Davide Fontana
Supervised by Dr. Chris Bouchard

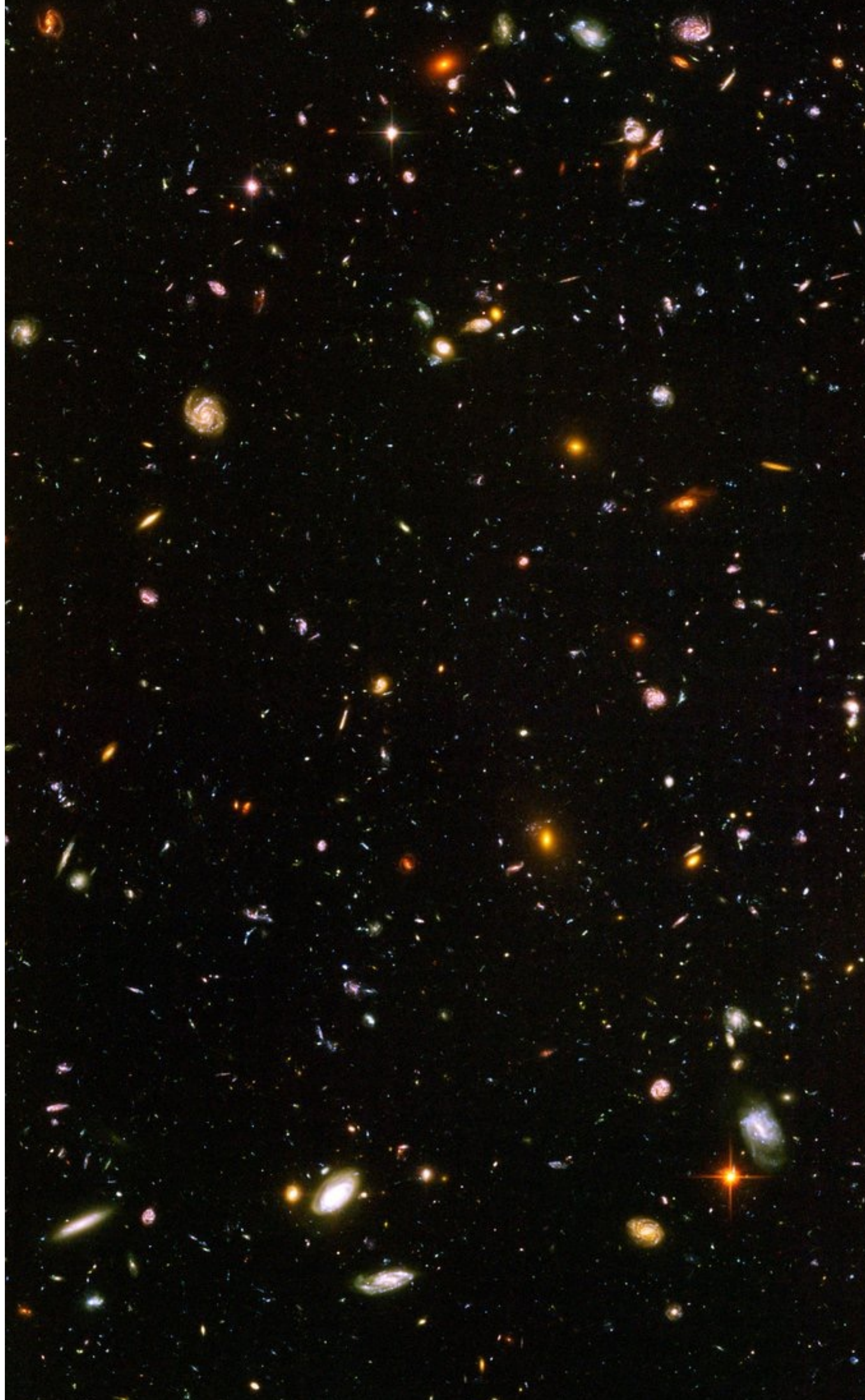
Monday, 06th March 2023

QUANTISED GRAVITY ON A LATTICE

MASTER'S PROJECT - PRESENTATION

University of Glasgow

VIA VERITAS VITA



General outline

Quantisation of gravity

- Path integral quantum gravity
- Positive action conjecture

Analytical calculations and deformation Δ^{μ}

- Derivation of the Ricci scalar curvature
- Boundary conditions

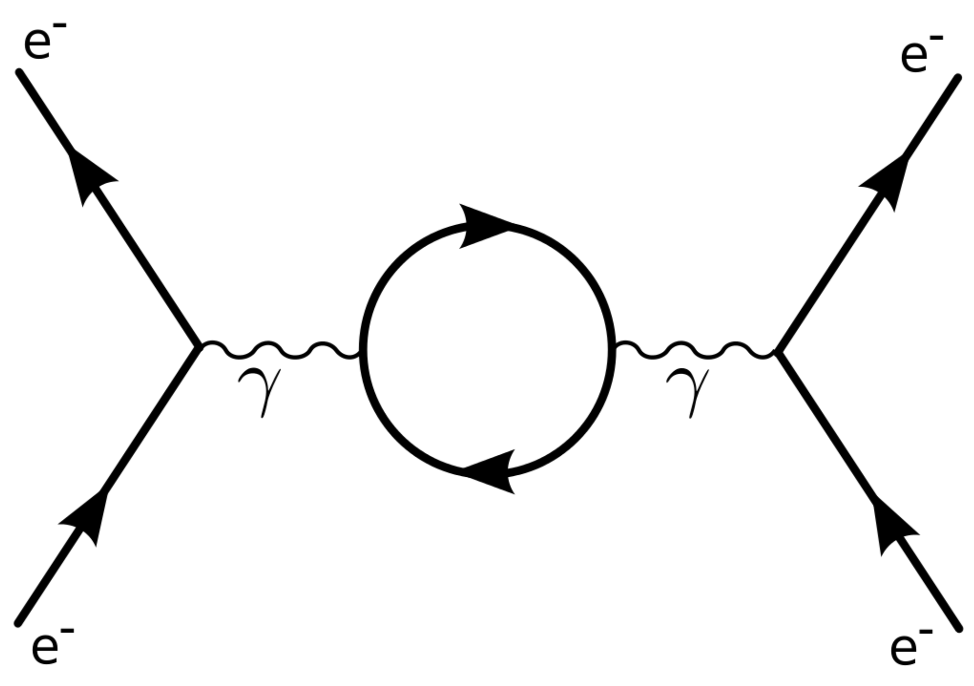
Dynamical spacetime on a lattice

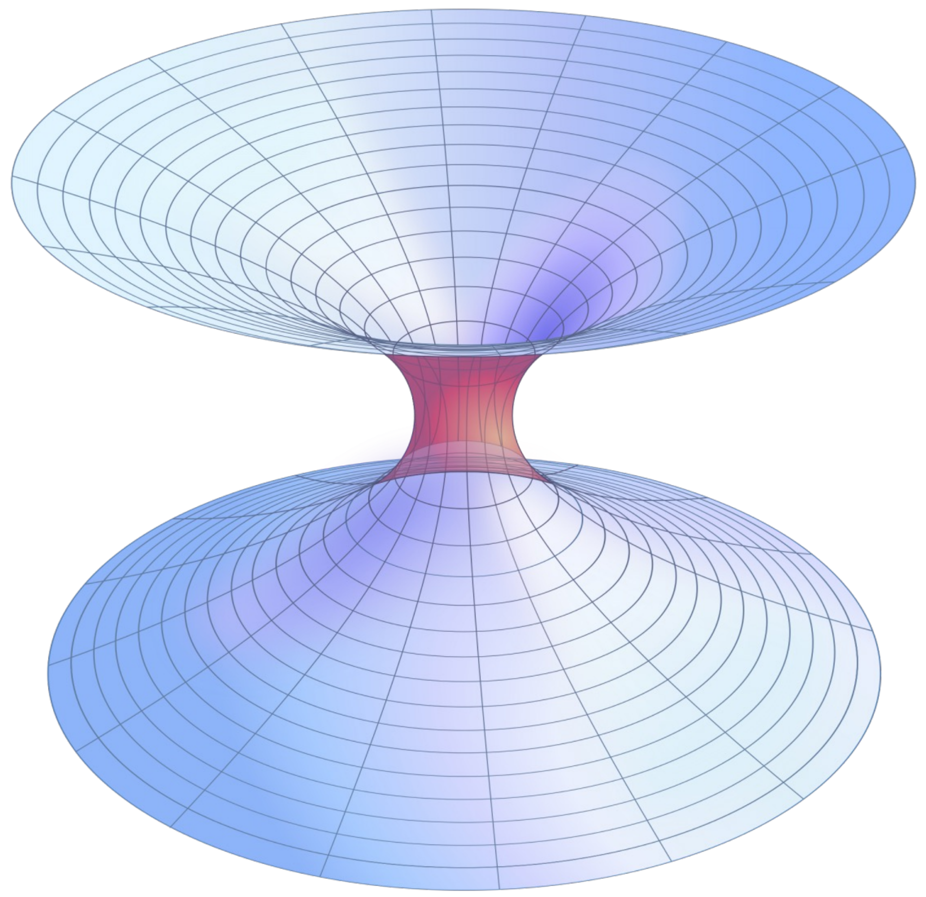
- Markov-chain of pure gravity
- Generating spacetime configurations
- Discussion

Conclusions

Motivation and background

Modern physics sits on two fundamental frameworks, quantum field theory (QFT) and general relativity (GR).





