AV Lab : Kalman Filter

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1 Introduction

The Kalman filter is a set of mathematical equations that provides an efficient computational (recursive) solution of the least-squares method. The filter is very powerful as it supports estimations of past, present, and even future states, and it can do so even when the precise nature of the modeled system is unknown. It is one of the most widely used methods for tracking and estimation due to its simplicity, optimality, tractability and robustness. The purpose of this work is to provide a practical introduction to the use of discrete Kalman filter applied to a voltage-driven DC motor.

2 Model Description

An input voltage $u(t)$ is used to drive a DC Motor. The angular position of the motor $\theta(t)$ is measured with an incremental encoder with the precision L = 512 angles per lap, which provides the measure $y(t)$ of $\theta(t)$. The angular velocity is $\omega(t)$. So, given $u(t)$ and $y(t)$, $\theta(t)$ and $\omega(t)$ are to be estimated in this work.

2.1 Input Voltage

A periodic zero-mean square wave with time period $100ms$ and peak-to-peak amplitude of $0.1V$ is sampled with a sampling time of $T_s = 1ms$. This sampled signal shown in Fig:1 is used as an input voltage to the DC motor.

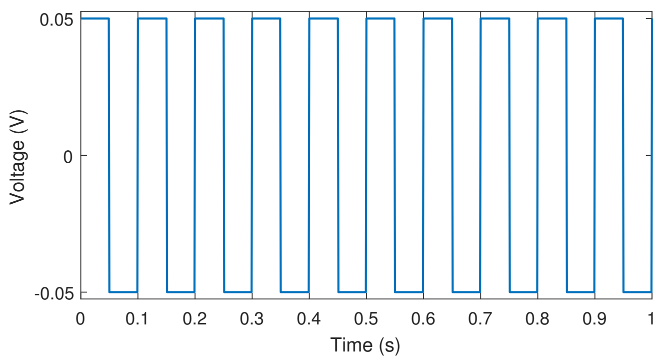
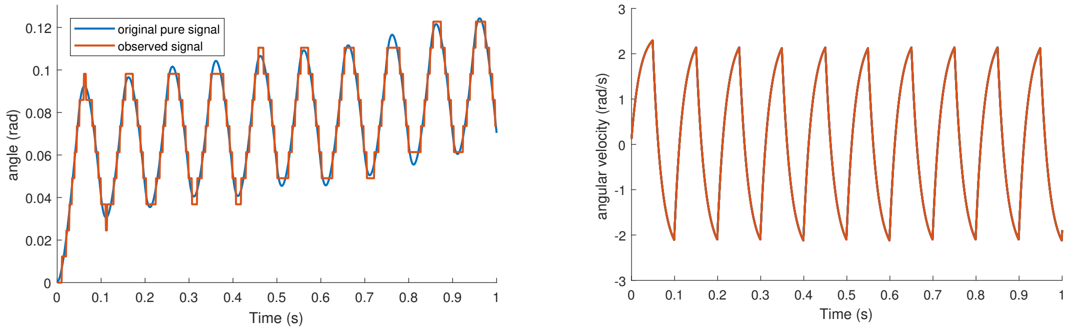


Figure 1: Input Voltage for DC Motor

2.2 System modeling and simulation

A state vector X(t)=$\begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix}$ representing the rotational angle and angular velocity, is used for the state estimation of the DC Motor. The state-space representation is given below:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{G}{T} \end{bmatrix} u \\ \theta &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} + [0]u \end{aligned} \tag{1}$$



(a) $\Theta(t)$ and sampled y_n (b) $\Omega(t)$

Figure 2: System simulation state vectors and output evolution

where the constant parameters of the motor are the time constant T=20ms, and the gain is G=50$rad.s^{-1}.V^{-1}$.

The system definition is a continuous time system. This is converted into a discrete time model using the zero-order-hold discretization method "zoh".

The step invariance sampling with time step T_s is:

$$\begin{cases} x_{n+1} = \tilde{A}x_n + \tilde{B}u_n \\ y_n = \tilde{C}x_n + \tilde{D}u_n \end{cases} \quad \text{where} \begin{cases} \tilde{A} = e^{AT_s} \\ \tilde{B} = \int_{T_s}^{0} e^{A\tau}Bd\tau \\ \tilde{C} = C \\ \tilde{D} = D \end{cases}$$

In MATLAB, this discretization is performed using the library function *c2dm*. The angular position $\theta(t)$ is quantized as y_n with precision L=512 angles per lap by the incremental encoders. It is done using equation 2

$$y = \frac{2\pi}{L} round\left(\frac{\theta L}{2\pi}\right) \tag{2}$$

We can observe the comparison of θ and y_n in figure 2a. The angle velocity Ω is shown on figure 2b.

