|HW 9| |DANIIL GLUKHOVSKIY|

Problem 2

Differentiating $f(x) := \frac{1}{2} x^T C x - m^T x$, get

$$\nabla f(x) = Cx - m^T.$$

We estimate partial derivative of f w.r.t. x_1:

$$\frac{\partial f}{\partial x_1} = 0.5113x_1 + 0.2451x_2 + 0.2175x_3 - 1.2938 \leq$$

$$\leq |0.5113x_1| + |0.2451x_2| + |0.2175x_3| - 1.2938 \leq$$

$$\leq 0.5113 + 0.2451 + 0.2175 - 1.2938 < 0,$$

so, unless $x_1 = 1$, we could increase x_1 and leave x_2 and x_3 the same in order to decrease f. Thus minimum is achieved with $x_1 = 1$.

Similarly, estimate $\frac{\partial f}{\partial x_2}$:

$$\frac{\partial f}{\partial x_2} = 0.2175x_1 + 0.1144x_2 + 0.4094x_3 + 1.1919 \geq$$

$$\geq -|0.2175x_1| - |0.1144x_2| - |0.4094x_3| + 1.1919 \geq$$

$$\geq -0.2175 - 0.1144 - 0.4094 + 1.1919 > 0,$$

so, unless $x_3 = -1$, we could decrease x_3 in order to decrease f. Thus minimum is achieved with $x_3 = -1$

Finally, with determined $x_1 = 1$ and $x_3 = -1$,

$$\frac{\partial f}{\partial x_2} = 0.2451x_1 + 0.3551x_2 + 0.1144x_3 - 0.1307 =$$

$$= 0.2451 + 0.3551x_2 - 0.1144 - 0.1307 =$$

$$= 0.3551x_2.$$

Thus minimum could not be achieved when $x_2 < 0$ (as we could decrease f by increasing x_2) or when $x_2 > 0$ (could decrease f by decreasing x_2). Thus minimum is at $x_2 = 0$.

That is, $x_{sol} = (1, 0, -1)$ is a minimizer.

Problem 3

Original problem is equivalent to the following:

$$\underset{y}{\text{minimize}}\ k^T y \text{ subject to } y \leq 0, \quad (*).$$
where $k = A^{-1^T} m$

Indeed, consider a bijection $\varphi : x \longrightarrow Ax - b$ from the set of admissible values of the original problem to the set of admissible values of problem (*).

- It is bijection since A is invertible
- $k^T\varphi(x) = k^T(Ax - b) = m^T A^{-1}(Ax - b) = m^T x - m^T A^{-1} b$, so objective funtion of original problem evaluated at x differs from objective function of problem (*) evaluated at $\varphi(x)$ by constant.

Thus, if x^* is a minimizer of the original problem, $y^* = \varphi(x^*)$ is a minimizer of problem (*) and vice versa. So we could solve problem (*).

minimize $k^T y$ subject to $y \le 0$. $(*)$

- If any entry of k is positive, minimizer would not exist. Indeed, suppose j-th entry of k is positive number a. Then $k^T \begin{pmatrix} 0 \\ \vdots \\ -n \\ \vdots \\ 0 \end{pmatrix} = -an \to -\infty$ as $n \to \infty$, and $y_n = \begin{pmatrix} 0 \\ \vdots \\ -n \\ \vdots \\ 0 \end{pmatrix}$ is admissible.

  j-th entry

- Thus, for minimizer to exist we must have $k \le 0$.. Then $\min_{y \le 0} k^T y = 0$ is achieved, in particular, at $y^* = 0$.

Thus original problem has finite minimizer whenever $\underbrace{k = (A^{-1})^T m \le 0}$. Optimal value of objective function is then $\underbrace{m^T A^{-1} b}$, achieved, in particular, at $x^* = \varphi^{-1}(0) = A^{-1} b$.

Problem 4

a) $x \in [2,4]$ are feasible (since quadratic function with positive leading coeff. is negative between its roots).