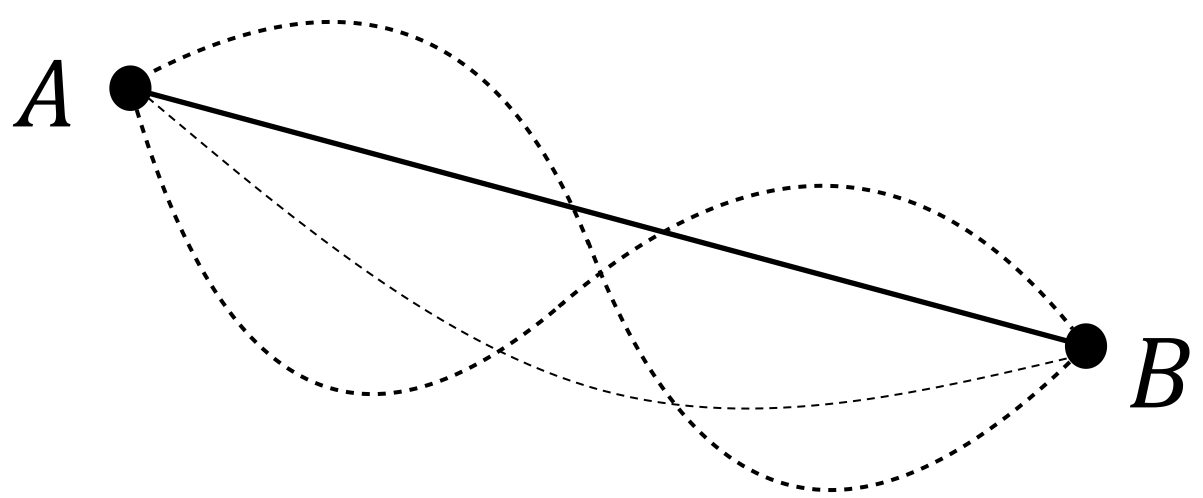
The two theories are, however, inconsistent with each other. **Quantum theory of gravity?**

In this project:

- Couple lattice quantum chromodynamics (QCD) to a dynamical spacetime avoiding no-go theorems [1] ,
- Gravity is quantised via the path integral formalism,
- This approach was proved non-renomalisable by ‘t Hooft [2] , but GR is regarded as an effective field theory.

Path integral quantization of gravity

Path integral quantization was first proposed by Richard Feynman in 1948 [3].



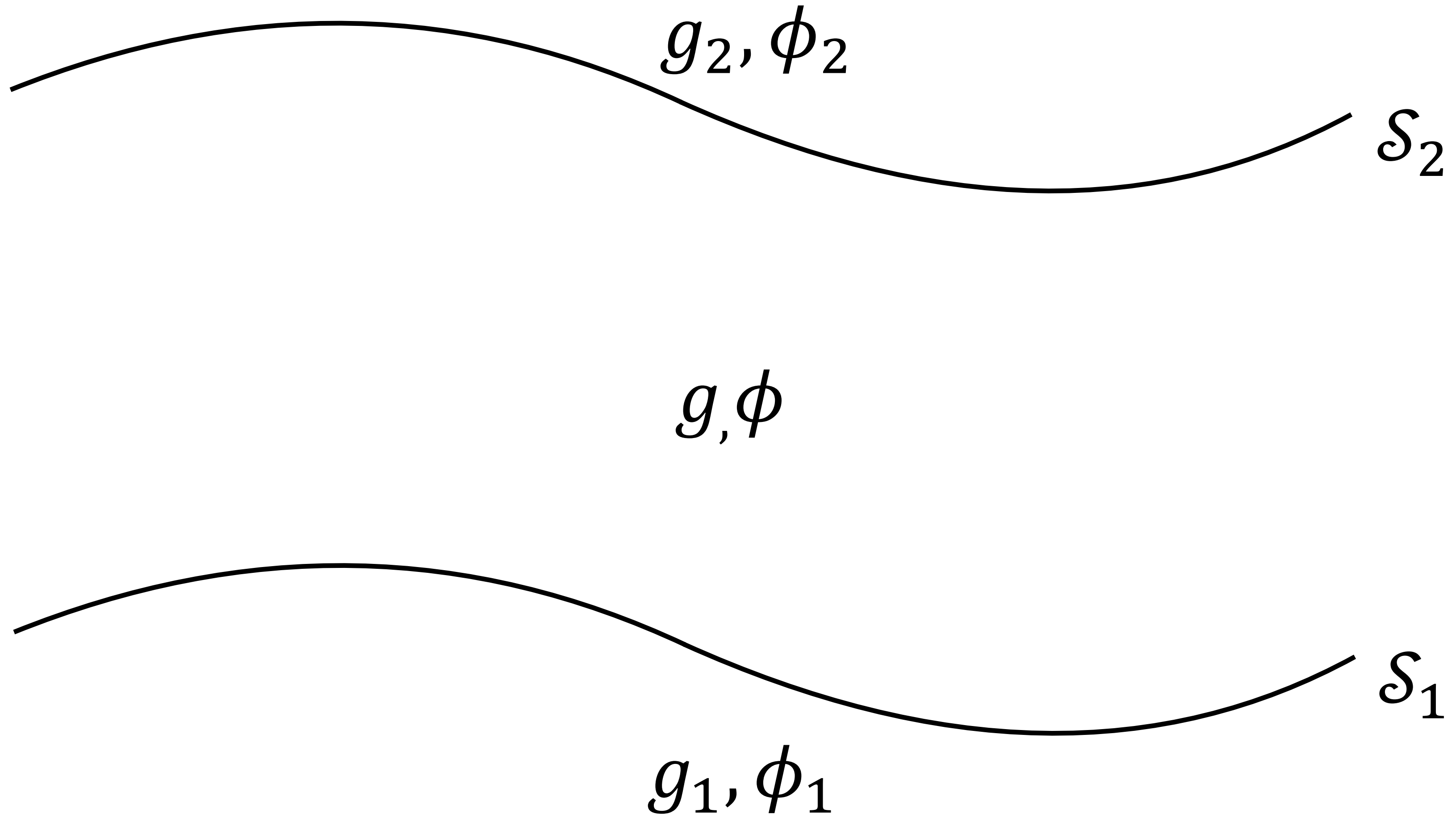
$$\mathrm{P}(A \longrightarrow \mathrm{B}) = \int f(p)\, dp\, e^{iS} \quad \text{where} \quad S = \int d^4x\mathcal{L}(p)$$

is the **action** of a system with $\{p\}$ degrees of freedom.

In 1957, Charles Misner suggested for the first time to quantize gravity within this framework [4] .

$$\langle g_2, \phi_2, \mathcal{S}_2 | g_1, \phi_1, \mathcal{S}_1 \rangle = \int D[g, \phi]\; e^{(\, iS[g,\, \phi]\,)}$$

Over all possible metrics g and fields ϕ.



The picture above has been taken from [5] .

Gravitational action and Euclidean signature $(+,+,+,+)$

For pure gravity, the action is taken to be the Einstein-Hilbert action S_{EH} ,

$$S_{EH} = \frac{1}{16\pi}\int d^4x\,\sqrt{-\det g}\;R \quad \text{with} \quad G = \hbar = c = 1\ .$$

Expressing this in flat spacetime coordinates, say x', then $\int d^4x \;\longrightarrow\; \int d^4x' = \int d^4\,x\,|\det \mathbf{J}\,|$.
This gives,

$$S_{EH} = \frac{1}{16\pi}\int d^4x'\,R\ .$$

The path integral then reads $Z = \int D[g]e^{iS_{EH}}$.

However, $Z \propto \int e^{iS_{EH}}$ will diverge.

This problem was solved by Gibbons and Hawking [6] via Wick rotation, i.e. $t \to it$.

$$\boxed{S_{EH} \to i\,\hat{S}_{EH} = -\frac{i}{16\pi}\int d^4x'\,R \quad \text{and the corresponding path integral} \quad Z = \int D[g]e^{-\hat{S}_{EH}}}$$

Positive action conjecture

In Euclidean signature, $Z = \int D[g] e^{-\hat{S}_{EH}}$ will converge $\Leftrightarrow \hat{S}_{EH} \geq 0$.

The positive action conjecture states that [7, 8] :

> The action of any asymptotically Euclidean 4-dimensional Riemannian metric must be positive, vanishing if and only if the space is flat.

In the context of this project, we considered two ways to satisfy this condition:

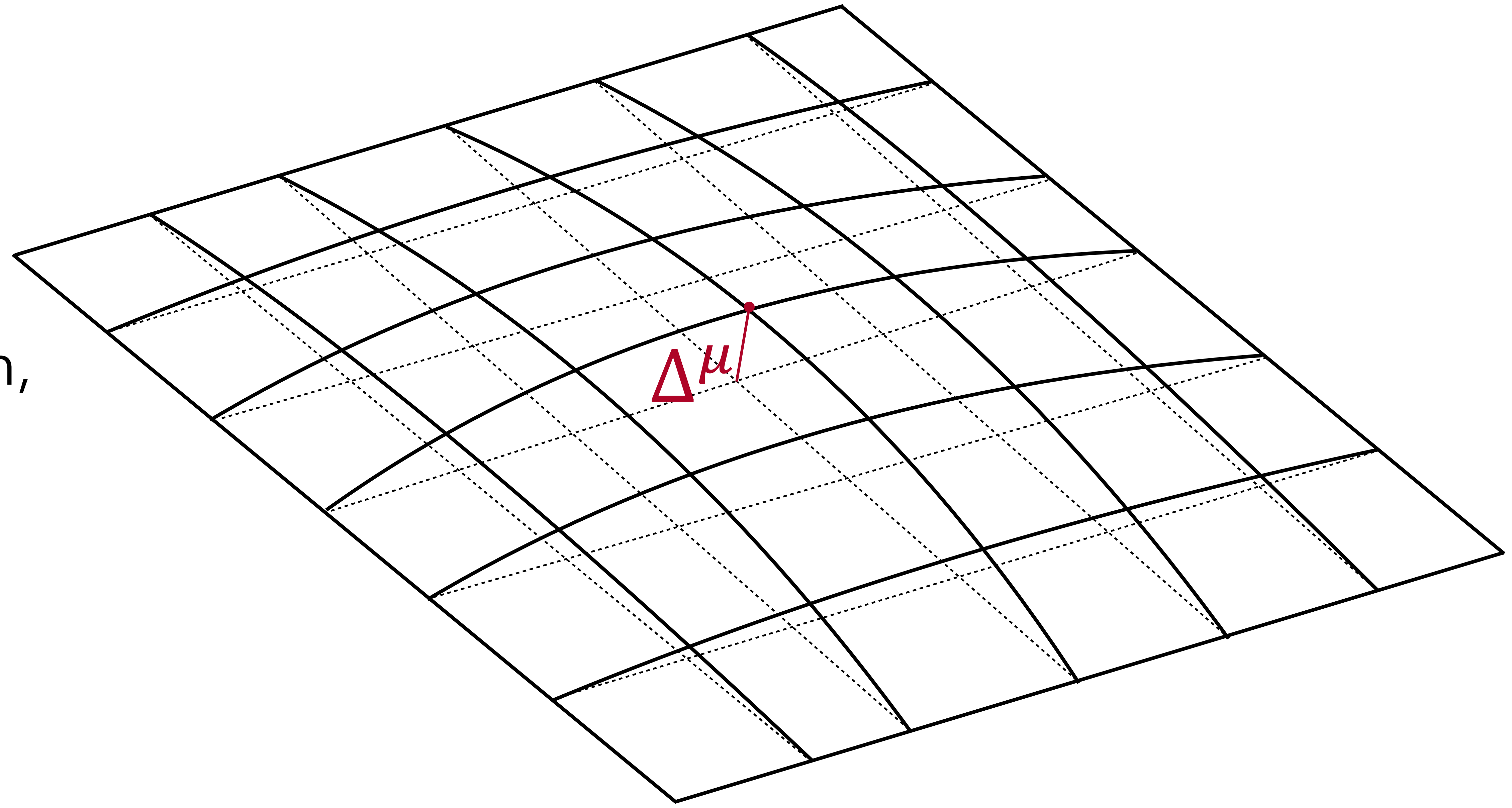
- By adding the so called Gibbons-Hawking-York boundary term to the action, such that $S_{EH} \rightarrow S_{EH} + S_{GHY}$.
- By constraining the metric at the boundaries of the lattice such that $\hat{g}_{\alpha\beta} = [g_{\alpha\beta}]_{\mathcal{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The weak-field limit

In the weak-field limit of GR, spacetime coordinates are:

$$x^{\mu} = x'^{\mu} + \Delta^{\mu}(x'^{\mu})$$

where the primed indices indicates flat spacetime.



To map $x \rightarrow x'$ the Jacobian $\mathbf{J}$ is calculated. In component form,

$$J_{\nu}^{\mu'} = \delta_{\mu'\nu'} - (\partial_{\nu'}\Delta_{\mu'}) + (\partial_{\nu'}\Delta_{\rho'})(\partial_{\rho'}\Delta_{\mu'}) + \mathcal{O}(\Delta^3)$$

from which, the metric can be derived. In fact, $\mathbf{g} = \mathbf{J}^T\eta\mathbf{J}$.
This finally gives,

$$g_{\alpha\beta} = \delta_{\alpha'\beta'} - \partial_{\alpha'}\Delta_{\beta'} - \partial_{\beta'}\Delta_{\alpha'} + (\partial_{\alpha'}\Delta_{\mu'})(\partial_{\mu'}\Delta_{\beta'}) + (\partial_{\beta'}\Delta_{\mu'})(\partial_{\mu'}\Delta_{\alpha'}) + (\partial_{\alpha'}\Delta_{\mu'})(\partial_{\beta'}\Delta_{\mu'}) + \mathcal{O}(\Delta^3)$$

which can clearly be expressed as,

$$g_{\alpha\beta} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'} + \mathcal{O}(\Delta^3).$$

Calculation of the Ricci scalar curvature

Having obtained the metric of the form $g_{\alpha\beta} = \eta_{\alpha'\beta'} + h_{\alpha\beta}$, it is possible to write

$$R \equiv g^{\alpha\beta} R_{\alpha\beta} = \partial_{\alpha\beta} h^{\alpha\beta} - \partial_{\alpha}^{\alpha} h_{\beta}^{\beta} \; ,$$

valid in the weak-field limit.

Recall that $g_{\alpha\beta}$ acts as a raising operator, for example $h^{\alpha\beta} = g^{\alpha\rho} g^{\beta\sigma} h_{\rho\sigma}$.

Finally, transforming from a flat to curved spacetime coordinates via $\partial_\alpha \to J_\alpha^{\mu'} \partial_{\mu'}$ we obtained:

$$R = J_\alpha^{\mu'} \partial_{\mu'} J_\beta^{\nu'} \partial_{\nu'} g^{\alpha\rho} g^{\beta\sigma} h_{\rho\sigma} - g^{\alpha\sigma} J_\sigma^{\nu'} \partial_{\nu'} J_\alpha^{\mu'} \partial_{\mu'} g^{\beta\rho} h_{\rho\beta}.$$

This eventually yields,

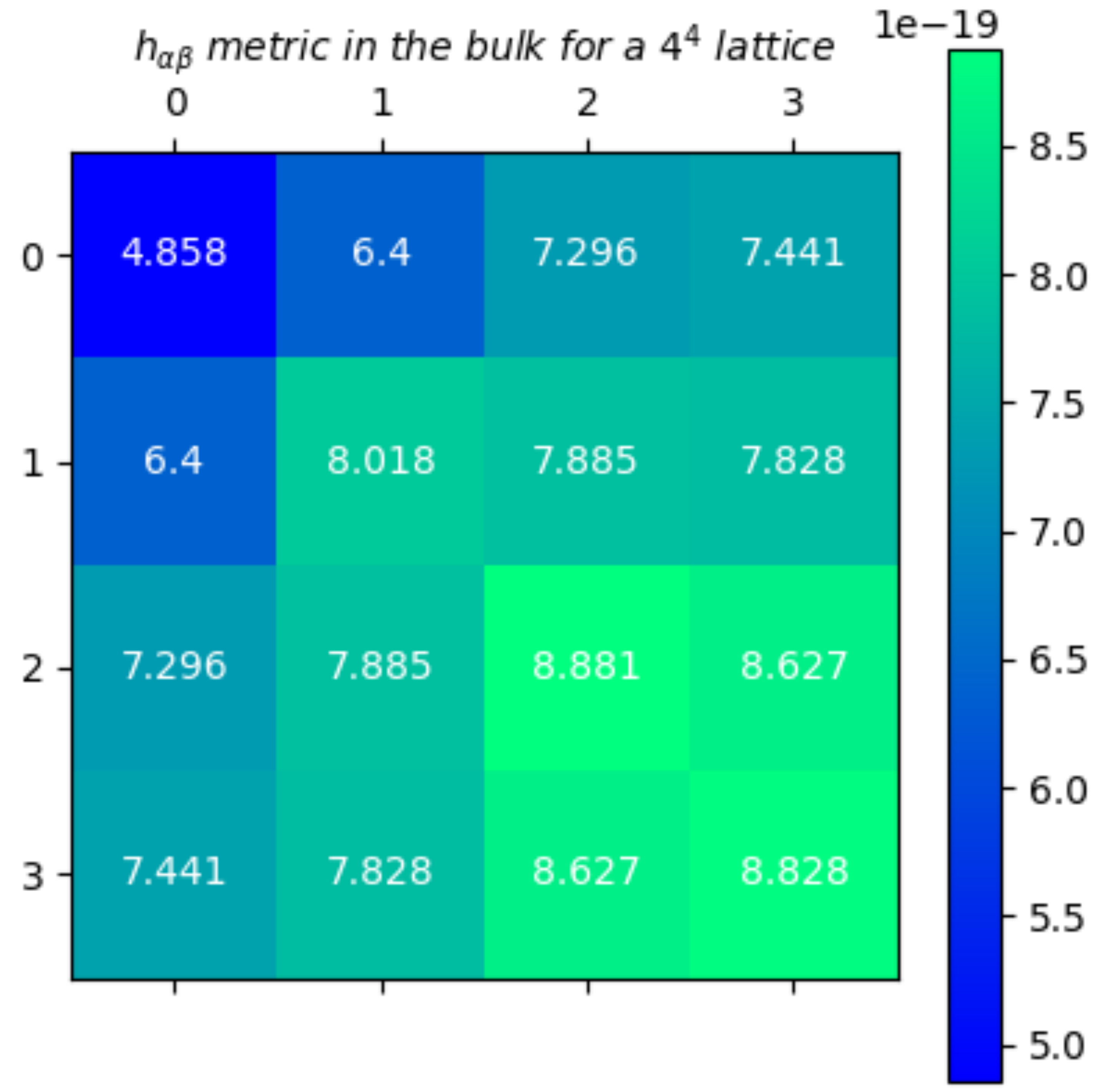
$$\begin{gathered} R = +(\partial_{\mu'\nu'}\Delta_{\rho'})(\partial_{\mu'\nu'}\Delta_{\rho'}) - 4(\partial_{\mu'\mu'}\Delta_{\nu'})(\partial_{\nu'\rho'}\Delta_{\rho'}) - 4(\partial_{\rho'}\Delta_{\nu'})(\partial_{\rho'\nu'\mu'}\Delta_{\mu'}) \\ -2(\partial_{\mu'\rho'}\Delta_{\mu'})(\partial_{\nu'\rho'}\Delta_{\nu'}) - 2(\partial_{\mu'\nu'}\Delta_{\rho'})(\partial_{\rho'\nu'}\Delta_{\mu'}) - (\partial_{\mu'\mu'}\Delta_{\nu'})(\partial_{\rho'\rho'}\Delta_{\nu'}) + \mathcal{O}(\Delta^3). \end{gathered}$$

