SAMPLE CRITERIA FOR TESTING OUTLYING OBSERVATIONS[1]

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1. Summary. The problem of testing outlying observations, although an old one, is of considerable importance in applied statistics. Many and various types of significance tests have been proposed by statisticians interested in this field of application. In this connection, we bring out in the Historical Comments notable advances toward a clear formulation of the problem and important points which should be considered in attempting a complete solution. In Section 4 we state some of the situations the experimental statistician will very likely encounter in practice, these considerations being based on experience. For testing the significance of the largest observation in a sample of size n from a normal population, we propose the statistic

$$\frac{S_n^2}{S^2} = \frac{\sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

where $x_1 \leq x_2 \leq \cdots \leq x_n$, $\bar{x}_n = \frac{1}{n-1} \sum_{i=1}^{n-1} x_i$ *and* $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

A similar statistic, S_1^2/S^2, can be used for testing whether the smallest observation is too low.

It turns out that

$$\frac{S_n^2}{S^2} = 1 - \frac{1}{n-1}\left(\frac{x_n - \bar{x}}{s}\right)^2 = 1 - \frac{1}{n-1} T_n^2,$$

where $s^2 = \frac{1}{n} \Sigma(x_i - \bar{x})^2$, and T_n is the studentized extreme deviation already suggested by E. Pearson and C. Chandra Sekar [1] for testing the significance of the largest observation. Based on previous work by W. R. Thompson [12], Pearson and Chandra Sekar were able to obtain certain percentage points of T_n without deriving the exact distribution of T_n. The exact distribution of S_n^2/S^2 (or T_n) is apparently derived for the first time by the present author.

For testing whether the two largest observations are too large we propose the statistic

$$\frac{S_{n-1,n}^2}{S^2} = \frac{\sum_{i=1}^{n-2} (x_i - \bar{x}_{n-1,n})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}, \qquad \bar{x}_{n-1,n} = \frac{1}{n-2} \sum_{i=1}^{n-2} x_i$$

[1] This paper has been extracted from a thesis approved for the Degree of PhD at the University of Michigan.

and a similar statistic, $S_{1,2}^2/S^2$, can be used to test the significance of the two smallest observations. The probability distributions of the above sample statistics

TABLE I

Table of Percentage Points for $\frac{S_n^2}{S^2}$ *or* $\frac{S_1^2}{S^2}$

Percentage Points

n	1%	2.5%	5%	10%
3	.0001	.0007	.0027	.0109
4	.0100	.0248	.0494	.0975
5	.0442	.0808	.1270	.1984
6	.0928	.1453	.2032	.2826
7	.1447	.2066	.2696	.3503
8	.1948	.2616	.3261	.4050
9	.2411	.3101	.3742	.4502
10	.2831	.3526	.4154	.4881
11	.3211	.3901	.4511	.5204
12	.3554	.4232	.4822	.5483
13	.3864	.4528	.5097	.5727
14	.4145	.4792	.5340	.5942
15	.4401	.5030	.5559	.6134
16	.4634	.5246	.5755	.6306
17	.4848	.5442	.5933	.6461
18	.5044	.5621	.6095	.6601
19	.5225	.5785	.6243	.6730
20	.5393	.5937	.6379	.6848
21	.5548	.6076	.6504	.6958
22	.5692	.6206	.6621	.7058
23	.5827	.6327	.6728	.7151
24	.5953	.6439	.6829	.7238
25	.6071	.6544	.6923	.7319

$$S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad \text{where} \quad \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$S_n^2 = \sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2 \quad \text{where} \quad \bar{x}_n = \frac{1}{n-1} \sum_{i=1}^{n-1} x_i$$

$$S_1^2 = \sum_{i=2}^{n} (x_i - \bar{x}_1)^2 \quad \text{where} \quad \bar{x}_1 = \frac{1}{n-1} \sum_{i=2}^{n} x_i$$

are derived for a normal parent and tables of appropriate percentage points are given in this paper (Table I and Table V). Although the efficiencies of the above tests have not been completely investigated under various models for outlying observations, it is apparent that the proposed sample criteria have considerable intuitive appeal. In deriving the distributions of the sample statistics for testing the largest (or smallest) or the two largest (or two smallest) observations, it was first necessary to derive the distribution of the difference between the extreme observation and the sample mean in terms of the population σ. This probability

TABLE IA

Table of Percentage Points for $T_n = \frac{x_n - \bar{x}}{s}$ *or* $T_1 = \frac{\bar{x} - x_1}{s}$

n	1%	2.5%	5%	10%
3	1.414	1.414	1.412	1.406
4	1.723	1.710	1.689	1.645.
5	1.955	1.917	1.869	1.791
6	2.130	2.067	1.996	1.894
7	2.265	2.182	2.093	1.974
8	2.374	2.273	2 172	2.041
9	2.464	2.349	2.237	2.097
10	2.540	2.414	2.294	2.146
11	2.606	2.470	2.343	2.190
12	2.663	2.519	2.387	2.229
13	2.714	2.562	2.426	2.264
14	2.759	2.602	2.461	2.297
15	2.800	2.638	2.493	2.326
16	2.837	2.670	2.523	2.354
17	2.871	2.701	2.551	2.380
18	2.903	2.728	2.577	2.404
19	2.932	2.754	2.600	2.426
20	2.959	2.778	2.623	2.447
21	2.984	2.801	2.644	2.467
22	3.008	2.823	2.664	2.486
23	3.030	2.843	2.683	2.504
24	3.051	2.862	2.701	2.520
25	3.071	2.880	2.717	2.537

$$x_1 \leq x_2 \leq x_3 \cdots \leq x_n$$

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \qquad \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

distribution was apparently derived first by A. T. McKay [11] who employed the method of characteristic functions. The author was not aware of the work of McKay when the simplified derivation for the distribution of $\frac{x_n - \bar{x}}{\sigma}$ outlined in Section 5 below was worked out by him in the spring of 1945, McKay's result being called to his attention by C. C. Craig. It has been noted also that K. R. Nair [20] worked out independently and published the same derivation of the distribution of the extreme minus the mean arrived at by the present author—see *Biometrika*, Vol. 35, May, 1948. We nevertheless include part of this derivation in Section 5 below as it was basic to the work in connection with the derivations given in Sections 8 and 9. Our table is considerably more extensive than Nair's table of the probability integral of the extreme deviation from the sample mean in normal samples, since Nair's table runs from $n = 2$ to $n = 9$, whereas our Table II is for $n = 2$ to $n = 25$. The present work is concluded with some examples.

2. Introduction. Scientific data are collected usually for purposes of interpretation and if proper use is to be made of the information thus obtained then some decision should be reached or some action taken as a result of analyzing the data. In many cases a critical examination of the data collected is necessary in order to insure that the results of sampling are representative of the thing or process we are examining. Quite frequently our observations do not appear to be consistent with one another, i.e. the data may seem to display non-homogeneities and the group of observations as a whole may not appear to represent a random sample from, say, a single normal population or universe. In particular, one or more of the observations may have the appearance of being "outliers" and we are interested here in determining once and for all whether such observations should be retained in the sample for interpreting results or whether they should be regarded as being inconsistent with the remaining observations. It is clear that rejection of the "outliers" in a sample will in a great number of cases lead to a different course of action than would have been taken had such observations been retained in the sample. Actually, the rejection of "outlying" observations may be just as much a practical (or common sense) problem as a statistical one and sometimes the practical or experimental viewpoint may naturally outweigh any statistical contributions. In this connection, the concluding remarks of Rider's survey [2] are pertinent: "In the final analysis it would seem that the question of the rejection or the retention of a discordant observation reduces to a question of common sense. Certainly the judgment of an experienced observer should be allowed considerable influence in reaching a decision. This judgment can undoubtedly be aided by the application of one or more tests based on the theory of probability, but any test which requires an inordinate amount of calculation seems hardly to be worth while, and the testimony of any criterion which is based upon a complicated hypothesis should be accepted with extreme caution." Hence, it would appear that statistical tests of significance for judging or testing "outliers" come into importance either in supporting doubtful practical viewpoints or in providing a basis for action in the absence of sufficient experimental knowledge of underlying causes in an investigation. Indeed, the latter two situations are met quite frequently in practice.

In the present treatment, we intend to throw some light beyond the work

TABLE II

Probability Integral of the Extreme Minus the Mean, u_n, in Normal Samples of n Observations (Pop. S.D. as unit) $P(u_n \leq u)$

u \ n	2	3	4	5	6	7	8	9	u
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00
.05	.05637	.00309	.00017	.00001	.00000	.00000	.00000	.00000	.05
.10	.11246	.01231	.00134	.00015	.00002	.00000	.00000	.00000	.10
.15	.16800	.02745	.00445	.00072	.00012	.00002	.00000	.00000	.15
.20	.22270	.04817	.01033	.00221	.00047	.00010	.00002	.00000	.20
.25	.27633	.07403	.01966	.00520	.00137	.00036	.00010	.00003	.25
.30	.32863	.10450	.03292	.01033	.00324	.00101	.00032	.00010	.30
.35	.37938	.13896	.05040	.01820	.00656	.00236	.00085	.00031	.35
.40	.42839	.17677	.07218	.02935	.01191	.00482	.00195	.00079	.40
.45	.47548	.21724	.09816	.04416	.01982	.00889	.00398	.00178	.45
.50	.52050	.25968	.12807	.06288	.03080	.01507	.00737	.00360	.50
.55	.56332	.30344	.16152	.08559	.04525	.02390	.01261	.00665	.55
.60	.60386	.34788	.19801	.11219	.06344	.03583	.02022	.01140	.60
.65	.64203	.39243	.23697	.14246	.08547	.05121	.03067	.01836	.65
.70	.67780	.43656	.27781	.17602	.11130	.07030	.04437	.02800	.70
.75	.71116	.47983	.31992	.21242	.14076	.09318	.06164	.04076	.75
.80	.74210	.52185	.36274	.25113	.17353	.11978	.08263	.05698	.80
.85	.77067	.56230	.40571	.29160	.20920	.14993	.10739	.07688	.85
.90	.79691	.60095	.44835	.33325	.24727	.18329	.13578	.10055	.90
.95	.82089	.63761	.49021	.37555	.28721	.21945	.16757	.12791	.95
1.00	.84270	.67214	.53093	.41795	.32847	.25791	.20240	.15877	1.00
1.05	.86244	.70448	.57020	.45999	.37050	.29815	.23980	.19280	1.05
1.10	.88021	.73459	.60777	.50125	.41276	.33961	.27927	.22957	1.10
1.15	.89612	.76248	.64346	.54136	.45478	.38173	.32025	.26858	1.15
1.20	.91031	.78817	.67713	.58001	.49611	.42401	.36220	.30931	1.20
1.25	.92290	.81174	.70870	.61697	.53638	.46595	.40457	.35117	1.25
1.30	.93401	.83325	.73812	.65205	.57525	.50712	.44685	.39362	1.30
1.35	.94376	.85280	.76540	.68513	.61249	.54716	.48857	.43613	1.35
1.40	.95229	.87049	.79055	.71612	.64788	.58574	.52933	.47822	1.40
1.45	.95970	.88644	.81364	.74497	.68129	.62263	.56878	.51945	1.45
1.50	.96611	.90075	.83472	.77170	.71261	.65762	.60663	.55944	1.50
1.55	.97162	.91355	.85390	.79632	.74180	.69058	.64265	.59789	1.55
1.60	.97635	.92495	.87127	.81890	.76885	.72143	.67668	.63456	1.60
1.65	.98038	.93506	.88693	.83949	.79378	.75013	.70862	.66925	1.65
1.70	.98379	.94400	.90099	.85820	.81664	.77666	.73839	.70184	1.70
1.75	.98667	.95187	.91358	.87513	.83750	.80107	.76597	.73225	1.75
1.80	.98909	.95877	.92480	.89037	.85646	.82341	.79139	.76046	1.80
1.85	.99111	.96480	.93476	.90405	.87360	.84376	.81469	.78647	1.85
1.90	.99279	.97005	.94358	.91628	.88903	.86220	.83593	.81032	1.90
1.95	.99418	.97461	.95135	.92716	.90288	.87885	.85522	.83207	1.95

