# Probability

## Contents



# 1 Probability

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### Configuring R

Functions from these packages will be used throughout this document:

```r
library(conflicted) # check for conflicting function definitions
# library(printr) # inserts help-file output into markdown output
library(rmarkdown) # Convert R Markdown documents into a variety of formats.
library(pander) # format tables for markdown
library(ggplot2) # graphics
library(ggfortify) # help with graphics
library(dplyr) # manipulate data
library(tibble) # `tibble`s extend `data.frame`s
library(magrittr) # `%>%` and other additional piping tools
library(haven) # import Stata files
library(knitr) # format R output for markdown
library(tidyr) # Tools to help to create tidy data
library(plotly) # interactive graphics
library(dobson) # datasets from Dobson and Barnett 2018
library(parameters) # format model output tables for markdown
library(haven) # import Stata files
library(latex2exp) # use LaTeX in R code (for figures and tables)
```

```r
library(fs) # filesystem path manipulations
library(survival) # survival analysis
library(survminer) # survival analysis graphics
library(KMsurv) # datasets from Klein and Moeschberger
library(parameters) # format model output tables for
library(webshot2) # convert interactive content to static for pdf
library(forcats) # functions for categorical variables ("factors")
library(stringr) # functions for dealing with strings
library(lubridate) # functions for dealing with dates and times
```

Here are some R settings I use in this document:

```r
rm(list = ls()) # delete any data that's already loaded into R

conflicts_prefer(dplyr::filter)
ggplot2::theme_set(
  ggplot2::theme_bw() +
        # ggplot2::labs(col = "") +
    ggplot2::theme(
      legend.position = "bottom",
      text = ggplot2::element_text(size = 12, family = "serif")))

knitr::opts_chunk$set(message = FALSE)
options('digits' = 6)

panderOptions("big.mark", ",")
pander::panderOptions("table.emphasize.rownames", FALSE)
pander::panderOptions("table.split.table", Inf)
conflicts_prefer(dplyr::filter) # use the `filter()` function from dplyr() by default
legend_text_size = 9
run_graphs = TRUE
```

---

Most of the content in this chapter should be review from UC Davis Epi 202.

## 1.1 Core properties of probabilities

### 1.1.1 Defining probabilities

**Definition 1.1** (Probability measure)**.** A **probability measure**, often denoted Pr() or P(), is a function whose domain is a $\sigma$-algebra[1] of possible outcomes, $\mathcal{S}$, and which satisfies the following properties:

1. For any statistical event $A \in \mathcal{S}$, $\Pr(A) \geq 0$.

2. The probability of the union of all outcomes ($\Omega \stackrel{\text{def}}{=} \cup \mathcal{S}$) is 1:

$$\Pr(\Omega) = 1$$

3. The probability of the union of disjoint events, $A_1 \cup A_2 : A_1 \cap A_2 = \emptyset$, is equal to the sum of their probabilities:

$$\Pr(A_1 \cup A_2) = \Pr(A_1) + \Pr(A_2)$$

---

**Theorem 1.1.** *If $A$ and $B$ are statistical events and $A \subseteq B$, then* $\Pr(A \cap B) = \Pr(A)$.

---

[1] https://en.wikipedia.org/wiki/%CE%A3-algebra

*Proof.* Left to the reader for now. □

---

**Theorem 1.2.**

$$\Pr(A) + \Pr(\neg A) = 1$$

---

*Proof.* By properties 2 and 3 of Definition 1.1. □

---

**Corollary 1.1.**

$$\Pr(\neg A) = 1 - \Pr(A)$$

---

*Proof.* By Theorem 1.2 and algebra. □

---

**Corollary 1.2.** *If the probability of an outcome $A$ is $\Pr(A) = \pi$, then the probability that $A$ does not occur is:*

$$\Pr(\neg A) = 1 - \pi$$

---

*Proof.* Using Corollary 1.1:

$$\begin{aligned}\Pr(\neg A) &= 1 - \Pr(A) \\ &= 1 - \pi\end{aligned}$$

□

## 1.2 Random variables

### 1.2.1 Binary variables

**Definition 1.2** (binary variable)**.** A **binary variable** is a random variable which has only two possible values in its range.