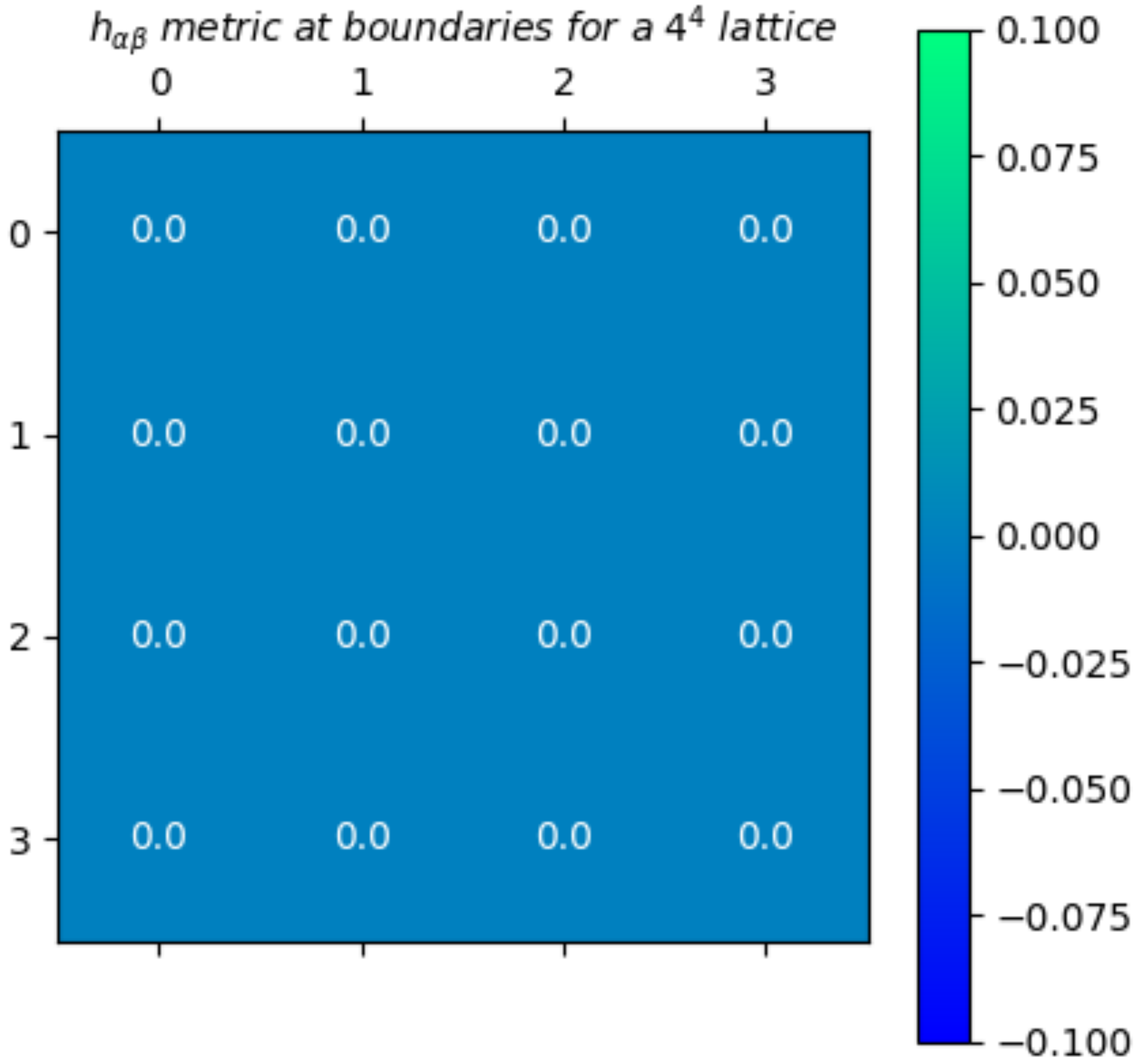
Enforcing the positive action conjecture

Equivalently, the metric asymptotes to flat at boundaries. This means that at boundaries, $g_{\alpha\beta} \to \eta_{\alpha'\beta'}$, then $h_{\alpha'\beta'} = 0$.

Consider the metric $\widehat{\boldsymbol{h}}^{(\mu)}$ approaching the boundary from μ - direction.

To first order, $\hat{h}^{(\mu)}_{a'b'} = -\partial_{a'}\Delta_{b'} - \partial_{b'}\Delta_{a'}$ where $a', b' \in \{\{0,1,2,3\} - \{\mu'\}\}$.





