Quantised gravity on Euclidean lattices

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Abstract

In this paper, a scheme to quantise gravity on a lattice of finite volume is constructed. Through the path integral approach, linearised weak-field general relativity is quantised, following Hawking's formulation. Within this framework, the Einstein-Hilbert action must obey the positive action conjecture, which ensures the convergence of the path integral. As a consequence, the Euclidean metric is required to asymptote to flat at the boundaries of the lattice. This quantisation of gravity has been proven to be non-renormalisable, but it should be a well-behaved effective theory at scales of order ~ 1 fm. Hence, the theory is discretised via a first-order forward finite difference method, and dynamical spacetimes are successfully generated on lattice configurations through a Metropolis update algorithm. This work constitutes the first important step towards the construction of a scheme that enables the coupling between the gravitational field and the vacuum state of quantum chromodynamics.

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1 Introduction

More than a century has passed since Einstein first published his work on General Relativity [1], replacing the Newtonian idea of a fixed background space with the dynamical geometry of spacetime. To take this seriously is to refuse the idea of a background altogether, and to accept the notion of matter fields interacting with spacetime, which is itself a field: the gravitational field [2]. Yet, spacetime differs from matter in that it seems to only make sense in the classical, continuous limit: the efforts to quantise it have so far failed. Thus, the formulation of a quantum theory of gravity remains an unsolved problem, and perhaps one of the greatest challenges of modern theoretical physics. Many have been the attempts to consistently construct one such theory, none of which has, however, conclusively proven to be right (the reader may refer to Schulz's paper for a review [3]). In the context of this work, Feynman's path integral approach, or "sum over histories", will be considered. The path integral quantisation of the gravitational field was first proposed by Charles Misner in 1957 [4] and majorly developed by Gibbons and Hawking [5, 6, 7], who realised that for the evaluation of the integral, it is convenient to Wick rotate time to the imaginary line and work in Euclidean signature. However, this approach has been proven to have non-renormalisable divergences at the first loop order when coupled with a matter field [8], and at the second loop order in the pure gravity case [9]; these divergences in the high-energy limit rule it out as a UV complete theory of quantum gravity. Nonetheless, at low energies, path integral general relativity can be regarded as a perfectly valid effective field theory.

In this work, weak-field linearised general relativity is quantised following the prescriptions of the path integral formalism. Thus, a Markov chain of discrete, curved spacetimes is generated on Euclidean hypercubic lattices via a Metropolis update algorithm. At this stage, only the gravitational field is implemented on the lattice and the behaviour of the resulting dynamical spacetime is studied. However, the motivation behind the project is to couple this effective field theory for gravity with quantum chromodynamics (QCD), whose lattice field implementation in Euclidean signature is well-understood and widely studied [10, 11]. Within this scheme, the gravitational field is coupled to the quantum fluctuations of QCD, in a fashion that avoids the issues related to the coupling of a classical and a quantum theory [12]. So far, field theories have been studied on curved backgrounds [13] but their response to a dynamical spacetime, such as the one generated in this simulation, is not well understood: this work provides an opportunity to gain insight into this issue. The interaction between QCD vacuum and the gravitational field is of particular interest, as a way to investigate the possibility that vacuum energy deforms spacetime. This would also allow us to speculate on the cosmological implications of one such interaction, specifically in relation to the cosmological constant problem and the origins of dark energy [14]. In this paper, however, the work is focused on constructing a consistent implementation for the pure gravity case, therefore momentarily ignoring the QCD field.

In Section 2, Feynman's sum over spacetime histories as a means for the quantisation of the gravitational field is discussed. In particular, as a consequence of Wick rotation, the metric space sampled by the path integral is constrained by the positive action conjecture [15], which ensures the definiteness of the integral. In Section 3, the Ricci scalar curvature is analytically derived within the linearised weak-field regime of general relativity. This allows the numerical evaluation of the path integral on the 4D lattices. As a possible way of enforcing the positive action conjecture, the Gibbons-Hawking-York boundary term of the gravitational action is also calculated. Section 4 provides a more detailed outlook on the implementation of the simulation: here, the generation of a Markov chain of lattice configurations of pure gravity is further discussed. Lattice ensembles with spacetime volume ${N_\sigma}^3 \times N_\tau$ are generated, and in Section 5, an initial study on their spacetime properties is carried out, together with an overview of the next steps in this work. Finally, Section 6 provides a summary of the main results, after which some conclusive remarks are presented.

2 Path integral quantisation of gravity

2.1 Sum over histories

The path integral formalism, or "sum over histories", was first proposed by Feynman [16, 17] as a way of quantising a classical field theory. This scheme provides an alternative but equivalent approach to canonical quantisation. The formulation is based on a well-known yet crucial notion of quantum mechanics: in a quantum system, the probability of a measurement C yielding outcome c when it follows a measurement A resulting in a can be expressed as

$$P(C = c \mid A = a) = P_{ac} = |\phi_{ac}|^2 \ , \tag{2.1}$$

where $\phi_{ac} \in \mathbb{C}$ is known as the probability amplitude. If an intermediate measurement B with possible outcomes b is considered, the probability amplitude associated with measurements A and C then becomes

$$\phi_{ac} = \sum_b \phi_{ab}\phi_{bc} \ , \tag{2.2}$$

provided that b is a non-degenerate state and given probability amplitudes ϕ_{ab}, $\phi_{bc} \in \mathbb{C}$. The physical meaning of the path integral formulation is enclosed in this equation. If a measurement B is performed and the outcome measured, then the probability P_{ac} is calculated classically as

$$P_{ac} = \sum_b P_{abc} = \sum_b P_{ab}P_{bc}. \tag{2.3}$$

However, if the measurement B is not performed, then the probability P_{ac} is computed from Eq. 2.1 and 2.2 as

$$P_{ac} = |\sum_b \phi_{ab}\phi_{bc}|^2 \ . \tag{2.4}$$

This will produce purely quantum mechanical interference terms that would not appear in a classical theory. Although very well established, these ideas provide a conceptual foundation for the path integral quantisation and, more importantly, have astounding physical implications: Eq. 2.3 and Eq. 2.4 lie at the core of the wave-particle duality of matter.
These notions can simply be extended [1] in the continuous limit to identify the probability amplitude $\phi(R)$ of finding a particle in a particular region of spacetime R. Specifically,

$$\phi(R) = \int_R \mathcal{D}[x(t)] \ \Phi[x(t)], \tag{2.5}$$

where $\mathcal{D}[x(t)]$ means that the integral is taken over the space of continuous paths $x(t)$. Hence, the value of $\Phi[x(t)]$ depends on functions $x(t)$ rather than on a variable x, as is the case in Riemann integration. The final step required for a complete picture is to find a suitable form for $\Phi[x(t)]$. In Feynman's paper [16], $\Phi[x(t)] \propto \exp\left(iS[x(t)]/\hbar\right)$ with the classical action S expressed as

$$S = \int dt \ L(x, \dot{x}) = \int d^4x \ \mathcal{L}(x, \dot{x}) \ , \tag{2.6}$$

where $L(x, \dot{x})$ is the Lagrangian and $\mathcal{L}(x, \dot{x})$ the Lagrangian density. From this, it is then possible to rewrite Eq. 2.5 such that, given some normalisation factor A,

$$\phi(R) = \frac{1}{A} \int_R \mathcal{D}[x(t)] \ \exp\left(\frac{i}{\hbar} S[x(t)]\right) . \tag{2.7}$$

The equation above is central to the path integral formalism: $|\phi(R)|^2$ is physically interpreted as the probability of a particle to be found in the region R. Furthermore, it should be noted that Planck's constant appears explicitly in Eq. 2.7, emphasizing the quantum mechanical nature of the approach. However, Planck's units shall henceforth be used, meaning that $c = G = \hbar = 1$. Einstein's summation convention is also assumed from this point forward.

2.2 Gravitational path integral

With quantum mechanics formulated as outlined in Section 2.1, it seems natural to attempt the quantisation of gravity following the prescriptions of the path integral approach. This formulation is manifestly Lorentz covariant since the action S is a Lorentz scalar and, therefore, invariant under Lorentz transformations. This was first considered by Misner [4] and subsequently developed by

[1] For the complete argument the reader shall consult Feynman's original paper, *Spacetime approach to non-relativistic quantum mechanics* [16] .

Hawking and others [7]. In general, a quantised version of gravity based on the path integral approach can be formulated as follows: the probability amplitude for a system that undergoes a transition from metric g_1 and matter fields ϕ_1 on a hypersurface $\mathcal{V}_1$ to metric g_2 and matter fields ϕ_2 on a hypersurface $\mathcal{V}_2$ is expressed as [18]

$$\langle g_2, \phi_2, \mathcal{V}_2 \mid g_1, \phi_1, \mathcal{V}_1 \rangle = \int \mathcal{D}[g, \phi] \ \exp\left(iS[g, \phi]\right) , \tag{2.8}$$

using Dirac's notation in the left-hand side of the equation, and taking $D[g, \phi]$ to be the measure of integration over all possible configurations of metric g and matter fields ϕ. It is clear that the central object of this formulation is then the probability amplitude for a spacetime history $\{g, \phi, \mathcal{V}\}$, in the case where the presence of matter fields ϕ is assumed. However, in this paper, the gravitational field alone undergoes quantum fluctuations resulting in a dynamical spacetime, thus meaning that matter fields ϕ are neglected. For pure gravity, the action is the Einstein-Hilbert action, and it reads [3]

$$S_{EH} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \ \sqrt{-\det g} R, \tag{2.9}$$

up to a boundary term. R is the Ricci scalar curvature, g the spacetime metric and the cosmological constant Λ is assumed to be zero. The integral is taken over the spacetime manifold $\mathcal{M}$ bounded by the hypersurface $\partial\mathcal{M}$. Therefore, it is possible to rewrite the path integral in Eq. 2.8 as

$$Z = \int \mathcal{D}[g] \ \exp(iS_{EH}[g]) \ . \tag{2.10}$$

This equation is central to our discussion, as it completely encodes the dynamical properties of a gravitational system through the action S_{EH}, and it samples the space of all possible metrics g for which the relevant equations of motion are satisfied. On a conceptual level, the evolution of spacetime can be described by the sum over all possible spacetime geometries defined by metrics $\{g\}$.