TABLE II—*Continued*

u \ n	2	3	4	5	6	7	8	9	u
2.00	.99532	.97854	.95818	.93682	.91526	.89381	.87264	.85183	2.00
2.05	.99626	.98193	.96416	.94536	.92627	.90721	.88832	.86968	2.05
2.10	.99702	.98483	.96938	.95289	.93605	.91916	.90236	.88574	2.10
2.15	.99764	.98731	.97392	.95949	.94468	.92977	.91490	.90012	2.15
2.20	.99814	.98942	.97785	.96527	.95229	.93917	.92604	.91296	2.20
2.25	.99854	.99121	.98125	.97032	.95897	.94746	.93591	.92438	2.25
2.30	.99886	.99273	.98418	.97470	.96482	.95476	.94462	.93448	2.30
2.35	.99911	.99400	.98669	.97850	.96992	.96114	.95229	.94340	2.35
2.40	.99931	.99507	.98883	.98178	.97435	.96672	.95900	.95125	2.40
2.45	.99947	.99596	.99066	.98461	.97819	.97158	.96487	.95812	2.45
2.50	.99959	.99670	.99222	.98703	.98151	.97580	.96999	.96412	2.50
2.55	.99969	.99732	.99353	.98911	.98436	.97944	.97443	.96935	2.55
2.60	.99976	.99782	.99464	.99088	.98681	.98259	.97827	.97389	2.60
2.65	.99982	.99824	.99557	.99238	.98891	.98529	.98158	.97781	2.65
2.70	.99987	.99858	.99635	.99365	.99070	.98761	.98443	.98120	2.70
2.75	.99990	.99886	.99701	.99473	.99223	.98959	.98688	.98411	2.75
2.80	.99992	.99909	.99755	.99564	.99352	.99128	.98897	.98661	2.80
2.85	.99994	.99928	.99800	.99640	.99461	.99272	.99075	.98874	2.85
2.90	.99996	.99943	.99838	.99704	.99553	.99393	.99227	.99056	2.90
2.95	.99997	.99955	.99868	.99757	.99631	.99496	.99355	.99211	2.95
3.00	.99998	.99964	.99894	.99801	.99696	.99582	.99464	.99342	3.00
3.05	.99998	.99972	.99914	.99838	.99750	.99655	.99555	.99453	3.05
3.10	.99999	.99978	.99931	.99868	.99795	.99716	.99632	.99546	3.10
3.15	.99999	.99983	.99945	.99893	.99832	.99766	.99697	.99625	3.15
3.20	.99999	.99987	.99956	.99913	.99863	.99808	.99750	.99690	3.20
3.25	1.00000	.99990	.99965	.99930	.99889	.99843	.99795	.99745	3.25
3.30		.99992	.99972	.99944	.99910	.99872	.99832	.99791	3.30
3.35		.99994	.99978	.99955	.99927	.99896	.99863	.99829	3.35
3.40		.99995	.99983	.99964	.99941	.99916	.99889	.99860	3.40
3.45		.99996	.99986	.99971	.99953	.99932	.99910	.99886	3.45
3.50		.99997	.99989	.99977	.99962	.99945	.99927	.99908	3.50
3.55		.99998	.99992	.99982	.99970	.99956	.99941	.99925	3.55
3.60		.99998	.99994	.99986	.99976	.99965	.99952	.99940	3.60
3.65		.99999	.99995	.99989	.99981	.99972	.99962	.99951	3.65
3.70		.99999	.99996	.99991	.99985	.99977	.99969	.99961	3.70
3.75		.99999	.99997	.99993	.99988	.99982	.99976	.99969	3.75
3.80		1.00000	.99998	.99995	.99991	.99986	.99981	.99975	3.80
3.85			.99998	.99996	.99993	.99989	.99985	.99980	3.85
3.90			.99999	.99997	.99994	.99991	.99988	.99984	3.90
3.95			.99999	.99997	.99995	.99993	.99990	.99987	3.95

TABLE II—*Continued*

u \ n	2	3	4	5	6	7	8	9	u
4.00			.99999	.99998	.99996	.99995	.99992	.99990	4.00
4.05			.99999	.99999	.99997	.99996	.99994	.99992	4.05
4.10			1.00000	.99999	.99998	.99997	.99995	.99994	4.10
4.15				.99999	.99998	.99997	.99996	.99995	4.15
4.20				.99999	.99999	.99998	.99997	.99996	4.20
4.25				.99999	.99999	.99998	.99998	.99997	4.25
4.30				1.00000	.99999	.99999	.99998	.99998	4.30
4.35					.99999	.99999	.99999	.99998	4.35
4.40					1.00000	.99999	.99999	.99999	4.40
4.45						.99999	.99999	.99999	4.45
4.50						1.00000	.99999	.99999	4.50
4.55							1.00000	.99999	4.55
4.60								1.00000	4.60

u \ n	10	11	12	13	14	15	16	17	u
.25	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.25
.30	.00003	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.30
.35	.00011	.00004	.00001	.00001	.00000	.00000	.00000	.00000	.35
.40	.00032	.00013	.00005	.00002	.00001	.00000	.00000	.00000	.40
.45	.00080	.00036	.00016	.00007	.00003	.00001	.00001	.00000	.45
.50	.00176	.00086	.00042	.00021	.00010	.00005	.00002	.00001	.50
.55	.00351	.00185	.00098	.00051	.00027	.00014	.00008	.00004	.55
.60	.00643	.00363	.00204	.00115	.00065	.00037	.00021	.00012	.60
.65	.01098	.00657	.00393	.00235	.00141	.00084	.00050	.00030	.65
.70	.01766	.01113	.00702	.00443	.00279	.00176	.00111	.00070	.70
.75	.02694	.01780	.01177	.00777	.00514	.00339	.00224	.00148	.75
.80	.03928	.02707	.01865	.01285	.00886	.00610	.00420	.00289	.80
.85	.05503	.03938	.02818	.02016	.01442	.01031	.00738	.00527	.85
.90	.07444	.05510	.04077	.03017	.02232	.01652	.01222	.00904	.90
.95	.09761	.07448	.05682	.04334	.03305	.02521	.01922	.01466	.95
1.00	.12452	.09763	.07655	.06000	.04703	.03687	.02889	.02265	1.00
1.05	.15497	.12454	.10008	.08041	.06460	.05190	.04169	.03348	1.05
1.10	.18867	.15503	.12737	.10464	.08595	.07060	.05799	.04762	1.10
1.15	.22520	.18879	.15825	.13263	.11116	.09315	.07806	.06541	1.15
1.20	.26407	.22542	.19240	.16420	.14013	.11957	.10203	.08706	1.20
1.25	.30475	.26442	.22941	.19901	.17263	.14973	.12987	.11264	1.25
1.30	.34666	.30525	.26876	.23662	.20830	.18336	.16140	.14207	1.30
1.35	.38924	.34734	.30992	.27650	.24667	.22005	.19629	.17509	1.35
1.40	.43196	.39011	.35229	.31810	.28721	.25931	.23411	.21135	1.40
1.45	.47430	.43302	.39529	.36082	.32934	.30058	.27433	.25036	1.45