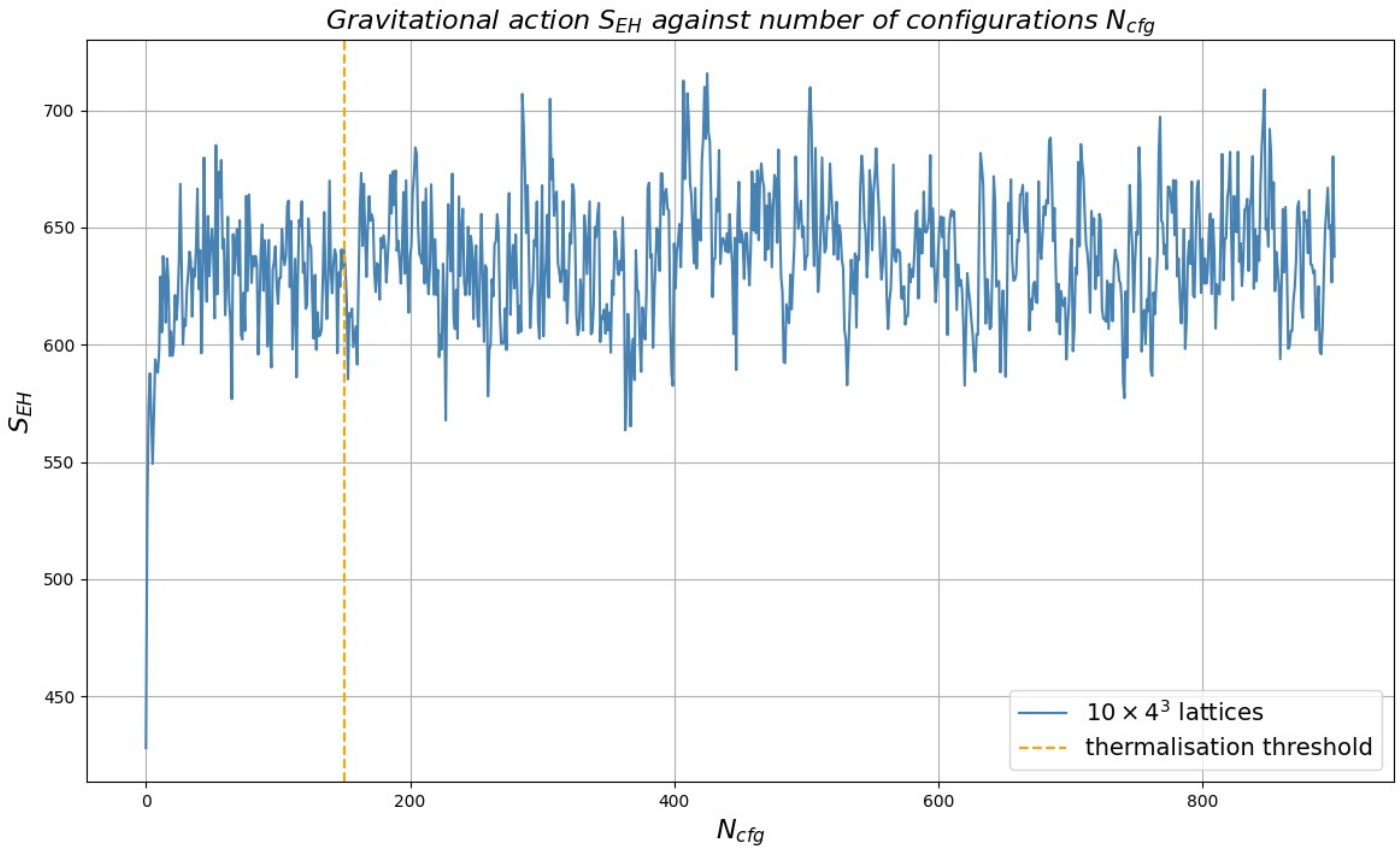
Therefore,

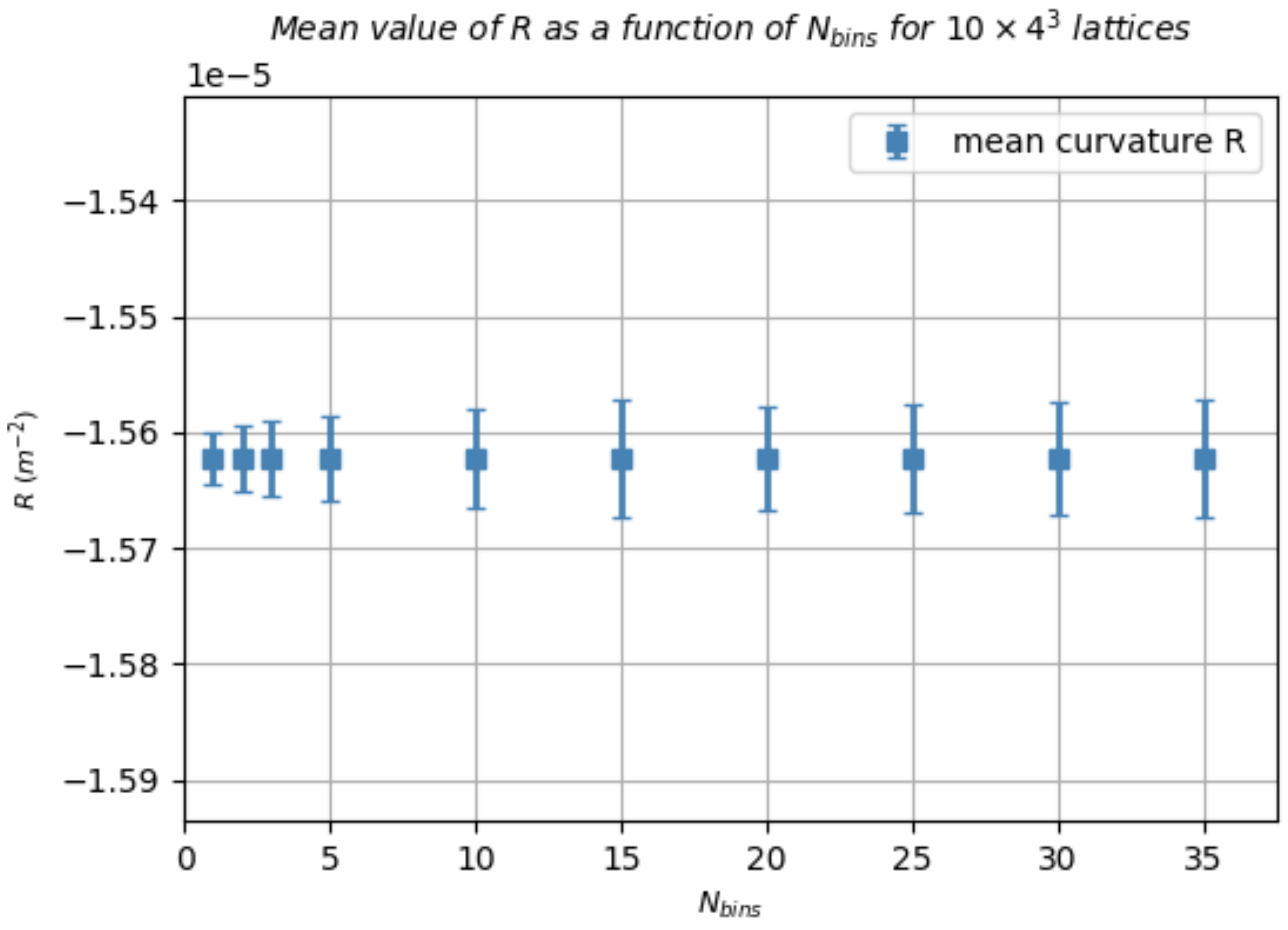
$\partial_{a'}\Delta_{a'} = 0$ and $\partial_{a'}\Delta_{b'} = -\partial_{b'}\Delta_{a'}$.

This was chosen for the simulation.

Markov-chain of curved spacetime configurations

With a Metropolis algorithm, a Markov-chain of pure gravity is generated.