2.3 Euclidean signature and positive action conjecture

Path integrals are arguably the most elegant way to define the behaviour of a given quantum system, but they suffer from at least two major shortcomings. Firstly, Eq. 2.10 is in general mathematically ill-defined [19], due to the unclear meaning of the measure $\mathcal{D}[g]$. Even restricting ourselves to situations where the measure is well understood, it is clear that the oscillatory behaviour of the integrand causes the integral to diverge.

Although in recent years it has been attempted to define a real-time path integral for gravity [20, 21], the traditional approach to tackle this problem is to Wick rotate the time axis by $\pi/2$ in the complex plane, effectively making time imaginary. Thus, under the rotation $t \to -i\tau$ (where τ is denoted as the Euclidean time), the gravitational action becomes imaginary yielding [22]

$$S_{EH}[g] \to \hat{S}[g] = -iS_{EH}[g] = -\frac{1}{16\pi} \int_{\mathcal{M}} d^4x \ \sqrt{\det g} \ R, \tag{2.11}$$

up to a boundary term. This is equivalent to a mapping from Lorentzian spacetime with signature $(-, +, +, +)$ to a Euclidean spacetime with signature $(+, +, +, +)$. This procedure reduces the integrand of Eq. 2.10 to a real-valued function. The path integral is therefore written as

$$Z = \int \mathcal{D}[g] \exp\left(-\hat{S}[g]\right). \tag{2.12}$$

It is straightforward to see that the integral is now well-behaved and can be evaluated, so long as $\hat{S}[g]$ is positive. In fact, provided that $\hat{S}[g] \geq 0$ then $\exp\left(-\hat{S}[g]\right) \in [0, 1]$. Furthermore, the argument of Eq. 2.12 can be physically interpreted as a probability corresponding to the metric g: it provides a value for the likelihood of a certain geometry to occur.

This constraint on the allowed values of the action is explicitly formulated by the positive action conjecture, which was first proposed by Hawking [5] and then proven by Schoen and Yau [23], as well as Witten [24]. In general, the positive action conjecture requires that for any asymptotically Euclidean 4-dimensional Riemannian metric, the action shall be positive. Further, the action is zero if and only if the space is flat. By asymptotically Euclidean, we mean that outside the bulk of the spacetime manifold $\mathcal{M}$, given some thickness $\mathcal{E}$ normal to the hypersurface $\partial\mathcal{M}$, the manifold $\partial\mathcal{M}\otimes\mathcal{E}$ is $\mathbb{R}^4$ and the metric tends to the flat Euclidean metric [7, 15]. In Euclidean signature, this conjecture is simply an implication of the positive energy conjecture to one higher dimension [7]: $\hat{S}\geq 0$ provides a lower bound to the gravitational energy, therefore preventing spacetime from decaying into a negative energy state. Fundamentally, the conjecture poses a constraint on the allowed metrics to ensure that the Einstein field equations are recovered.

In this respect, let us consider the variation of the Einstein-Hilbert action[2] as presented in Eq. 2.11. By the principle of least action,

$$\begin{aligned}(16\pi)\delta\hat{S} &= 0\\ &= -\int_{\mathcal{M}} d^4x\ \delta\Big(g^{\mu\nu}R_{\mu\nu}\sqrt{\det g}\Big)\\ &= -\int_{\mathcal{M}} d^4x\ \Big(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\Big)\delta g^{\mu\nu}\sqrt{\det g}-\int_{\mathcal{M}} d^4x\ g^{\mu\nu}\delta R_{\mu\nu}\sqrt{\det g}\ ,\end{aligned}\tag{2.13}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the spacetime metric and $g^{\mu\nu}$ its inverse. In the third line, the first term of the right-hand side recovers the Einstein field equations for the pure gravity case with cosmological constant $\Lambda=0$. However, the second term prevents this from happening. Therefore, a counterterm will need to be added to the Einstein-Hilbert action, in order to recover the equations of motion for classical gravity. To understand what this counterterm should look like, let us investigate further the second term of Eq. 2.13. In Poisson's textbook, it is shown (here without proof) that [25]

$$\begin{aligned}g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\alpha V^\alpha\\ &= \nabla_\alpha\Big(g^{\mu\nu}\delta\Gamma^{\alpha}_{\ \mu\nu}-g^{\mu\alpha}\delta\Gamma^{\nu}_{\ \mu\nu}\Big),\end{aligned}\tag{2.14}$$

given the covariant derivative ∇_α and Christoffel symbols $\Gamma^i_{\ jk}$. Note that the action of a covariant derivative on any vector A^β is given by $\nabla_\alpha A^\beta=\partial_\alpha A^\beta+\Gamma^{\beta}_{\ \delta\alpha}A^\delta$. At this point, it is possible to rewrite the second term of the right-hand side of Eq. 2.13 as

$$\begin{aligned}\int_{\mathcal{M}} d^4x\ g^{\mu\nu}\delta R_{\mu\nu}\sqrt{\det g} &= \int_{\mathcal{M}} d^4x\ \nabla_\alpha V^\alpha\sqrt{\det g}\\ &= \epsilon\oint_{\partial\mathcal{M}} d^3x\ N_\alpha V^\alpha\sqrt{\det\hat{g}},\end{aligned}\tag{2.15}$$

for a hypersurface $\partial\mathcal{M}$ at the boundaries of the spacetime manifold $\mathcal{M}$. N_α is the unit vector normal to $\partial\mathcal{M}$, $\hat{g}$ is the induced metric at boundaries and $\epsilon=N^\alpha N_\alpha$. The contraction of the normal $\epsilon=1$ if N^α is spacelike, conversely $\epsilon=-1$ if N^α is timelike [26]. Furthermore, it should be noted that the second line of this equation is obtained by applying the divergence theorem. Therefore, the counterterm to be added to the action will have to be a boundary term. Finally, $N_\alpha V^\alpha$ can be derived: even though the calculations for this term are not explicitly shown here, the result can be plugged back into Eq. 2.15, and this into Eq. 2.13 to obtain

$$\begin{aligned}(16\pi)\delta\hat{S} &= I_1+I_2\\ &= -\int_{\mathcal{M}} d^4x\ G_{\mu\nu}\delta g^{\mu\nu}\sqrt{\det g}+\epsilon\oint_{\partial\mathcal{M}} d^3x\ \hat{g}^{\mu\nu}\sqrt{\det\hat{g}}\,N^\alpha\partial_\alpha\delta g_{\mu\nu},\end{aligned}\tag{2.16}$$

where $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. Therefore, it can be concluded that to recover the appropriate equations of motion, a boundary term $\hat{S}_B$ needs to be added to the action $\hat{S}$, that satisfies $\delta(\hat{S}_B)=-I_2$. If this condition is met, then adding this term to the action means that $\hat{S}\to\hat{S}'=\hat{S}+\hat{S}_B$ and thus

$$(16\pi)\delta(\hat{S}')=(16\pi)\delta(\hat{S}+\hat{S}_B)=I_1=0\ .\tag{2.17}$$

[2] The following relies on standard textbook material, with the exception of a sign flip in the action due to Wick rotation. In particular, I follow E. Poisson's *A relativist's toolkit: the mathematics of Black-Hole mechanics* [25].

This yields the correct form for the Einstein field equations.

Let us think back to the positive action conjecture. The core point of the argument above is that by adding a boundary term to the action, the correct equations of motion are recovered. This, in turn, means that the term $\hat{S}_B$ ensures that the positive action conjecture is satisfied at boundaries, where the induced metric asymptotes to the Euclidean flat metric. Finally, it should be noted that the statement of the positive action conjecture is not a double implication: a positive action on a given spacetime does not guarantee that said spacetime is asymptotically Euclidean. It is important to keep this in mind when implementing it in a simulation, as it will be necessary to explicitly make sure that the metric asymptotes to flat at boundaries.

3 Linearised weak-field gravity

In the previous section, the theoretical background of this work has been outlined. Henceforth, the path integral needs to be evaluated, which means to evaluate the gravitational action $\hat{S}$ in Euclidean signature and subsequently Z as expressed in Eq. 2.12. For this purpose, the weak-field regime of general relativity is assumed. To understand the motivation behind this choice, the reader shall think back to the driving motive of this research, which is to couple the gravitational field with the QCD vacuum: clearly, even assuming that the quantum fluctuations of the gluon field will produce spacetime deformations, they can be approximated to perturbations over a flat background, as prescribed by the weak-field limit. Within this domain, any point x^μ in curved spacetime can be expressed as a small deviation from a flat background such that,

$$x^\mu = x'^\mu + \Delta^\mu(x'), \tag{3.1}$$

where x' is a point in flat spacetime coordinates and the deformation Δ is a function of x'. To make the calculations analytically accessible, all the quantities will be calculated as functions of the flat spacetime coordinate system. A different and perhaps more relevant reason to do so is that spacetime dynamics in the simulation is encoded in the deformations Δ: this also justifies the following sections, where the scalar curvature R and the trace of extrinsic curvature K are derived as functions of Δ. From now on, any variable expressed with respect to the flat coordinate system will be indicated with a primed index, as shown above. It should also be noted that for any $n-$dimensional Euclidean space, one need not distinguish between subscripts and superscripts, as there is no difference in the components of a vector and its one-form counterpart. For this reason, subscripts only will be used for all indices when working in Euclidean signature.

