Homework 4

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Problem 1

$$P_0(\text{2 year coupon bond maturing at } \$100) = \frac{7}{(1+r_{0,1})} + \frac{107}{(1+r_{0,2})}, \text{ where}$$

$r_{0,1}, r_{0,2}$ – discount rates for 1 and 2 years from now.

We know $$100 = \frac{7}{(1+r_{0,1})} + \frac{107}{(1+r_{0,2})}$$

and $$95.238\ldots = \frac{100}{(1+r_{0,1})}$$

From second equation get $1+r_{0,1} = 1.05$, so $r_{0,1} = 0.05$. (5%)

Then from first get $1+r_{0,2} = \frac{107}{100 - \frac{7}{1.05}} \approx 1.146$, so $r_{0,2} \approx 0.146$. (14.6%)

implied 2-year zero rate.

One-year zero rate one year from now is then calculated by the fact that

$$(1+r_{0,1})(1+r_{1,2}) = (1+r_{0,2}), \text{ so}$$

looking for this $$r_{1,2} = \frac{1,146}{1.05} - 1 \approx 0.092 = 9.2\%$$

Problem 2

Code with graphs attached.

- Percentage of trace from the first 3 eigenvalues
  ~ 93%.
- Do eigenvectors look like Frye eigenvectors?
  Well, to some extent. Lines are not so smooth, by the trends are similar – first is flat, second is increasing, third is twisted.
  I believe picture is less nice because we only have 250 data points, compared to 1543 for Frye's picture.

Problem 3

Code with graphs attached.
Fitted parameters: $\beta_0 = 2.76, \beta_1 = -1.20, \beta_2 = -1.13,$
$\tau = 57.1$.

For maturity going to infinity, Nelson–Siegel curve approaches $\beta_0 = 2.76$. Thus perpetual interest rate is 2.76.

Problem 4

We have $P_t(r) = \sum_{i=1}^{n} (100r+s)(1+r)^{t-t_i} = \sum_{i=1}^{n} (100r+s)(1+r)^{-i}$

since flat curve + def. of floater. since annual coupons

We compute modified duration (same sign as Macalay duration):

$$-\frac{P_t'}{P_t} = -\frac{\sum_{i=1}^{n} 100(1+r)^{-i} - i(100r+s)(1+r)^{-i-1}}{\sum_{i=1}^{n}(100r+s)(1+r)^{-i}} =$$

$$= -\frac{100}{100r+s} + \frac{\sum_{i=1}^{n}(1+r)^{-i-1}\cdot i}{\sum_{i=1}^{n}(1+r)^{-i}} =$$ standard computation with derivative of geometric series $=$

$$= -\frac{100}{100r+s} + \frac{\frac{(1+r)^{-n-1}\left((1+r)^n + r\left((1+r)^n - n - 1\right) - 1\right)}{r^2}}{\frac{1-(1+r)^{-n}}{r}} =$$

$$= -\frac{100}{100r+s} + \frac{1 - r(n+1)(1+r)^{-n-1} - (1+r)^{-n-1}}{r\left(1-(1+r)^{-n}\right)}$$

for it to be negative, we must have

$$\frac{100}{100r+s} > \frac{1 - r(n+1)(1+r)^{-n-1} - (1+r)^{-n-1}}{r - r(1+r)^{-n}} \iff$$

$$\iff 100r - 100r(1+r)^{-n} > (100r+s)\left(1 - (1+r)^{-n-1}(r(n+1)-1)\right)$$

$$-100r(1+r)^{-n} > -100r(1+r)^{-n-1}\left(r(n+1)-1\right) +$$

$$+S\left(1-(1+r)^{-n-1}\left(r(n+1)-1\right)\right)$$

$$\Updownarrow$$

$$S < \frac{100r(1+r)^{-n-1}\left(r(n+1)-1-1-r\right)}{1-(1+r)^{-n-1}\left(r(n+1)-1\right)} = \frac{100r(1+r)^{-n-1}\left(rn-2\right)}{1-(1+r)^{-n-1}\left(r(n+1)-1\right)}$$