TABLE II—*Continued*

u \ n	10	11	12	13	14	15	16	17	u
1.50	.51583	.47555	.43838	.40408	.37244	.34327	.31636	.29156	1.50
1.55	.55615	.51726	.48104	.44733	.41595	.38676	.35960	.33434	1.55
1.60	.59495	.55774	.52282	.49004	.45930	.43046	.40342	.37807	1.60
1.65	.63196	.59668	.56332	.53178	.50199	.47384	.44726	.42216	1.65
1.70	.66699	.63380	.60221	.57216	.54358	.51641	.49058	.46602	1.70
1.75	.69991	.66892	.63925	.61086	.58370	.55773	.53289	.50915	1.75
1.80	.73063	.70189	.67424	.64763	.62204	.59744	.57380	.55108	1.80
1.85	.75912	.73264	.70704	.68229	.65838	.63528	.61297	.59144	1.85
1.90	.78538	.76113	.73758	.71472	.69254	.67102	.65016	.62992	1.90
1.95	.80945	.78737	.76584	.74486	.72443	.70453	.68516	.66630	1.95
2.00	.83141	.81140	.79183	.77269	.75399	.73571	.71786	.70042	2.00
2.05	.85133	.83330	.81560	.79824	.78121	.76453	.74819	.73218	2.05
2.10	.86932	.85314	.83721	.82155	.80614	.79101	.77614	.76153	2.10
2.15	.88550	.87105	.85678	.84271	.82885	.81519	.80174	.78849	2.15
2.20	.89998	.88713	.87440	.86183	.84941	.83715	.82505	.81311	2.20
2.25	.91290	.90151	.89021	.87902	.86795	.85699	.84616	.83545	2.25
2.30	.92437	.91431	.90432	.89441	.88458	.87484	.86518	.85563	2.30
2.35	.93453	.92568	.91688	.90812	.89943	.89081	.88224	.87375	2.35
2.40	.94348	.93572	.92799	.92030	.91264	.90504	.89748	.88997	2.40
2.45	.95134	.94457	.93781	.93106	.92435	.91766	.91101	.90440	2.45
2.50	.95823	.95233	.94644	.94055	.93468	.92883	.92300	.91720	2.50
2.55	.96424	.95912	.95400	.94887	.94376	.93866	.93357	.92850	2.55
2.60	.96948	.96504	.96060	.95616	.95172	.94728	.94285	.93844	2.60
2.65	.97401	.97019	.96635	.96251	.95866	.95482	.95098	.94715	2.65
2.70	.97793	.97464	.97134	.96802	.96471	.96139	.95807	.95475	2.70
2.75	.98131	.97849	.97565	.97280	.96995	.96709	.96423	.96137	2.75
2.80	.98422	.98180	.97937	.97693	.97448	.97203	.96957	.96712	2.80
2.85	.98671	.98464	.98257	.98048	.97839	.97629	.97418	.97208	2.85
2.90	.98883	.98708	.98531	.98353	.98174	.97995	.97816	.97636	2.90
2.95	.99064	.98915	.98765	.98614	.98462	.98309	.98156	.98003	2.95
3.00	.99218	.99092	.98965	.98837	.98708	.98578	.98448	.98318	3.00
3.05	.99348	.99242	.99134	.99026	.98917	.98807	.98697	.98587	3.05
3.10	.99458	.99369	.99278	.99187	.99095	.99002	.98909	.98816	3.10
3.15	.99551	.99476	.99400	.99323	.99245	.99167	.99089	.99010	3.15
3.20	.99628	.99566	.99502	.99437	.99372	.99307	.99241	.99175	3.20
3.25	.99694	.99641	.99588	.99534	.99479	.99424	.99369	.99314	3.25
3.30	.99748	.99704	.99660	.99615	.99569	.99523	.99477	.99431	3.30
3.35	.99793	.99757	.99720	.99682	.99644	.99606	.99568	.99529	3.35
3.40	.99831	.99801	.99770	.99739	.99707	.99676	.99644	.99611	3.40
3.45	.99862	.99837	.99812	.99786	.99760	.99733	.99707	.99680	3.45

TABLE II—*Continued*

u \ n	10	11	12	13	14	15	16	17	u
3.50	.99888	.99867	.99846	.99825	.99803	.99781	.99759	.99737	3.50
3.55	.99909	.99892	.99875	.99857	.99839	.99821	.99803	.99785	3.55
3.60	.99926	.99912	.99898	.99884	.99869	.99854	.99839	.99824	3.60
3.65	.99940	.99929	.99917	.99906	.99894	.99881	.99869	.99857	3.65
3.70	.99952	.99943	.99933	.99924	.99914	.99904	.99894	.99883	3.70
3.75	.99961	.99954	.99946	.99938	.99930	.99922	.99914	.99905	3.75
3.80	.99969	.99963	.99957	.99950	.99944	.99937	.99930	.99923	3.80
3.85	.99975	.99970	.99965	.99960	.99955	.99949	.99944	.99938	3.85
3.90	.99980	.99976	.99972	.99968	.99964	.99959	.99955	.99950	3.90
3.95	.99984	.99981	.99978	.99974	.99971	.99967	.99964	.99960	3.95
4.00	.99988	.99985	.99982	.99980	.99977	.99974	.99971	.99968	4.00
4.05	.99990	.99988	.99986	.99984	.99982	.99979	.99977	.99974	4.05
4.10	.99992	.99991	.99989	.99987	.99985	.99983	.99981	.99979	4.10
4.15	.99994	.99993	.99991	.99990	.99988	.99987	.99985	.99984	4.15
4.20	.99995	.99994	.99993	.99992	.99991	.99990	.99988	.99987	4.20
4.25	.99996	.99995	.99995	.99994	.99993	.99992	.99991	.99990	4.25
4.30	.99997	.99996	.99996	.99995	.99994	.99993	.99993	.99992	4.30
4.35	.99998	.99997	.99997	.99996	.99996	.99995	.99994	.99993	4.35
4.40	.99998	.99998	.99997	.99997	.99996	.99996	.99995	.99995	4.40
4.45	.99999	.99998	.99998	.99998	.99997	.99997	.99996	.99996	4.45
4.50	.99999	.99999	.99998	.99998	.99998	.99998	.99997	.99997	4.50
4.55	.99999	.99999	.99999	.99999	.99998	.99998	.99998	.99997	4.55
4.60	.99999	.99999	.99999	.99999	.99999	.99998	.99998	.99998	4.60
4.65	1.00000	.99999	.99999	.99999	.99999	.99999	.99999	.99998	4.65
4.70		1.00000	.99999	.99999	.99999	.99999	.99999	.99999	4.70
4.75			1.00000	1.00000	.99999	.99999	.99999	.99999	4.75
4.80					1.00000	.99999	.99999	.99999	4.80
4.85						1.00000	1.00000	1.00000	4.85

u \ n	18	19	20	21	22	23	24	25	u
.50	.00001	.00000	.0000	.0000	.0000	.0000	.0000	.0000	.50
.55	.00002	.00001	.0000	.0000	.0000	.0000	.0000	.0000	.55
.60	.00007	.00004	.0000	.0000	.0000	.0000	.0000	.0000	.60
.65	.00018	.00011	.0001	.0000	.0000	.0000	.0000	.0000	.65
.70	.00044	.00028	.0002	.0001	.0001	.0000	.0000	.0000	.70
.75	.00098	.00065	.0004	.0003	.0002	.0001	.0001	.0001	.75
.80	.00199	.00137	.0009	.0007	.0004	.0003	.0002	.0001	.80
.85	.00377	.00270	.0019	.0014	.0010	.0007	.0005	.0004	.85
.90	.00669	.00494	.0037	.0027	.0020	.0015	.0011	.0008	.90
.95	.01118	.00853	.0065	.0049	.0038	.0029	.0022	.0017	.95

TABLE II—*Continued*

u \ n	18	19	20	21	22	23	24	25	u
1.00	.01775	.01391	.0109	.0085	.0067	.0052	.0041	.0032	1.00
1.05	.02690	.02161	.0174	.0139	.0112	.0090	.0072	.0058	1.05
1.10	.03911	.03212	.0264	.0217	.0178	.0146	.0120	.0099	1.10
1.15	.05481	.04592	.0385	.0322	.0270	.0226	.0190	.0159	1.15
1.20	.07428	.06338	.0541	.0461	.0394	.0336	.0287	.0244	1.20
1.25	.09769	.08472	.0735	.0637	.0553	.0479	.0416	.0360	1.25
1.30	.12504	.11005	.0969	.0853	.0750	.0660	.0581	.0512	1.30
1.35	.15618	.13930	.1242	.1108	.0988	.0882	.0786	.0701	1.35
1.40	.19080	.17225	.1555	.1404	.1267	.1144	.1033	.0932	1.40
1.45	.22848	.20851	.1903	.1736	.1585	.1446	.1320	.1204	1.45
1.50	.26869	.24761	.2282	.2103	.1938	.1786	.1646	.1516	1.50
1.55	.31084	.28899	.2687	.2498	.2322	.2159	.2007	.1866	1.55
1.60	.35430	.33202	.3111	.2916	.2732	.2560	.2399	.2248	1.60
1.65	.39845	.37607	.3549	.3349	.3162	.2984	.2816	.2658	1.65
1.70	.44269	.42052	.3994	.3794	.3604	.3424	.3252	.3089	1.70
1.75	.48645	.46476	.4440	.4242	.4053	.3872	.3699	.3534	1.75
1.80	.52924	.50827	.4881	.4687	.4502	.4323	.4152	.3987	1.80
1.85	.57065	.55058	.5312	.5125	.4945	.4771	.4603	.4441	1.85
1.90	.61031	.59130	.5729	.5549	.5377	.5209	.5047	.4890	1.90
1.95	.64796	.63011	.6127	.5958	.5794	.5634	.5479	.5328	1.95
2.00	.68340	.66678	.6506	.6348	.6193	.6042	.5895	.5752	2.00
2.05	.71650	.70114	.6861	.6714	.6570	.6429	.6291	.6156	2.05
2.10	.74719	.73311	.7193	.7058	.6924	.6793	.6665	.6540	2.10
2.15	.77545	.76262	.7500	.7375	.7254	.7133	.7015	.6899	2.15
2.20	.80132	.78971	.7782	.7670	.7558	.7448	.7340	.7234	2.20
2.25	.82486	.81440	.8041	.7938	.7838	.7738	.7640	.7543	2.25
2.30	.84616	.83679	.8275	.8184	.8093	.8003	.7914	.7827	2.30
2.35	.86533	.85699	.8487	.8405	.8324	.8244	.8164	.8085	2.35
2.40	.88251	.87511	.8678	.8605	.8533	.8461	.8390	.8319	2.40
2.45	.89783	.89129	.8848	.8784	.8720	.8656	.8593	.8530	2.45
2.50	.91142	.90568	.9000	.8943	.8887	.8831	.8775	.8719	2.50
2.55	.92345	.91842	.9134	.9084	.9035	.8985	.8936	.8888	2.55
2.60	.93404	.92965	.9253	.9209	.9166	.9123	.9080	.9037	2.60
2.65	.94332	.93951	.9357	.9319	.9282	.9244	.9207	.9169	2.65
2.70	.95144	.94814	.9448	.9416	.9382	.9351	.9318	.9286	2.70
2.75	.95852	.95567	.9528	.9500	.9472	.9444	.9415	.9387	2.75
2.80	.96466	.96220	.9598	.9573	.9549	.9524	.9500	.9476	2.80
2.85	.96997	.96787	.9658	.9637	.9616	.9595	.9574	.9553	2.85
2.90	.97456	.97275	.9710	.9692	.9674	.9656	.9638	.9620	2.90
2.95	.97850	.97696	.9754	.9739	.9724	.9709	.9693	.9678	2.95