Under the transformation $x \to x'$, the integral and measure in Eq. 2.11 become

$$\int_{\mathcal{M}'} d^4x' = \int_{\mathcal{M}} d^4x |\det J| \tag{3.2}$$

for the Jacobian J corresponding to this transformation, which will be explicitly derived below. This leads to a fortunate simplification in Eq. 2.11 from realising that $g = J^T \eta J$, where $\eta = \mathbb{1}_{4\times4}$ is the flat spacetime metric in Euclidean signature. It follows that $(\det g)^{1/2} = \det J$. Furthermore, the volume of the spacetime region $\mathcal{M}$ over which the integral is taken, is invariant under a change of coordinates [27], thus meaning that $\mathcal{M} = \mathcal{M}'$.

Finally, consider a scalar function f. Straightforwardly, $f(x) = f(x')$, as the value of the function at a point is independent of the coordinate system used to express it. If f is evaluated at spacetime point $x = P = x' + \Delta$, it is then possible to write $f(x = P) = f(x' = P - \Delta)$. Taking the function f to be the Ricci scalar curvature R and Taylor expanding it in Δ, the following is obtained:

$$R(x) = R(x') - \Delta_{\rho'}(x')\partial_{\rho'}R(x') + \mathcal{O}(\Delta^2)\ . \tag{3.3}$$

Notice that the right-hand side of this equation is a function of flat spacetime coordinates only, which is the desired result. Plugging this back into the action, Eq. 2.11 is rewritten as

$$\hat{S}(\Delta) = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x'\ [R(x') - \Delta_{\rho'}(x')\partial_{\rho'}R(x')] + \mathcal{O}(\Delta^2), \tag{3.4}$$

up to the first order in Δ. Since $\Delta \ll 1$, any term of order Δ^3 or higher is neglected in the calculation of this integral. As it will be shown below, $R(x')$ contains terms of order Δ^2 and for this reason, the second term of Eq. 3.4 will be elided in the numerical evaluation of the integral. In summary, the action $\hat{S}$ has been expressed as a function of flat spacetime coordinates, a simplification that makes the computation of the integral much easier. In the next section, the Ricci scalar curvature is calculated up to order Δ^2, starting from the assumption of a weak gravitational field. At this stage, we note that many of the equations in Sections 3.1 and 3.2 were derived by the author. Specifically, all quantities expressed as functions of Δ were calculated using the Mathematica software [28] and constituted a substantial part of the project.

3.1 Derivation of the Ricci scalar curvature R

As mentioned above, the only missing piece to the puzzle is to analytically compute the scalar curvature $R(x')$. For this calculation, it is convenient to start from the inverse Jacobian J^{-1}, which in component form is written as

$$(J^{-1})^{\alpha}{}_{\nu'} = \frac{\partial x^{\alpha}}{\partial x'^{\nu}} = \frac{\partial}{\partial x'^{\nu}}(x'^{\alpha} + \Delta'^{\alpha}) = \delta_{\alpha'\nu'} + \partial_{\nu'}\Delta_{\alpha'} \tag{3.5}$$

where $\delta_{\alpha'\nu'}$ is the Kronecker delta and the formula for x^{α} is provided by Eq. 3.1. Having defined J^{-1}, the Jacobian J can be straightforwardly calculated by taking its inverse. The result yields

$$J^{\mu'}{}_{\nu} = \delta_{\mu'\nu'} - (\partial_{\nu'}\Delta_{\mu'}) + (\partial_{\nu'}\Delta_{\rho'})(\partial_{\rho'}\Delta_{\mu'}) + \mathcal{O}(\Delta^3). \tag{3.6}$$

The derivation of the Jacobian allows the calculation of the deformed spacetime metric $g_{\alpha\beta}$, which can now be calculated relative to the flat spacetime metric $\eta_{\alpha\beta}$. In fact, in Euclidean signature, $\eta = \mathbb{1}_{4\times 4}$ and the deformed spacetime metric $g = J^T \eta J$ which in component form reads,

$$g_{\alpha\beta} = \eta_{\mu'\nu'} J^{\mu'}{}_{\alpha} J^{\nu'}{}_{\beta}. \tag{3.7}$$

Once again, this can be easily computed. In component form, the metric g can be written as

$$\begin{aligned} g_{\alpha\beta} = {} & \delta_{\alpha'\beta'} - \partial_{\alpha'}\Delta_{\beta'} - \partial_{\beta'}\Delta_{\alpha'} \\ & + (\partial_{\alpha'}\Delta_{\mu'})(\partial_{\mu'}\Delta_{\beta'}) + (\partial_{\beta'}\Delta_{\mu'})(\partial_{\mu'}\Delta_{\alpha'}) + (\partial_{\alpha'}\Delta_{\mu'})(\partial_{\beta'}\Delta_{\mu'}) + \mathcal{O}(\Delta^3). \end{aligned} \tag{3.8}$$

From the result, it is immediately evident that the deformed spacetime metric can be written as

$$g_{\alpha\beta} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'}, \tag{3.9}$$

which is exactly the result one would expect, having started from the assumption of a weak-field limit. It then follows that the perturbation metric $h_{\alpha\beta}$ describing the deformation around the background can be written in terms of flat spacetime coordinates as

$$\begin{aligned} h_{\alpha\beta} = {} & -\partial_{\alpha'}\Delta_{\beta'} - \partial_{\beta'}\Delta_{\alpha'} \\ & + (\partial_{\alpha'}\Delta_{\mu'})(\partial_{\mu'}\Delta_{\beta'}) + (\partial_{\beta'}\Delta_{\mu'})(\partial_{\mu'}\Delta_{\alpha'}) + (\partial_{\alpha'}\Delta_{\mu'})(\partial_{\beta'}\Delta_{\mu'}) + \mathcal{O}(\Delta^3). \end{aligned} \tag{3.10}$$

At this point, it is also possible to compute the inverse of the spacetime metric, $g^{\alpha\beta}$ which will have components

$$g^{\alpha\beta} = \delta_{\alpha'\beta'} + \partial_{\alpha'}\Delta_{\beta'} + \partial_{\beta'}\Delta_{\alpha'} + (\partial_{\mu'}\Delta_{\alpha'})(\partial_{\mu'}\Delta_{\beta'}) + \mathcal{O}(\Delta^3). \tag{3.11}$$

From this result, the Ricci scalar curvature R can be computed. In fact, $R = g^{\mu\nu}R_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu}$, where the second equality only holds in the weak-field limit, as the metric can be expressed as given by Eq. 3.9 and the terms of order h^2 are omitted. One may note that the Ricci tensor has not been calculated yet. However, in the weak-field approximation, it can be expressed solely in terms of the perturbation metric $h_{\alpha\beta}$, namely [29]

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_{\nu\alpha}h^{\alpha}_{\mu} + \partial_{\mu\alpha}h^{\alpha}_{\nu} - \partial^{\alpha}_{\alpha}h_{\mu\nu} - \partial_{\mu\nu}h^{\alpha}_{\alpha}\right), \tag{3.12}$$

where the notation $\partial_{\alpha\beta}$ indicates the second derivative of the quantity on the right with respect to components α and β. Moreover, noticing that within this approximation scheme the metric $\eta_{\mu\nu} = \eta^{\mu\nu}$

acts as a raising and lowering operator, then $h^{\alpha}_{\beta} = \eta^{\alpha\gamma} h_{\gamma\beta}$ and $\partial^{\alpha}_{\alpha} = \eta^{\alpha\beta}\partial_{\beta\alpha}$. The Ricci scalar curvature is then given by

$$\begin{aligned} R &= \eta^{\mu\nu} R_{\mu\nu} \\ &= \frac{1}{2}[\eta^{\mu\nu}\partial_{\nu\alpha}(\eta^{\alpha\beta}h_{\beta\mu}) + \eta^{\mu\nu}\partial_{\mu\alpha}(\eta^{\alpha\beta}h_{\beta\nu}) - \eta^{\mu\nu}\eta^{\alpha\sigma}\partial_{\alpha\sigma}(h_{\mu\nu}) - \eta^{\mu\nu}\partial_{\mu\nu}(\eta^{\alpha\beta}h_{\alpha\beta})] \ . \end{aligned} \tag{3.13}$$

The components of the Euclidean metric $\eta = \mathbb{1}_{4\times4}$ are just constants and can therefore be moved to the right or to the left of the derivative without changing the meaning of the equation. Thus,

$$\begin{aligned} R &= \frac{1}{2}[\partial_{\nu\alpha}(\eta^{\mu\nu}\eta^{\alpha\beta}h_{\beta\mu}) + \partial_{\mu\alpha}(\eta^{\mu\nu}\eta^{\alpha\beta}h_{\beta\nu}) - \eta^{\alpha\sigma}\partial_{\alpha\sigma}(\eta^{\mu\nu}h_{\mu\nu}) - \eta^{\mu\nu}\partial_{\mu\nu}(\eta^{\alpha\beta}h_{\alpha\beta})] \\ &= \frac{1}{2}[\partial_{\alpha\nu}h^{\alpha\nu} + \partial_{\alpha\mu}h^{\alpha\mu} - \partial^{\alpha}_{\alpha}h^{\mu}_{\mu} - \partial^{\mu}_{\mu}h^{\alpha}_{\alpha}] \\ &= \partial_{\alpha\beta}h^{\alpha\beta} - \partial^{\alpha}_{\alpha}h^{\beta}_{\beta} \ . \end{aligned} \tag{3.14}$$