3 Kalman Filter

The encoder remove some information about the angle. We want to reconstruct the real output angle. A model of our system is already available. This model can be incorrect. Moreover we know how the encoder work and then how it lose information. A probabilistic approach is carried out to obtain an approximation of the real angle of our system.

3.1 Markov model

The noise components associated with our model are:

- w_n : Quantization error of the incremental encoder. The constant piecewise output y_n is indeed not perfect. The variance of this noise is r. So the noisy output is $y_n = \theta_n + w_n$. This noise is assumed to have a uniform distribution because the real angle output can equally be distributed in the encoder step.
- v_n : A white noise considered which doesn't exist in reality, but will be useful to take into account modeling errors. Variance of this noise is q. So the noisy input is $u'_n = u_n + v_n$.

The Markov equations with input u_n and output y_n can be listed as:

$$\begin{cases} x[n+1] = \tilde{A}x[n] + \tilde{B}(u[n] + v[n]) \\ y[n] = \tilde{C}x[n] + w[n] \end{cases} \tag{3}$$

3.2 Evaluation of encoder variance

The encoder give a round approximation of the angle. From a continuous physic output, it give a constant piecewise information without reference. As there is L value possible in one lap, between to step, the output from the encoder is between $\theta_i - \frac{\pi}{L} < \theta_i < \theta_i + \frac{\pi}{L}$. We assume that the probability density function is constant in this interval. The noise taken into account using the signal w_n is a uniform distribution with a mean value being zero and a variance being then :

$$r = \frac{(2\pi)^2}{12L^2} \tag{4}$$

3.3 Evaluation of input noise variance

For what it main concern the variance of the noise added in Markov model to the input signal, it is unknown and depend on the belief we have in the model. For the simulation we will compare different value to see its impact when the model is not correct. The value q is the variance of the noise $v[n]$ added to the input $u[n]$. In the Kalman filter, it will result in a variance $Q = var(\tilde{B}v[n]) = \tilde{B}q\tilde{B}^T$.

3.4 Initial guess

The initial guess for $\hat{X}_{1/0}$ is determined by the initial position and velocity of the motor. For the initial position, this value could be between $-\pi$ and π. Indeed it depend on the reference we take on the stator and rotor. This angle is but define modulo 2π. The reference being taken arbitrary. It can then been decided to take the middle value for the position $\hat{X}_{1/0}(1) = 0$. The real value is distributed with a uniform distribution with medium value equal to zero and variance equal to $P_{1/0}(0,0) = \frac{(2\pi)^2}{12}$.

The initial velocity of the motor is null by assumption. We consider it as being sure, the variance associated is then zero. The velocity and position are totally uncorrelated (every position can be taken with all velocity). We have then all the remaining term of $P_{1/0}$ equal to zero. Those value are represented on equation 5 and 6.

$$\hat{X}_{1/0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

$$P_{1/0} = \begin{bmatrix} \frac{(2\pi)^2}{12} & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

4 Kalman Filter

The Kalman filter addresses the general problem of trying to estimate the state $x \in R^n$ of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_n = Ax_{n-1} + Bu_{n-1} + v_{n-1} \tag{7}$$

with a measurement $y \in R^m$ that is

$$y_n = Hx_n + w_n \tag{8}$$

The random variables v_n and w_n represent the process and measurement noise (respectively). They are assumed to be independent (of each other), white, and with normal probability distributions $p(v) = N(0, R)$, $p(w) = N(0, Q)$ where R is the process noise covariance and Q is the measurement noise covariance. As followed from the earlier section, we can see that our process can be very accurately estimated by such a filter. Thus, it is used to recursively estimate the state of our process, by producing an optimal estimate in the sense that the mean value of the sum of the estimation errors gets a minimal value.

Algorithm overview

The Kalman filter estimates a process by using a form of feedback control [1]. The filter estimates the process state at some time and then obtains feedback in the form of (noisy) measurements. As such, the equations for the Kalman filter fall into two groups: time update equations and measurement update equations. The time update equations are responsible for projecting forward (in time) the current state and error covariance estimates to obtain the a priori estimates for the next time step. The measurement update equations are responsible for the feedback which incorporate a new measurement into the a priori estimate to obtain an improved a posteriori estimate.