TABLE II—*Continued*

u \ n	18	19	20	21	22	23	24	25	u
3.00	.98187	.98057	.9793	.9780	.9767	.9753	.9741	.9728	3.00
3.05	.98476	.98365	.9825	.9814	.9803	.9793	.9781	.9771	3.05
3.10	.98722	.98629	.9853	.9844	.9835	.9826	.9816	.9807	3.10
3.15	.98931	.98852	.9877	.9869	.9862	.9853	.9846	.9838	3.15
3.20	.99108	.99042	.9898	.9891	.9884	.9878	.9871	.9865	3.20
3.25	.99258	.99202	.9915	.9909	.9904	.9898	.9893	.9887	3.25
3.30	.99384	.99337	.9929	.9924	.9920	.9915	.9911	.9906	3.30
3.35	.99490	.99451	.9941	.9937	.9933	.9930	.9926	.9922	3.35
3.40	.99579	.99546	.9951	.9948	.9945	.9942	.9939	.9936	3.40
3.45	.99653	.99626	.9960	.9957	.9955	.9952	.9949	.9947	3.45
3.50	.99715	.99693	.9967	.9965	.9963	.9961	.9958	.9956	3.50
3.55	.99766	.99748	.9973	.9971	.9969	.9968	.9966	.9964	3.55
3.60	.99809	.99794	.9978	.9976	.9975	.9973	.9972	.9971	3.60
3.65	.99844	.99832	.9982	.9981	.9979	.9978	.9977	.9976	3.65
3.70	.99873	.99863	.9985	.9984	.9983	.9982	.9982	.9981	3.70
3.75	.99897	.99889	.9988	.9987	.9986	.9986	.9985	.9984	3.75
3.80	.99917	.99910	.9990	.9990	.9989	.9988	.9988	.9988	3.80
3.85	.99933	.99927	.9992	.9992	.9991	.9991	.9990	.9990	3.85
3.90	.99946	.99941	.9994	.9993	.9993	.9993	.9992	.9992	3.90
3.95	.99956	.99953	.9995	.9995	.9994	.9994	.9994	.9994	3.95
4.00	.99965	.99962	.9996	.9996	.9995	.9995	.9995	.9995	4.00
4.05	.99972	.99969	.9997	.9996	.9996	.9996	.9996	.9996	4.05
4.10	.99977	.99975	.9997	.9997	.9997	.9997	.9997	.9997	4.10
4.15	.99982	.99980	.9998	.9998	.9998	.9998	.9998	.9998	4.15
4.20	.99986	.99984	.9998	.9998	.9998	.9998	.9998	.9998	4.20
4.25	.99989	.99987	.9999	.9999	.9999	.9999	.9999	.9999	4.25
4.30	.99991	.99990	.9999	.9999	.9999	.9999	.9999	.9999	4.30
4.35	.99993	.99992	.9999	.9999	.9999	.9999	.9999	.9999	4.35
4.40	.99994	.99994	.9999	.9999	.9999	.9999	.9999	.9999	4.40
4.45	.99995	.99995	1.0000	.9999	.9999	.9999	.9999	.9999	4.45
4.50	.99996	.99996		1.0000	1.0000	1.0000	.9999	.9999	4.50
4.55	.99997	.99997					1.0000	1.0000	4.55
4.60	.99998	.99997							4.60
4.65	.99998	.99998							4.65
4.70	.99998	.99998							4.70
4.75	.99999	.99998							4.75
4.80	.99999	.99999							4.80
4.85	.99999	.99999							4.85
4.90	1.00000	1.00000							4.90

that has already been done [1], [2], [3], [4], [11], [12], [20] on the problem of testing outlying observations statistically and to see just where our contributions fit into this corner of mathematical statistics. First, however, we give a very brief history of the problem.

3. Historical comments. A survey of statistical literature indicates that the problem of testing the significance of outlying observations received considerable attention prior to 1937. Since this date, however, published literature on the subject seems to have been unusually scant—perhaps because of inherent difficulties in the problem as pointed out by E. S. Pearson and C. Chandra Sekar [1]. These authors made some important contributions to the problem of outlying observations by bringing clearly into the foreground the concept of efficiency of tests which may be used in view of admissible alternative hypotheses.

In 1933, P. R. Rider [2] published a rather comprehensive survey of work on the problem of testing the significance of outlying observations up to that date. The test criteria surveyed by Rider appear to impose as an initial condition that the standard deviation, σ, of the population from which the items were drawn should be known accurately. In connection with such tests requiring accurate knowledge of σ, we mention (1) Irwin's criteria [3] which utilize the difference between the first two individuals or the difference between the second and third individuals in random samples from a normal population and (2) the range[2] or maximum dispersion [4], [5], [6], [7], [8], [9], [10], [18] of a sample which has been advocated by "Student" [4] and others for testing the significance of outlying observations. We remark further that a natural statistic to use for testing an "outlier" is the difference between such an extreme observation and the sample mean. In 1935, McKay [11] published a note on the distribution of the last-mentioned statistic and by means of a rather elaborate procedure obtained a recurrence relation between the distribution of the extreme minus the mean in samples of n from a normal universe and the distribution of this statistic in samples of $n - 1$ from the same parent. McKay gave also an approximate expression for the upper percentage points of the distribution but did not tabulate the exact distribution due to the complicity of the multiple integrals involved. McKay pointed out that if K_p denotes the p-th semi-invariant of the distribution of $x_n - \bar{x}$ (where x_n is the largest observation) and K'_p refers similarly to the distribution of x_n, then $K_1 = K'_1 - \mu$, $K_2 = K'_2 - \frac{1}{n}$ and $K_p = K'_p$ ($p \geq 3$ where $\mu = E(x_i)$. Nair [20] has tabulated the distribution of the difference between the extreme and sample mean for $n = 2$ to $n = 9$.

Under certain circumstances, accurate knowledge concerning σ may be available as, for example, in using "daily control" tests [4], [18] the population standard deviation may be estimated in some cases with sufficient precision from past

[2] The derivation for the exact distribution of the range is given in reference [9], 1942; however, Dr. L. S. Dederick of the Ballistic Research Laboratory also derived the exact distribution of the range in an unpublished Aberdeen Proving Ground Report (1926).

data. In general, however, an accurate estimate of σ may not be available and it becomes necessary to estimate the population standard deviation from the single sample involved or "Studentize" [18], [20] the statistic to be used, thus providing a true measure of the risks involved in the significance test advocated for testing outlying observations. W. R. Thompson [12] apparently had this very point in mind when he devised an exact test in his paper, "On a Criterion for the Rejection of Observations and the Distribution of the Ratio of the Deviation to the Sample Standard Deviation," which appeared in 1935. Thompson showed that if

$$T_i = \frac{x_i - \bar{x}}{s}$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$, $s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$ and x_i is an observation selected arbitrarily from a random sample of n items drawn from a normal parent, then the probability density function of

$$t = \frac{T\sqrt{n-2}}{\sqrt{n-1-T^2}}$$

is given by "Student's" t-distribution with $f = n - 2$ degrees of freedom.

Pearson and Chandra Sekar have given a rather comprehensive study of Thompson's criterion in an interesting and important paper [1] which appeared in 1936. They discussed also some very important viewpoints which should be taken into consideration when dealing with the problem of testing outlying observations. By setting up alternatives to the null-hypothesis H_0 that all items in the sample come from the same population, Pearson and Chandra Sekar point out that if only one of the observations actually came from a population with divergent mean, then Thompson's criterion would be very useful, whereas if two or more of the observations are truly outlying then the criterion $|x_i - \bar{x}| \geq T_0 s$ may be quite ineffective, particularly if the sample contains less than about 30 or 40 observations.

A point of major interest concerning Thompson's work nevertheless, is that he proposed an *exact* test for the hypothesis that all of the observations came from the same normal population. With regard to the use of an arbitrary observation in Thompson's test, however, it should be borne in mind that the problem of finding the probability that an arbitrary observation will be outlying is different from that of finding the probability that a particular observation (the largest, for example) will be outlying with respect to the other $n - 1$ observations of the sample.

As a final point concerning the paper of Pearson and Chandra Sekar [1], we see that for the n values of T_i arranged in order of magnitude taking account of sign, say

$$T^{(1)}, T^{(2)}, \cdots T^{(n)},$$

then

$$T^{(1)} \geq T^{(2)} \geq T^{(3)} \cdots \geq T^{(n)}.$$

The above authors show that the form of the total distribution of all the T_i at its extremes depend only on $T^{(1)}$ and $T^{(n)}$. This is because for some combinations of sample size and percentage points the algebraic upper limit for $T^{(2)}$ and algebraic lower limit for $T^{(n-1)}$ do not extend into the "tails" of the total distribution. Hence, the following probability law holds for $T^{(1)}$ when $T^{(1)} \geq$ the algebraic maximum of $T^{(2)}$:

$$p\{T^{(1)}\} = Np(T).$$

Likewise,

$$p\{T^{(n)}\} = Np(T)$$

for $T^{(n)} \leq$ algebraic minimum of $T^{(n-1)}$. Therefore, Pearson and Chandra Sekar were able to use Thompson's table [12] and give (for some sample sizes) upper probability limits for $T^{(1)} = \dfrac{x_i - \bar{x}}{s}$ for the highest observation and lower probability limits for $T^{(n)} = \dfrac{x_i - \bar{x}}{s}$ for the lowest observation without actually obtaining the exact probability distribution of $T^{(1)}$ and $T^{(n)}$. Hence, the appearance of the table of percentage points on page 318 of their paper [1] was a substantial contribution to the problem of testing outlying observations since an exact test for the significance of a single outlying observation was provided for the case where an accurate estimate of σ is not available. (The exact distribution of $T^{(1)}$ or $T^{(n)}$ is derived later in this work.)

With the above highlights of historical background in mind, we turn now to a consideration of the types of problems the experimenter may be faced with in testing "outlying" observations.

4. Statement of hypotheses in tests of outliers. Once the sample results of an experiment are available, the practicing statistician may be confronted with one or more of the following distinct situations as regards discordant observations: (a) To begin with, a very frequent or perhaps prevalent situation is that either the greatest observation or the least observation in a sample may have the appearance of belonging to a different population than the one from which the remaining observations were drawn. Here we are confronted with tests for a single outlying observation. (b) Then again, both the largest and the smallest observations may appear to be "different" from the remaining items in the sample. Here we are interested in testing the hypothesis that both the largest and the smallest observations are truly "outliers." (c) Another frequent situation is that either the two largest or the two smallest observations may have the appearance of being discordant. Here we are interested in reaching a decision as to whether we should reject the two largest or the two smallest observations as not being representative of the thing we are sampling.