With this result in hand, the Ricci scalar curvature can now be calculated. However, the expression above is still written in terms of curved spacetime coordinates. This can be solved by realising that to move from curved spacetime coordinates x to flat spacetime coordinates x' is to map $\partial_{\alpha} \to {J^{\mu'}}_{\alpha}\partial_{\mu'}$. Moreover, now using g as a raising and lowering operator, $g^{\alpha\sigma}\partial_{\sigma} = \partial^{\alpha}$ and $g^{\beta\rho}h_{\rho\beta} = {h^{\beta}}_{\beta}$. As a result, we rewrite Eq. 3.14 as

$$R = {J^{\mu'}}_{\alpha}\partial_{\mu'}{J^{\nu'}}_{\beta}\partial_{\nu'}g^{\alpha\rho}g^{\beta\sigma}h_{\rho\sigma} - g^{\alpha\sigma}{J^{\nu'}}_{\sigma}\partial_{\nu'}{J^{\mu'}}_{\alpha}\partial_{\mu'}g^{\beta\rho}h_{\rho\beta}, \tag{3.15}$$

which now has the Ricci curvature expressed as a function of flat spacetime coordinates only. The attentive reader will have noticed that for this calculation, the metric g is used as a raising and lowering operator, whereas to derive Eq. 3.14 only the flat Euclidean metric η was used. The justification for this is two-fold: firstly, the equation above was computed via Mathematica, and the physical quantity used for the calculation was indeed the metric g, which is therefore kept in the expression of the Ricci scalar curvature in Eq. 3.15. Conversely, the derivation of Eq. 3.14 was carried out exactly and followed directly the standard assumptions in the weak-field limit, such as the statement given in the first line of Eq. 3.13. Secondly, it should be recalled that only terms up to Δ^2 will be kept in the final expression for R. Hence, the approximation scheme is consistent.
Back to the scalar curvature, the two terms of Eq. 3.15 are calculated separately for clarity. The first term can be written as,

$$\begin{aligned} {J^{\mu'}}_{\alpha}\partial_{\mu'}{J^{\nu'}}_{\beta}\partial_{\nu'}h^{\alpha\beta} &= -2\partial_{\mu'\mu'\nu'}\Delta_{\nu'} - 2(\partial_{\mu'\mu'}\Delta_{\nu'})(\partial_{\nu'\rho'}\Delta_{\rho'}) - 4(\partial_{\rho'}\Delta_{\nu'})(\partial_{\rho'\nu'\mu'}\Delta_{\mu'}) \\ &\quad - 2(\partial_{\nu'}\Delta_{\rho'})(\partial_{\mu'\mu'\nu'}\Delta_{\rho'}) - 2(\partial_{\mu'\rho'}\Delta_{\mu'})(\partial_{\nu'\rho'}\Delta_{\nu'}) - 2(\partial_{\mu'\nu'}\Delta_{\rho'})(\partial_{\rho'\nu'}\Delta_{\mu'}) \\ &\quad - (\partial_{\mu'\nu'}\Delta_{\rho'})(\partial_{\mu'\nu'}\Delta_{\rho'}) - (\partial_{\mu'\mu'}\Delta_{\nu'})(\partial_{\rho'\rho'}\Delta_{\nu'}) + \mathcal{O}(\Delta^3), \end{aligned} \tag{3.16}$$

where $\partial_{\alpha\beta\gamma}$ indicates the triple derivative with respect to components α, β and γ. Similarly, the second term reads:

$$\begin{aligned} g^{\alpha\sigma}{J^{\nu'}}_{\sigma}\partial_{\nu'}{J^{\mu'}}_{\alpha}\partial_{\mu'}g^{\beta\rho}h_{\rho\beta} &= -2(\partial_{\alpha'\alpha'\sigma'}\Delta_{\sigma'}) - 2(\partial_{\alpha'\beta'}\Delta_{\sigma'})(\partial_{\alpha'\beta'}\Delta_{\sigma'}) \\ &\quad - 2(\partial_{\alpha'}\Delta_{\sigma'})(\partial_{\alpha'\beta'\beta'}\Delta_{\sigma'}) + 2(\partial_{\alpha'\alpha'}\Delta_{\sigma'})(\partial_{\sigma'\beta'}\Delta_{\beta'}) + \mathcal{O}(\Delta^3). \end{aligned} \tag{3.17}$$

Finally, joining Eq. 3.16 and Eq. 3.17 an equation for the Ricci scalar is found,

$$\begin{aligned} R = &+(\partial_{\mu'\nu'}\Delta_{\rho'})(\partial_{\mu'\nu'}\Delta_{\rho'}) - 4(\partial_{\mu'\mu'}\Delta_{\nu'})(\partial_{\nu'\rho'}\Delta_{\rho'}) - 4(\partial_{\rho'}\Delta_{\nu'})(\partial_{\rho'\nu'\mu'}\Delta_{\mu'}) \\ &- 2(\partial_{\mu'\rho'}\Delta_{\mu'})(\partial_{\nu'\rho'}\Delta_{\nu'}) - 2(\partial_{\mu'\nu'}\Delta_{\rho'})(\partial_{\rho'\nu'}\Delta_{\mu'}) - (\partial_{\mu'\mu'}\Delta_{\nu'})(\partial_{\rho'\rho'}\Delta_{\nu'}) + \mathcal{O}(\Delta^3). \end{aligned} \tag{3.18}$$

This result finally allows the calculation of the action for a spacetime of pure gravity as expressed in Eq. 3.4 up to order Δ^2.

3.2 The Gibbons-Hawking-York boundary term

Having derived the Ricci scalar curvature in terms of small spacetime deformations Δ, the naïve Einstein-Hilbert action can now be calculated. However, as specified in Section 2.3, the Euclidean path integral only converges if $\hat{S} \geq 0$. In the same section, this constraint was formally expressed in terms of the positive action conjecture. It was argued that to ensure that the Einstein field equations are recovered and that the action is positive, a boundary term $\hat{S}_B$ ought to be added to the action, such that its variation yields

$$(16\pi)\delta\hat{S}_B = -\epsilon \oint_{\partial\mathcal{M}} d^3x\ \hat{g}^{\mu\nu}\sqrt{\det\hat{g}}\, N^\alpha \partial_\alpha \delta g_{\mu\nu}\ . \tag{3.19}$$

Let us consider as an ansatz the Gibbons-Hawking-York boundary term [30, 26], which is defined in Euclidean spacetime as

$$\hat{S}_{GHY} = -\frac{\epsilon}{8\pi}\oint_{\partial\mathcal{M}} d^3x K\sqrt{\det\hat{g}}, \tag{3.20}$$

where $K = \nabla_\alpha N^\alpha$ is the trace of the extrinsic curvature [31] of the hypersurface $\partial\mathcal{M}$. Although it is not explicitly proved here, Poisson's textbook [25] shows that the variation of this term recovers Eq. 3.19, since the variation of K yields

$$\delta K = \frac{1}{2}\hat{g}^{\mu\nu}N^\alpha\partial_\alpha\delta g_{\mu\nu}\ . \tag{3.21}$$

Therefore, $\hat{S}_{GHY}$ is an appropriate boundary term. In order to add it to the naïve Einstein-Hilbert action, it needs to be mapped to flat coordinates and the extrinsic curvature ought to be calculated. In analogy with the argument outlined to justify Eq. 3.2, $\int_{\partial\mathcal{M}'} d^3x' = \int_{\partial\mathcal{M}} d^3x\sqrt{\det\hat{g}}$ in flat coordinates. Furthermore, in the weak-field limit, the extrinsic curvature K can be expressed as

$$\begin{aligned} K &= \nabla_\alpha N^\alpha \\ &= \partial_\alpha N^\alpha + \Gamma^\alpha_{\alpha\mu}N^\mu \\ &= \partial_\alpha N^\alpha + \frac{1}{2}(\partial_\mu h^\alpha{}_\alpha)N^\mu, \end{aligned} \tag{3.22}$$

since $\Gamma^\alpha_{\alpha\mu} = \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu h_{\alpha\beta} + \partial_\alpha h_{\mu\beta} - \partial_\beta h_{\alpha\mu})$ within this regime, and $\eta_{\alpha\beta} = \mathbb{1}_{4\times4}$ in Euclidean signature. Moreover, the unit vector normal to the hypersurface $\partial\mathcal{M}$ is defined as

$$N_\alpha \equiv \epsilon\frac{\partial_\alpha S}{|g^{\mu\nu}(\partial_\mu S)(\partial_\nu S)|^{1/2}} = \epsilon\frac{J^{\mu'}{}_\alpha\partial_{\mu'}S}{|g^{\mu\nu}(J^{\rho'}{}_\mu\partial_{\rho'}S)(J^{\sigma'}{}_\nu\partial_{\sigma'}S)|^{1/2}}, \tag{3.23}$$

where S is a function that defines the hypersurface $\partial\mathcal{M} = \{x \in \mathcal{M}|S(x) = 0\}$ [32]. To reflect the geometry of a hypercubic lattice, S is chosen to be $S = x_0x_1x_2x_3$.

In Eq. 3.23, the first equality defines the unit vector, whereas the second is only true for N'_α expressed in flat spacetime coordinates. Finally, the trace of the extrinsic curvature K' in flat coordinates can be written as

$$K' = J^{\mu'}{}_\alpha\partial_{\mu'}N'^\alpha + \frac{1}{2}(J^{\nu'}{}_\mu\partial_{\nu'}h^\alpha{}_\alpha)N'^\mu\ , \tag{3.24}$$

with all the quantities expressed as functions of a flat coordinate system. The equation above, as well as Eq. 3.23 were computed on a Mathematica script. The value of K' was found to depend on the direction from which the boundary is approached. In general, assuming that the boundary of the geometry is approached from the μ-direction, which means that the component $x'_\mu = 0$, the following is obtained:

$$K'_\mu = \epsilon[-\partial_{\nu'\nu'}\Delta_\mu - \partial_{\mu\nu'}\Delta_{\nu'} - \frac{1}{x'_a}\partial_{a'}\Delta_\mu - -\frac{1}{x'_b}\partial_\mu\Delta_{b'}] + \mathcal{O}(\Delta^2)\ . \tag{3.25}$$

Notice that the Greek letter $\nu' \in \{0,1,2,3\}$ as usual, whereas the letters from the English alphabet a', $b' \in \{\{0,1,2,3\} - \{\mu\}\}$. Moreover, the factor of ϵ in Eq. 3.25 is multiplied by the one in Eq. 3.20 and it simplifies to $\epsilon^2 = 1$ regardless of the surface being spacelike or timelike. This result, in principle, allows us to compute the Gibbons-Hawking-York term, add it to the naïve Einstein-Hilbert action and satisfy the positive action conjecture. In the following section, however, we will devise an alternative way to enforce the conjecture and comment on the validity of each.