The time update equations can also be thought of as predictor equations, while the measurement update equations can be thought of as corrector equations. The specific equations are discussed below:

1. Time update - "Predict"

- Project the state ahead $\hat{x}_n^- = A\hat{x}_{n-1} + Bu_{n-1}$
- Project the error covariance ahead $P_n^- = AP_{n-1}A^T + Q$

2. Measurement update - "Correct"

- Compute the Kalman gain $K_n = P_n^- H^T (HP_n^- H^T + R)^{\check{}1}$
- Update estimate with measurement $\hat{x}_n = \hat{x}_n^- + K_n(z_n + H\hat{x}_n^-)$
- Update the error covariance $P_n = (I - K_n H)P_n^-$

In these measurement update equations, $\hat{x}_n^-$ which is the prior estimate, which in a way, means the rough estimate before the measurement update correction. And P_n^- is the prior error covariance.

The Kalman Filter gain is usually a time-varying gain matrix. In this work, we explore both time-varying as well as stationary gain Kalman Filter.

5 Simulations

In this section, we compare the estimation of our model by the two types of Kalman gain under two settings for the model - *Perfect* and *Rough*. The parameter q represents the uncertainty of our model. In case of an accurate model, q is expected to be zero. In the case of errors in the model, q will adjust to correspond to the uncertainty of the model. The values chosen for q in our work, are 10^{-2},10^{-4} and 10^{-6}.

5.1 Perfect model

In the first case we assume the model to be perfectly known. The gain G and time constant T for the model simulation and in Kalman Filter are equal:

$$\begin{cases} G_{actual} = G_{filter} = 50rad.s^{-1} \\ T_{actual} = T_{filter} = 20ms \end{cases} \tag{9}$$

In the following we compare the time-varying and the constant gain Kalman filter applied to above.

5.1.1 Effect of q with varying gain

The first case studied is the output of the Kalman filter with a varying gain. This is theoretically the best Kalman filter. We plot on figure 3 four different plot corresponding to the angle of the motor 3A, the error of position estimation 3B, the rotational velocity of the motor 3C and its error estimation. For the plot 3A (respectively 3C), four different curve are represented. In black the real position or pure position (respectively velocity) of the rotor. This correspond to the one

