Variety	Density $k/hectare$	Sample		
1	10	7.9	9.2	10.5
2	10	8.1	8.6	10.1
3	10	15.3	16.1	17.5
1	20	11.2	12.8	13.3
2	20	11.5	12.7	13.7
3	20	16.6	18.5	19.2
1	30	12.1	12.6	14.0
2	30	13.7	14.4	15.4
3	30	18.0	20.8	21.0
1	40	9.1	10.8	12.5
2	40	11.3	12.5	14.5
3	40	17.2	18.4	18.9

Statistical model?

$$Y_{ijk} =$$

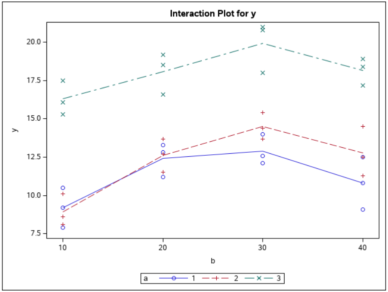
ANOVA table

```
                         The SAS System                              1
                        The GLM Procedure

                Class          Levels    Values
                a                   3    1 2 3
                b                   4    10 20 30 40

                                    Sum of
Source                    DF       Squares    Mean Square   F Value   Pr > F

Model                     11   422.3155556     38.3923232     24.22   <.0001
Error                     24    38.0400000      1.5850000
Corrected Total           35   460.3555556

Source                    DF     Type I SS    Mean Square   F Value   Pr > F
a                          2   327.5972222    163.7986111    103.34   <.0001
b                          3    86.6866667     28.8955556     18.23   <.0001
a*b                        6     8.0316667      1.3386111      0.84   0.5484
```

Level of a	N	y Mean	y Std Dev
1	12	11.3333333	1.88309867
2	12	12.2083333	2.34887142
3	12	18.1250000	1.73369023

Level of b	N	y Mean	y Std Dev
10	9	11.4777778	3.75458978
20	9	14.3888889	2.96835158
30	9	15.7777778	3.36480972
40	9	13.9111111	3.53250777

Level of a	Level of b	N	y Mean	y Std Dev
1	10	3	9.2000000	1.30000000
1	20	3	12.4333333	1.09696551
1	30	3	12.9000000	0.98488578
1	40	3	10.8000000	1.70000000
2	10	3	8.9333333	1.04083300
2	20	3	12.6333333	1.10151411
2	30	3	14.5000000	0.85440037
2	40	3	12.7666667	1.61658075
3	10	3	16.3000000	1.11355287
3	20	3	18.1000000	1.34536240
3	30	3	19.9333333	1.67729942
3	40	3	18.1666667	0.87368949



A conventional look at main effects is just to make pairwise comparisons among marginal means, after averaging over other factors. Pairwise comparisons of density means using Tukey's procedure with $\alpha = 0.05$ are given below. (Use `means b/tukey;`. to obtain the output.)

```
                         The GLM Procedure

              Tukey's Studentized Range (HSD) Test for y

 NOTE: This test controls the Type I experimentwise error rate, but it
       generally has a higher Type II error rate than REGWQ.

             Alpha                                         0.05
             Error Degrees of Freedom                        24
             Error Mean Square                            1.585
             Critical Value of Studentized Range        3.90126
             Minimum Significant Difference              1.6372


      Means with the same letter are not significantly different.


         Tukey Grouping           Mean       N     b

                      A        15.7778       9    30
                      A
                 B    A        14.3889       9    20
                 B
                 B             13.9111       9    40

                      C        11.4778       9    10
```

$q = \text{qtukey}(.95, 4, 24)$

$q = 3.9$

$HSD = 3.9\sqrt{\frac{1.58}{9}}$

A three-factor example

In a balanced, complete, crossed design, $N = 36$ ~~shrimp~~ tanks of shrimp were randomized to $abc = 12$ treatment combinations from the factors below:

A1: Temperature at 25^o C
A2: Temperature at 35^o C
B1: Density of shrimp population at 80 shrimp/40l
B2: Density of shrimp population at 160 shrimp/40l
C1: Salinity at 10 units
C2: Salinity at 25 units
C3: Salinity at 40 units

The response variable of interest is weight gain Y_{ijkl} after four weeks.

Three-way ANOVA Model:

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkl}$$

$$i = 1,2 \quad j = 1,2 \quad k = 1,2,3 \quad l = 1,2,3$$

$$E_{ijkl} \stackrel{iid}{\sim} N(0, \sigma^2)$$

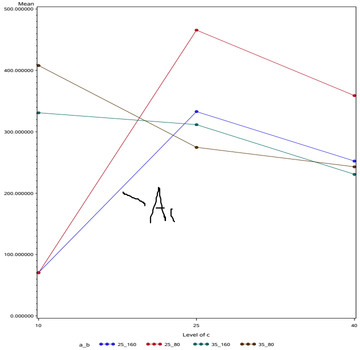
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	11	467636.3333	42512.3939	14.64	<.0001
Error	24	69690.6667	2903.7778		
Corrected Total	35	537327.0000			

sqrt(2MS(E)/3) ~= 44

Source	DF	Type I SS	Mean Square	F Value	Pr > F
a	1	15376.0000	15376.0000	5.30	0.0304
b	1	21218.7778	21218.7778	7.31	0.0124
a*b	1	8711.1111	8711.1111	3.00	0.0961
c	2	96762.5000	48381.2500	16.66	<.0001
a*c	2	300855.1667	150427.5833	51.80	<.0001
b*c	2	674.3889	337.1944	0.12	0.8909
a*b*c	2	24038.3889	12019.1944	4.14	0.0285