3.3 Enforcing the positive action conjecture: a conceptual aside

In Section 2.3, we argued that for an asymptotically Euclidean metric, the action $\hat{S}$ must be positive, which simultaneously provides a lower bound to the gravitational energy and ensures that the equations of motions are recovered. This positive action conjecture was shown to be satisfied with the addition of an extra boundary term to the naïve Einstein-Hilbert action formulated in Eq. 2.11.
Crucially, the conjecture can also be satisfied by directly constraining the metric, as it is a claim about the allowed geometries in Euclidean space. First of all, spacetime is required to be asymptotically Euclidean, which means that at the boundaries of the manifold $\mathcal{M}$, the metric is constrained by

$$[\delta g_{\mu\nu}]_{\partial\mathcal{M}} = 0 \implies [g_{\mu\nu}]_{\partial\mathcal{M}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3.26}$$

where $[g_{\mu\nu}]_{\partial\mathcal{M}} = \hat{g}_{\mu\nu}$ is the 3×3 metric on the hypersurface $\partial\mathcal{M}$. The second condition may appear less straightforward, but it is fundamental to recover the equations of motion. Consider the variation of the Einstein-Hilbert action in Eq. 2.16: the second term is the one preventing the correct formulation of the field equations, which is why the addition of a boundary term was discussed in the first place. However, it is apparent that imposing the constraint

$$[\hat{g}^{\mu\nu} N^\alpha \partial_\alpha \delta g_{\mu\nu}]_{\partial\mathcal{M}} = 0 \implies [\partial_\alpha g_{\mu\nu}]_{\partial\mathcal{M}} = 0 \tag{3.27}$$

on the metric recovers the correct equations of motion. The right-hand side of the implication above comes from Eq. 3.26: if the metric is constant at the boundary, then the derivative of $g_{\mu\nu}$ as $\partial\mathcal{M}$ is approached from the α-direction must be zero. At this stage, it is important to stress that these two ways to satisfy the positive action conjecture are theoretically similar. In fact, at the level of the equations of motion, the addition of a boundary term $\hat{S}_{GHY}$ to the action produces the same effect as the two boundary conditions defined above.
Nonetheless, one could conceptually make an argument in favour of the boundary term, since the constraint is imposed on the equations of motion by adding $\hat{S}_{GHY}$, rather than on the space of possible metrics $\mathcal{D}[g]$ directly. Conversely, enforcing Eq. 3.26 and 3.27 poses a constraint on the fluctuations of the metric, directly affecting $\mathcal{D}[g]$. On the other hand, however, these two boundary conditions explicitly ensure that the space is asymptotically Euclidean and that the Einstein field equations are recovered. Therefore, even though the former approach seems to be more attuned to the spirit of path integral quantisation (as it does not alter the metric fluctuations directly), the latter scheme is implemented in the simulation and boundary conditions are imposed. Section 3.2 has however been included for completeness, and future work might investigate further the behaviour of the action as the Gibbons-Hawking-York term is added.

4 Quantising gravity on a lattice

In this chapter, the lattice implementation of the physics discussed in Section 2 and 3 is presented. At this stage, it should be noted that the following work relies on the code written by Dr Chris Bouchard [33].
In the lattice approximation, spacetime is discretised as a box of finite size and well-defined spacing a between any two adjacent spacetime points. For fixed spacing, the size of the lattice is then defined in each direction by the number of steps that need to be taken to get from one edge of the lattice to the other. The number of steps in each spatial direction is denoted by N_x, N_y and N_z in Cartesian coordinates. Analogously, the number of steps in the Euclidean time direction is indicated by N_τ. In general, a spacetime point on one such lattice can be written as $P = a(n_\tau,\ n_x,\ n_y,\ n_z)$ with $n_i \in \{0, 1, 2, ..., N_i - 1\}$ and $i \in \{\tau, x, y, z\}$ [11]. For a cubic shape in 3D space, which implies that $N_x = N_y = N_z = N_\sigma$, the spacetime volume $V_{\tau xyz}$ is given by

$$V_{\tau xyz} = (aN_\tau)(aN_\sigma)^3. \tag{4.1}$$

This work concerns the analysis of lattice configurations whose volume can be written as Eq. 4.1. It should be noted, however, that the simulation is implemented in a way that does not limit the user to a volume of this form, as N_x, N_y and N_z can be chosen arbitrarily and independently of each other. The lattice spacing a can also be chosen, by specifying different values of parameter β, which sets the energy scale on the lattice. A further discussion on the relationship between a and β is given in Section 5.2. The description that follows provides an overview of the concepts upon which the simulation is based.

4.1 Generating a dynamical spacetime

On a 4-dimensional lattice constructed as above, spacetime is flat. This means that $x = x'$ holds for all spacetime points on the lattice. Therefore, every point ought to be deformed by a random, small perturbation Δ in each of the four spacetime directions. This is, in principle, how the flat spacetime acquires a curvature. Consequently, the Ricci scalar R is calculated at each point, according to Eq. 3.18. To carry out this calculation, a first-order forward finite difference scheme is employed: for a general function f, its derivative with respect to the μ coordinate is then defined as

$$\delta_\mu f = a^{-1}[f(x' + a\hat{\mu}) - f(x')] \,, \tag{4.2}$$

where δ_μ is used to indicate the derivative instead of the conventional notation since this is a numerical first-order approximation. Moreover, $\hat{\mu}$ is a unit vector parallel to the direction of differentiation. The gravitational action can then be calculated straightforwardly by summing the values of R over all spacetime points on the lattice and multiplying the result by a constant $\kappa = c^4/(16\pi G) \approx 7.6 \times 10^{37}$ fm^{-2}, with a relative minus sign that comes from Wick rotation, as specified by Eq. 3.4. Nevertheless, the procedure outlined so far does not ensure that the Ricci curvature R is negative at all points on the lattice or, equivalently, that the action is positive. This means that it is possible to violate the positive action conjecture. This issue is solved by checking the value of the curvature at every spacetime point: imagine, for example, that a given point P on the lattice has been perturbed by Δ and the resulting curvature is positive. Then, a new small deformation is proposed and the curvature is calculated again until an allowed, negative value is found. At this point, it is important to be careful and consider the non-local properties of general relativity: if the value of the curvature changes at some spacetime point P, it ought to change at nearby points as well. In the simulation, this is accounted for by updating the values of R three lattice spacings away from point P in all directions. For a proposed update of Δ at point P, the values of the resulting curvatures are simultaneously checked. This method is clearly very costly computationally but ensures that R is negative at all points on the lattice as well as accounting for the non-locality of the theory.

At this point, the reader might realise that so far, the perturbation Δ has been treated as the central quantity of the simulation. However, nowhere in the expression for the scalar curvature R or the metric $g_{\mu\nu}$, Δ appears on its own. Rather, only its first derivative $\partial_\nu \Delta_\mu$ show up in the calculations, or higher derivatives. For this reason, the code was implemented such that the fundamental quantity would be $\partial_\nu \Delta_\mu$. This procedure limits the computational cost of the algorithm and reduces discretisation effects: the non-local effects of general relativity only appear two lattice spacings away after this implementation, resulting in a much faster runtime. A consequence of this optimised scheme, however, is that each spacetime coordinate in the lattice is associated with 16 different values of $\partial_\nu \Delta_\mu$, which can be regarded as a 4×4 deformation matrix. This may appear worrisome at first glance, as general relativity is known to be a theory with two degrees of freedom [34], whereas 16 appear here. Fortunately, this is not an actual problem: 6 out of the total 16 degrees of freedom are cancelled by the symmetry of the metric $g_{\mu\nu}$, whose entries (assuming matrix representation) can be calculated from $\partial_\nu \Delta_\mu$. Of the remaining 10, 4 are taken care of by the Bianchi identities, and 4 by gauge fixing, since general relativity is a gauge theory. Therefore, the expected 2 degrees of freedom are recovered.

The procedure described thus far allows for the generation of a dynamical spacetime on the lattice, such that the condition $\hat{S} \geq 0$ is satisfied. Even though a positive action is a requirement for the path integral to be well-behaved, the spacetime could be unphysical, as it might not be possible to recover Einstein's field equations. As discussed in Section 3.3, this can be fixed by either adding a boundary term S_{GHY} to the action or by constraining the metric directly as dictated by Eq. 3.26 and 3.27. In the same section, it was concluded that the latter method would be implemented in the simulation. Let us then start by considering the first condition, given by Eq. 3.26, which is equivalent to requiring that the perturbation metric $h_{\alpha\beta}$ (in Eq. 3.10) goes to zero at boundaries. As the boundary is approached from the μ direction, this constraint reduces to $\partial_b \Delta_c + \partial_c \Delta_b = 0$ with b, $c \in \{\{0,1,2,3\} - \{\mu\}\}$, if terms of order higher than Δ are truncated. Using a first-order forward finite difference scheme, this can further be implemented as

$$\delta_b \Delta_b(x') = 0, \tag{4.3}$$

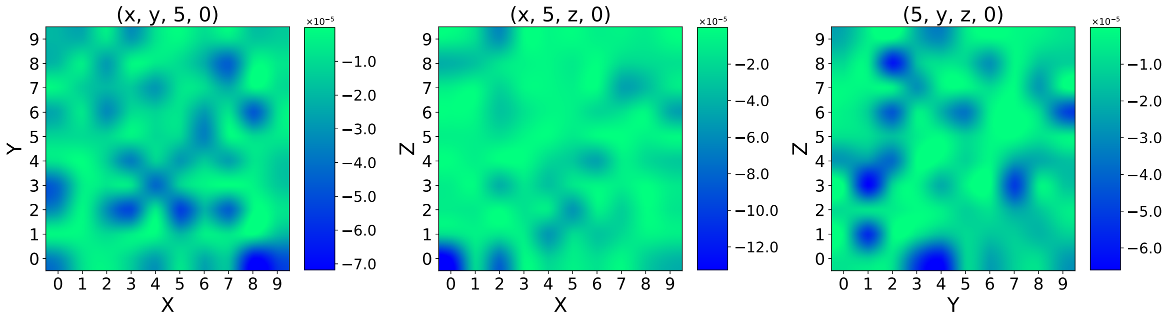

Figure 1: The graphs display a representation of the spacetime curvature R on a lattice. From left to right, the plots show R on the xy-plane, the xz-plane and yz-plane respectively. Spatial slices are taken through the middle of a 10^4 lattice at fixed time $\tau = 0$. The discrete values of R at each lattice point on the plane have been extrapolated to the continuum via spline interpolation and hence plotted as a heat map. It should be noticed that R is negative everywhere in spacetime, therefore satisfying the positive action conjecture.

