ST518, Analysis of Covariance

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November 7, 2019

Analysis of covariance, ANCOVA

Covariates are ____________ ____________. Associations between covariates z and main response variable of interest y can be used to reduce unexplained variation σ^2.

An nutrition example

Nutrition scientist conducted expt. to evaluate effects of four vitamin supplements on weight gain of lab animals. Experiment conducted in a CRD with $N = 20$ animals randomized to $a = 4$ supplement groups, each with sample size $n \equiv 5$. Response variable of interest is weight gain, but calorie intake z measured concomitantly.

Diet	$y(g)$	Diet	y	Diet	y	Diet	y
1	48	2	65	3	79	4	59
1	67	2	49	3	52	4	50
1	78	2	37	3	63	4	59
1	69	2	75	3	65	4	42
1	53	2	63	3	67	4	34
1	$\overline{y}_{1\cdot} = 63$	2	$\overline{y}_{2\cdot} = 57.8$	3	$\overline{y}_{3\cdot} = 65.2$	4	$\overline{y}_{4\cdot} = 48.8$
1	$s_1 = 12.3$	2	$s_2 = 14.9$	3	$s_3 = 9.7$	4	$s_4 = 10.9$

Q: Is there evidence of a vitamin supplement effect?

```
proc glm ;
   class diet;
   model y=diet;
run;
```

```
The GLM Procedure

Class          Levels     Values
diet                4     1 2 3 4

Dependent Variable: y

                              Sum of
Source               DF      Squares    Mean Square   F Value   Pr > F
Model                 3   797.800000     265.933333      1.82   0.1836
Error                16  2334.400000     145.900000
Corrected Total      19  3132.200000
```

But calorie intake z was measured concomitantly:

Diet	y	z	Diet	y	z	Diet	y	z	Diet	y	z
1	48	350	2	65	400	3	79	510	4	59	530
1	67	440	2	49	450	3	52	410	4	50	520
1	78	440	2	37	370	3	63	470	4	59	520
1	69	510	2	73	530	3	65	470	4	42	510
1	53	470	2	63	420	3	67	480	4	34	430

Q: How and why could these new data be incorporated into analysis?
A: ANCOVA can be used to reduce unexplained variation.
Model, given z_i,

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_z z_i + E_i \quad \text{for } i = 1, \dots, 20$$

where x_{ij} is an indicator variable for subject i receiving vitamin supplement j:

$$x_{ij} = \begin{cases} 1 & \text{subject } i \text{ receives supplement } j \\ 0 & \text{else} \end{cases}$$

and errors $E_i \overset{iid}{\sim} N(0, \sigma^2)$.

Exercise: specify the parametric mean weight gain for the first subject in each treatment group, conditional on their caloric intakes.

Exercise

$$
\begin{aligned}
E(Y_1) &= ? \\
E(Y_6) &= ? \\
E(Y_{11}) &= ? \\
E(Y_{16}) &= ?
\end{aligned}
$$

Proceeding with MLR analysis of this general linear model:

```
                              The GLM Procedure
                          Class Level Information
                      Class          Levels    Values
                      diet                4    1 2 3 4

Dependent Variable: y

                                     Sum of
 Source                     DF      Squares   Mean Square  F Value  Pr > F
 Model                       4  1951.680373    487.920093     6.20  0.0038
 Error                      15  1180.519627     78.701308
 Corrected Total            19  3132.200000

           R-Square     Coeff Var      Root MSE       y Mean
           0.623102      15.11308      8.871376     58.70000

 Source                     DF    Type I SS   Mean Square  F Value  Pr > F
 diet                        3   797.800000    265.933333     3.38  0.0463
 z                           1  1153.880373   1153.880373    14.66  0.0016

 Source                     DF  Type III SS   Mean Square  F Value  Pr > F
 diet                        3  1537.071659    512.357220     6.51  0.0049
 z                           1  1153.880373   1153.880373    14.66  0.0016
```

To test for a diet effect: $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$, use the type III F-ratio, on 3 and 15 numerator and denominator degrees of freedom. (Note that this is a comparison of nested models.)

Q: Conclusion?

FYI: model was fit with the following code:

```
proc glm;
   class diet;
   model y=diet z;
   means diet;
   lsmeans diet/stderr;
run;
```

NOTE: the drop in $\sqrt{MSE}$ (was $\hat{\sigma} \approx 12g$ is $\hat{\sigma} \approx 9g$)

Adjusted and unadjusted means

Recall the sample mean weight gains for the four diets (generated by the `means diet;` statement in `proc glm`):

The GLM Procedure

Level of diet	N	y Mean	y Std Dev	z Mean	z Std Dev
1	5	63.0000000	12.2678441	442.000000	58.9067059
2	5	57.8000000	14.8727940	434.000000	61.0737259
3	5	65.2000000	9.6540147	468.000000	36.3318042
4	5	48.8000000	10.8949530	502.000000	40.8656335

These means y are computed without taking z into account, so they are called ______________ means.

Unadjusted means do not make any adjustment for the facts that

1. caloric intake may vary by diet (presumably by chance, not because of diet)
2. weight gain depends on caloric intake

Adjusted means

Adjusted means are estimated mean weight gains at a common reference value (sample mean, $\bar{z}$) of the covariate, z.