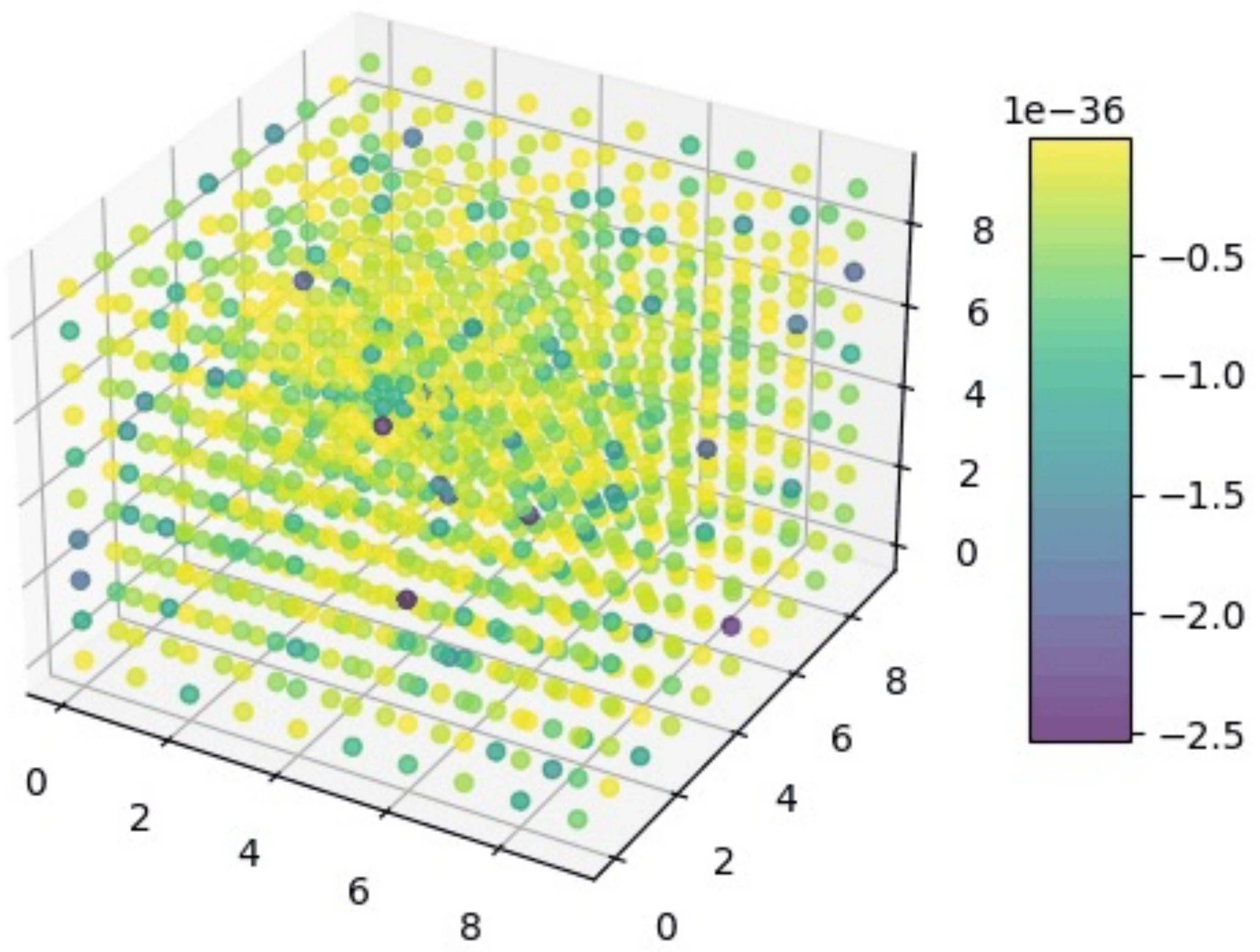


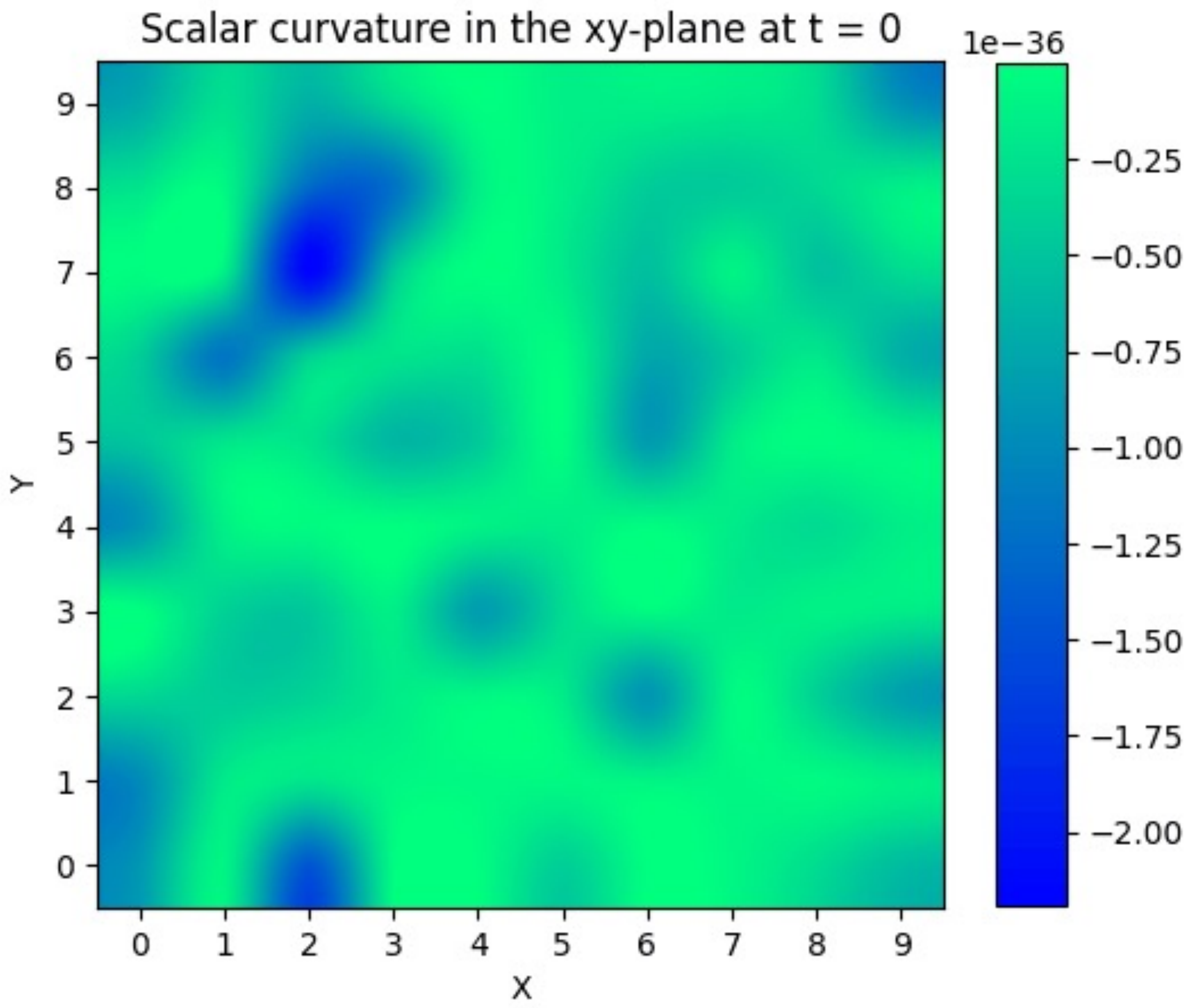
$N_{cfg} = 900,$

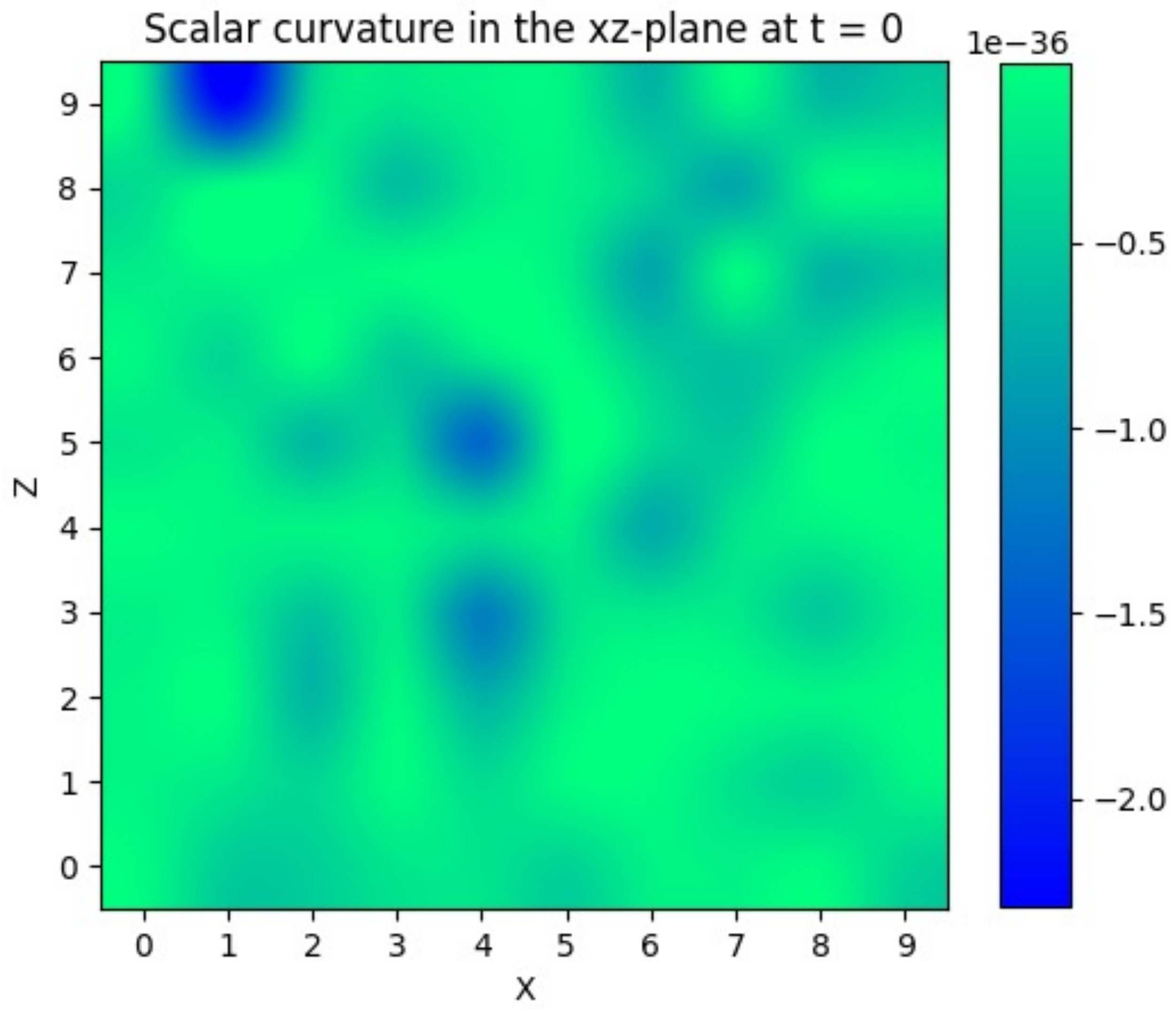
$\langle R \rangle = -1.5623(48){\times}10^{-5}\, m^{-2}$

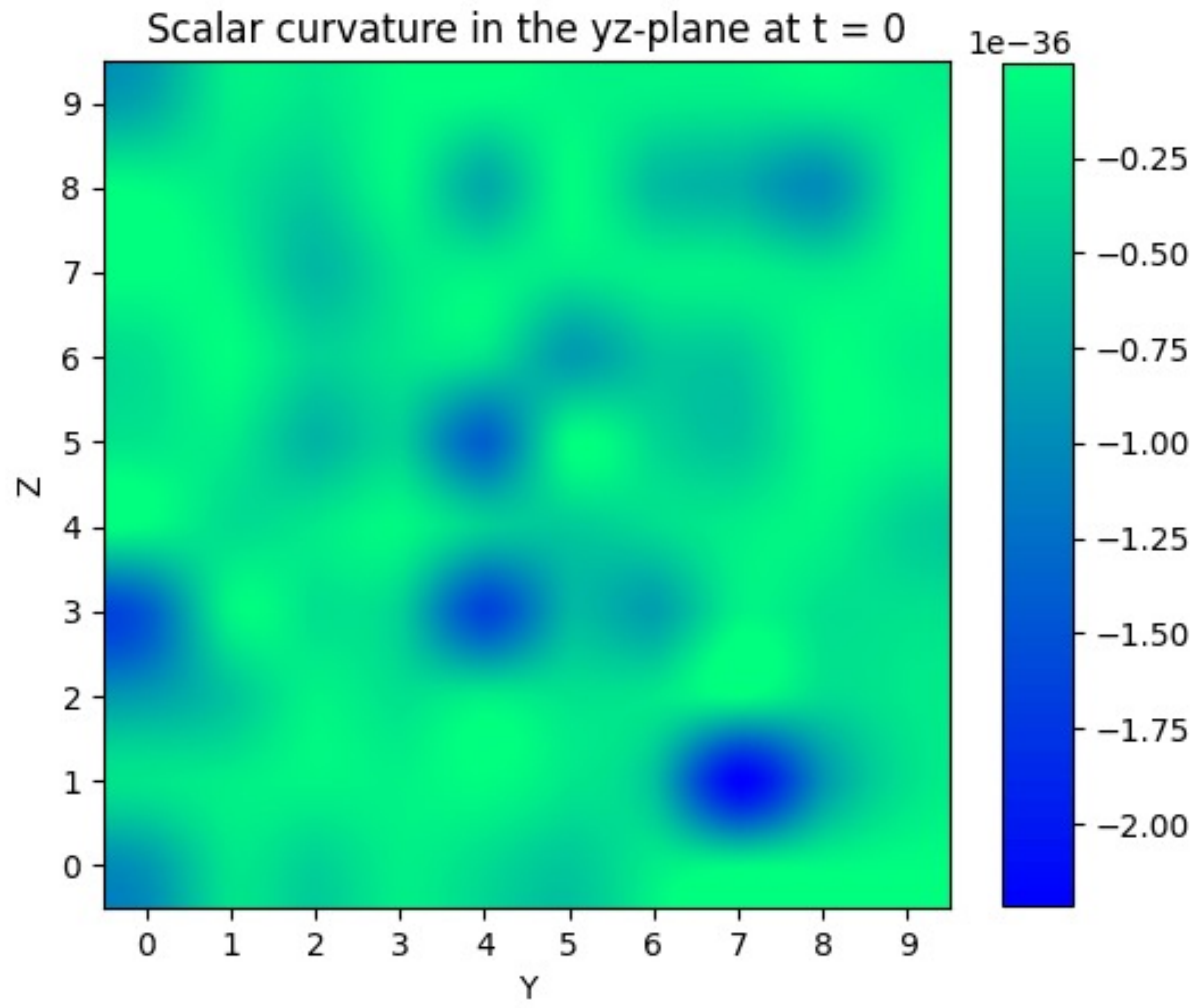
Generating dynamical spacetime on a lattice

- A dynamical spacetime can be generated on a lattice of arbitrary volume,
- Different parameters can be tuned,
- $\partial_{\mu'}\Delta_{\nu'}$ is used as the fundamental degree of freedom.