$$\delta_b \Delta_c(x') = -\delta_c \Delta_b(x') \tag{4.4}$$

for diagonal and off-diagonal terms, respectively. Analogously, the equations from the second constraint, given by Eq. 3.27, can be derived to be

$$[\delta_\mu g_{\alpha\beta}(x')]_{\partial\mathcal{M}} = a^{-1}[g_{\alpha\beta}(x' + a\hat{\mu}) - g_{\alpha\beta}(x')]_{\partial\mathcal{M}} = 0 \tag{4.5}$$

at first-order finite difference approximation. Since $[g_{\alpha\beta}(x')]_{\partial\mathcal{M}} = \mathbb{1}_{3\times 3}$, then $[g_{\alpha\beta}(x'+a\hat{\mu})]_{\partial\mathcal{M}} = \mathbb{1}_{3\times 3}$. Finally this yields,

$$\delta_b \Delta_b(x' + a\hat{\mu}) = 0, \tag{4.6}$$

$$\delta_b \Delta_c(x' + a\hat{\mu}) = -\delta_c \Delta_b(x' + a\hat{\mu}) \tag{4.7}$$

for diagonal and off-diagonal terms, respectively. The b and c indices are defined as above, and $\hat{\mu}$ is the unit vector parallel to the direction from which the boundary is approached. Constraining the boundaries of the lattice according to equations 4.3, 4.4, 4.6 and 4.7 ensures that the Einstein field equations are recovered. The positive action conjecture is therefore satisfied, and a dynamical spacetime can be generated on a 4-dimensional lattice. A visualisation of such spacetime is provided in Fig. 1, where the Ricci scalar curvature R (in SI units of m^{-2}) is plotted over 2-dimensional spatial slices through the middle of a 10^4 lattice, at fixed time $\tau = 0$.

4.2 Markov-chain of pure gravity

We now wish to generate an ensemble of N_{cfg} lattice configurations, each with a particular geometry defined by metrics $g^{(\alpha)}_{\mu\nu} \equiv \{g^{(\alpha)}_{\mu\nu}(\tau, x, y, z)\}$, or equivalently by spacetime deformations $\partial_\nu \Delta^{(\alpha)}_\mu \equiv \{\partial_\nu \Delta^{(\alpha)}_\mu(\tau, x, y, z)\}$ where $\alpha = 0, 1, ..., N_{\text{cfg}} - 1$ and the τ, x, y, z coordinates span the whole lattice. The space of possible configurations is sampled such that the probability of obtaining a certain geometry $\partial_\nu \Delta^{(\alpha)}_\mu$ is given by

$$P[\partial_\nu \Delta^{(\alpha)}_\mu] \propto \exp\left(-\hat{S}[\partial_\nu \Delta^{(\alpha)}_\mu]\right), \tag{4.8}$$

as dictated by the path integral in Eq. 2.12. In order to probe this space correctly, a Metropolis update algorithm is used and a Markov chain of pure gravity lattice configurations is generated. Starting from $\alpha = 0$, a collection $\partial_\nu \Delta^{(0)}_\mu$ is arbitrarily chosen: this means that a 4×4 deformation matrix $\partial_\nu \Delta_\mu$ is assigned to each spacetime point P of the lattice. To go from configuration $\alpha = 0$ to $\alpha = 1$, each entry of the deformation matrix ought to be updated at all sites. This then needs to be repeated up to configuration $N_{\text{cfg}} - 1$. Clearly, updating the elements of the deformation matrix must reflect the probability distribution given in Eq. 4.8. To ensure that this is the case, the following procedure is implemented. Consider a single element of $\partial_\nu \Delta_\mu$ at spacetime point P. A random value $\omega \in [-\lambda, +\lambda]$

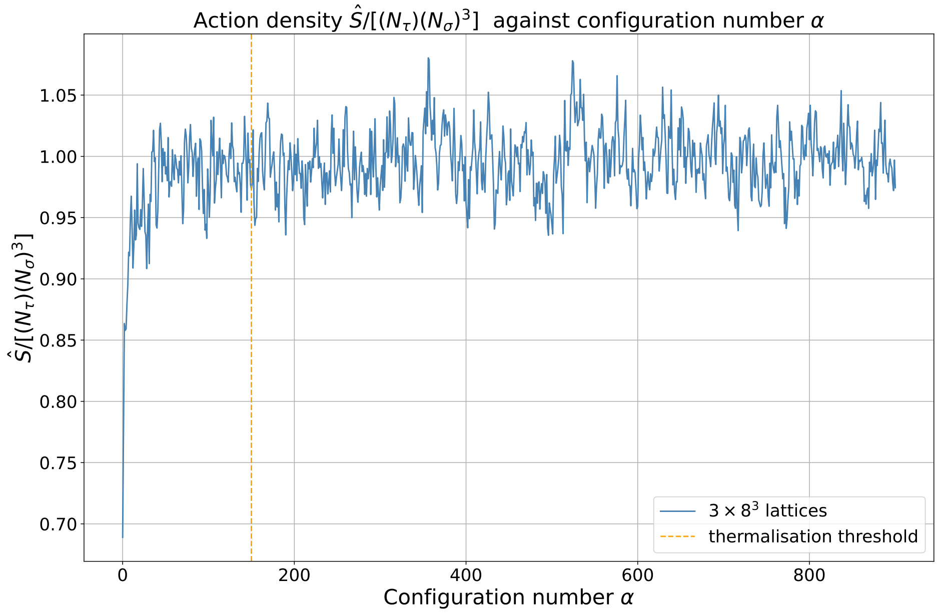

Figure 2: Via Metropolis update algorithm, a Markov chain of pure gravity is generated. The lattice configurations reach an equilibrium state only after a number of attempts, as testified by the large rate of change in the action density at low α's. This prompts the definition of a thermalisation threshold (orange dashed line) below which the samples are discarded. In this plot, a total number of configurations $N_{\text{cfg}} = 900$ is created for 3×8^3 lattices and the threshold is set at $\alpha = 150$. At this value of α, it is straightforward to see that the ensemble has thermalised, meaning that the lattices above this value provide reliable statistics.

for some[3] number λ is added to this matrix element and the action resulting from this change is calculated. Thus, let us call $\Delta\hat{S}$ the difference between the updated and original values of the action at that point. If $\Delta\hat{S} < 0$, meaning that the action has decreased, the matrix element is updated, and we proceed to the next entry of the deformation matrix. Conversely, if $\Delta\hat{S} > 0$, a random number $\chi \in [0, 1]$ is generated, and two outcomes are possible: if $\exp(-\Delta\hat{S}) > \chi$ then the new value of the matrix element is updated, otherwise, the update is discarded, and the original value is kept [10].
Through this procedure, ensembles of N_{cfg} lattice configurations are generated. Fig. 2 for example, shows a collection of $N_{\text{cfg}} = 900$ lattices. The value of the "action density" $\hat{S}/[(N_\tau)(N_\sigma)^3]$ is plotted for each lattice of the ensemble. This graph highlights a key feature of the Markov chain in the context of lattice field theories, called thermalisation: at low values of configuration number α, the action density varies quite widely before eventually plateauing to a value $\hat{S}/[(N_\tau)(N_\sigma)^3] \sim 1$. The oscillations around this plateau are due to the Monte Carlo randomisation. This evolution of the ensemble towards a steady state is named lattice thermalisation, and it is due to the properties of the Markov chain. This implies that the early configurations are not going to produce meaningful statistics and should therefore be discarded: in Fig. 2, the thermalisation threshold is set at $\alpha = 150$. It is worth pointing out that there is no universal guideline that defines where the thermalisation threshold should be set. Rather, it may vary depending on the observable and the lattices that are being simulated. In the analysis carried out in section 5, the threshold is held fixed at $\alpha = 150$, as it was observed to be appropriate for all lattice ensembles that were generated and analysed.
Before we move on to study the behaviour of the gravitational field on the lattice, a remark on the statistical analysis of the Markov chain. It is clear that any α and $\alpha - 1$ configurations are correlated, by virtue of the updating process outlined above. For this reason, we need to be careful about computing the errors relative to a certain observable. If the error is computed naïvely as proportional to the standard deviation over the whole ensemble, the output value may be much lower than the actual

[3] The choice of λ plays a relevant role in the simulation. I suggest reading *Lattice QCD for novices* by Lepage [10] for further insight into this.

