Homework 6. DANIIL GLUKHOVSKIY

Problem 1

Since C is assumed to be positive definite symmetric matrix, it is diagonalizable with orthogonal eigenvectors

i.e. $C = U^T \Lambda U$ with unitary U and

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \phi \\ & \lambda_2 & & \\ & & \ddots & \\ \phi & & & \lambda_n \end{pmatrix}$$ where λ_i's are eigenvectors of U

Then $\max\limits_{\|w\|^2=k} \frac{1}{2} w^T C w = \frac{1}{2} \max\limits_{\|w\|^2=k} (Uw)^T \Lambda (Uw) = \left\{ \begin{array}{l} \text{since} \\ U \text{ is unitary} \end{array} \right\}$

$$= \frac{1}{2} \max_{\|u\|^2=k} u^T \Lambda u = \frac{1}{2} \max_{\|u\|^2=k} \sum u_i^2 \cdot \lambda_i$$

$$\left\{ \begin{array}{l} u = Uw \\ \|u\| = \|w\| = k \end{array} \right.$$

$\sum u_i^2 \lambda$ is a linear combination of eigenvalues with total weight K. It clearly attains its maximum when $u_i^2 = \begin{cases} K & \text{if } \lambda_i \text{ is largest eig.} \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{\text{That means that } w = U^T u = \pm\sqrt{K}\, v_{max}, \text{ where } v_{max} \text{ is an eigenvector of } C \text{ corresponding to largest eigenvalue}}$$

Problem 2

By Bayes' Rule, $P(\text{turbulent} \mid r = 1.03^4 - 1) =$

$$= \frac{P(r = 1.03^4 - 1 \mid \text{turbulent}) \cdot P(\text{turbulent})}{P(r = 1.03^4 - 1)} =$$

$$= \frac{P(r = 1.03^4 - 1 \mid \text{turbulent})\, P(\text{turbulent})}{P(r = 1.03^4 - 1 \mid \text{turbulent})\, P(\text{turbulent}) + P(r = 1.03^4 - 1 \mid \text{non turb})\, P(\text{non turb})} =$$

$$= \frac{0.05 \cdot \binom{6}{4} / 64}{0.05 \cdot \binom{6}{4} / 64 + 0.95 \cdot \binom{6}{5} / 64} = \frac{0.75}{0.75 + 5.7} \approx \boxed{11.63\%}$$