As to why the discordant observations in a sample may be outliers, this may be due to errors of measurement in which case we would naturally want to reject or at least "correct" such observations. On the other hand, it may be that the population we are sampling is not homogeneous in the uni-modal sense and it will consequently be desirable to know this so that we may carry out further development work on our product if possible or desirable.

Although there may be many models for outliers, we believe that an important practical case involves the situation where all the observations in the sample may be subject to the same standard error, whereas it may happen that the largest or smallest observations result from shifts in level. For example, if one observation appears unusually high compared to the others in the sample we may want to consider the hypothesis that all the observations come from a normal parent with mean μ and standard deviation σ as against the alternative hypothesis that the largest observation comes from a normal population with mean $\mu + \lambda\sigma$ $(\lambda > 0)$ and standard deviation σ, whereas the remaining observations are from $N(\mu, \sigma)$.

Another case involves the situation where the largest and/or smallest observations may be from $N(\mu, \lambda\sigma)$, $\lambda > 1$, whereas the remaining observations of the sample are from the normal parent $N(\mu, \sigma)$.

Although we have not investigated the power of the tests proposed herein for various models, it is believed that the exact test of Section 8 for the largest (or smallest) observation and the test of Section 9 for the two largest (or two smallest) observations possess considerable intuitive appeal for the practical situations described above.[3]

5. Distribution of the difference between the extreme and mean in samples of n from a normal population. The simultaneous density function of n independent observations from a normal parent with zero mean and variance σ^2 which are arranged in order of magnitude is given by

$$(1) \qquad dF(x_1, x_2, \cdots, x_n) = \frac{n!}{(\sqrt{2\pi}\,\sigma)^n} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right] \quad dx_1\, dx_2 \cdots dx_n$$

subject to $x_1 \leq x_2 \leq \cdots \leq x_n$.

Since

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{n}{n-1}(x_n - \bar{x})^2 + \sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2$$

where

$$\bar{x}_n = \frac{1}{n-1}\sum_{i=1}^{n-1} x_i,$$

[3] The author is indebted to J. W. Tukey and S. S. Wilks for calling attention to an incorrect distribution function in the originally submitted manuscript on which several yet-to-be proved or disproved statements concerning optimum properties of statistics in this paper were based.

then

$$\sum_{i=1}^{n} x_i^2 = n\bar{x}^2 + \frac{n}{n-1}(x_n - \bar{x})^2 + \frac{n-1}{n-2}(x_{n-1} - \bar{x}_n)^2$$

$$(2) \qquad + \frac{n-2}{n-3}(x_{n-2} - \bar{x}_{n,n-1})^2 + \cdots + \frac{3}{2}\left(x_3 - \frac{x_1 + x_2 + x_3}{3}\right)^2 + \frac{2}{1}\left(x_2 - \frac{x_1 + x_2}{2}\right)^2$$

where

$$\bar{x}_{n,n-1} = \frac{1}{n-2}\sum_{i=1}^{n-2} x_i, \quad \text{etc.}$$

and consequently we find that we are particularly interested in the following Helmert orthogonal transformation:

$$(3) \qquad \begin{aligned} \sqrt{2\cdot 1}\,\sigma\eta_2 &= -x_1 + x_2, \\ \sqrt{3\cdot 2}\,\sigma\eta_3 &= -x_1 - x_2 + 2x_3, \\ &\vdots \\ \sqrt{n(n-1)}\,\sigma\eta_n &= -x_1 - x_2 - x_3 - x_4 - \cdots - x_r \\ &\qquad - \cdots - x_{n-1} + (n-1)x_n, \\ \sqrt{n}\,\sigma\eta_{n+1} &= x_1 + x_2 + x_3 + x_4 + \cdots + x_r + \cdots + x_{n-1} + x_n. \end{aligned}$$

The above transformation will lead to the distribution of the difference between the extreme and sample mean in terms of the unknown population σ for samples of n from a normal parent. Since, however, K. R. Nair (*Biometrika*, May, 1948) has already published the details independently, we will only record here for later reference that the density function of $\eta_2, \eta_3, \cdots, \eta_n$ (after integrating η_{n+1} over $-\infty \leq \eta_{n+1} \leq +\infty$) is

$$(4) \qquad dF(\eta_2, \eta_3, \cdots \eta_n) = \frac{n!}{(\sqrt{2\pi})^{n-1}} \exp\left[-\frac{1}{2}\sum_{i=2}^{n}\eta_i^2\right] d\eta_2\, d\eta_3 \cdots d\eta_n$$

where the η_i are restricted by the relations

$$(5) \qquad \infty \geq \eta_2 \geq 0, \qquad \sqrt{\frac{r}{r-2}}\,\eta_r \geq \eta_{r-1}.$$

Upon making the transformations

$$(6) \qquad \frac{\sqrt{r(r-1)}}{r}\,\eta_r = \frac{x_r - \bar{x}}{\sigma} = u_r, \qquad (r = 2, 3, \cdots, n),$$

defining

$$(7) \qquad F_n(u) = \int_0^u dF(u_n) = \text{probability } u_n \leq u,$$

and integrating the u_n over their appropriate ranges we find the cumulative probability integrals of the extreme deviation from the sample mean (in terms of the population σ) for $n = 2, 3, \cdots$ to be

$$F_2(u) = 2\sqrt{2}\int_0^u \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(2x^2)}\,dx = \frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}\,dx,$$

a well-known result, where for $n = 2$, x is either the sample standard deviation, the difference between the extreme and sample mean, the mean deviation or the semi-range.

$$F_3(u) = \frac{3\sqrt{3}}{\sqrt{2}}\int_0^u \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{3}{2}x^2)} F_2(\tfrac{3}{2}x)\,dx,$$

$$\vdots \tag{8}$$

$$F_n(u) = \frac{n\sqrt{n}}{\sqrt{n-1}}\int_0^u \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((n)/(n-1)x^2)} F_{n-1}\left(\frac{n}{n-1}x\right)dx.$$

This is equivalent to the result of McKay (11), although the derivation indicated is a considerably simpler one.

Now $F_{n-1}(u)$ increases from 0 to 1 as u increases from 0 to ∞. Hence, if $F_{n-1}\left(\frac{n}{n-1}u\right)$ is practically unity, i.e. for $\frac{n}{n-1}u$ numerically large, the upper percentage points of u_n may be approximated by the normal integral

$$\begin{aligned}\int_{u_n}^{\infty} dF(u_n) &= \frac{n}{\sqrt{2\pi}}\int_{u_n}^{\infty}\exp\left[-\frac{1}{2}\frac{n}{n-1}u_n^2\right]\frac{\sqrt{n}}{\sqrt{n-1}}\,du_n \\ &= \frac{n}{\sqrt{2\pi}}\int_{\sqrt{n/(n-1)}\,u_n}^{\infty}\exp\left[-\frac{t^2}{2}\right]dt.\end{aligned} \tag{9}$$

Formula (9) was found to be particularly useful in checking the higher probabilities in Table II.

The cumulative distribution functions (8) may be put into another form by setting

$$u_r = \frac{1}{r}v_r; \qquad r = 2, 3, \cdots, n.$$

Then $F_n(u)$ becomes

$$\begin{aligned}F_n(u) = \frac{\sqrt{n}}{(\sqrt{2\pi})^{n-1}}\int_0^{nu}\int_0^{v_n}\int_0^{v_{n-1}}\cdots\int_0^{v_4}\int_0^{v_3} \\ \cdot\exp\left[-\frac{1}{2}\sum_{i=2}^{n}\frac{v_i^2}{i(i-1)}\right]dv_2\,dv_3\cdots dv_n.\end{aligned} \tag{10}$$

Define the following functions:

$$H_1(x) = 1,$$

$$H_2(x) = \sqrt{2}\int_0^x \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\cdot\frac{t^2}{2\cdot 1}\right] H_1(t)\,dt,$$

$$\vdots$$

$$H_n(x) = \sqrt{\frac{n}{n-1}}\int_0^x \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\cdot\frac{t^2}{n(n-1)}\right] H_{n-1}(t)\,dt.$$

Hence, the probability that the difference between the extreme and the mean in samples of n from a normal population is less than $u\sigma$ is given by the alternative forms

$$P\{u_n \leq u\sigma\} = F_n(u) = H_n(nu).$$

Of course, $H_n(nu) \rightarrow 1$ as $u \rightarrow \infty$ for any given n.

In the November 1945 issue of *Biometrika*, Godwin [13] arrived at a series of functions closely related to the $H_r(x)$ in connection with the distribution of the mean deviation in samples of n from a normal parent. In Godwin's work, he defines functions $G_r(x)$ which are related to the $H_r(x)$ by the equation

$$(2\pi)^{r/2} H_{r+1}(x) = G_r(x).$$

The $G_r(x)$ functions were computed by H. O. Hartley [15] for $r = 2, 3, \cdots 9$ only. Computations on the functions $F_n(u)$, i.e. (8), were well under way by the author before Godwin's article on the mean deviation appeared. The $H_r(x)$ or $G_r(x)$ can be used to obtain both the distribution of the difference between the extreme and mean and also the probability integral of the mean deviation. Indeed, it is believed that these functions may have a useful place in tabulating distributions of order statistics.

6. Tabulation of the distribution function, $F_n(u)$.

The tabulation of the $F_n(u)$ with ordinary computing equipment is quite laborious. However, a table model computing machine was used initially to obtain the $F_n(u)$ for $n = 2$ to $n = 15$ using formulae (8) and a numerical quadrature process.

In view of the possible general usefulness of the $H_r(x)$, these functions were also computed as a sample problem on a high-speed computing device, the ENIAC (Electronic Numerical Integrator and Computor) of the Ballistic Re-

[4] The author suggested the problem of tabulating the functions $F_n(u)$ or $H_n(nu)$ to the Computing Laboratory of the Ballistic Research Laboratories in the fall of 1945; however, due to problems of higher priority, these functions were not computed on the ENIAC until March, 1948.

search Laboratories of the Ordnance Department.[4] In this connection, the $H_r(u)$ have been computed for $r = 2$ to $r = 25$ at the Ballistic Research Laboratories. For $n = 2$, the functions $H_r(x)$ were computed to nine decimal places of accuracy on the ENIAC and at $n = 25$ about five decimal places of accuracy were obtained. In Table II we have tabulated $F_n(u)$ or $H_n(nu)$, i.e. the probability integral of the extreme minus the mean, at intervals of $u = .05\sigma$. Values computed on the table model computing machine agreed to five decimal places at $n = 15$ with values from the ENIAC. Percentage Points of the distribution are given in Table III and the moment constants may be found in Table IV. Moment constants for $n = 60$, 100, 200, 500 and 1000 were obtained by use of McKay's formulae [11] (which relate the semi-invariants of $x_n - \bar{x}$ with those of x_n) and Tippetts moments [5] for the largest observation x_n.