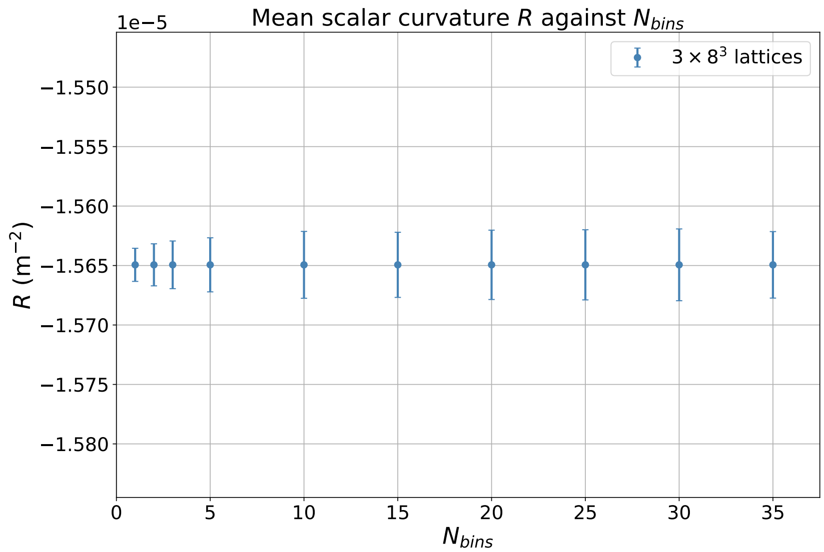

Figure 3: This graph shows the mean value of the Ricci scalar curvature R taken over 900 lattices with volume 3×8^3. The error associated with this mean is plotted against the number of bins, which is equivalent to varying bin sizes. For $N_{\text{bins}} < 5$, the error evidently increases due to the correlation of the samples. However, as larger N_{bins} are considered, the error appears to be independent, which means that the individual samples have been decorrelated and the error associated with one such measurement is therefore reliable.

error, as a consequence of the correlation between individual samples. The so-called process of binning typically provides a solution to this issue and a way to decorrelate the configurations in the ensemble. An example of this procedure is shown in Fig. 3. Take $R^{(\alpha)}$ to be the mean value of the scalar curvature for the lattice configuration α and $\{R^{(\alpha)}\}$ to be the collection of these values over the whole ensemble. To avoid an unreliable estimate of the error, $\{R^{(\alpha)}\}$ can be split into a number N_{bins} of bins, each of bin size N_{cfg}/N_{bins}. Performing statistical analysis on a collection of measurements arranged in such a way leaves the mean value unchanged but decorrelates the individual samples in the ensemble. For N_{bins} large enough, the error associated with the measurement becomes independent from the bin size, which indicates that the configurations have been fully decorrelated and that the error can be used to do reliable statistical analysis. In Fig. 3, it is clear that for $N_{\text{bins}} > 15$ the error is decoupled from the bin size. For the statistical analysis carried out in this work, the binning procedure was performed for every ensemble of $N_\tau \times N_\sigma^3$ lattices and it was found that for all of them, the error at $N_{\text{bins}} = 30$ had appropriately decoupled from the bin size and could be used. As a final observation, note that for N_{bins} too large compared to the number of lattice configurations in the ensemble, this procedure will not work, as each bin will contain too few values to allow for a reliable statistical analysis. Much like the thermalisation case before, there is not a well-defined rule that allows to exactly define which N_{bins} will output the optimal result. It is, therefore, appropriate to generate plots like the one we have just described and take a reasonable guess from there.

5 Results

In the previous section, the features of the simulation and its implementation have been discussed. In this chapter, an initial study of the quantised, dynamical spacetime generated on the lattice configurations is conducted. Straightforwardly, there are at least three independent parameters that can be tuned on the lattice, namely lattice spacing a, spatial extent N_σ and temporal extent N_τ. The following sections cover the results obtained by varying each of these parameters individually. For this analysis, a number of different lattice ensembles were generated. Each collection was set to have $N_{\text{cfg}} = 900$, and a thermalisation threshold at $\alpha = 150$ was found to be appropriate independently of the spacetime volume $V_{\tau xyz}$ or the spacing a. Statistical analysis was carried out as outlined in Section 4.2. Here, the results are presented.

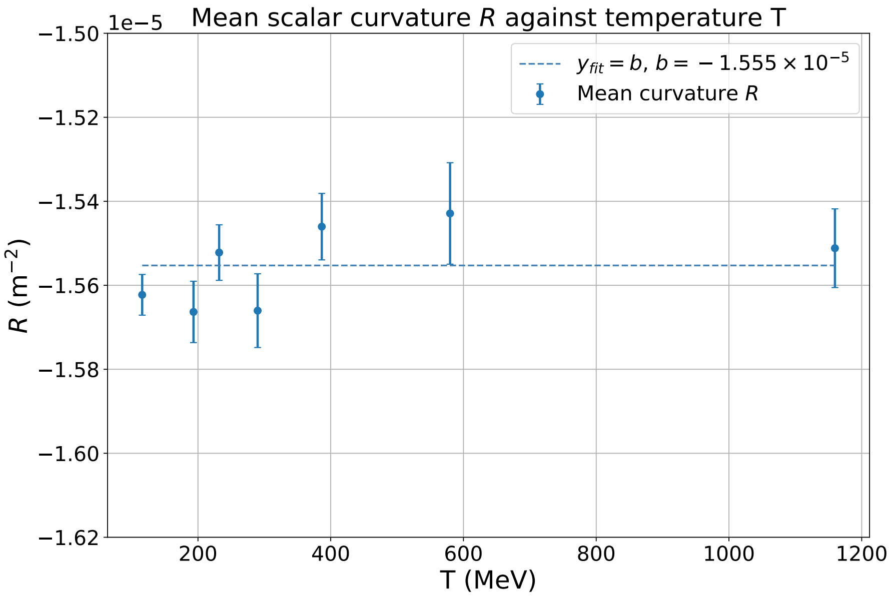

Figure 4: The mean Ricci scalar curvature is plotted against the change in temperature T on the lattice. The temperature is inversely proportional to the time extent N_τ. Each data point is obtained by taking the mean of R over all spacetime points on a single lattice, and then averaging over all the lattices of the ensemble. Each ensemble has a different N_τ and, therefore, a different temperature.

5.1 Temperature and volume dependence

The temperature of the lattice is expressed as [35]

$$T = (aN_\tau)^{-1}, \tag{5.1}$$

for spacing a and temporal extent N_τ. At this stage, the lattice spacing is fixed at $a = 0.17$ fm and temperature T is modified by varying N_τ on the lattices. The spatial extent $N_\sigma = 4$ is held constant as well. Fig. 4 shows the relationship between the Ricci scalar curvature R (m^{-2}) and the temperature T (MeV) for lattice ensembles with $N_\tau \in \{1, 2, 3, 4, 5, 6, 10\}$: this spans a range that goes from a temperature $T = 115.97$ MeV to a temperature $T = 1159.7$ MeV, exactly 10 times as large. The analysis would benefit from more data points, but given the computational cost of the lattice generation, we had to restrict ourselves to a limited number of ensembles. From this first study, no visible dependence between the scalar curvature and the temporal extent is observed. An assumption was made that the Ricci scalar curvature is independent of N_τ and the data were fitted with a straight line $y = b$, with $b = 1.555 \times 10^{-5}$ m^{-2}. This result was not expected, and we plan to study this dependence in more detail.
An analogous analysis was carried out by varying the volume $(aN_\sigma)^3$, at constant temporal extent $N_\tau = 3$ and lattice spacing $a = 0.17$ fm. More precisely, lattice ensembles of spatial extent $N_\sigma \in \{2, 3, 4, 5, 6, 7, 8\}$ were generated. The results of this study are plotted in Fig. 5, which does not highlight any noticeable dependence between volume and the mean spacetime curvature. Once again, assuming that the two quantities are independent, the data is fit to a straight line $y = b$: in this case, $b = 1.567 \times 10^{-5}$ m^{-2}, which is similar to the result obtained above. It should be noted that the error bars associated with each measurement drastically decrease as we move to higher values of N_σ. This is readily understood by realising that as N_σ is increased, the number of lattice sites grows cubically. This in turn implies that the error decreases by a factor of $(N_\sigma)^{-3/2}$, which explains what is observed in Fig. 5.

5.2 Divergence of the path integral quantisation of gravity

In the final part of this analysis, the lattice spacing a is modified and studied on 4^4 lattice ensembles. As a consequence of the relationship between temperature and lattice spacing, namely $T \propto a^{-1}$, this

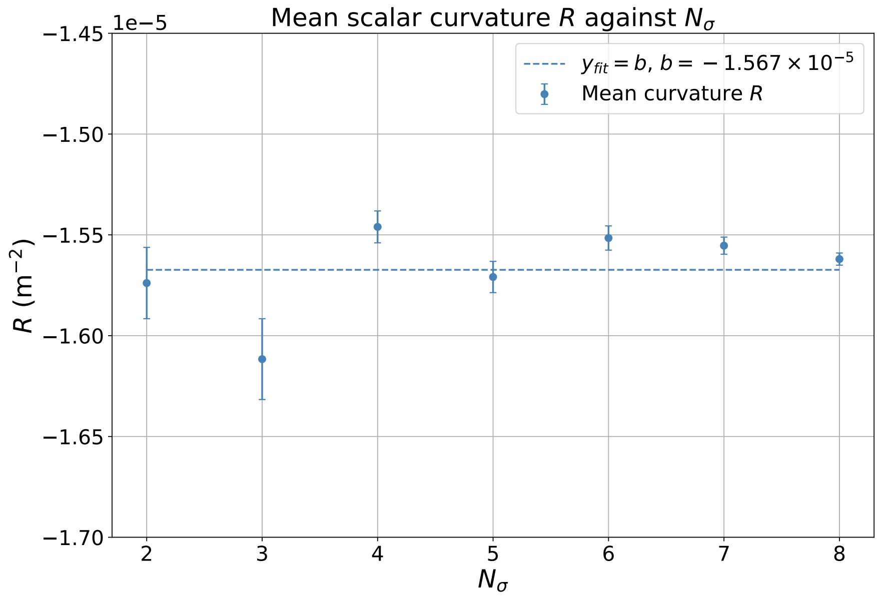

Figure 5: The mean curvature is here plotted as a function of the spatial extent N_σ. Both lattice spacing a and temporal extent N_τ are kept constant. As before, each point on the graph is obtained by taking the average of R over each spacetime point on a single lattice and then averaging over the whole ensemble. Since the volume increases as we move further to the right, the error associated with each measurement decreases.

