Kalman Filter

- The Kalman Filter (KF) is an optimal estimator.
- It evinces quantities of interest from indirect, inaccurate and uncertain observations.
- It is used for:
  - Spatio-temporal estimation
  - Tracking in computer vision applications
  - Tracking targets
  - Autonomous navigation
  - Robotic application

Kalman Filter - properties

- Estimator: it performs a stochastic estimation of the system state from noisy sensor measurements.
- Model based: it is based on (it makes use of) a system model described by a state equation and an output (measurement) equation.
- Linear: in the standard form all the equations are linear and the noises are Gaussian with zero mean.
- Least square: the KF minimizes the mean square error of the estimated parameters. In this respect, the KF is an optimal estimator.

Kalman Filter - properties

- Recursive: it can process new measurements as they arrive. Thus the KF eliminates the need for storing the entire past observed data and is computationally more efficient.
- Stochastic: the confidence about Kalman quantities are expressed in terms of probability distributions.

Whether some of the KF hypothesis fails (e.g. nonlinear system), we can consider different forms of the KF to take into account these failures (e.g. Extended Kalman Filter).

Kalman Filter

It can be used if the probability distribution $p(\mathbf{X}_t \mid Y_t)$ is gaussian

$\mathbf{X}_t$ is the state at time t

$Y_t = (\mathbf{Y}_1, \ldots, \mathbf{Y}_t)$ spans the measures till time t

KF implements a "predictor-corrector" estimator. It aims to minimize the covariance of the estimation

Kalman Filter

GENERAL STATEMENT:

KF aim to estimate the state $x \in \Re^n$ of a discrete-time system from a stochastic difference equation (***process equation***):

$$\boxed{x_k = \mathbf{A}x_{k-1} + \mathbf{b}u_k + w_{k-1}}$$

Transition matrix input Process noise

given a noisy measure $y \in \Re^m$ (***measurement equation***):

$$\boxed{y_k = \mathbf{H}x_k + v_k}$$

Measurement matrix Measurement noise

Kalman Filter

- Process and measurement noise are mutually independent, white and normally distributed:

$$p(w) \approx N(0, \mathbf{Q})$$

$$p(v) \approx N(0, \mathbf{R})$$

- The process noise covariance **Q** and the measurement noise covariance **R** can be time-variant, in a first approximation they can be assumes constant.
- We assume both the measurement (**H**) and the process matrix (**A**) constant during the filtering stage.

Kalman Filter

$\boxed{\hat{x}_k^- \in \Re^n}$ **a priori estimate** of the state at time k given the knowledge of the process at time k-1.

$\boxed{\hat{x}_k \in \Re^n}$ **a posteriori estimate** of the state at time k given the measurement at time k.

We can define the error on the a *priori* estimate:

$$\boxed{e_k^- = x_k - \hat{x}_k^-} \text{ with covariance } P_k^- = E[e_k^- e_k^{-T}]$$

and on the *a posteriori* estimate:

$$\boxed{e_k = x_k - \hat{x}_k} \text{ with covariance } P_k = E[e_k e_k^T]$$

Kalman Filter

It is possible to compute the a posteriori estimate $\hat{x}_k$ as a linear combination of the a priori estimate $\hat{x}_k^-$ and of a weighted difference between the actual measure y_k and a measurement prediction:

$$\boxed{\hat{x}_k = \hat{x}_k^- + \mathbf{K}(y_k - \mathbf{H}\hat{x}_k^-)}$$

Kalman gain

innovation

Kalman Filter

TEMPORAL UPDATE:

Prediction of the state (a priori estimate):

$$\hat{x}_k^- = \mathbf{A}\hat{x}_{k-1} + \mathbf{B}u_k$$

Prediction of the error covariance:

$$P_k^- = \mathbf{A}P_{k-1}\mathbf{A}^T + \mathbf{Q}$$

MEASUREMENT UPDATE:

Kalman gain

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}^T(\mathbf{H}\mathbf{P}_k^-\mathbf{H}^T + \mathbf{R})^{-1}$$