**Exercise 1.1** (Examples of binary variables)**.** What are some examples of binary variables in the health sciences?

---

*Solution.* Examples of binary outcomes include:

- exposure (exposed vs unexposed)
- disease (diseased vs healthy)
- recovery (recovered vs unrecovered)
- relapse (relapse vs remission)
- return to hospital (returned vs not)
- vital status (dead vs alive)

---

### 1.2.2 Count variables

**Definition 1.3** (Count variable)**.** A **count variable** is a random variable whose possible values are some subset of the non-negative integers; that is, a random variable $X$ such that:

$$\mathcal{R}(X) \in \text{N}$$

---

**Exercise 1.2.** What are some examples of count variables?

---

*Solution.*

- Number of fish in a pond
- Number of cyclones per season
- Seconds of tooth-brushing per session (if rounded)[2]
- Infections per person-year
- Visits to ER per person-month
- Car accidents per 1000 miles driven

---

**Definition 1.4** (Exposure magnitude)**.** For many count outcomes, there is some sense of an **exposure magnitude**, such as **population size**, or **duration of observation**, which multiplicatively rescales the expected (mean) count.

---

**Exercise 1.3.** What are some examples of exposure magnitudes?

---

*Solution.*

Table 1: Examples of exposure units

| outcome | exposure units |
|---|---|
| disease incidence | number of individuals exposed; time at risk |
| car accidents | miles driven |
| worksite accidents | person-hours worked |
| population size | size of habitat |

Exposure units are similar to the number of trials in a binomial distribution, but **in non-binomial count outcomes, there can be more than one event per unit of exposure**.

We can use $t$ to represent continuous-valued exposures/observation durations, and $n$ to represent discrete-valued exposures.

---

**Definition 1.5** (Event rate)**.**

For a count outcome $Y$ with exposure magnitude $t$, the **event rate** (denoted $\lambda$) is defined as the mean of $Y$ divided by the exposure magnitude. That is:

$$\mu \stackrel{\text{def}}{=} \text{E}[Y|T = t]$$

$$\lambda \stackrel{\text{def}}{=} \frac{\mu}{t} \tag{1}$$

[2] https://pubmed.ncbi.nlm.nih.gov/35587489/

Event rate is somewhat analogous to odds in binary outcome models; it typically serves as an intermediate transformation between the mean of the outcome and the linear component of the model. However, in contrast with the odds function, the transformation $\lambda = \mu/t$ is *not* considered part of the Poisson model's link function, and it treats the exposure magnitude covariate differently from the other covariates.

---

**Theorem 1.3** (Transformation function from event rate to mean)**.** *For a count variable with mean $\mu$, event rate $\lambda$, and exposure magnitude $t$:*

$$\therefore \mu = \lambda \cdot t \tag{2}$$

---

*Solution.* Start from definition of event rate and use algebra to solve for $\mu$.

---

Equation 2 is analogous to the inverse-odds function for binary variables.

---

**Theorem 1.4.** *When the exposure magnitude is 0, there is no opportunity for events to occur:*

$$E[Y|T = 0] = 0$$

---

*Proof.*

$$\mathrm{E}[Y|T = 0] = \lambda \cdot 0 = 0$$

□

---

**Probability distributions for count outcomes**

- Poisson distribution
- Negative binomial distribution

---

## 1.3 Key probability distributions

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Some distributions are typically used for outcome models (Table 2); other distributions are typically used for test statistics (Table 3).

Table 2: Distributions typically used for outcome models

| Distribution | Uses |
|---|---|
| Bernoulli | Binary outcomes |
| Binomial | Sums of Bernoulli outcomes |
| Poisson | unbounded count outcomes |
| Geometric | Counts of non-events before an event occurs |

| Distribution | Uses |
|---|---|
| Negative binomal | Mixtures of Poisson distributions, counts of non-events until a given number of events occurs |
| Normal (Gaussian) | Continuous outcomes without a more specific distribution |
| exponential | Time to event outcomes |
| Gamma | Time to event outcomes |
| Weibull | Time to event outcomes |
| Log-normal | Time to event outcomes |