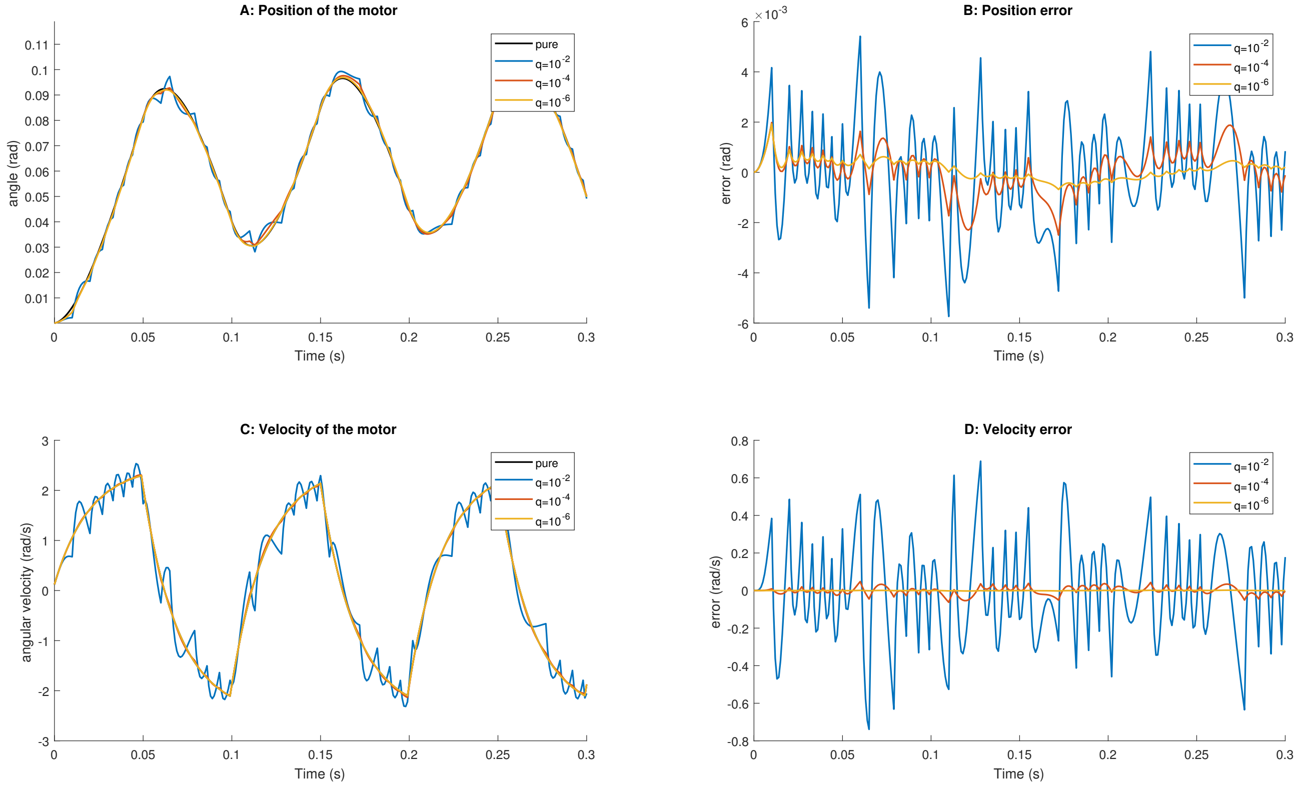


Figure 3: Position and velocity behavior with a varying Gain Kalman filtering for three different variance q and with a perfect model of the system

presented on figure 2a (respectively 2b). The three other one correspond to the estimated output made with the the three variance q.

The two figure remaining (3B and 3D) correspond to the error. This error is in reality never accessible but in simulation, it will permit to interpret the effect of Kalman filter. Those error have been calculated with the following formula :

$$\mathbf{e} = \mathbf{x_{real}} - \mathbf{x_{estimated}}(q) \tag{10}$$

The first remark is that the behavior of the Kalman filter is heavily dependent of the variance q. Here, for the perfect model, as expected, the lowest variance is the better one. There is indeed no error in the model. The maximum error obtained with the variance $q = 10^{-6}$ is of $10^{-3} rad$ for the position and $3.10^{-3} rad.s^{-1}$ for the speed. For the gain equal to $q = 10^{-4}$ it is of $3.10^{-3} rad$ for the position and $0.1 rad.s^{-1}$ for the speed. Finally for the gain equal to $q = 10^{-2}$ it is of $7.10^{-3} rad$ for the position and $0.7 rad.s^{-1}$ for the speed. The error is raising as the variance chosen is increasing. This is particularly true for the velocity error. The frequency of oscillation is not changed with the variance. We see on the graph 3B that the error increase and decrease at the same time. This is due to the output discretization made by the encoder. We see moreover that the error of initial at initial pose is nearly null. The initial guess of $\hat{\theta}_{1/0}$ was yet very bad with an error of 0.05. (This point has not been printed on this graph.) The varying gain gave the ability to correct it quickly with the covariance matrix $P_{1/0}$ defined in 6 explaining the uncertainty of this value. We see then that this uncertainty was corrected step by step by seeing on graph 3B that the maximum error was obtained at the beginning at $t = 0.005s$ with $q = 10^{-6}$ and not after.

Finally, this show that if the model is accurate and we know it, the variance q should be reduced. Moreover the effect of the varying gain is visible, particularly at the beginning of the graph. Anyway, the filter worked correctly and the error have been heavily reduced.