Minimal value is 5, achieved at $x = 2$ (since x^2+1 is increasing).

b) Lagrange dual function:

$$L(\lambda) = \inf_{x \in [2,4]} \{ x^2 + 1 + \lambda (x-2)(x-4) \}$$

For $\lambda > 0$ parabola points up, so we perform the minimization as following: $\frac{d}{dx}\left(x^2+1+\lambda(x-2)(x-4)\right)=0$.

$\Leftrightarrow 2x+2\lambda x-6\lambda=0 \Leftrightarrow x=\frac{3\lambda}{1+\lambda}$. This extremum point is between 2 and 4 if $2 \leq \frac{3\lambda}{1+\lambda} \leq 4 \Leftrightarrow$

$\Leftrightarrow 2+2\lambda \leq 3\lambda \leq 4+4\lambda \Leftrightarrow \lambda \geq 2$. So for $\lambda \geq 2$ minimum is achieved at $\frac{3\lambda}{1+\lambda}$ and $L(\lambda)=\left(\frac{3\lambda}{1+\lambda}\right)^2+1+\lambda\left(\frac{\lambda-2}{1+\lambda}\right)\left(\frac{-\lambda-4}{1+\lambda}\right)=$

$$=\frac{9\lambda^2+1+2\lambda+\lambda^2-\lambda^3-2\lambda^2+8\lambda}{(1+\lambda)^2}=\frac{-\lambda^3+8\lambda^2+10\lambda+1}{(1+\lambda)^2}=\frac{1+9\lambda-\lambda^2}{1+\lambda}$$

For $0 \leq \lambda < 2$, $\frac{3\lambda}{1+\lambda}<2$, so function is increasing on $[2,4]$. Then minimum is at $x=2$, so $L(\lambda)=5$.

Summarizing, $L(\lambda)=\begin{cases}\frac{1+9\lambda-\lambda^2}{1+\lambda}, & \text{if } \lambda \geq 2\\ 5, & \text{if } 0 \leq \lambda < 2\end{cases}$

Now solve dual problem:

minimize $-L(\lambda)$ subject to $\lambda \geq 0$

$\frac{dL}{d\lambda}=0 \Leftrightarrow \left[\begin{array}{l}0<\lambda \leq 2 \text{ or}\\ (-2\lambda+9)(1+\lambda)-(1+9\lambda-\lambda^2)=0,\ \lambda>2\end{array}\right. \Leftrightarrow$

$\Leftrightarrow \left[\begin{array}{l}0 \leq \lambda \leq 2 \text{ or}\\ \left[\begin{array}{l}\lambda=-4 \text{ or}\\ \lambda=2\end{array}\right.\end{array}\right.$ So feasible minima are achieved on $[0,2]$

$-L(2)=5$

$-L(\infty)=+\infty$

Since $\min(\text{dual})=-L(2)=5=\min(\text{original})$, strong duality holds.

Problem 5:

We aim to show that $F(tx+(1-t)y) \le tF(x)+(1-t)F(y)$ for all $t \in [0,1]$. For it write:

$$tF(x)+(1-t)F(y) = t\int_0^x f(s)\,\frac{ds}{x} + (1-t)\int_0^y f(s)\,\frac{ds}{y} = \begin{cases} \omega = s/x \text{ in } 1^{st} \text{ int.} \\ \omega = s/y \text{ in } 2^{nd} \text{ int.} \end{cases}$$

$$= t\int_0^1 f(\omega x)\,d\omega + (1-t)\int_0^1 f(\omega y)\,dy = \int_0^1 tf(\omega x)+(1-t)f(\omega y)\,d\omega \ge$$

$$\underset{\substack{\text{convexity} \\ \text{of } f}}{\ge} \int_0^1 f(t(\omega x)+(1-t)\omega y)\,d\omega \underset{s=\omega(tx+(1-t)y)}{=} \int_0^{tx+(1-t)y} f(s)\,\frac{ds}{tx+(1-t)y} = \frac{1}{tx+(1-t)y}\int_0^{tx+(1-t)y} f(s)\,ds =$$

$= F(tx+(1-t)y)$, as desired. $\square$

Problem 1

Turned out to be the hardest problem, but I managed! "Idee Adyzn" is the image of "Jeff Bezos" under the letter wise action of the backward-shift operator acting on English alphabet.