Table 3: Distributions typically used for test statistics

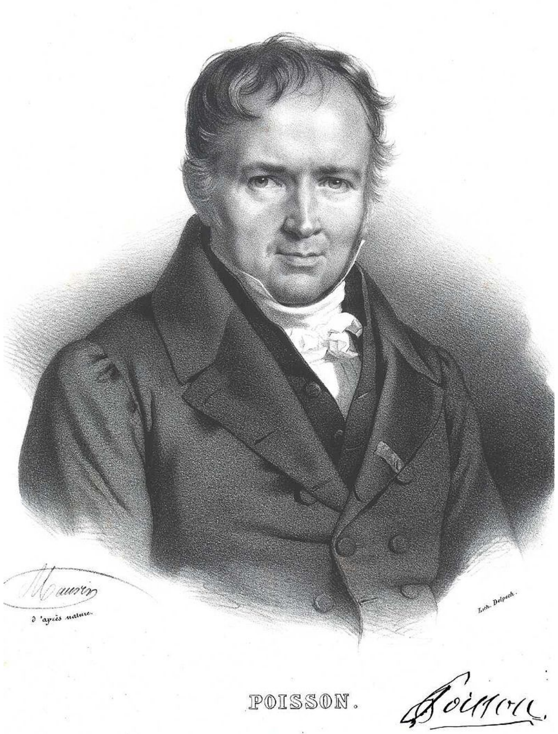
| Distribution | Uses |
|---|---|
| $\chi^2$ | Regression comparisons (asymptotic), contingency table independence tests, goodness-of-fit tests |
| $F$ | Gaussian model comparisons (exact) |
| $Z$ (standard normal) | Proportions, means, regression coefficients (asymptotic) |
| $T$ | Means, regression coefficients in Gaussian outcome models (exact) |

#### 1.3.1 The Bernoulli distribution

**Definition 1.6** (Bernoulli distribution)**.** The **Bernoulli distribution** family for a random variable $X$ is defined as:

$$\begin{aligned}\Pr(X = x) &= 1_{x \in \{0,1\}} \pi^x (1-\pi)^{1-x} \\ &= \begin{cases} \pi, x = 1 \\ 1-\pi, x = 0 \end{cases}\end{aligned}$$

### 1.3.2 The Poisson distribution




(a) Siméon Denis Poisson


(b) Les Poissons[a]

[a]https://youtu.be/UoJxBEQRLd0?t=12

Figure 1: "Les Poissons"

---

**Exercise 1.4.** Define the Poisson distribution.

---

*Solution* 1.1.

*Definition* 1.7 (Poisson distribution).

$$\mathrm{P}(Y = y) = \frac{\mu^y e^{-\mu}}{y!}, y \in \mathrm{N} \tag{3}$$

(see Figure 2)

---

**Exercise 1.5.** What is the range of possible values for a Poisson distribution?

---

*Solution* 1.2.

$$\mathcal{R}(Y) = \{0, 1, 2, ...\} = \mathrm{N}$$

---

**Theorem 1.5** (CDF of Poisson distribution)**.**

$$P(Y \leq y) = e^{-\mu} \sum_{j=0}^{\lfloor y \rfloor} \frac{\mu^j}{j!} \tag{4}$$

(see Figure 3)

```r
library(dplyr)
pois_dists <- tibble(
  mu = c(0.5, 1, 2, 5, 10, 20)
) |>
  reframe(
    .by = mu,
    x = 0:30
  ) |>
  mutate(
    `P(X = x)` = dpois(x, lambda = mu),
    `P(X <= x)` = ppois(x, lambda = mu),
    mu = factor(mu)
  )

library(ggplot2)
library(latex2exp)

plot0 <- pois_dists |>
  ggplot(
    aes(
      x = x,
      y = `P(X = x)`,
      fill = mu,
      col = mu
    )
  ) +
  theme(legend.position = "bottom") +
  labs(
    fill = latex2exp::TeX("$\\mu$"),
    col = latex2exp::TeX("$\\mu$"),
    y = latex2exp::TeX("$\\Pr_{\\mu}(X = x)$")
  )

plot1 <- plot0 +
  geom_segment(yend = 0) +
  facet_wrap(~mu)

print(plot1)
```