5.1.2 Effect of q with stationary gain

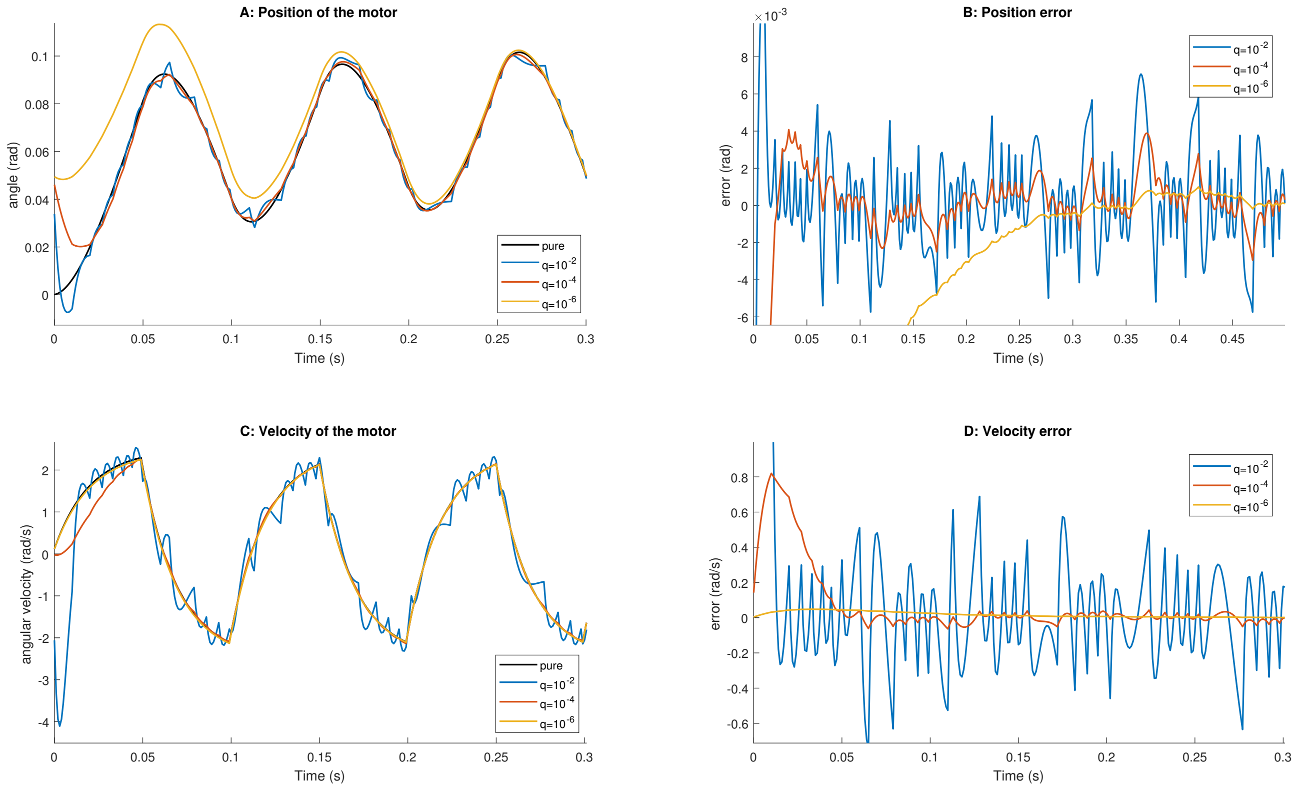


Figure 4: Position and velocity behavior with a stationary Gain Kalman filtering for three different variance q and with a perfect model of the system

On figure 4, the output behavior of the Kalman filter with a stationary gain is depicted. The four graphs are defined the same way as the previous one. The stationary gain give here some differences compared to the stationary gain. Particularly for small time.

For time superior to $t = 0.35s$, the result are very similar to the case with varying gain. The gain had converged to the one of the stationary gain. The same conclusion could then be made, the smaller variance you give, the smaller the error is. But this is not true at the beginning of the simulation.

This time, we clearly see the that the initial guess was wrong on figure 4A. Indeed the covariance matrix $P_{1/0}$ is not taken into account, the covariance taken is the limit one $P[\infty]$. This results in a wrong estimation during the first ms for all variance q. The time of convergence depend but in the value of q. For $q = 10^{-6}$, being week, the certainty was taken high even if it was not in reality. This result in a very long correction of the error. It needed $0.3s$ to converge. On figure 4D, with $q = 10^{-6}$ we see that the error remain small, the model is followed quite accurately. It is indeed supposed to be accurate with a small q. As q rise to $q = 10^{-4}$, we see on figure 4A that the time of convergence is significantly reduced. With very high q, with $q = 10^{-2}$, there is even an overtaking, synonym of an instability. If we look the figure 4C, the velocity become indeed very high, the error becoming important.

On this part, the conclusion is that the Kalman filter with stationary gain give bad result if there is a high error in initial estimation. In this case, to have quick convergence to the good value, the variance could be rise. The problem is that when it converged, the model is perfect and a smaller variance should be taken.

5.2 Rough model

The model of the system is know supposed to be wrong. To do that, we take the following value for the gain G and time constant T used respectively in the simulation and in Kalman Filter.

$$\begin{cases} G_{actual} = G_{filter} = 50 rad.s^{-1} \\ T_{actual} = 20ms \\ T_{filter} = 25ms \end{cases} \tag{11}$$

This error in ideal case should be corrected by the supposed error introduced in the model in the Kalman filter. This will be verified in the next section.

