Homework 11. | DANIIL GLUKHOVSKIY |

Problem 1

$M = rJ + (1-r)I$. Let $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

a) $M\vec{e} = \lambda \vec{e} \iff rJ\vec{e} + (1-r)I\vec{e} = \lambda\vec{e} \iff$

$$\iff r(u^T e)\vec{u} = (\lambda - 1 + r)\vec{e}.$$

For that to happen we must have either

(i) $(u^T e)r = 0 = (\lambda - 1 + r)$

- If $r \neq 0$

This corresponds to eigenvalue $\lambda = 1 - r$ with multiplicity $\dim\{e \in \mathbb{R}^n \mid u^T e = 0\} = n-1$.
Corresponding normalized eigenvectors could be taken to be

$$\vec{e}_i = \begin{pmatrix} \vdots \\ \sqrt{2}/2 \\ \vdots \\ -\sqrt{2}/2 \end{pmatrix} \begin{array}{l} \\ \leftarrow i\text{-th place} \\ \\ \leftarrow n\text{-th place.} \end{array}, \quad i = 1, \dots, n-1$$

- If $r = 0$, $M = I$, eigenvalue 1 has multiplicity n, eigenvectors are normal basis vectors.

(ii) $\vec{e} = \frac{\vec{u}}{\sqrt{n}}$, $r(u^T e) = (\lambda - 1 + r) \cdot \frac{1}{\sqrt{n}}$

$$r\sqrt{n} = (\lambda - 1 + r) \cdot \frac{1}{\sqrt{n}}$$

$$\lambda = 1 - r + rn$$

Thus eigenvalue $\lambda = 1 - r + rn$ has multiplicity one and eigenvector $\vec{e}_n = \frac{\vec{u}}{\sqrt{n}}$.

Note that case $r=0$ reduces to general one, so final answer is

$$\begin{cases} \lambda_i = 1 - r, & \vec{e}_i = \left(0\,0 \ldots \frac{\sqrt{2}}{2} \ldots 0 \ldots -\frac{\sqrt{2}}{2}\right)^T, \quad 0 < i < n. \\ \lambda_n = 1 - r + rn, & \vec{e}_n = \frac{\vec{u}}{\sqrt{n}} \end{cases}$$

b) Matrix M is clearly symmetric. Thus, we only need to check when is it positive-semidefinite.

CHOLESKY $\iff$ POSITIVE-SEMIDEF $\iff \lambda_i \geq 0 \iff$

$$\iff \begin{cases} 1 - r \geq 0 \\ 1 - r + rn \geq 0 \end{cases} \iff \begin{cases} r \leq 1 \\ 1 + r(n-1) \geq 0 \end{cases} \iff \begin{cases} r \leq 1, \\ r \geq \frac{1}{n-1} \end{cases}$$

(For $n = 1$ matrix is just (1) and there are no conditions).

c) $\begin{pmatrix} 1 & r & r & r & \\ r & 1 & r & r & \\ r & r & 1 & r & \\ r & r & r & 1 & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & \emptyset & \\ a_{31} & a_{32} & a_{33} & & \\ a_{41} & a_{42} & a_{43} & a_{44} & \\ - & - & - & - & - \\ - & - & & & \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} & a_{41} & | \\ & a_{22} & a_{32} & a_{42} & | \\ & & a_{33} & a_{43} & | \\ & & & a_{44} & | \\ & \emptyset & & & \end{pmatrix}$

- $a_{11} \cdot a_{11} = 1 \Rightarrow a_{11} = 1$ (non-negative in Cholesky)
- $a_{21} \cdot a_{11} = r \Rightarrow a_{21} = r$
- $a_{21}^2 + a_{22}^2 = 1 \Rightarrow a_{22} = \sqrt{1-r^2}$
- $a_{31} \cdot a_{11} = r \Rightarrow a_{31} = r$
- $a_{31} \cdot a_{21} + a_{32} \cdot a_{22} = r \Rightarrow a_{32} = \dfrac{r-r^2}{\sqrt{1-r^2}}$

- $a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \Rightarrow a_{33} = \sqrt{1 - r^2 - \dfrac{(r-r^2)^2}{1-r^2}} =$

$$= \sqrt{\frac{(1-r^2)^2 - (r-r^2)^2}{1-r^2}} = \sqrt{\frac{((1+r)^2 - r^2)(1-r)}{(1+r)}} = \sqrt{\frac{(2r+1)(1-r)}{(1+r)}}$$

- $a_{41} \cdot a_{11} = r \Rightarrow a_{41} = r$
- $a_{41} \cdot a_{21} + a_{42} \cdot a_{22} = r \Rightarrow a_{42} = \dfrac{r-r^2}{\sqrt{1-r^2}}$
- $a_{41} \cdot a_{31} + a_{42} \cdot a_{32} + a_{43} \cdot a_{33} = r \Rightarrow$

$$\Rightarrow a_{43} = \frac{r - r^2 - \frac{(r-r^2)^2}{1-r^2}}{\sqrt{\frac{(2r+1)(1-r)}{(1+r)}}} = r\sqrt{\frac{1-r}{(2r+1)(1+r)}}$$

$\cdot\ a_{41}^2 + a_{42}^2 + a_{43}^2 + a_{44}^2 = 1 \Rightarrow$

$$\Rightarrow a_{44} = \sqrt{1 - r^2 - \frac{(r-r^2)^2}{1-r^2} - r^2 \frac{1-r}{(2r+1)(1+r)}} =$$

$$= \sqrt{\frac{(2r+1)(1-r)}{1+r} - r^2 \frac{1-r}{(2r+1)(1+r)}} =$$

$$= \sqrt{\frac{(1-r)(3r^2+4r+1)}{(2r+1)(1+r)}} = \sqrt{\frac{(1-r)(3r+1)}{(2r+1)}}$$