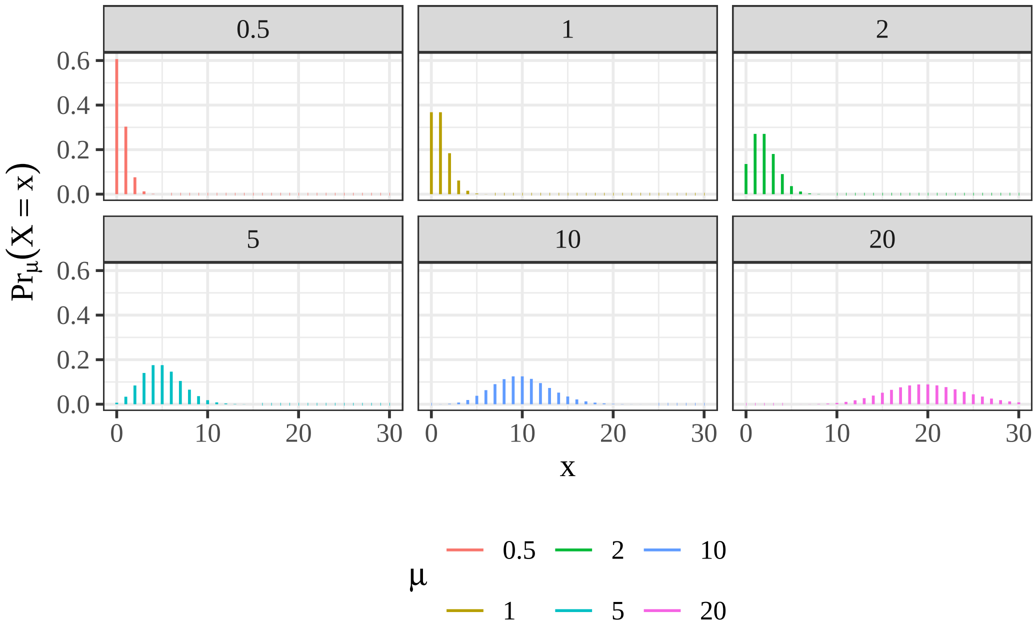



Figure 2: Poisson PMFs, by mean parameter $\mu$

---

```
library(ggplot2)

plot2 <-
  plot0 +
  geom_step(alpha = 0.75) +
  aes(y = `P(X <= x)`) +
  labs(y = latex2exp::TeX("$\\Pr_{\\mu}(X \\leq x)$"))

print(plot2)
```

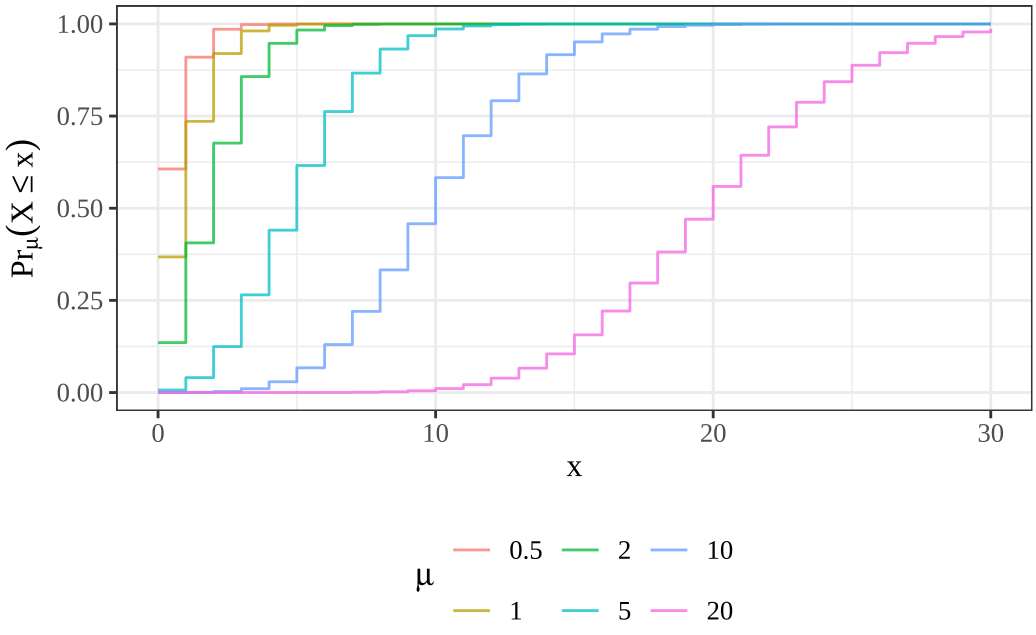



Figure 3: Poisson CDFs

---

**Exercise 1.6** (Poisson distribution functions)**.** Let $X \sim \text{Pois}(\mu = 3.75)$.

Compute:

- $\text{P}(X = 4|\mu = 3.75)$
- $\text{P}(X \leq 7|\mu = 3.75)$
- $\text{P}(X > 5|\mu = 3.75)$

---

*Solution.*

- $\text{P}(X = 4) = 0.19378$
- $\text{P}(X \leq 7) = 0.962379$
- $\text{P}(X > 5) = 0.177117$

---

**Theorem 1.6** (Properties of the Poisson distribution)**.** *If* $X \sim Pois(\mu)$*, then:*

- $E[X] = \mu$
- $Var(X) = \mu$
- $P(X = x) = \frac{\mu}{x} P(X = x - 1)$
- *For* $x < \mu$*,* $P(X = x) > P(X = x - 1)$
- *For* $x = \mu$*,* $P(X = x) = P(X = x - 1)$
- *For* $x > \mu$*,* $P(X = x) < P(X = x - 1)$
- $\arg\max_x P(X = x) = \lfloor \mu \rfloor$