TABLE III

Percentage Points for Extreme Minus Mean

n	90%	95%	99%	99.5%
2	1.163	1.386	1.821	1.985
3	1.497	1.738	2.215	2.396
4	1.696	1.941	2.431	2.618
5	1.835	2.080	2.574	2.764
6	1.939	2.184	2.679	2.870
7	2.022	2.267	2.761	2.952
8	2.091	2.334	2.828	3.019
9	2.150	2.392	2.884	3.074
10	2.200	2.441	2.931	3.122
11	2.245	2.484	2.973	3.163
12	2.284	2.523	3.010	3.199
13	2.320	2.557	3.043	3.232
14	2.352	2.589	3.072	3.261
15	2.382	2.617	3.099	3.287
16	2.409	2.644	3.124	3.312
17	2.434	2.668	3.147	3.334
18	2.458	2.691	3.168	3.355
19	2.480	2.712	3.188	3.375
20	2.500	2.732	3.207	3.393
21	2.519	2.750	3.224	3.409
22	2.538	2.768	3.240	3.425
23	2.555	2.784	3.255	3.439
24	2.571	2.800	3.269	3.453
25	2.587	2.815	3.282	3.465

TABLE IV

Moment Constants for Extreme Minus Mean

n	Mean	Std. Dev.	α_3	α_4
2	.5642	.4263	.9953	3.8692
3	.8463	.4755	.8296	3.7135
4	1.0294	.4916	.7675	3.6717
5	1.1630	.4974	.7372	3.6560
6	1.2672	.4993	.7165	3.6511
7	1.3522	.4991	.7042	3.6503
8	1.4236	.4979	.6959	3.6518
9	1.4850	.4962	.6900	3.6546
10	1.5388	.4943	.6857	3.6582
11	1.5864	.4923	.6827	3.6622
12	1.6292	.4902	.6804	3.6663
13	1.6680	.4881	.6788	3.6705
14	1.7034	.4861	.6777	3.6746
15	1.7359	.4841	.6770	3.6787
20	1.867	.475	.677	3.700
60	2.319	.436	.699	3.801
100	2.508	.418	.712	3.855
200	2.746	.395	.737	3.932
500	3.037	.368	.771	4.033
1000	3.241	.350	.794	4.105

7. Relation between the distribution of the largest minus the mean of all n observations and the largest minus the mean of the remaining n-1 items. The following relation is of interest concerning these two statistics:

Let

$$u_n = x_n - \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$= \frac{1}{n}\{(n-1)\,x_n - x_1 - x_2 - \cdots - x_{n-1}\}.$$

Let

$$v_n = x_n - \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}$$

$$= \frac{1}{n-1}\{(n-1)\,x_n - x_1 - x_2 - \cdots - x_{n-1}\}.$$

Hence,

$$v_n = \frac{n}{n-1}\,u_n$$

or

$$P(v_n \leq t_0) = P\left(\frac{n}{n-1} u_n \leq t_0\right) = P\left\{u_n \leq \frac{n-1}{n} t_0\right\},$$

i.e. the probability integral of the largest minus the mean of the other observations may be obtained by interpolation on the distribution of the largest minus the mean of all n items in the sample.

8. The distribution of S_n^2/S^2 and S_1^2/S^2. As indicated in the Summary, we proposed the sample criterion

$$\frac{S_n^2}{S^2} = \frac{\sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \leq k, \qquad \bar{x}_n = \frac{1}{n-1} \sum_{i=1}^{n-1} x_i,$$

for testing the significance of the largest observation and the criterion

$$\frac{S_1^2}{S^2} = \frac{\sum_{i=2}^{n} (x_i - \bar{x}_1)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \leq k, \qquad \bar{x}_1 = \frac{1}{n-1} \sum_{i=2}^{n} x_i,$$

for testing whether the smallest observation is outlying. We now find the probability distribution of S_n^2/S^2; hence, also that of S_1^2/S^2.

Returning to the density function

$$dF(\eta_2, \eta_3, \cdots, \eta_n) = \frac{n!}{(\sqrt{2\pi})^{n-1}} \exp\left[-\frac{1}{2} \sum_{i=2}^{n} \eta_i^2\right] d\eta_2\, d\eta_3 \cdots d\eta_n$$

of Section 5, we make the polar transformation

$$\begin{aligned}
\eta_2 &= r \sin \theta_n \sin \theta_{n-1} \cdots \sin \theta_4 \sin \theta_3, \\
\eta_3 &= r \sin \theta_n \sin \theta_{n-1} \cdots \sin \theta_4 \cos \theta_3, \\
\eta_4 &= r \sin \theta_n \sin \theta_{n-1} \cdots \cos \theta_4, \\
&\;\;\vdots \\
\eta_{n-1} &= r \sin \theta_n \cos \theta_{n-1}, \\
\eta_n &= r \cos \theta_n .
\end{aligned} \tag{11}$$

Now

$$\sum_{i=2}^{n} \eta_i^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 = r^2$$

and

$$\sum_{i=2}^{n-1} \eta_i^2 = \sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2 = r^2 \sin^2 \theta_n .$$

Hence,

$$\sin^2 \theta_n = \frac{\sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

The Jacobian of the above transformation is

$$r^{n-2} \sin^{n-3}\theta_n \sin^{n-4}\theta_{n-1} \cdots \sin^3 \theta_6 \sin^2 \theta_5 \sin \theta_4,$$

and since $0 \leq r \leq \infty$

$$\begin{aligned} &dF(\theta_n, \theta_{n-1}, \cdots, \theta_5, \theta_4, \theta_3) \\ (12) \quad &= \frac{n!}{(2\pi)^{(n-1)/2}} 2^{(n-3)/2} \Gamma\left(\frac{n-1}{2}\right) \sin^{n-3} \theta_n \cdots \sin^2 \theta_5 \sin \theta_4 \, d\theta_n \cdots d\theta_5 \, d\theta_4 \, d\theta_3 . \end{aligned}$$

Since the restrictions on the η_i are

$$\eta_2 \geq 0, \qquad \sqrt{\frac{r}{r-2}}\, \eta_r \geq \eta_r - 1, \qquad r \geq 3,$$

we have

$$\tan \theta_n \cos \theta_{n-1} = \frac{\eta_{n-1}}{\eta_n}, \qquad n \geq 4,$$

or

$$\tan \theta_n \leq \sqrt{\frac{n}{n-2}} \sec \theta_{n-1}, \qquad n \geq 4,$$

and

$$0 \leq \theta_3 \leq \frac{\pi}{3}.$$

Thus, letting $K_n = \frac{n!}{(2\pi)^{(n-1)/2}} 2^{(n-3)/2} \Gamma\left(\frac{n-1}{2}\right)$, we see that

$$(13) \quad K_n \int_0^{\pi/3} \int_0^{l_3} \cdots \int_0^{l_{n-2}} \int_0^{l_{n-1}} \sin^{n-3} \theta_n \cdots \sin^2 \theta_5 \sin \theta_4 \, d\theta_n \cdots d\theta_4 \, d\theta_3 = 1,$$

where $l_r = \tan^{-1} \sqrt{\frac{r+1}{r-1}} \sec \theta_r$.

Upon reversing the order of integration (the variable limits are monotonic) we get for $n = 3$

$$K_3 \int_0^{\pi/3} d\theta_3 = 1,$$

so that

$$(14) \qquad P(\theta_3 \leq \theta) = K_3 \int_0^{\theta} d\theta_3 \qquad 0 \leq \theta \leq M_3 = \tan^{-1} \sqrt{3 \cdot 1}.$$

When $n = 4$, we obtain

$$K_4 \int_0^{m_4} \int_0^{\pi/3} \sin \theta_4 \, d\theta_3 \, d\theta_4 + K_4 \int_{m_4}^{M_4} \int_{L_4}^{\pi/3} \sin \theta_4 \, d\theta_3 \, d\theta_4 = 1$$

where

$$m_r = \tan^{-1} \sqrt{\frac{r}{r-2}}, M_r = \tan^{-1} \sqrt{r(r-2)} \text{ and } L_r = \sec^{-1} \sqrt{\frac{r-2}{r}} \tan \theta_r,$$

so that

$$\text{(15a)} \quad P(\theta_4 \leq \theta) = \frac{K_4}{K_3} \int_0^{\theta} \sin \theta_4 \, d\theta_4 \qquad \text{when } 0 \leq \theta \leq m_4 = \tan^{-1} \sqrt{\frac{4}{2}}$$

and

$$\text{(15b)} \quad P(\theta_4 \leq \theta) = \frac{K_4}{K_3} \int_0^{m_4} \sin \theta_4 \, d\theta_4 + K_4 \int_{m_4}^{\theta} \int_{L_4}^{\pi/3} \sin \theta_4 \, d\theta_3 \, d\theta_4$$

$$\text{when } m_4 = \tan^{-1} \sqrt{\frac{4}{2}} \leq \theta \leq M_4 = \tan^{-1} \sqrt{4 \cdot 2}.$$

When $n = 5$, we get,

$$K_5 \int_0^{m_5} \int_0^{m_4} \int_0^{\pi/3} \sin^2 \theta_5 \sin \theta_4 \, d\theta_3 \, d\theta_4 \, d\theta_5 + K_5 \int_0^{m_5} \int_{m_4}^{M_4} \int_{L_4}^{\pi/3} \sin^2 \theta_5 \sin \theta_4 \, d\theta_3 \, d\theta_4 \, d\theta_5$$

$$+ K_5 \int_{m_5}^{M_5} \int_{L_5}^{M_4} \int_{L_4}^{\pi/3} \sin^2 \theta_5 \sin \theta_4 \, d\theta_3 \, d\theta_4 \, d\theta_5 = 1$$

$\left(\text{where } L_4 = \sec^{-1} \sqrt{\frac{2}{4}} \tan \theta_4 \text{ is to be taken as 0 whenever } \theta_4 \leq m_4 = \tan^{-1} \sqrt{\frac{4}{2}}\right)$ so that

$$\text{(16a)} \quad P(\theta_5 \leq \theta) = \frac{K_5}{K_4} \int_0^{\theta} \sin^2 \theta_5 \, d\theta_5 \qquad \text{when } 0 \leq \theta \leq m_5 = \tan^{-1} \sqrt{\frac{5}{3}}$$

and

$$\text{(16b)} \quad P(\theta_5 \leq \theta) = \frac{K_5}{K_4} \int_0^{m_5} \sin^2 \theta_5 \, d\theta_5 + K_5 \int_{m_5}^{\theta} \int_{L_5}^{M_4} \int_{L_4}^{\pi/3} \sin^2 \theta_5 \sin \theta_4 \, d\theta_3 \, d\theta_4 \, d\theta_5$$

where $m_5 = \tan^{-1} \sqrt{\frac{5}{3}} \leq \theta \leq M_5 = \tan^{-1} \sqrt{5 \cdot 3}$,
and we put $L_4 = \sec^{-1} \sqrt{\frac{2}{4}} \tan \theta_4 = 0$ whenever $\theta_4 \leq m_4 = \tan^{-1} \sqrt{\frac{4}{2}}$.