5.2.1 Effect of q with varying gain

The small variance $q = 10^{-6}$ show the error coming of the model. On figure 5A, the shifting can be seen due to the error of the time constant T. Moreover the amplitude of θ is reduced. On figure 5C, the same phenomenon is observed. This can be seen on figure 5B and 5D with oscillation of period Δ and large amplitude. q being small, the compensation on the model error are small. The behavior of the wrong model is visible in the output.

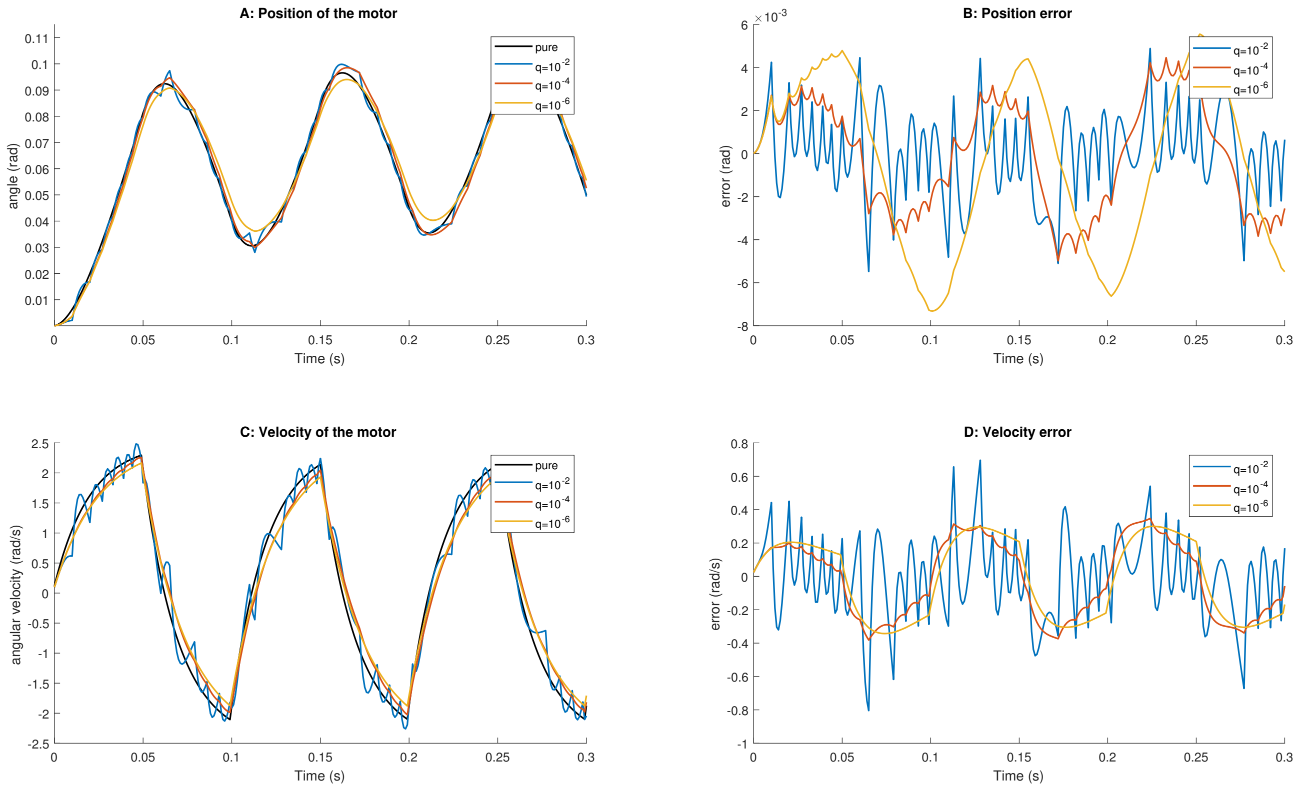


Figure 5: Position and velocity behavior with a varying Gain Kalman filtering for three different variance q and with a rough model of the system

We use first the varying gain Kalman filter with this model. The results are presented on figure 5. The results are here less good than previously.

As the input variance is increased, the correction of the model error is visible. The amplitude of the error with $q = 10^{-4}$ on figure 5B is smaller than the one with $q = 10^{-6}$. The filtering is better. The amplitude is but similar for the speed. The model cannot been completely corrected.

If we increase again the variance, like with $q = 10^-2$, we ss on figure 5B that the oscillation due to the error of the encoder appear. The previous behavior with exact model appears. A small variation with a period of Δ due to the error of model appear to. Its amplitude seem a bit smaller.