**Exercise 1.7.** Prove Theorem 1.6.

---

*Solution.*

$$
\begin{aligned}
\mathrm{E}[X] &= \sum_{x=0}^{\infty} x \cdot P(X=x) \\
&= 0 \cdot P(X=0) + \sum_{x=1}^{\infty} x \cdot P(X=x) \\
&= 0 + \sum_{x=1}^{\infty} x \cdot P(X=x) \\
&= \sum_{x=1}^{\infty} x \cdot P(X=x) \\
&= \sum_{x=1}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \sum_{x=1}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x \cdot (x-1)!} && \text{[definition of factorial (”!”) function]} \\
&= \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!} \\
&= \sum_{x=1}^{\infty} \frac{(\lambda \cdot \lambda^{x-1}) e^{-\lambda}}{(x-1)!} \\
&= \lambda \cdot \sum_{x=1}^{\infty} \frac{(\lambda^{x-1}) e^{-\lambda}}{(x-1)!} \\
&= \lambda \cdot \sum_{y=0}^{\infty} \frac{(\lambda^{y}) e^{-\lambda}}{(y)!} && \text{[substituting } y \stackrel{\text{def}}{=} x-1\text{]} \\
&= \lambda \cdot 1 && \text{[because PDFs sum to 1]} \\
&= \lambda
\end{aligned}
$$

See also https://statproofbook.github.io/P/poiss-mean.

For the variance, see https://statproofbook.github.io/P/poiss-var.

---

**Accounting for exposure**

If the exposures/observation durations, denoted $T = t$ or $N = n$, vary between observations, we model:

$$\mu = \lambda \cdot t$$

$\lambda$ is interpreted as the “expected event rate per unit of exposure”; that is,

$$\lambda = \frac{\mathrm{E}[Y|T=t]}{t}$$

> **! Important**
>
> The exposure magnitude, $T$, is *similar* to a covariate in linear or logistic regression. However, there is an important difference: in count regression, **there is no intercept corresponding to E**$[Y|T=0]$. In other words, this model assumes that if there is no exposure, there can't be any events.

**Theorem 1.7.** *If* $\mu = \lambda \cdot t$*, then:*

$$\log \mu = \log \lambda + \log t$$

**Definition 1.8** (Offset)**.** When the linear component of a model involves a term without an unknown coefficient, that term is called an **offset**.

---

**Theorem 1.8.** *If $X$ and $Y$ are independent Poisson random variables with means $\mu_X$ and $\mu_Y$, their sum, $Z = X + Y$, is also a Poisson random variable, with mean $\mu_Z = \mu_X + \mu_Y$.*

---

*Proof.* See https://web.stanford.edu/class/archive/cs/cs109/cs109.1206/lectureNotes/LN12_independent_rvs.pdf, Example 3. □

---

### 1.3.3 The Negative-Binomial distribution

**Definition 1.9** (Negative binomial distribution)**.**

$$\mathrm{P}(Y = y) = \frac{\mu^y}{y!} \cdot \frac{\Gamma(\rho + y)}{\Gamma(\rho) \cdot (\rho + \mu)^y} \cdot \left(1 + \frac{\mu}{\rho}\right)^{-\rho}$$

where $\rho$ is an overdispersion parameter and $\Gamma(x) = (x - 1)!$ for integers $x$.

You don't need to memorize or understand this expression.

As $\rho \to \infty$, the second term converges to 1 and the third term converges to $\exp\{-\mu\}$, which brings us back to the Poisson distribution.

---

**Theorem 1.9.** *If $Y \sim NegBin(\mu, \rho)$, then:*

- $E[Y] = \mu$
- $Var(Y) = \mu + \frac{\mu^2}{\rho} > \mu$

### 1.3.4 Weibull Distribution

$$\begin{aligned} p(t) &= \alpha\lambda x^{\alpha-1} \mathrm{e}^{-\lambda x^{\alpha}} \\ \lambda(t) &= \alpha\lambda x^{\alpha-1} \\ \mathrm{S}(t) &= \mathrm{e}^{-\lambda x^{\alpha}} \\ E(T) &= \Gamma(1 + 1/\alpha) \cdot \lambda^{-1/\alpha} \end{aligned}$$

When $\alpha = 1$ this is the exponential. When $\alpha > 1$ the hazard is increasing and when $\alpha < 1$ the hazard is decreasing. This provides more flexibility than the exponential.