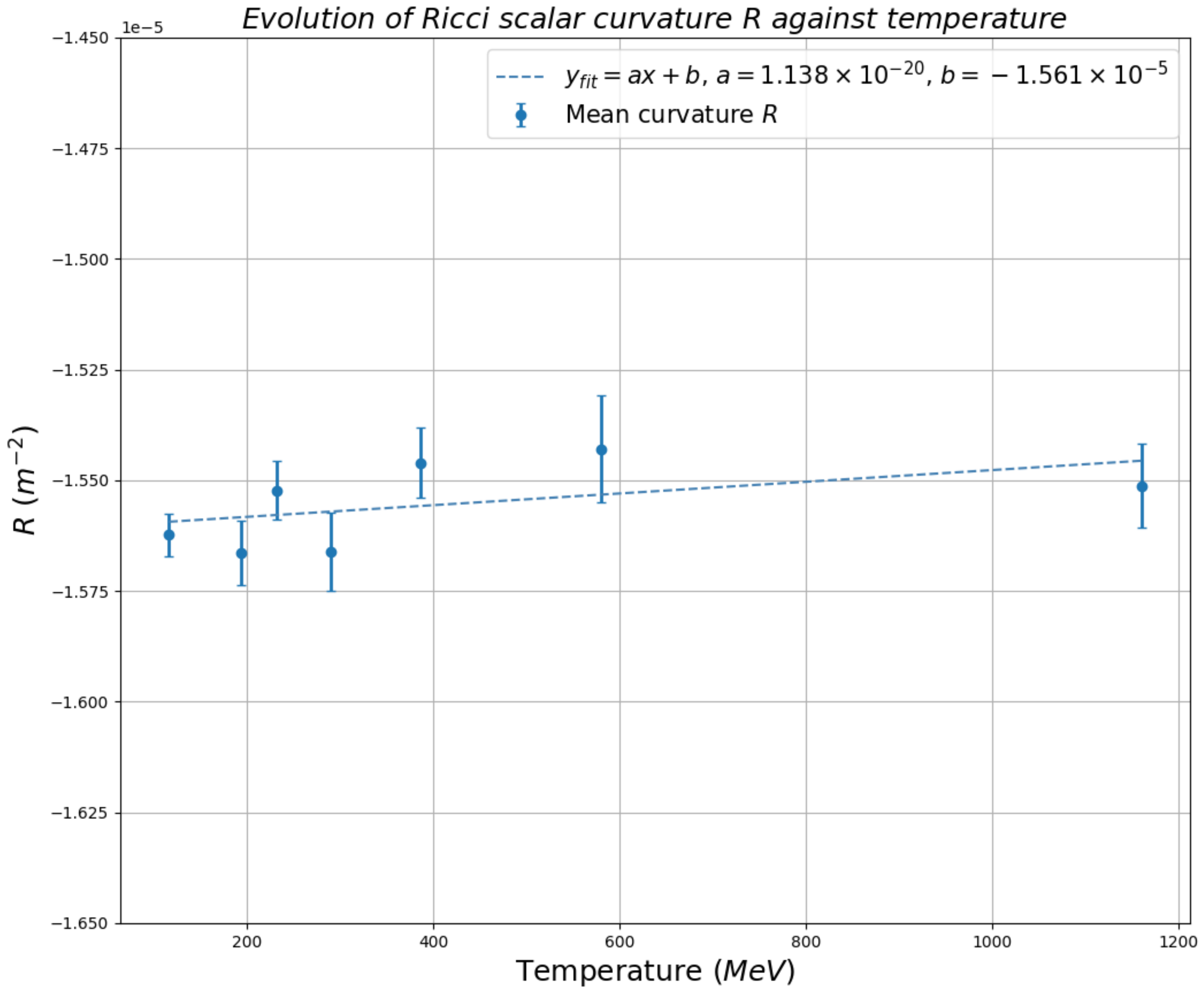


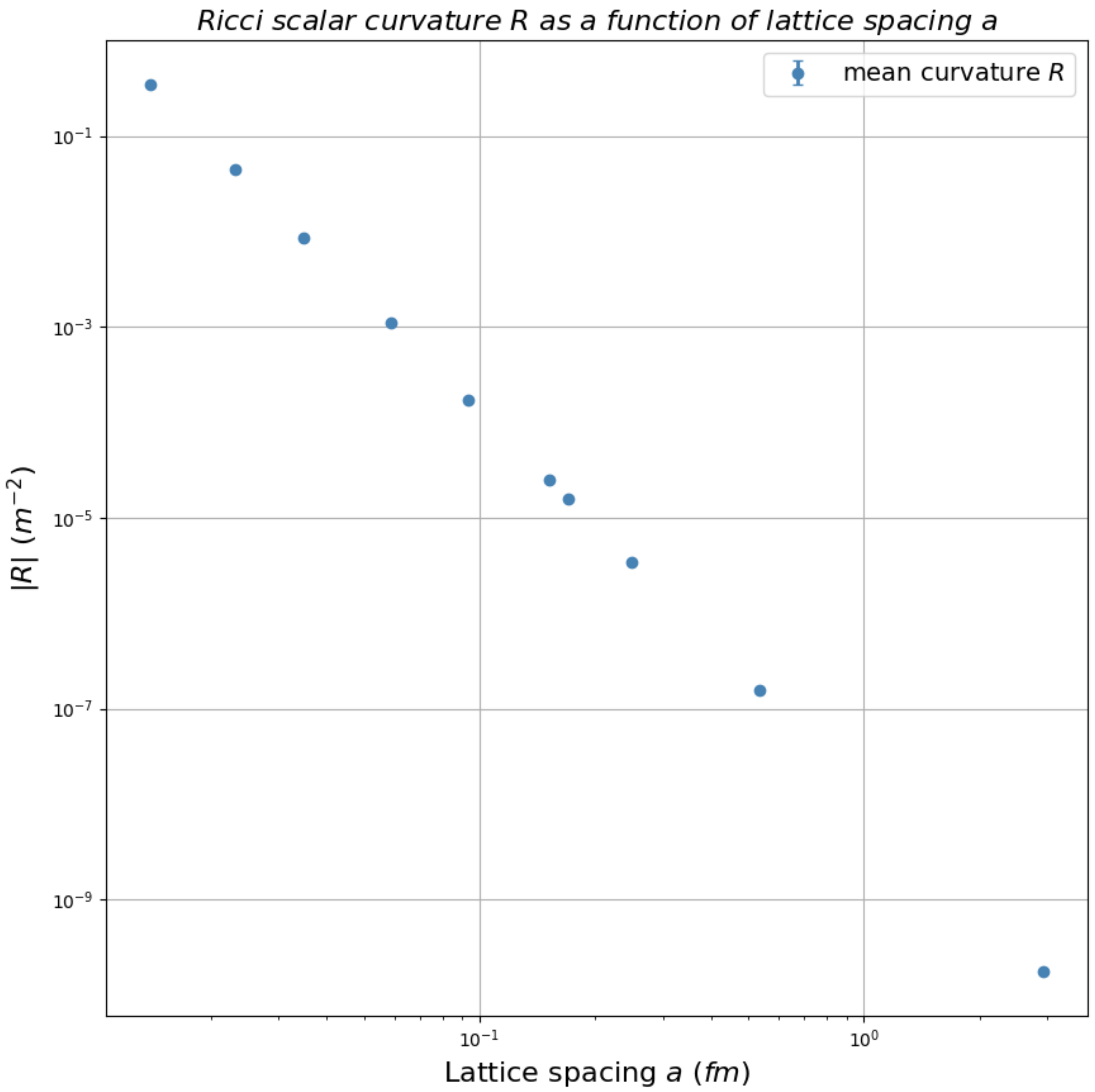




Properties of spacetime on Euclidean lattices

The behaviour of the gravitational field was studied by analysing the Ricci scalar curvature.





Conclusions

GR is regarded as an effective field theory and quantized via path integral formalism. Though the limits of this quantisation are known, dynamical spacetimes are successfully generated on a lattice.

In this project:

- Fundamental quantities were calculated as functions of $\partial_{\mu'}\Delta_{\nu'}$,
- Positive action conjecture was satisfied,
- Pure gravity lattices were analysed.

Future work includes:

- Test further the consistency of the simulation,
- Error sources are discretization effects and boundary terms,
- Coupling gravity with the quantum fluctuations of the QCD field.

Bibliography

[1] J. Oppenheim, High Energy Physics - Theory, arXiv (2018)

[2] G. 't Hooft, Ann. Inst. H. Poincare Phys. Theor. A, **20**:69-94 (1974)

[3] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948)

[4] C. W. Misner, Rev. Mod. Phys. **29**, 497-509 (1957)

[5] S. W. Hawking, Euclidean Quantum Gravity, pages 29-72 (1979)

[6] G. W. Gibbons, S. W. Hawking, Phys. Rev. D **15**, 2752-2756 (1977)

[7] G. W. Gibbons, C. N. Pope, Commun. Math. Phys. **66**, 267 (1979)

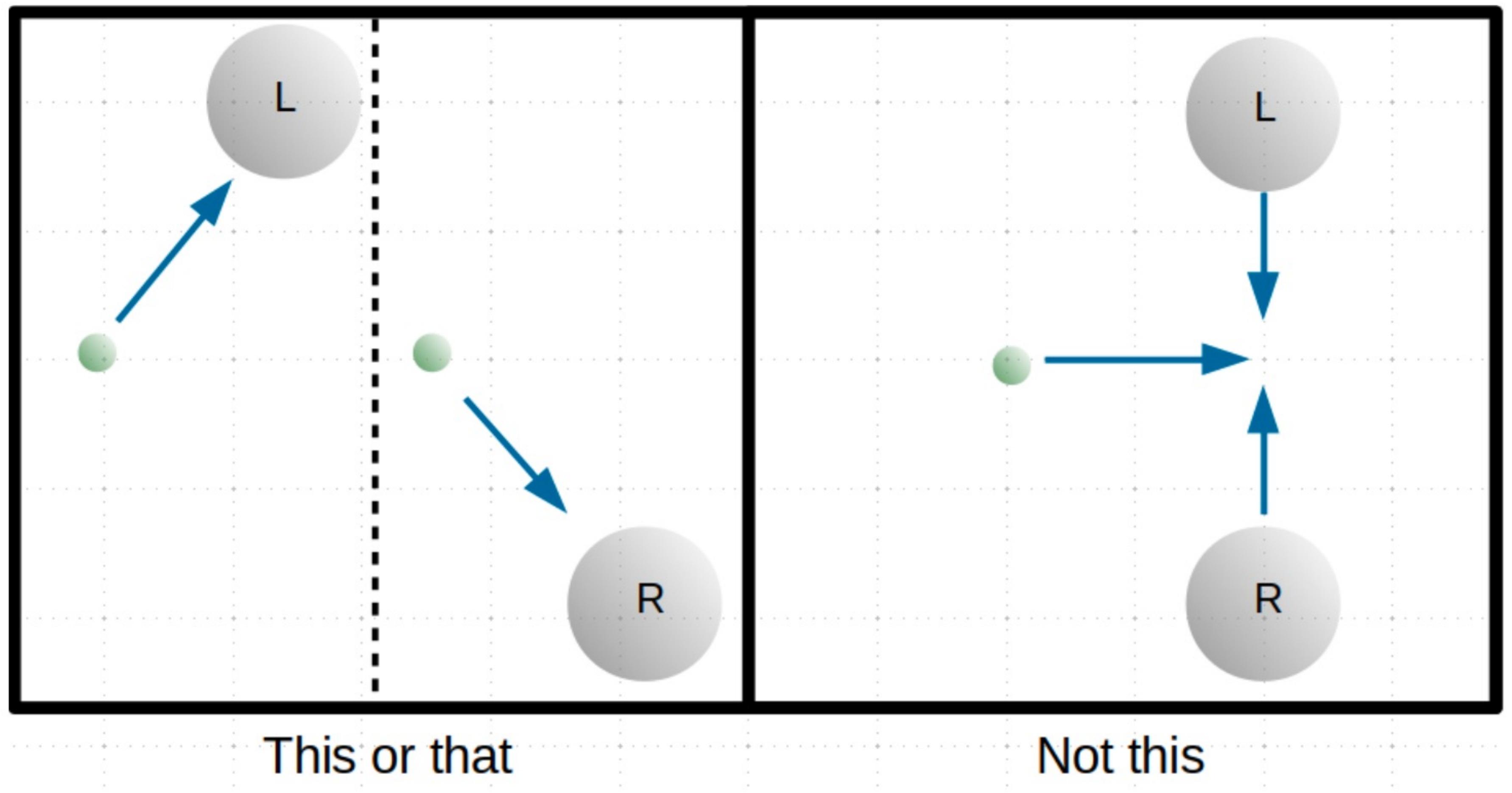
[8] E. Witten, Commun. Math. Phys. **80**, 381 (1981)