Finally, a compromise between a high variance leading to instability due to the encoder, and a small variance leading to an absence of correction of the model should be find. The intermediate variance permit to correct the model and avoid important variation due to the encoder. The error is better reduced with a variance of $q = 10^{-4}$.

5.2.2 Effect of q with stationary gain

On figure 6, the last case is commented. The model is taken approximated and the Kalman filter with stationary gain is selected. The behavior of the stationary Kalman filter is quite similar to the one depicted in part 5.1.2. Indeed, the four graphs can be separated in two parts. A first part where the position converge to a better approximation as the initial guess is wrong. And then a part where the two Kalman filter seem to have the same behavior. On graph 6B, we see again that the time of response depend on the variance chosen. With a low variance, the time of convergence is around $0.25s$. With a higher variance as seen previously, the correction is bring by the correction of the input and the time of convergence is reduced to less than $0.05s$ for $q = 10^{-4}$ and less than $0.02s$ for $q = 10^{-2}$. The problem being that an overtaking appear on the velocity as we see on figure 6D as it can't be corrected on the gain of the Kalman filtering. The best compromise for the convergence part is then a properly defined variance neither to high or to low. Once converged, oscillation due to error on the model with a period Δ, and oscillation due to encoder with an higher frequency. A low variance ($q = 10^{-6}$) won't correct the model and high amplitude oscillation with period Δ appear on figure 6B. But here the high frequency are better removed and doesn't appear. When a compensation is brought to the input by increasing the variance, a correction on the model appear but high frequency oscillation appear to. On the extreme case, the model is completely corrected but the correction on the encoder output is reduced and high frequency signal amplitude is increased. Finally, here again, the value should be for the second part again chosen carefully and be in keeping with the error of the model.

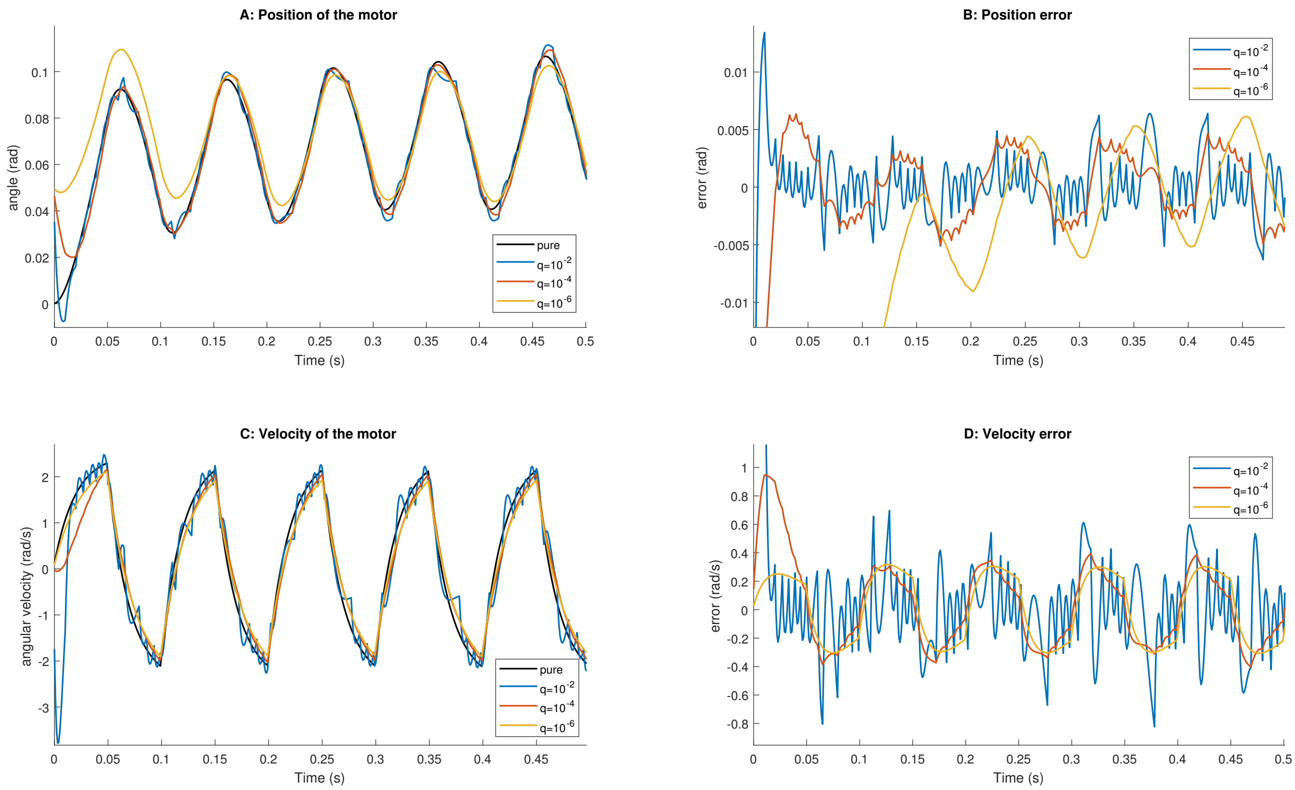


Figure 6: Position and velocity behavior with a stationary Gain Kalman filtering for three different variance q and with a rough model of the system