We will see more of this distribution later.

## 1.4 Characteristics of probability distributions

### 1.4.1 Probability density function

**Definition 1.10** (probability density)**.** If $X$ is a continuous random variable, then the **probability density** of $X$ at value $x$, denoted $f(x)$, $f_X(x)$, $\mathrm{p}(x)$, $\mathrm{p}_X(x)$, or $\mathrm{p}(X = x)$, is defined as the limit of the probability (mass) that $X$ is in an interval around $x$, divided by the width of that interval, as that width reduces to 0.

$$f(x) \stackrel{\text{def}}{=} \lim_{\Delta \to 0} \frac{\mathrm{P}(X \in [x, x + \Delta])}{\Delta}$$

See also Rothman et al. (2021) (Chapter 22, p. 535) and https://en.wikipedia.org/wiki/Probability_density_function#Formal_definition

---

**Theorem 1.10** (Density function is derivative of CDF)**.** *The density function $f(t)$ or $p(T = t)$ for a random variable $T$ at value $t$ is equal to the derivative of the cumulative probability function $F(t) \stackrel{def}{=} P(T \le t)$; that is:*

$$f(t) \stackrel{def}{=} \frac{\partial}{\partial t} F(t)$$

---

**Theorem 1.11** (Density functions integrate to 1)**.** *For any density function $f(x)$,*

$$\int_{x \in \mathcal{R}(X)} f(x) dx = 1$$

---

### 1.4.2 Hazard function

**Definition 1.11** (Hazard function, hazard rate, hazard rate function)**.**

The **hazard function**, **hazard rate**, **hazard rate function**, for a random variable $T$ at value $t$, typically denoted as $\mathrm{h}(t)$ [3] or $\lambda(t)$, [4] is the conditional density[5] of $T$ at $t$, given $T \ge t$. That is:

$$\lambda(t) \stackrel{\text{def}}{=} \mathrm{p}(T = t | T \ge t)$$

If $T$ represents the time at which an event occurs, then $\lambda(t)$ is the probability that the event occurs at time $t$, given that it has not occurred prior to time $t$.

---

Table 4: Probability distribution functions

| Name | Symbols | Definition |
|---|---|---|
| Probability density function (PDF) | $\mathrm{f}(t), \mathrm{p}(t)$ | $\mathrm{p}(T = t)$ |
| Cumulative distribution function (CDF) | $\mathrm{F}(t), \mathrm{P}(t)$ | $\mathrm{P}(T \le t)$ |
| Survival function | $\mathrm{S}(t), \bar{\mathrm{F}}(t)$ | $\mathrm{P}(T > t)$ |
| Hazard function | $\lambda(t), \mathrm{h}(t)$ | $\mathrm{p}(T = t \| T \ge t)$ |
| Cumulative hazard function | $\Lambda(t), \mathrm{H}(t)$ | $\int_{u=-\infty}^{t} \lambda(u) du$ |
| Log-hazard function | $\eta(t)$ | $\log\{\lambda(t)\}$ |

---

$$\mathrm{f}(t) \xleftarrow[\mathrm{S}(t)\lambda(t)]{-S'(t)} \mathrm{S}(t) \xleftarrow{\exp\{-\Lambda(t)\}} \Lambda(t) \xleftarrow{\int_{u=0}^{t} \lambda(u) du} \lambda(t) \xleftarrow{\exp\{\eta(t)\}} \eta(t)$$

$$\mathrm{f}(t) \xrightarrow[\int_{u=t}^{\infty} \mathrm{f}(u) du]{\mathrm{f}(t)/\lambda(t)} \mathrm{S}(t) \xrightarrow[-\log \mathrm{S}(t)]{} \Lambda(t) \xrightarrow[\Lambda'(t)]{} \lambda(t) \xrightarrow[\log\{\lambda(t)\}]{} \eta(t)$$

---

[3] for example in Dobson and Barnett (2018), Vittinghoff et al. (2012), Klein and Moeschberger (2003), and Kleinbaum and Klein (2012)

[4] for example, in Rothman et al. (2021) and Kalbfleisch and Prentice (2011)