Here, $\bar{z} = (442 + 434 + 468 + 502)/4 = 461.5$. The adjusted means are then just

$$\begin{aligned} \bar{y}_{1,a} &= \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_z(461.5) \\ \bar{y}_{2,a} &= \hat{\beta}_0 + \hat{\beta}_2 + \hat{\beta}_z(461.5) \\ \bar{y}_{3,a} &= \hat{\beta}_0 + \hat{\beta}_3 + \hat{\beta}_z(461.5) \\ \bar{y}_{4,a} &= \hat{\beta}_0 + \hat{\beta}_z(461.5) \end{aligned}$$

"means adjusted to the average calorie intake, $\overline{z} = 461.5$"

SOLUTION option in MODEL statement of PROC GLM produces (nonuniquely estimable) parameter estimates that correspond to parameterization with diet 4 effect set to 0:

Parameter		Estimate		Standard Error	t Value	Pr > |t|
Intercept		-35.66310108	B	22.41252629	-1.59	0.1324
diet	1	24.29519136	B	6.19932022	3.92	0.0014
diet	2	20.44121688	B	6.35678835	3.22	0.0058
diet	3	22.12060844	B	5.80625371	3.81	0.0017
diet	4	0.00000000	B	.	.	.
z		0.16825319		0.04394140	3.83	0.0016

```
NOTE: The X'X matrix has been found to be singular, and a generalized
      inverse was used to solve the normal equations.  Terms whose
 estimates are followed by the letter 'B' are not uniquely estimable.
```

Substitution of $\hat{\beta}$ into the expressions for adjusted means yields

$$
\begin{aligned}
\bar{y}_{1,a} &= -35.7 + 24.3 + 0.17(461.5) = 66.3 \\
\bar{y}_{2,a} &= -35.7 + 20.4 + 0.17(461.5) = 62.4 \\
\bar{y}_{3,a} &= -35.7 + 22.1 + 0.17(461.5) = 64.1 \\
\bar{y}_{4,a} &= -35.7 + +0.17(461.5) = 42.0
\end{aligned}
$$

Standard errors of $\bar{y}_{j,a}$

Consider $\bar{y}_{2,a}$. What vector c is needed so that $c'\hat{\beta} = \bar{y}_{2,a}$?

What is the standard error of $c'\hat{\beta}$?

To get SAS to produce the adjusted means and estimated standard errors, use an `LSMEANS` statement for the factor `diet` and a `STDERR` option:

```
                          The GLM Procedure
                         Least Squares Means

                                                Standard
           diet            y LSMEAN                Error    Pr > |t|

           1             66.2809372            4.0588750      <.0001
           2             62.4269627            4.1473443      <.0001
           3             64.1063543            3.9776677      <.0001
           4             41.9857458            4.3482563      <.0001
```

Concerns:

Aside from the usual residual-based checks for model adequacy, does treatment affect the covariate? To check this, one could carry out a one-way ANOVA treating z as a response variable and check for a diet effect on the mean of z:

The GLM Procedure

Dependent Variable: z

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model (diet)	3	14095.00000	4698.33333	1.84	0.179
Error	16	40760.00000	2547.50000		
Corrected Total	19	54855.00000			

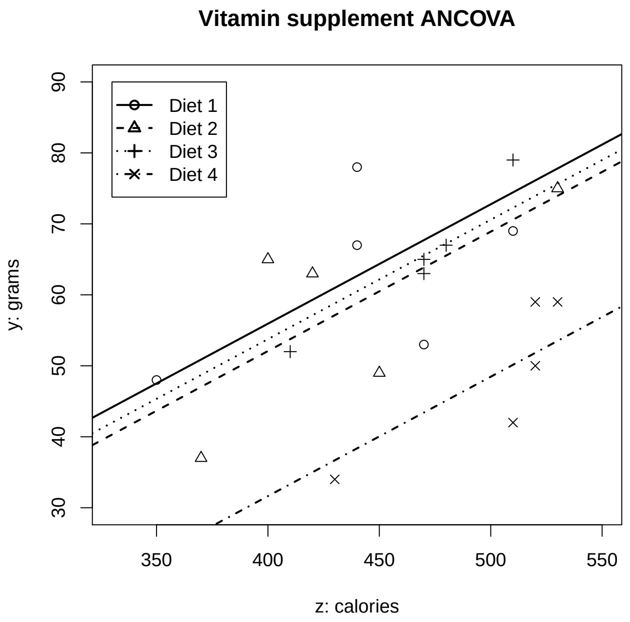
A: No evidence that treatment affects covariate.

Q: Among the diets, which we've concluded are different, what are the differences? (Look at the means, have a guess.)

Q: If you are a lab animal and you want to gain weight, which diet(s) would you choose?

Q: Why are the standard errors for the adjusted means different?

Q: Which adjusted means require the most adjustment?


Vitamin supplement ANCOVA
y: grams
z: calories
Diet 1
Diet 2
Diet 3
Diet 4
30
40
50
60
70
80
90
350
400
450
500
550

```
vitsupp.dat <- read.table("vitsupp.txt",header=TRUE)
vitsupp.dat$z <- 10*vitsupp.dat$z
pdf(file="vitsupp1.pdf")
par(cex=1.2)
attach(vitsupp.dat)
plot(z,y,pch=Diet,main="Vitamin supplement ANCOVA",xlab="z: calories",
     ylab="y: grams", xlim=c(330,550),ylim=c(30,90))
legend(330,90,legend=c("Diet 1","Diet 2","Diet 3","Diet 4"),pch=1:4,
     lty=1:4,lwd=2)

vitsupp.fit <- lm(y~as.factor(Diet)+z)
betahat <- coef(vitsupp.fit)
abline(betahat[1],betahat[5],lwd=2)
abline(sum(betahat[1:2]),betahat[5],lwd=2,lty=2)
abline(sum(betahat[c(1,3)]),betahat[5],lwd=2,lty=3)
abline(sum(betahat[c(1,4)]),betahat[5],lwd=2,lty=4)

dev.off()
```