Advanced Machine Learning Week 6

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Derivation for the Entropy:

For MK 31.1:

$$H = -\frac{dF}{dT} \tag{1}$$

We know that the free energy in the replica model is equal to:

$$F(m,q) = \frac{1}{2}J_0 m^2 - \frac{\beta}{4}J^2(q-1)^2 - \frac{1}{\sqrt{2\pi}\beta}\int dz e^{-\frac{z^2}{2}} \log(2\cosh(\beta J_0 m^2 + \beta J\sqrt{q}z) \tag{2}$$

Derivating the above with respect to $T = \frac{1}{\beta}$ yields:

$$\begin{aligned}\frac{dF}{dT} =& \frac{\beta^2}{4}J^2(q-1)^2 - \frac{1}{\sqrt{2\pi}}\int dz e^{-\frac{z^2}{2}} \log\left(2\cosh(\beta J_0 m^2 + \beta J\sqrt{q}z)\right) \\ &- \frac{1}{\sqrt{2\pi}\beta}\int dz e^{-\frac{z^2}{2}} \frac{d\log\left(2\cosh(\beta J_0 m^2 + \beta J\sqrt{q}z)\right)}{dT}\end{aligned} \tag{3}$$

Applying the chain rule, the last term $\alpha(\beta) = -\frac{1}{\sqrt{2\pi}\beta}\int dz e^{-\frac{z^2}{2}} \frac{d\log\left(2\cosh(\beta J_0 m^2 + \beta J\sqrt{q}z)\right)}{dT}$ becomes:

$$\begin{aligned}\alpha(\beta) &= -\frac{1}{\sqrt{2\pi}\beta}\int dz\, e^{-\frac{z^2}{2}} \frac{\sinh(\beta J_0 m^2 + \beta J\sqrt{q}z)}{\cosh(\beta J_0 m^2 + \beta J\sqrt{q}z)}(J_0 m\beta^2 + J\sqrt{q}z\beta^2) \\ &= -\frac{1}{\sqrt{2\pi}\beta}\int dz\, e^{-\frac{z^2}{2}} \tanh(\beta J_0 m^2 + \beta J\sqrt{q}z) J_0 m\beta^2 \\ &\quad - \frac{1}{\sqrt{2\pi}\beta}\int dz\, e^{-\frac{z^2}{2}} \tanh(\beta J_0 m^2 + \beta J\sqrt{q}z) Jz\sqrt{q}\beta^2\end{aligned}$$

Next, by using the definition of m to the first term and solving the second term by parts, we obtain:

$$\begin{aligned}\alpha(\beta) &= -\beta J_0 m^2 + \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{z^2}{2}} \tanh(\beta J_0 m^2 + \beta J \sqrt{q} z) J \Big|_{-\infty}^{+\infty} \\ &\quad - \beta^2 J^2 q \frac{1}{\sqrt{2\pi}} \int dz\, e^{-\frac{z^2}{2}} \sec^2(\beta J_0 m^2 + \beta J \sqrt{q} z) \\ &= -\beta J_0 m^2 - J^2 \beta^2 q(q-1)\end{aligned}$$

where the last inequality follows from the definition of q and the fact that the term $e^{-\frac{z^2}{2}} \tanh(\beta J_0 m^2 + \beta J \sqrt{q} z) J \Big|_{-\infty}^{+\infty}$ is 0 as it is the product of a finite quantity - the hyperbolic tangent - and a negative exponential, which approaches 0.

Therefore, we have obtained:

$$\begin{aligned}H = &\frac{\beta^2}{4} J^2 (q-1)^2 - \frac{1}{\sqrt{2\pi}} \int e^{-\frac{z^2}{2}} \log(2\cosh(\beta J_0 m^2 + \beta J \sqrt{q} z))\, dz \\ &- \beta J_0 m^2 - J^2 \beta^2 q(1-q)\end{aligned} \tag{4}$$

Derivation for average energy:

Also for MK 31.1:

$$H = \log(Z(\beta)) + \beta \bar{E}(\beta) \tag{5}$$

as such

$$\bar{E}(\beta) = \frac{H}{\beta} + F \tag{6}$$

Replacing Equation 2 in F and Equation 4 in H, we obtain:

$$\begin{aligned}\bar{E}(\beta) = &-\frac{\beta}{4} J^2 (q-1)^2 + \frac{1}{\sqrt{2\pi}\beta} \int dz e^{-\frac{z^2}{2}} \log\left(2\cosh(\beta J_0 m^2 + \beta J \sqrt{q} z)\right) \\ &- J_0 m^2 - \beta J^2 q(1-q) + \frac{1}{2} J_0 m^2 - \frac{1}{4}(q-1)^2 \\ &- \frac{1}{\sqrt{2\pi}\beta} \int dz e^{-\frac{z^2}{2}} \log\left(2\cosh(\beta J_0 m^2 + \beta J \sqrt{q} z)\right) \\ = &-\frac{\beta}{2} J^2 (q-1)^2 + \beta J^2 q - \frac{1}{2} J_0 m^2 - \beta J^2 q + \beta J^2 q^2 \\ = &-\frac{1}{2} J_0 m^2 - \frac{1}{2} \beta J^2 (1 - q^2)\end{aligned}$$

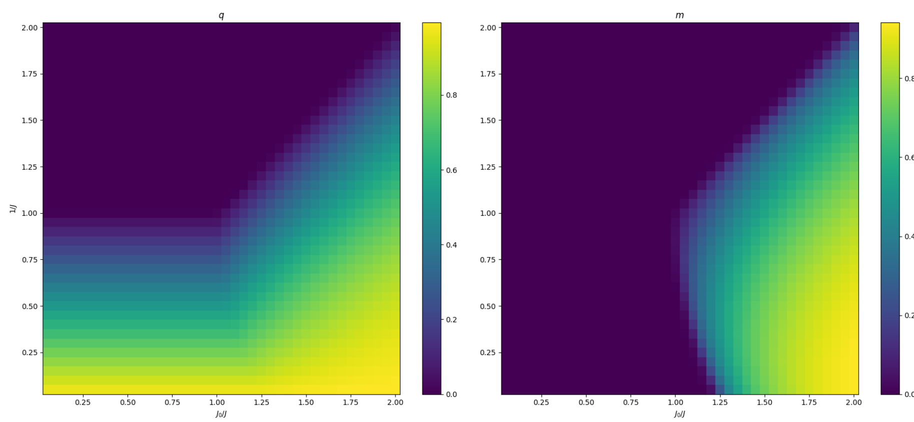

Figure 1: The solution q (left) and m (right) of the RS equations as a function of J0/J and 1/J

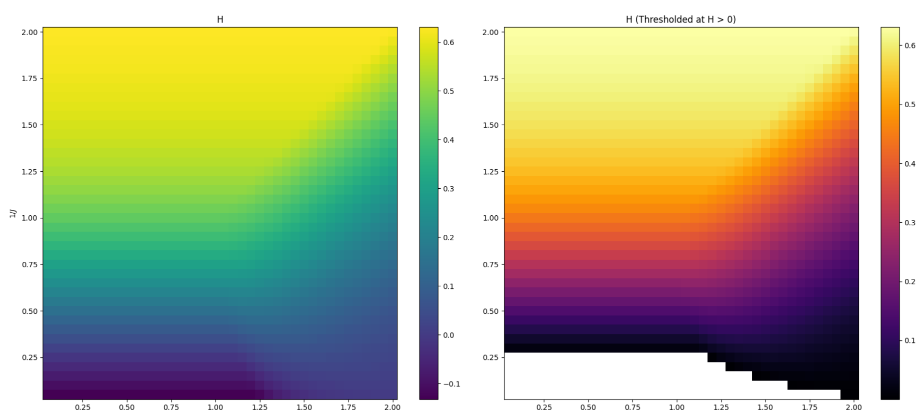

Figure 2: The entropy H of the system in the RS approximation as a function of $J0/J$ and $1/J$ (left) and the thresholded value ($H > 0$) (right) showing the AT line ($\beta = 1$)