[5] probability.qmd#def-pdf

### 1.4.3 Expectation

**Definition 1.12** (Expectation, expected value, population mean ). The **expectation**, **expected value**, or **population mean** of a *continuous* random variable $X$, denoted $\text{E}[X]$, $\mu(X)$, or $\mu_X$, is the weighted mean of $X$'s possible values, weighted by the probability density function of those values:

$$\text{E}[X] = \int_{x \in \mathcal{R}(X)} x \cdot \text{p}(X = x)dx$$

The **expectation**, **expected value**, or **population mean** of a *discrete* random variable $X$, denoted $\text{E}[X]$, $\mu(X)$, or $\mu_X$, is the mean of $X$'s possible values, weighted by the probability mass function of those values:

$$\text{E}[X] = \sum_{x \in \mathcal{R}(X)} x \cdot \text{P}(X = x)$$

(c.f. https://en.wikipedia.org/wiki/Expected_value)

---

**Theorem 1.12** (Expectation of the Bernoulli distribution). *The expectation of a Bernoulli random variable with parameter $\pi$ is:*

$$E[X] = \pi$$

---

*Proof.*

$$\begin{aligned}
\text{E}[X] &= \sum_{x \in \mathcal{R}(X)} x \cdot \text{P}(X = x) \\
&= \sum_{x \in \{0,1\}} x \cdot \text{P}(X = x) \\
&= (0 \cdot \text{P}(X = 0)) + (1 \cdot \text{P}(X = 1)) \\
&= (0 \cdot (1 - \pi)) + (1 \cdot \pi) \\
&= 0 + \pi \\
&= \pi
\end{aligned}$$

□

---

**Theorem 1.13** (Expectation of time-to-event variables). *If $T$ is a non-negative random variable, then:*

$$\mu(T|\tilde{X} = \tilde{x}) = \int_{t=0}^{\infty} S(t)dt$$

---

### 1.4.4 Variance and related characteristics

**Definition 1.13** (Variance). The variance of a random variable $X$ is the expectation of the squared difference between $X$ and $\text{E}[X]$; that is:

$$\text{Var}(X) \stackrel{\text{def}}{=} \text{E}[(X - \text{E}[X])^2]$$

---

**Theorem 1.14** (Simplified expression for variance).

$$Var(X) = E[X^2] - (E[X])^2$$

*Proof.* By linearity of expectation, we have:

$$\begin{aligned}
\text{Var}(X) &\stackrel{\text{def}}{=} \text{E}[(X - \text{E}[X])^2] \\
&= \text{E}\Big[X^2 - 2X\text{E}[X] + (\text{E}[X])^2\Big] \\
&= \text{E}[X^2] - \text{E}[2X\text{E}[X]] + \text{E}\Big[(\text{E}[X])^2\Big] \\
&= \text{E}[X^2] - 2\text{E}[X]\text{E}[X] + (\text{E}[X])^2 \\
&= \text{E}[X^2] - (\text{E}[X])^2
\end{aligned}$$

□

**Definition 1.14** (Precision)**.** The **precision** of a random variable $X$, often denoted $\tau(X)$, $\tau_X$, or shorthanded as $\tau$, is the inverse of that random variable's variance; that is:

$$\tau(X) \stackrel{\text{def}}{=} (\text{Var}(X))^{-1}$$

**Definition 1.15** (Standard deviation)**.** The standard deviation of a random variable $X$ is the square-root of the variance of $X$:

$$\text{SD}(X) \stackrel{\text{def}}{=} \sqrt{\text{Var}(X)}$$

**Definition 1.16** (Covariance)**.** For any two one-dimensional random variables, $X, Y$:

$$\text{Cov}(X, Y) \stackrel{\text{def}}{=} \text{E}[(X - \text{E}[X])(Y - \text{E}[Y])]$$

**Theorem 1.15.**

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

*Proof.* Left to the reader. □

**Lemma 1.1** (The covariance of a variable with itself is its variance)**.** *For any random variable X:*

$$Cov(X, X) = Var(X)$$

*Proof.*

$$\begin{aligned}
\text{Cov}(X, X) &= E[XX] - E[X]E[X] \\
&= E[X^2] - (E[X])^2 \\
&= \text{Var}(X)
\end{aligned}$$

□

**Definition 1.17** (Variance/covariance of a $p \times 1$ random vector)**.** For a $p \times 1$ dimensional random vector $\tilde{X}$,

$$\begin{aligned}
\text{Var}\Big(\tilde{X}\Big) &\stackrel{\text{def}}{=} \text{Cov}\Big(\tilde{X}\Big) \\
&\stackrel{\text{def}}{=} \text{E}\Big[\Big(\tilde{X} - \text{E}\tilde{X}\Big)^\top \Big(\tilde{X} - \text{E}\tilde{X}\Big)\Big]
\end{aligned}$$