Thank you for listening.

Any question?

EXTRA: No-go theorems

This discussion on no-go theorems in the coupling between semi-classical GR and a quantum field theories follows Oppenheim's paper [1] .



The picture above has been taken from [1] .

What is the back-reaction that quantum matter produces on spacetime? Generally,

$$G^{\mu\nu} = 8\pi \ < \boldsymbol{T}^{\mu\nu} >$$

But this is only valid for small fluctuations.

Conversely, by coupling GR and QCD via the path integral, $Z = \int d^4x \, e^{i\,(S_{EH} + S_{QCD})}$ the gravitational field is sensitive to the quantum fluctuations of the QCD field.

PHILOSOPHICALLY, when GR is coupled with QCD, this formalism describes a semi-classical theory of gravity.

EXTRA: R as a function of $h_{\alpha\beta}$

In the following, a complete derivation of the equation in slide 8.
The standard definition of R in weak-field limit is (with standard GR textbook notation):

$$R = g^{\mu\nu} R_{\mu\nu} \text{ where } R_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu\ ,\nu\alpha} + h^{\alpha}_{\nu\ ,\mu\alpha} - h_{\mu\nu\ ,\alpha}^{\ \ ,\alpha} - h^{\alpha}_{\alpha\ ,\mu\nu}\right)$$

However, since the the metric can be written as $g_{\alpha\beta} = \eta_{\alpha'\beta'} + h_{\alpha\beta}$ and terms in $\mathcal{O}(h^2)$ are omitted, we obtain:

$$R = \frac{1}{2}\left[\eta^{\mu\nu}\left(\eta^{\alpha\beta} h_{\beta\mu}\right)_{,\nu\alpha} + \eta^{\mu\nu}\left(\eta^{\alpha\beta} h_{\beta\nu}\right)_{,\mu\alpha} - \eta^{\mu\nu}\eta^{\alpha\sigma}\left(h_{\mu\nu,\alpha}\right)_{\sigma} - \eta^{\mu\nu}\left(\eta^{\alpha\beta} h_{\alpha\beta}\right)_{,\mu\nu}\right]$$

Finally, considering that in Euclidean spacetime $\eta = \mathbb{I}_4$,

$$R = \frac{1}{2}\left[\left(h^{\alpha\nu}\right)_{,\nu\alpha} + \left(h^{\alpha\mu}\right)_{,\mu\alpha} - \left(h^{\mu}_{\mu}\right)^{,\alpha}_{,\alpha} - \left(h^{\alpha}_{\alpha}\right)^{,\mu}_{,\mu}\right].$$

This is straightforwardly equivalent to, $R = \partial_{\alpha\beta} h^{\alpha\beta} - \partial^{\alpha}_{\alpha} h^{\beta}_{\beta}$ as required.

EXTRA: Gibbons-Hawking-York boundary term

The Gibbons-Hawking-York boundary term S_{GHY} is defined as,

$$S_{GHY} = +\frac{1}{8\pi}\oint_{\mathcal{S}} d^3x\sqrt{\det \hat{g}}\, K = +\frac{1}{8\pi}\oint_{\mathcal{S}} d^3x' K'$$

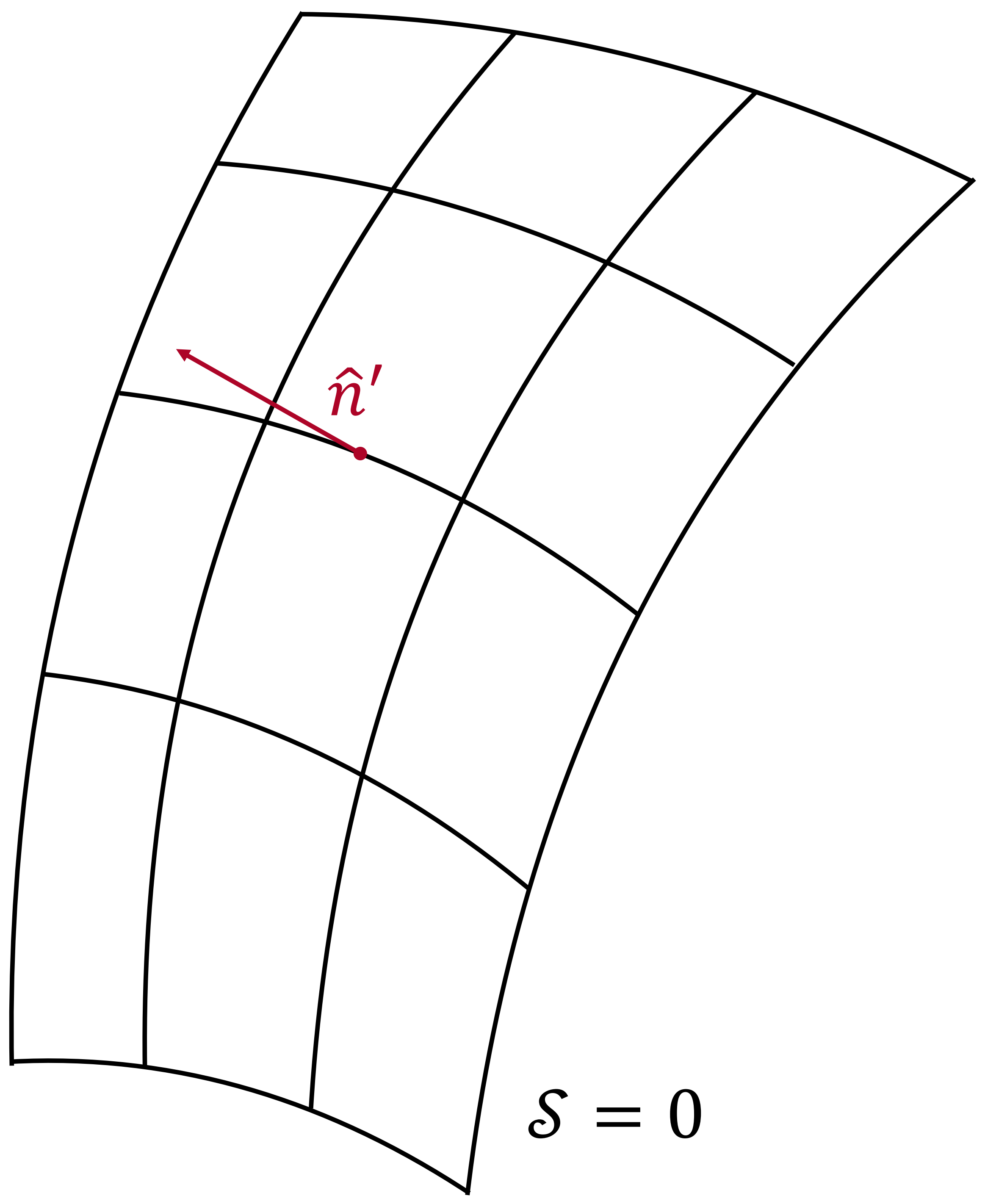
in Euclidean signature [7] .

In flat spacetime coordinates, the second fundamental trace K can be written as,

$$K' = J_{\alpha}^{\mu'}\partial_{\mu'} n^{\alpha'} + \frac{1}{2}(J_{\mu}^{\nu'}\partial_{\nu'} h_{\alpha}^{\alpha}) n^{\mu'}$$

where the normal $\hat{n}'$ to the hypersurface $\mathcal{S}$.

The hypersurface $\mathcal{S} = x^0 x^1 x^2 x^3 = 0$ reflects the geometry of the lattice. At $x^{\mu} = 0$,



$$K'|_{x^{\mu}=0} = \partial_{\nu'\nu'}\Delta_{\mu'} - \partial_{\mu'\nu'}\Delta_{\nu'} - \frac{1}{x^{a'}}\partial_{a'}\Delta_{\mu'} - \frac{1}{x^{b'}}\partial_{\mu'}\Delta_{b'} + \mathcal{O}(\Delta^2)$$

with μ', $\nu' \in \{0,1,2,3\}$ and a', $b' \in \{\{0,1,2,3\} - \{\mu'\}\}$.

EXTRA: Infinite volume limit

Some analysis was made in the infinite volume limit of the lattice.

- As the lattice increases in volume, the number of proposed non-allowed curvatures decays exponentially.
- As we extrapolate to the infinite volume limit, the positive action conjecture seems to be satisfied for all the proposed spacetime curvatures.