For a sample of n items

$$(17a)\quad P(\theta_n \leq \theta) = \frac{n}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \int_0^\theta \sin^{n-3}\theta_n \, d\theta_n$$

$$= \frac{n}{2} I_{\sin^2\theta}\left(\frac{n-2}{2}, \frac{1}{2}\right) \qquad \text{when } 0 \leq \theta \leq \tan^{-1}\sqrt{\frac{n}{n-2}}\,^{5}$$

and

$$(17b)\quad P(\theta_n \leq \theta) = \frac{n}{2} I_{n/(2(n-1))}\left(\frac{n-2}{2}, \frac{1}{2}\right)$$

$$+ K_n \int_{m_n}^{\theta} \int_{L_n}^{M_{n-1}} \int_{L_{n-1}}^{M_{n-2}} \cdots \int_{L_4}^{\pi/2} \sin^{n-3}\theta_n \cdots \sin\theta_4 \, d\theta_3 \, d\theta_4 \cdots d\theta_n$$

for

$$m_n = \tan^{-1}\sqrt{\frac{n}{n-2}} \leq \theta \leq M_n = \tan^{-1}\sqrt{n(n-2)}$$

where $I_x(p, q)$ is K. Pearson's Incomplete Beta Function Ratio [19]. It is to be understood in (17) that

$$L_i = \sec^{-1}\sqrt{\frac{i-2}{i}}\tan\theta_i \quad \text{for} \quad i = 4, 5, \cdots, n-1$$

is to be taken as zero when $\theta_i \leq \tan^{-1}\sqrt{\frac{i}{i-2}}$.

Percentage points for the sample statistic

$$\sin^2\theta_n = \frac{S_n^2}{S^2} = \frac{\sum_{i=1}^{n-1}(x_1 - \bar{x}_n)^2}{\sum_{i=1}^{n}(x_1 - \bar{x})}$$

or the statistic S_1^2/S_2 are given in Table I and were obtained by inverse interpolation on the tabulation of the probability integral (17) above. Percentage points for the Pearson and Chandra Sekar statistics, $T_n = \frac{x_n - \bar{x}}{s}$ or $T_1 = \frac{\bar{x} - x_1}{s}$ (where $s^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$), are given in Table IA. The statistics S_n^2/S^2 and T_n are related by the formula

$$\frac{S_n^2}{S^2} = 1 - \frac{T_n^2}{n-1}.$$

---

[5] It has been noted that (17a) gives a good approximation to (17b) when $\theta \geq \tan\sqrt{\frac{n}{n-2}}$ provided we are interested in the important practical region $P \leq .10$, at least for $n \leq 25$.

The statistic T_n (or T_1) is easier to compute than S_n^2/S^2 (or S_1^2/S^2). The tabulation of the multiple integral (17) was carried out on the Bell Relay Computors at the Ballistic Research Laboratories.

9. The distribution of $S_{n-1,n}^2/S^2$ and $S_{1,2}^2/S^2$. As indicated in the Summary, the proposed criterion for judging the significance of the two largest observations is

$$S_{n-1,n}^2/S^2 = \frac{\sum_{i=1}^{n-2} (x_i - \bar{x}_{n-1,n})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \leq k \qquad \text{where } \bar{x}_{n-1,n} = \frac{1}{n-2} \sum_{i=1}^{n-2} x_i,$$

and that for testing the two smallest observations is

$$S_{1,2}^2/S^2 = \frac{\sum_{i=3}^{n} (x_i - \bar{x}_{1,2})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \leq k \qquad \text{where } \bar{x}_{1,2} = \frac{1}{n-2} \sum_{i=3}^{n} x_i.$$

From the preceding section, we note that

$$\sum_{i=2}^{n} \eta_i^2 = r^2, \qquad \sum_{i=2}^{n-2} \eta_i^2 = r^2 \sin^2 \theta_n \sin^2 \theta_{n-1}.$$

Hence,

$$\sin^2\theta_n \sin^2\theta_{n-1} = \frac{\sum_{i=1}^{n-2} (x_i - \bar{x}_{n-1,n})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}, \tag{18}$$

so that if we find the distribution of

$$\sin^2 \theta_n \sin^2 \theta_{n-1} = \sin^2 \Delta_n, \text{ say,}$$

then we have the distribution of $S_{n-1,n}^2/S^2$ and hence also that of $S_{1,2}^2/S^2$, i.e.

$$P\{\sin^2 \Delta_n \leq k\} = P\{\Delta_n \leq \sin^{-1} \sqrt{k}\}. \tag{19}$$

Returning to the multiple integral (13), let

$$\sin \Delta_n = \sin \theta_n \sin \theta_{n-1},$$

$$\Delta_i = \theta_i, \qquad 3 \leq i \leq n-1.$$

The Jacobian of this transformation is given by

$$\frac{\partial(\theta_n, \cdots, \theta_3)}{\partial(\Delta_n, \cdots, \Delta_3)} = \frac{\cos \Delta_n}{\sqrt{\sin^2 \Delta_{n-1} - \sin^2 \Delta_n}}.$$

The limits of integration for Δ_n are given by

$$0 \leq \Delta_n \leq \sin^{-1} \frac{\sqrt{n} \sin \Delta_{n-1}}{\sqrt{2(n-1) - (n-2) \sin^2 \Delta_{n-1}}}$$

and, of course, those for $\Delta_{n-1}, \cdots, \Delta_3$ are the same as the limits for $\theta_{n-1}, \cdots, \theta_3$ respectively. Hence, substituting in (13), we obtain

$$K_n \int_0^{\pi/3} \int_0^{\tan^{-1}\sqrt{4/2}\sec\Delta_3} \cdots \int_0^{\tan^{-1}\sqrt{n-1/n-3}\sec\Delta_{n-2}} \cdot \int_0^{\sin^{-1}\frac{\sqrt{n}\sin\Delta_{n-1}}{\sqrt{2(n-1)-(n-2)\sin^2\Delta_{n-1}}}} \frac{\sin^{n-3}\Delta_n \sin^{n-4}\Delta_{n-1}\cdots\sin^2\Delta_5\sin\Delta_4\cos\Delta_n\,d\Delta_n\cdots d\Delta_3}{\sin^{n-3}\Delta_{n-1}\sqrt{\sin^2\Delta_{n-1}-\sin^2\Delta_n}} = 1. \tag{20}$$

Reversing the order of integration, we have

$$K_n \int_0^{\sin^{-1}\sqrt{\frac{n(n-3)}{(n-1)(n-2)}}} \int_{\sin^{-1}\frac{\sqrt{2(n-1)}\sin\Delta_n}{\sqrt{n+(n-2)\sin^2\Delta_n}}}^{\tan^{-1}\sqrt{(n-1)(n-3)}} \int_{\sec^{-1}\sqrt{\frac{n-3}{n-1}}\tan\Delta_{n-1}}^{\tan^{-1}\sqrt{(n-2)(n-4)}} \cdots \int_{\sec^{-1}\sqrt{2/4}\tan\Delta_4}^{\pi/3} \frac{\sin^{n-3}\Delta_n\sin^{n-4}\Delta_{n-1}\cdots\sin\Delta_4\cos\Delta_n\,d\Delta_3\cdots d\Delta_n}{\sin^{n-3}\Delta_{n-1}\sqrt{\sin^2\Delta_{n-1}-\sin^2\Delta_n}} = 1 \tag{21}$$

$\left(\text{for } \Delta_i \leq \tan^{-1}\sqrt{\frac{i}{i-2}}, \text{ then } \sec^{-1}\sqrt{\frac{i-2}{i}}\tan\Delta_i \text{ is to be put equal to zero where } i \geq 4\right)$ so that for $n = 4$,

$$P(\Delta_4 \leq \Delta) = K_4 \int_0^{\Delta} \int_{\sin^{-1}\frac{\sqrt{3}\sin\Delta_3}{\sqrt{2+\sin^2\Delta_3}}}^{\pi/3} \frac{\sin\Delta_4\cos\Delta_4\,d\Delta_3\,d\Delta_4}{\sin\Delta_3\sqrt{\sin^2\Delta_3-\sin^2\Delta_4}} \tag{22}$$

where $0 \leq \Delta \leq \sin^{-1}\sqrt{\frac{2}{3}}$,

and for $n = 5$,

$$P(\Delta_5 \leq \Delta) = K_5 \int_0^{\Delta} \int_{\sin^{-1}\frac{\sqrt{4\cdot 2}\sin\Delta_5}{\sqrt{5+3\sin^2\Delta_5}}}^{\tan^{-1}\sqrt{4\cdot 2}} \cdot \int_{\sin^{-1}\sqrt{2/4}\tan\Delta_4}^{\pi/3} \frac{\sin^2\Delta_5\cos\Delta_5\,d\Delta_3\,d\Delta_4\,d\Delta_5}{\sin\Delta_4\sqrt{\sin^2\Delta_4-\sin^2\Delta_5}} \tag{23}$$

where $0 \leq \Delta \leq \sin^{-1}\sqrt{\frac{5}{6}}$, etc.

We remark that an obvious extension of the above principles should lead to the distributions of

$$S^2_{n-2,n-1,n}/S^2 \quad\text{and}\quad S^2_{1,2,3}/S^2,$$
$$S^2_{n-3,n-2,n-1,n}/S^2 \quad\text{and}\quad S^2_{1,2,3,4}/S^2,$$

etc. although the tabulation of such probability integrals may be exceedingly difficult.