Problem 2

a) First solve as in (6.21) for each currency, with $\mu = 0$, $\sigma = 1$, $\omega_1 = 0.95$

$$\begin{cases} K = 3\dfrac{(0.95\sigma_1^4 + 0.05\sigma_2^4)}{(0.95\sigma_1^2 + 0.05\sigma_2^2)} - 3 \\ 1 = 0.95\sigma_1^2 + 0.05\sigma_2^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0.95\sigma_1^4 + 0.05\sigma_2^4 = \dfrac{K}{3} + 1 \\ 0.95\sigma_1^2 + 0.05\sigma_2^2 = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0.95\sigma_1^2 + 0.05\sigma_2^2 = 1 \\ 0.95\sigma_1^4 + 0.05\left[\dfrac{1 - 0.95\sigma_1^2}{0.05}\right]^2 = \dfrac{K}{3} + 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \sigma_2^2 = \dfrac{1-0.95\sigma_1^2}{0.05} = 20 - 19\sigma_1^2 \\ 19\sigma_1^4 - 38\sigma_1^2 + \left(20 - \frac{k}{3} - 1\right) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \sigma_1^2 = \dfrac{38 \pm \sqrt{38^2 - 4 \times 19 \left(20 - \frac{k}{3} - 1\right)}}{2 \cdot 19} \\ \sigma_2^2 = 20 - 19\sigma_1^2 \end{cases} \quad (*)$$

We then compute empirical kurtosis to be

CHF: 14.2741 (see notebook for computation)
GBP: 7.2467
JPY: 9.522

Plugging in (see notebook again):

CHF: $\sigma_1 = 0.7068$, $\sigma_2 = 3.2416$
GBP: $\sigma_1 = 0.8021$, $\sigma_2 = 2.7883$
JPY: $\sigma_1 = 0.7689$, $\sigma_2 = 2.9607$.

Part b) is coded in notebook

Problem 3

a) Process is stationary if all roots of characteristic equation have absolute value smaller then 1.

In our case, characteristic equation is

$$z^2 - \frac{1}{2}z - b = 0 \iff z = \frac{1/2 \pm \sqrt{1/4 + 4b}}{2} = \frac{1}{4} \pm \sqrt{\frac{1}{16} + b}$$

If $b \geq -1/16$, $|z| < 1 \iff \left| \frac{1}{4} \pm \sqrt{\frac{1}{16}+b} \right| < 1 \iff$

$$\iff -1 < \frac{1}{4} \pm \sqrt{\frac{1}{16}+b} < 1 \iff -5/4 < \pm\sqrt{\frac{1}{16}+b} < 3/4 \iff$$

$$\iff \frac{1}{16} + b < \frac{9}{16} \iff b < \frac{1}{2}$$

If $b < -1/16$, $|z| < 1 \iff \left(\frac{1}{4}\right)^2 - \left(b + \frac{1}{16}\right) < 1 \iff b > -1$

Combining, process is stationary iff

$$b \in (-1, 1/2).$$

b) We write Yule-Walker equations:

$$\begin{pmatrix} r(1) \\ r(2) \end{pmatrix} = \begin{pmatrix} 1 & r(1) \\ r(1) & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ b \end{pmatrix}$$

$$\begin{cases} r(1) = \frac{1}{2} + r(1) \cdot b \\ r(2) = \frac{1}{2} r(1) + b \end{cases} \iff \begin{cases} r(1) = \frac{1}{2(1-b)} \\ r(2) = b + \frac{1}{4(1-b)} \end{cases}$$

Expected correlation between X_{100} and X_{98} is

$$r(100-98) = b + \frac{1}{4(1-b)}$$

Problem 4

$$X_t = X_{t-1} + \varepsilon_t . \quad \mathbb{E}(X_t) = X_{t-1}$$

a) We compute expectation at time t-1.

$$\mathbb{E}(X_t^3) = \mathbb{E}((X_{t-1}+\varepsilon_t)^3) = \mathbb{E}(X_{t-3}^{3}) + 3\,\mathbb{E}(X_t^2\varepsilon_t) +$$

$$+ 3\,\mathbb{E}(X_t\varepsilon_t^2) + \mathbb{E}(\varepsilon_t^3) = X_{t-1}^3 + 3X_{t-1}^3\underbrace{\mathbb{E}(\varepsilon_t)}_{\substack{0 \\ \text{(mean of normal)}}} + 3X_{t-1}\underbrace{\mathbb{E}(\varepsilon_t^2)}_{\substack{\sigma^2 \text{ (variance} \\ \text{of normal).}}} +$$

$$+ \underbrace{\mathbb{E}(\varepsilon_t^3)}_{\substack{0 \text{ (skewness} \\ \text{of normal)}}} = X_{t-1}^3 + 3X_{t-1}\sigma^2$$

Thus X_t^3 is not a martingale.

b) At time t-1:

$$\mathbb{E}(X_t^3 - btX_t) = \mathbb{E}(X_t^3) - bt\,\mathbb{E}(X_t) =$$

$= X_{t-1}^3 + 3X_{t-1}\sigma^2 - btX_{t-1}$. For it to be a martingale, we must have for all t

$$\mathbb{E}(X_t^3 - btX_t) = X_{t-1}^3 - b(t-1)X_{t-1}, \text{ so}$$

$$X_{t-1}^3 + X_{t-1}(3\sigma^2 - bt) = X_{t-1}^3 + X_t(-b(t-1))$$

$$3\sigma^2 - bt = -bt + b \iff 3\sigma^2 = b$$