$\mu[A,BC]$

A_2



A_1

Level of a	Level of b	N	y Mean	y Std Dev
25	80	9	298.333333	185.106051
25	160	9	218.666667	128.739077
35	80	9	308.555556	85.475305
35	160	9	291.111111	57.953525

Level of a	Level of c	N	y Mean	y Std Dev
25	10	6	70.500000	15.109600
25	25	6	399.333333	114.206246
25	40	6	305.666667	69.987618
35	10	6	369.500000	56.450864
35	25	6	293.166667	45.375838
35	40	6	236.833333	38.096807

Level of b	Level of c	N	y Mean	y Std Dev
80	10	6	239.166667	188.065326
80	25	6	370.166667	122.218520
80	40	6	301.000000	77.415761
160	10	6	200.833333	144.240655
160	25	6	322.333333	74.529636
160	40	6	241.500000	32.788718

Level of a	Level of b	Level of c	N	y Mean	y Std Dev
25	80	10	3	70.333333	17.156146
25	80	25	3	465.666667	87.648921
25	80	40	3	359.000000	59.858166
25	160	10	3	70.666667	16.623277
25	160	25	3	333.000000	108.282039
25	160	40	3	252.333333	11.372481
35	80	10	3	408.000000	51.117512
35	80	25	3	274.666667	47.961790
35	80	40	3	243.000000	36.166283
35	160	10	3	331.000000	30.116441
35	160	25	3	311.666667	42.665365
35	160	40	3	230.666667	46.971623

Interpretation of third order interaction

Interpretation of second order interaction

$\boxed{1^{st} \text{ order}}$ interaction is between two factors

$\boxed{2^{nd} \text{ order}}$ interaction is between three factors

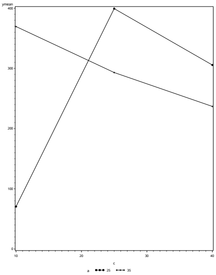
Upon inspection of the interaction plot, what do you see?
What is the primary two-factor/first-order interaction? A*C
Consider the means for low temperature (red and blue). Do you see evidence of BC interaction for temperature is low? Characterize it. yes. No effect of B when C=1
Neg eff of B when C=2 or C=3
Do you see evidence of BC interaction for temperature is high?
Not so much

If there is a BC interaction at one level of A but not the other, this is a second-order interaction.

Characterization of a three-factor interaction may not be unique. Here we first fixed A, but another analyst might first fix some other factor and characterize factorial effects in a different order.

```
%let d=divisor;   *an example of a macro variable;
data one;
   drop i;            /* a=temp, b=density, c=salinity */
   input a b c @;     * @ hold the line. prevent DATA step from loading ;
   do i=1 to 3;       *   new record when next INPUT encountered;
      input y @;      * @ hold the line;   *love isn't always on time! (Toto) ;
      y0=sqrt(y);
      output;
   end;
   cards;
   25 80 10 86 52 73
   25 80 25 544 371 482
   25 80 40 390 290 397
   25 160 10 53 73 86
   25 160 25 393 398 208
   25 160 40 249 265 243
   35 80 10 439 436 349
   35 80 25 249 245 330
   35 80 40 247 277 205
   35 160 10 324 305 364
   35 160 25 352 267 316
   35 160 40 188 223 281
;
run;

proc glimmix data=one;
   class a b c;
   model y=a|b|c;
   lsmeans a*b*c/slicediff=a*c;
run;
```

```
proc glimmix data=one;
   class a b c;
   model y=a|b|c;
   estimate "temp effect at c=1" a -1 1 a*c  -1 0 0   1 0 0;
   estimate "temp effect at c=2" a -1 1 a*c  0 -1 0   0 1 0;
   estimate "temp effect at c=3" a -1 1 a*c  0 0 -1   0 0 1;
   estimate "avg of temp effects at c=2,3" a -2 2  a*c 0 -1 -1  0 1 1/&d=2;
   estimate "mu[AC1]-.5(mu[AC2]+mu[AC3]) " a*c -2 1 1  2 -1 -1/divisor=2;
run;
```



Estimates

Label	Estimate	Standard Error	DF	t Value	Pr > |t|
temp effect at c=1	299.00	31.1115	24	9.61	<.0001
temp effect at c=2	-106.17	31.1115	24	-3.41	0.0023
temp effect at c=3	-68.8333	31.1115	24	-2.21	0.0367
avg of temp effects at c=2,3	-87.5000	21.9992	24	-3.98	0.0006
mu[AC1]-.5(mu[AC2]+mu[AC3])	386.50	38.1037	24	10.14	<.0001

We've characterized the $A \times C$ interaction. Note $SS(AC)$.