|HW12| DANIIL GLUKHOVSKIY

## Problem 1

$$\begin{cases} y_t = \sigma_t \, d\beta_t \\ \sigma_t^2 = c + b_1 \sigma_{t-1}^2 + a_1 y_{t-1}^2 \end{cases}$$

Writing $v_k := \mathbb{E}_t[\sigma_{t+k}^2]$, compute

$$\underline{v_{k+1}} = \mathbb{E}_t[\sigma_{t+k+1}^2] = \mathbb{E}_t[c + b_1 \sigma_{t+k}^2 + a_1 y_{t+k}^2] =$$

$$= c + b_1 \mathbb{E}_t[\sigma_{t+k}^2] + a_1 \mathbb{E}_t[\sigma_{t+k}^2 \, d\beta_{t+k}^2] =$$

$$= c + b_1 \mathbb{E}_t[\sigma_{t+k}^2] + a_1 \mathbb{E}_t[\sigma_{t+k}^2] \mathbb{E}[d\beta_{t+k}^2] =$$

independent, as $\sigma_{t+k}$ only depends on $d\beta_{t+k-1}, d\beta_{t+k-2}, \ldots\ldots$ Also, $\mathbb{E}_t[d\beta_{t+k}^2] = \mathrm{Var}(d\beta) = 1$.

$$= c + b_1 v_k + a_1 v_k = \underline{c + (b_1 + a_1) v_k},$$

This could be rewritten as

$$\left(v_{k+1} - \frac{c}{1-a_1-b_1}\right) = (a_1+b_1)\left(v_k - \frac{c}{1-a_1-b_1}\right), \quad v_t = \sigma_t^2$$

Last recurrence is easily solved:

$$\left(v_k - \frac{c}{1-a_1-b_1}\right) = \left(\sigma_t^2 - \frac{c}{1-a_1-b_1}\right)(a_1+b_1)^k.$$

So finally, $\underbrace{\mathbb{E}_t\left[\sigma_{t+k}^2\right] = \frac{c}{1-a_1-b_1} + \left(\sigma_t^2 - \frac{c}{1-a_1-b_1}\right)(a_1+b_1)^k}$.

Note that as $k \to \infty$ $\mathbb{E}[\sigma_{t+k}^2] \longrightarrow \frac{c}{1-a_1-b_1}$ if $a_1+b_1<1$.

## Problem 3

a. $H(t+1) = \frac{1}{2} + H(t) + \varepsilon_{t+1}$, where $\varepsilon_t = \begin{cases} \frac{1}{2} & \text{w/ } p=\frac{1}{2} \\ -\frac{1}{2} & \text{w/ } p=\frac{1}{2}. \end{cases}$

$T(t+1) = \frac{1}{2} + T(t) + \eta_{t+1}$ $\quad$ $\varepsilon_t$ i.i.d w/ mean 0, $\eta_t = -\varepsilon_t$ i.i.d m. 0.

$X(t+1) = H(t+1) - T(t+1) = H(t) - T(t) + \varepsilon_{t+1} - \eta_{t+1}$

$= X(t) + \underbrace{2\varepsilon_t}$. $\leftarrow$ i.i.d mean 0

So $H_t, T_t, X_t$ are all AR(1).

b. $\mathbb{E}_t(H_{t+1}) = \frac{1}{2} + H_t + \mathbb{E}_t(\varepsilon_{t+1}) = H_t + \frac{1}{2}$.

$\mathbb{E}_t(T_{t+1}) = \frac{1}{2} + T_t + \mathbb{E}_t(\eta_{t+1}) = T_t + \frac{1}{2}$ $\Big\}$ not martingales.

$\mathbb{E}_t(X_{t+1}) = X_t + \mathbb{E}(2\varepsilon_{t+1}) = X_t$ — martingale

c. All AR(1) processes are Markov, so $X_t, H_t, X_t$ are Markov.

d. Note the distribution of $\xi_t^n := (2\varepsilon_t)^n$

if $n$ is even $P(\xi_t^n=1)=1$, and $\mathbb{E}_s[\xi_t^n]=1$ for $s<t$

if $n$ is odd $P(\xi_t=1)=1/2$, $P(\xi_t=-1)=1/2$, $\mathbb{E}_s[\xi_t^n]=0$

Then,
$$\mathbb{E}_t[X_{t+1}^n]=\mathbb{E}_t[(X_t+\xi_{t+1})^n]=\mathbb{E}_t\left(\sum_{k=0}^{n} X_t^{n-k}\xi_{t+1}^k\binom{n}{k}\right)=$$

$$=\sum_{k=0}^{n}\mathbb{E}_t\left(X_t^{n-k}\xi_{t+1}^k\binom{n}{k}\right)=\sum_{k=0}^{n}X_t^{n-k}\binom{n}{k}\mathbb{E}_t(\xi_{t+1}^k)=$$

$$=\sum_{\substack{k=0\\ k\,\text{even}}}^{n}X_t^{n-k}\binom{n}{k}=\frac{(X_t+1)^n+(X_t-1)^n}{2}\neq X_t \text{ for } n>1.$$

## Problem 3

IN JUPYTER NOTEBOOK

## Problem 4

a) $dX_t=X_t\alpha\,dt+X_t\sigma(t,X_t)\,d\beta$

By Ito's lemma with $f(t,x)=\log(x)$,

$$d(\log X_t)=\left(X_t\alpha\frac{1}{X_t}-\frac{\sigma^2X_t^2}{2X_t^2}\right)dt+X_t\sigma\frac{1}{X}d\beta$$

$$\underline{d(\log X_t)=\left(\alpha-\frac{\sigma^2}{2}\right)dt+\sigma\,d\beta},$$

b) Intuitively, as $\varphi$ increases, variance should be "shaking more". This should mean that it gets large more often, so there are more outliers. Thus, kurtosis should increase.

c) IN JUPYTER NOTEBOOK