# Homework 5

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## Problem 1

We solve minimization problem with Lagrange multipliers:

$$\mathcal{L}(\omega,\lambda) = \frac{1}{2}\omega^T C\omega - \lambda(\omega^T m + r_f(1-\omega^T u) - \mu)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \omega} = C\omega - \lambda(m - ur_f) \Rightarrow \omega = \lambda C^{-1}(m - ur_f)$$

We seek $\lambda$ and $\mu$ such that this minimizer will lie on the line of efficient frontier. In particular it means that $u^T\omega = 1$. Plugging in,

$$1 = u^T\omega = u^T\lambda C^{-1}(m - ur_f) \Rightarrow \lambda = \frac{1}{u^T C^{-1}(m - ur_f)}$$

$$\Rightarrow \omega = \lambda C^{-1}(m - ur_f) = \frac{C^{-1}(m - ur_f)}{u^T C^{-1}(m - ur_f)}.$$

This is indeed tangency portfolio

$$\omega_{tp} = \frac{C^{-1}m_e}{u^T C^{-1} m_e}.$$

## Problem 2

We have $\omega_{minv} = \dfrac{C^{-1}u}{u^T C^{-1} u}$ and any other efficient portfolio has the form

$$\omega_1 = \left[I - \frac{C^{-1}uu^T}{u^T C^{-1} u}\right] C^{-1} m + \omega_{minv}$$

We compute $Cov(\omega_{minv}, \omega_1) = \omega_{minv}^T C \omega_1 =$

$$= \omega_{minv}^T C \omega_{minv} + \frac{u^T C^{-1}}{u^T C^{-1} u} \cdot C \cdot \lambda \left[ I - \frac{C^{-1} u u^T}{u^T C^{-1} u} \right] C^{-1} m =$$

$$= Var(\omega_{minv}) + \frac{\lambda u^T}{u^T C^{-1} u} \left[ I - \frac{C^{-1} u u^T}{u^T C^{-1} u} \right] C^{-1} m =$$

$$= Var(\omega_{minv}) + \frac{\lambda}{(u^T C^{-1} u)^2} \left[ u^T (u^T C^{-1} u) - u^T C^{-1} u u^T \right] C^{-1} m =$$

$$= Var(\omega_{minv}) + \frac{\lambda}{(u^T C^{-1} u)^2} \left[ u^T (u^T C^{-1} u) - (u^T C^{-1} u) u^T \right] C^{-1} m =$$

$$= Var(\omega_{minv}) + \frac{\lambda}{(u^T C^{-1} u)^2} \left[ (u^T C^{-1} u) u^T - (u^T C^{-1} u) u^T \right] C^{-1} m =$$

$$= Var(\omega_{minv}) + 0 = Var(\omega_{minv}). //$$

We have $0 = Cov(w_1, w_2) = w_1^T C((1-\alpha)\omega_{minv} + \alpha w_1) =$

$$= (1-\alpha) Cov(w_1, \omega_{minv}) + \alpha Cov(w_1, w_1) =$$

$$= (1-\alpha) Var(\omega_{minv}) + \alpha Var(\omega_1).$$

$\alpha = \dfrac{Var(\omega_{minv})}{Var(\omega_{minv}) - Var(\omega_1)} < 0$ since $\omega_{minv}$ has smallest variance of all efficient ptf$_0$s.

Problems 3 & 4 are in IPNB.