could be regarded as a study of the temperature dependence of R. However, the lattice spacing a is also related to a dimensionless quantity β, which effectively sets the energy scale on the lattice. Their relationship can be expressed explicitly as [36]

$$a(\beta) = r_0 \exp\Bigg(-1.6804 - 1.7331(\beta-6) + 0.7849(\beta-6)^2 - 0.4428(\beta-6)^3 \Bigg) \tag{5.2}$$

where $r_0 = 0.5$ fm and is called the Sommer parameter [37]. Thus, varying the lattice spacing a can alternatively be regarded as a variation of the energy scale at which the theory is probed. This investigation is of particular interest since the path integral quantisation of gravity is expected to diverge in the ultraviolet limit. More precisely, even though this approach is appropriate at low energies as an effective field theory, it was proven to be non-renormalisable at two-loop order, with divergence Γ_∞ that can be calculated exactly as [9]

$$\Gamma_\infty = \frac{209}{2880(4\pi)^4}\frac{1}{\epsilon}\int d^4x\,\sqrt{-\det g}\; R_{\alpha\beta}{}^{\gamma\delta}R_{\gamma\delta}{}^{\epsilon\zeta}R_{\epsilon\zeta}{}^{\alpha\beta}. \tag{5.3}$$

In the equation above, $R_{ij}{}^{kl}$ is the Riemann tensor with the raised third and fourth indices, and ϵ is the renormalisation parameter. From dimensional regularisation, it should be noted that $\epsilon \propto \log(p^2_{\max})$, where $p_{\max} = \pi/a$ is the momentum cut-off for the lattice [38]. As a result of the relationship between ϵ and a, it is reasonable to imagine that an observable would diverge logarithmically with respect to the lattice spacing. To verify this divergent behaviour, the spacing is varied in a range from $a = 0.014$ fm to $a = 2.921$ fm, spanning roughly two orders of magnitude for a total of 10 different values of a, corresponding to $\beta \in \{4.9,\ 5.3,\ 5.55,\ 5.7,\ 5.75,\ 6.0,\ 6.3,\ 6.7,\ 7.0,\ 7.3\}$. In Fig. 6, the absolute value of the curvature $|R|$ is plotted against the lattice spacings a on a "log-log" scale. The data points were fitted to a function $y = bx^{-c}$ and the fit parameters were found to be $b = 1.3 \times 10^{-8}$ and $c = 4.00057$. In order to justify this result, let us recall that to first order finite difference scheme, the action can be rewritten as

$$\hat{S} = -\kappa a^4 \sum_{x'} R\ , \tag{5.4}$$

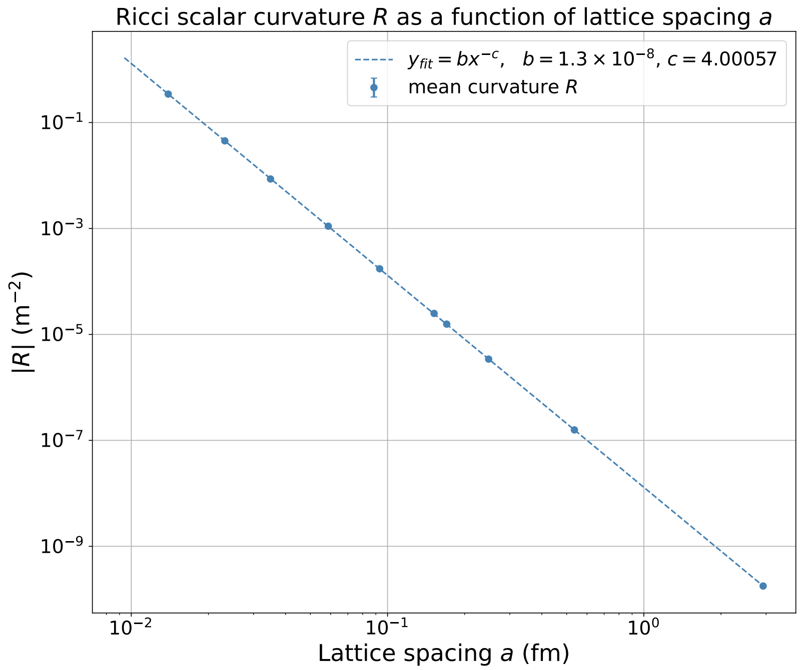

Figure 6: The path integral quantization of general relativity has been shown to have non-renormalizable divergences at second loop order by Goroff and Sagnotti in 1985 [9]. The plot appears to be in agreement with this prediction. In fact, as the lattice spacing a tends to zero, the Ricci scalar curvature asymptotes to infinity. In this plot, the values of R are fitted to the function $y = bx^{-(4+\mathcal{C})}$ with $b = 1.3 \times 10^{-8}$ and $\mathcal{C} = 0.57 \times 10^{-3}$.

where $\kappa \approx 7.6 \times 10^{37}$ fm^{-2} and the sum is taken over all spacetime points of the lattice. From this equation, it is then possible to define the action density as

$$\hat{S}/[(N_\tau)(N_\sigma)^3] = -\kappa a^4 \langle R \rangle, \tag{5.5}$$

where $\langle R \rangle$ is the mean value of the Ricci scalar curvature. Crucially, it was observed that the value of the action density is approximately equal to unity, independently of N_σ and N_τ. This finally suggests that $\langle R \rangle \propto a^{-4}$, which is consistent with the data displayed in Fig. 6. This result is however just a byproduct of the lattice discretisation scheme, and does not match with the expected logarithmic dependence in a of the divergence term Γ_∞. This apparent discrepancy is reconciled by considering that the fit function $y \propto a^{-4.00057}$ can be rewritten as the negative quartic power of a multiplied by a very small term $a^{-\mathcal{C}}$, with $\mathcal{C} = 0.57 \times 10^{-3}$. This seemingly innocuous factor can then be Taylor expanded to first order in $\mathcal{C}$ and expressed as

$$a^{-\mathcal{C}} = 1 - \mathcal{C} \log(a) + \mathcal{O}(\mathcal{C}^2), \tag{5.6}$$

which recovers the logarithmic dependence expected from theory. It is then possible to conclude that the factor $a^{-\mathcal{C}}$ might be a signature of the theoretical divergence of the path integral quantisation scheme. At this stage, it is impossible to claim this with certainty, as the integral of Eq. 5.3 is yet to be solved within this framework. Nonetheless, the results obtained so far seem to indicate that the quantum theory living on the lattice is sensitive to the expected theoretical divergences, which is an important consistency check and offers validation of the model.

5.3 Future work

Even though Hawking's approach appears to work for linearised, weak-field general relativity at scales of order ~ 1 fm, the consistency of this framework on the lattice still needs to be thoroughly tested. Following the method outlined in Section 3, the Christoffel symbols Γ^i_{jk} should be calculated. This

result would allow the explicit derivation of the Riemann tensor $R^{\mu}_{\alpha\beta\gamma}$ and the Ricci tensor $R_{\mu\nu}$. From this, the Ricci scalar curvature R could be calculated without having to resort to the approximation given in Eq. 3.12. From the Riemann tensor $R^{\mu}_{\alpha\beta\gamma}$, the divergent term of Eq. 5.3 could be exactly quantified, therefore allowing to check whether the results of Fig. 6 are consistent with the theoretical predictions. Finally, these quantities would allow the derivation of the left-hand side of the Einstein field equations. The significance of this result is two-fold: firstly, the lattice theory could be checked and further constrained by the Bianchi identities. Secondly, the explicit calculation of the Einstein field equations would provide a powerful consistency check, verifying whether the positive action conjecture is satisfied by the generated geometries.
Furthermore, the dynamical spacetime devised in this paper could be coupled with the QCD vacuum. To this day, the coupling of gauge theories with general relativity constitutes a theoretical challenge [12] and, for this reason, simulating the interplay of QCD and gravity on a lattice is on its own a compelling result. Moreover, the presence of a matter field in the lattice would allow the calculation of the right-hand side of the Einstein field equations, the energy-momentum tensor $T^{\mu\nu}$: effectively, this would correspond to investigate the response of gravity to vacuum fluctuations in otherwise empty space. These fluctuations are believed to make up for around 70% of the energy budget of the Universe, according to ΛCDM, the standard model of cosmology [39]. However, the theoretical estimate for the value of this energy and the current observations do not match. This discrepancy has come to be known as the cosmological constant problem [40] and it is an open question of modern cosmology. The coupling of quantum fluctuations of vacuum with a dynamical background, which can be achieved using our simulation, could provide insight into this problem.

6 Conclusions

In this work, we defined a consistent framework for the quantisation of gravity on lattices of finite volume. In the Euclidean path integral formalism, time is complexified, resulting in a real-valued functional integral that can be evaluated and interpreted as the probability associated with allowed spacetime geometries. As a consequence of Wick rotation, the positive action conjecture ought to be imposed, to ensure that the path integral remains well-defined and that the Einstein field equations can be retrieved from the action. Working in weak-field, linearised general relativity, the Ricci scalar curvature is derived as a function of the spacetime deformation Δ, which in turn enables the calculation of the Einstein-Hilbert action. Having devised this theoretical framework, we set the first derivative of the spacetime deformation Δ as the fundamental quantity of the simulation and evaluate the gravitational action numerically, using first-order finite difference methods. As a result, we successfully generate a dynamical spacetime on 4-dimensional lattice configurations via a Metropolis update algorithm.
At this stage, matter fields are neglected, and we study the properties of the dynamical spacetime alone. The lattices have a well-defined temperature and energy scale, which depend on the temporal extent N_τ and the spacing a, respectively. The variation of these two parameters is studied: while the mean curvature R does not seem to depend on time, it asymptotes to $-\infty$ as the lattice spacing $a \to 0$. This last result constitutes a powerful validation of the model since the theory is expected to diverge as the energy scale reaches the ultraviolet limit. Furthermore, a study of the scalar curvature R at different lattice volumes suggests the absence of dependence between these two quantities. In conclusion, this lattice quantisation scheme for gravity was shown to work and produce physical results. Even though it still requires a number of conceptual and practical consistency checks, the successful generation of quantised, dynamical spacetimes on the lattice constitutes an exciting first step towards the coupling of the gravitational field with lattice gauge theories.

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