Comparison of the two types of Kalman Gain

In the following figure 7 we observe the nature of the two types of Kalman gain used in our work. The bold lines represent the gain obtained in the perfect model and the dotted lines represent the nature in a rough model.

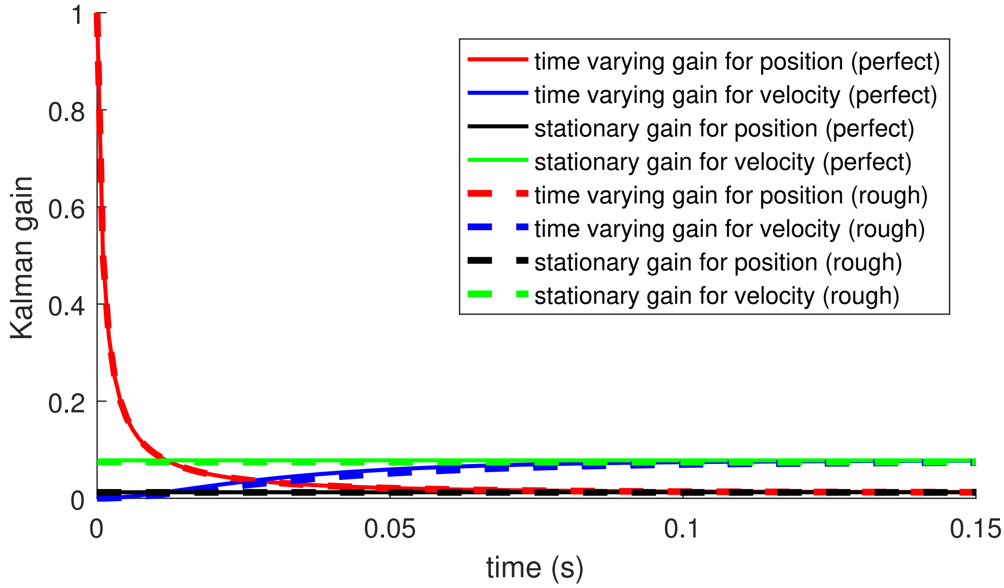


Figure 7: Comparison of Time-Varying Kalman Gain Vs Stationary Kalman Gain

We observe that the time-varying Kalman Gain catches up with the stationary Kalman gain value after a few iterations, and it stays stable for the rest of the estimation. For the time-varying gain, initially it is unity for the angle component and zero for the velocity component. This is expected since for angle, the initial variance is non-zero and it reduces as the error falls and finally reaches a constant value, as the estimation progresses. However, initially the variance for velocity is zero and it advances during the estimation till reaching a constant value.

The nature of the gains do not depend on the type of model, perfect or rough, as we can see from the same shape of the curves in both cases. However, the rough model takes longer time to converge the time-varying gain to the stationary value, than in case of the perfect model.

6 Conclusion

This work presents an application of the Kalman filtering on a voltage-driven DC motor with an encoder. The error on the output can be due to the input error, an modeling error and due to the encoder characteristics. The time-varying gain Kalman filter gives an estimation of the actual output, accounting for these noises. It has the ability to adapt itself to the input. The stationary Kalman filter cannot adapt but is computationally less complex. It has been demonstrated that the adaptation of the Kalman filter to a non-exact model of DC motor drive to difficulty to settle a parameter. A virtual variance added to the input has the ability to correct slightly the model but never completely. Indeed with high variance, the model begin to be corrected but perturb the correction of encoder output. The choice of the value of q depends on the modeling error - lower values trust the knowledge of the system, that the modeling and initializations are done accurately, whereas a higher value of q indicates that a large error is expected in either of the two, or both.

7 References

1. Bishop, Gary, and Greg Welch. "An introduction to the kalman filter." Proc of SIGGRAPH, Course 8.27599-23175 (2001): 41.