The problem of tabulating the probability integral (21) involves a double quadrature process and has been carried out on the Bell Relay Computors at the Ballistic Research Laboratories for $n = 4$ to $n = 20$, inclusive. Table V gives some useful percentage points for these sample sizes.

TABLE V

Table of Percentage Points for $\frac{S^2_{n-1,n}}{S^2}$ *or* $\frac{S^2_{1,2}}{S^2}$

n	1%	2.5%	5%	10%
4	.0000	.0002	.0008	.0031
5	.0035	.0090	.0183	.0376
6	.0186	.0349	.0565	.0921
7	.0440	.0708	.1020	.1479
8	.0750	.1101	.1478	.1994
9	.1082	.1492	.1909	.2454
10	.1415	.1865	.2305	.2863
11	.1736	.2212	.2666	.3226
12	.2044	.2536	.2996	.3552
13	.2333	.2836	.3295	.3843
14	.2605	.3112	.3568	.4106
15	.2859	.3367	.3818	.4345
16	.3098	.3603	.4048	.4562
17	.3321	.3822	.4259	.4761
18	.3530	.4025	.4455	.4944
19	.3725	.4214	.4636	.5113
20	.3909	.4391	.4804	.5269

$$S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad \text{where} \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$S^2_{n-1,n} = \sum_{i=1}^{n-2} (x_i - \bar{x}_{n-1,n})^2 \quad \text{where} \quad \bar{x}_{n-1,n} = \frac{1}{n-2}\sum_{i=1}^{n-2} x_i$$

$$S^2_{1,2} = \sum_{i=3}^{n} (x_i - \bar{x}_{1,2})^2 \quad \text{where} \quad \bar{x}_{1,2} = \frac{1}{n-2}\sum_{i=3}^{n} x_i$$

10. Comment on the distribution of $S^2_{1,n}/S^2$. In connection with the distribution of the statistic

$$\frac{S^2_{1,n}}{S^2} = \frac{\sum_{i=2}^{n-1} (x_i - \bar{x}_{1,2})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}, \quad \text{where} \quad \bar{x}_{1,n} = \frac{1}{n-2}\sum_{i=2}^{n-1} x_i,$$

for testing simultaneously whether the smallest and largest observations are outlying, an investigation indicates that since

$$\sum x_i^2 = n\bar{x}^{-2} + \frac{n}{n-1}(x_n - \bar{x})^2 + \frac{n-1}{n-2}(x_1 - \bar{x}_n)^2 + \frac{n-2}{n-3}(x_{n-1} - \bar{x}_{1,n})^2$$
$$+ \cdots + \frac{3}{2}\left(x_4 - \frac{x_2 + x_3 + x_4}{3}\right)^2 + 2\left(x_3 - \frac{x_2 + x_3}{2}\right)^2$$

then the transformation

$$
(24)\qquad
\begin{aligned}
\sqrt{2\cdot 1}v_2 &= -x_2 + x_3,\\
\sqrt{3\cdot 2}v_3 &= -x_2 - x_3 + 2x_4,\\
\sqrt{4\cdot 3}v_4 &= -x_2 - x_3 - x_4 + 3x_5,\\
&\ \ \vdots\\
\sqrt{(n-2)(n-3)}v_{n-2} &= -x_2 - x_3 - \cdots - x_{n-2} + (n-3)x_{n-1},\\
\sqrt{(n-1)(n-2)}v_{n-1} &= -(n-2)x_1 + x_2 + x_3 + \cdots + x_{n-1},\\
\sqrt{n(n-1)}v_n &= -x_1 - x_2 - x_3 - \cdots - x_{n-1} + (n-1)x_n,\\
\sqrt{n}v_{n+1} &= x_1 + x_2 + \cdots + x_n,
\end{aligned}
$$

followed by transformations of the type (11) and that of Section 9 may lead to the distribution of $S_{1,n}^2/S^2$. However, the limits of integration do not turn out to be functions of single variables and the task of computing the resulting multiple integral may be rather difficult.

11. Examples on testing outlying observations for rejection. We now turn to the problem of applying our theory to particular practical examples of data which appear to have outlying observations. Apparently, in the following examples there were not sufficient practical or experimental grounds to reject the suspected outliers and hence some statistical judgement became necessary either to support retaining the "outliers" in the sample or leave little doubt that certain of the observations should be questioned.

EXAMPLE 1. Our first example has almost become a classical one as Irwin [3], Rider [2], and other writers on the subject including Chauvenet, Peirce, Gould, etc. (see Rider's survey [2]) all refer to it, applying their various tests. The example consists of a sample of 15 observations of the vertical semi-diameters of Venus made by Lieut. Herndon in 1846 and is given in William Chauvenet's, *A Manual of Spherical and Practical Astronomy*, II (5th ed., 1876), p. 562. The individual residuals or deviations from the mean are:

−0.30″	0.48	0.63	−0.22	0.18
−0.44	−0.24	−0.13	−0.05	0.39
1.01	0.06	−1.40	0.20	0.10

Arranging the observations in increasing order of magnitude, we have:

−1.40″	−0.24	−0.05	0.18	0.48
−0.44	−0.22	0.06	0.20	0.63
−0.30	−0.13	0.10	0.39	1.01

and it is seen that two of the residuals, -1.40 and 1.01, appear to be outliers. Rider [2] indicates that the above observations have been referred to by previous writers as "residuals"; nevertheless their sum is 0.27, so that the sample mean, $\bar{x} = .018$. Let us apply the exact test, i.e. T_1 of Pearson and Chandra Sekar or S_1^2/S^2 as developed in Section 8 for a single outlier to the least observation, -1.40. We find $x_1 = -1.40$, $\bar{x} = .018$ and $s = .532$ (alternatively, we find $S^2 = 4.2496$ using all 15 observations and $S_1^2 = 2.0953$ which is based on 14 observations, the suspected outlier -1.40 not being included). Further, $T_1 = \dfrac{\bar{x} - x_1}{s} = \dfrac{.018 + 1.40}{.532} = 2.665$ (or $S_1^2/S^2 = 0.4931$) and from Table IA (or Table I) we see that $0.01 \leq P \leq 0.025$ so that we would reject the observation -1.40 when using the 5% level of significance. Having rejected -1.40, we now have left a sample of 14 observations and test the greatest one, i.e. 1.01. For T_n based on the remaining 14 observations, we have $n = 14$, $x_n = 1.01$, $\bar{x} = .119$ and $s = .387$ (alternatively, for the new sums of squares, we find $S_n^2 = 1.2409$ leaving out 1.01 and $S^2 = 2.0953$ including the observation 1.01). Hence, $T_n = \dfrac{x_n - \bar{x}}{s} = \dfrac{1.01 - .119}{.387} = 2.302$ (or $S_n^2/S^2 = 0.5922$) and from Table IA (or I), we find P slightly less than .10, so that we decide to retain the observation 1.01.

It would have been interesting nevertheless to see whether or not the test $S_{1,n}^2/S^2$ would have rejected simultaneously the observations -1.40 and 1.01 if percentage points for the distribution of this statistic were available.

It is of interest to remark that for this particular example Irwin [3, page 245], using the difference between the first two individuals divided by an estimate of σ, i.e. $\dfrac{x_2 - x_1}{\sigma}$, concluded also that -1.40 but not 1.01 should be rejected. In testing *both* of these observations, Irwin used the single biased estimate for σ,

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} = = .5326 \qquad \text{(assuming } \bar{x} = 0\text{)},$$

based on all 15 observations. It is a mere coincidence, of course, that for this example Irwin's test gives the same result as the exact test T_1 or the test based on the ratio S_1^2/S^2. In this connection, Irwin rightly calls attention to the fact that in dealing with a sample of only 15 observations the standard deviation of the sample is a very unreliable estimate of the population standard deviation.

It is remarked that here we would, of course, hesitate to apply the test $\dfrac{\bar{x} - x_1}{\sigma}$ to the observation -1.40 as we do not have available and accurate estimate of σ from past data.

EXAMPLE 2. The following ranges (horizontal distances from gun muzzle to point of impact) were obtained in firing projectiles from a weapon at a constant angle of elevation and at the same weight of charge of propellant powder:

Distances in yards

4782	4420
4838	4803
4765	4730
4549	4833

It is desired to know whether the projectiles exhibit uniformity in ballistic behavior or if some of the ranges, such as 4549 and 4420, are not consistent with the others.

Arranging the distances or ranges in increasing order of magnitude,

4420	4782
4549	4803
4730	4833
4765	4838

we suspect the presence of two outliers, i.e. 4420 and 4549. Having no available knowledge of σ from past data for this example, an intuitively efficient test to apply would be that of Section 9, i.e. $S_{1,2}^2/S^2$.
We find

$$\frac{S_{1,2}^2}{S^2} = \frac{\sum_{i=3}^{8} (x_i - \bar{x}_{1,2})^2}{\sum_{=1}^{8} (x_i - \bar{x})^2} = .054$$

which is significant at the .01 level (Table V) and consequently we would judge the distances 4420 and 4549 yds. as being unusually low.

As a matter of interest and as a recommended temporary practical expedient for testing several "outliers", consider for example the last seven of the above ordered observations,

4549	4803
4730	4833
4765	4838
4782	

and apply the exact test, S_1^2/S^2, to the smallest observation, 4549. We find $S_1^2/S^2 = .145$ so that $.01 < P < .025$ from Table I and we should thus reject 4549 from the sample of seven. Moreover, we should now surely reject 4420 as being outlying, arriving at the same result we had for the test $S_{1,2}^2/S^2$. Thus, as a general temporary expedient in testing for "outliers" one could rank the observations, and apply the tests S_1^2/S^2 (or S_n^2/S^2) and $S_{1,2}^2/S^2$ (or $S_{1,n}^2/S^2$), thus working from the "inside" observations of the ranked sample in order to establish consistency of the observations.

12. Additional comments. Although we have used a significance level of .05 in the examples, it may be preferable from a practical viewpoint to reject outlying observations only at a lower level, such as .01 or .005.

Extensions of the ideas for testing outlying observations presented in this paper may lead to efficient sample criteria for testing the significance of various numbers of high, low, or simultaneously high and low sample values. However, the mathematical details would probably be complicated. In this connection, it is remarked nevertheless that the advent of high-speed computing devices may have considerable bearing on establishing experimentally any probability distribution. That is to say high-speed electronic computing devices could probably be programmed to generate random numbers with frequencies equal to those of the normal (or any other) distribution, to compute various functions (such as ratios in this paper) of sample values, etc., and establish frequency distributions to a desired order of accuracy.

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