Update of the estimate (a posteriori estimate):

$$\hat{x}_k = \hat{x}_k^- + \mathbf{K}_k(y_k - \mathbf{H}\hat{x}_k^-)$$

Update of the error covariance

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_k^-$$

Initial estimate of the state and of the covariance

Convergence

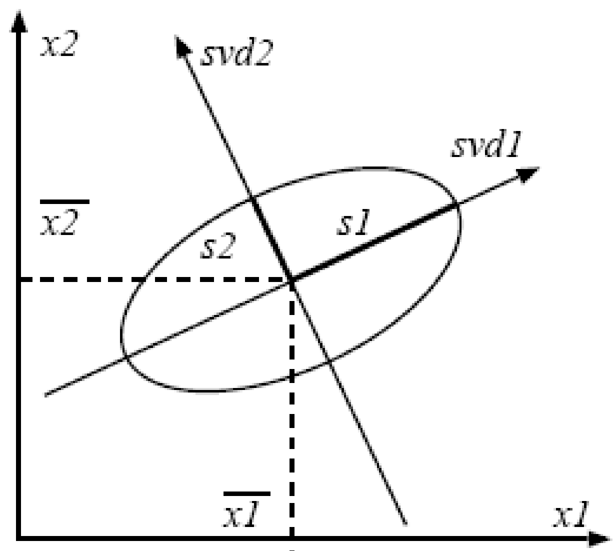
- **Convergence** is related to the reduction of the uncertainty of the state estimate.
  - The a posteriori estimate error covariance matrix *P* is a measure for the uncertainty of the state estimate
  - In particular, the main diagonal of the matrix represents the mean squared error of the state estimate, thus we can use these values to monitor how the uncertainty of the state estimate changes as a function of the recursion time: the new measurements should reduce this uncertainty.
  - It is worthy to note that we can monitor the behavior of each variable of the state.

Convergence

– Convergence test for the KF: by using a singular value decomposition of the error covariance matrix.

– In this case we can consider the singular directions as a new state space and the singular values as the semi-axes of an ellipsoid describing the uncertainty of the state estimate

– Convergence of the estimate means that the ellipsoid contracts in all directions.



Consistency

- **Consistency** is used to catch discrepancies between the model and the measurement.
- The error covariance matrix provides us only information about the properties of convergence of the KF and not whether the KF converges to the correct values.
- Since, the Kalman filter strongly relies on the model of the system, wrong models result in estimate errors, so the KF output is not reliable.

Consistency

- We have to check the consistency between the innovation and the model (between observed and predicted values) in statistical terms.
- A measure of the reliability of the KF output is the Normalized Innovation Squared (NIS) that has a chi-square distribution with *m* degrees of freedom:

$$NIS_k = a_k{}^T P_k{}^{-1} \alpha_\kappa$$

- where m is the number of statistically independent measurements.
- The consistency test is performed by checking whether the NIS is within a given confidence interval

See MATLAB examples

Extended Kalman Filter - EKF

The KF gives an estimate of the state $x \in \Re^n$ for a discrete-time process described by a linear difference stochastic equation.

What about measurements or processes described by non-linear equations?

$\Downarrow$

$\boxed{\textbf{Extended Kalman Filter (EKF)}}$

A linear approximation of the estimate is given by considering the partial derivatives of the process and measure equations

Extended Kalman Filter - EKF

The state vector of the process is $x \in \Re^n$
is described by the following non-linear equation

$$\boxed{x_k = f(x_{k-1}, u_k, w_{k-1})}$$

non-linear transition function

process noise

and the measure $y \in \Re^m$ by:

$$\boxed{y_k = h(x_k, v_k)}$$

non-linear measure function

measure noise

The EKF uses a first-order Taylor expansion. This could affect the statistic distribution in the a priori estimates.

Other estimators

- **Unscented Kalman Filter (UKF):** instead of a linear approximation of the measure and process functions, approximates the distribution of the unknown by a finite number of samples.
- It assumes a gaussian distribution for the a posteriori estimate.

- **Particle Filters**: used to estimate time-varying quantities, with a non-gaussian and multimodal distribution.

HOMEWORK

- Reading Assignments:
  - Notes about Kalman Filter (AulaWeb)
  - MATLAB examples (AulaWeb)

- Next class: Monday, November 7