**Theorem 1.16** (Alternate expression for variance of a random vector)**.**

$$Var(X) = E[X^\top X] - E[X]^\top E[X]$$

*Proof.*

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X^\top - E[X]^\top)(X - E[X])] \\
&= E[X^\top X - E[X]^\top X - X^\top E[X] + E[X]^\top E[X]] \\
&= E[X^\top X] - E[X]^\top E[X] - E[X]^\top E[X] + E[X]^\top E[X] \\
&= E[X^\top X] - 2E[X]^\top E[X] + E[X]^\top E[X] \\
&= E[X^\top X] - E[X]^\top E[X]
\end{aligned}$$

□

**Theorem 1.17** (Variance of a linear combination)**.** *For any vector of random variables $\tilde{X} = (X_1, \ldots, X_n)$ and corresponding vector of constants $\tilde{a} = (a_1, \ldots, a_n)$, the variance of their linear combination is:*

$$\begin{aligned}
Var\left(\tilde{a} \cdot \tilde{X}\right) &= Var\left(\sum_{i=1}^{n} a_i X_i\right) \\
&= \tilde{a}^\top Var\left(\tilde{X}\right)\tilde{a} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov(X_i, X_j)
\end{aligned}$$

*Proof.* Left to the reader… □

**Corollary 1.3.** *For any two random variables $X$ and $Y$ and scalars $a$ and $b$:*

$$Var(aX + bY) = a^2\, Var(X) + b^2\, Var(Y) + 2(a \cdot b)\, Cov(X, Y)$$

*Proof.* Apply Theorem 1.17 with $n = 2$, $X_1 = X$, and $X_2 = Y$.

Or, see https://statproofbook.github.io/P/var-lincomb.html □

**Definition 1.18** (homoskedastic, heteroskedastic)**.** A random variable $Y$ is **homoskedastic** (with respect to covariates $X$) if the variance of $Y$ does not vary with $X$:

$$\mathrm{Var}(Y|X = x) = \sigma^2, \forall x$$

Otherwise it is **heteroskedastic**.

**Definition 1.19** (Statistical independence)**.** A set of random variables $X_1, \ldots, X_n$ are **statistically independent** if their joint probability is equal to the product of their marginal probabilities:

$$\Pr(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} \Pr(X_i = x_i)$$

Tip

The symbol for independence, $\perp\!\!\!\perp$, is essentially just $\prod$ upside-down. So the symbol can remind you of its definition (Definition 1.19).

---

**Definition 1.20** (Conditional independence)**.** A set of random variables $Y_1, \dots, Y_n$ are **conditionally statistically independent** given a set of covariates $X_1, \dots, X_n$ if the joint probability of the $Y_i$s given the $X_i$s is equal to the product of their marginal probabilities:

$$\Pr(Y_1 = y_1, \dots, Y_n = y_n | X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^{n} \Pr(Y_i = y_i | X_i = x_i)$$

---

**Definition 1.21** (Identically distributed)**.** A set of random variables $X_1, \dots, X_n$ are **identically distributed** if they have the same range $\mathcal{R}(X)$ and if their marginal distributions $\text{P}(X_1 = x_1), \dots, \text{P}(X_n = x_n)$ are all equal to some shared distribution $\text{P}(X = x)$:

$$\forall i \in \{1 : n\}, \forall x \in \mathcal{R}(X) : \text{P}(X_i = x) = \text{P}(X = x)$$

---

**Definition 1.22** (Conditionally identically distributed)**.** A set of random variables $Y_1, \dots, Y_n$ are **conditionally identically distributed** given a set of covariates $X_1, \dots, X_n$ if $Y_1, \dots, Y_n$ have the same range $\mathcal{R}(X)$ and if the distributions $\text{P}(Y_i = y_i | X_i = x_i)$ are all equal to the same distribution $\text{P}(Y = y | X = x)$:

$$\text{P}(Y_i = y | X_i = x) = \text{P}(Y = y | X = x)$$

---

**Definition 1.23** (Independent and identically distributed)**.** A set of random variables $X_1, \dots, X_n$ are **independent and identically distributed** (shorthand: "$X_i$ iid") if they are statistically independent and identically distributed.

---

**Definition 1.24** (Conditionally independent and identically distributed)**.** A set of random variables $Y_1, \dots, Y_n$ are **conditionally independent and identically distributed** (shorthand: "$Y_i | X_i$ ciid" or just "$Y_i | X_i$ iid") given a set of covariates $X_1, \dots, X_n$ if $Y_1, \dots, Y_n$ are conditionally independent given $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are identically distributed given $X_1, \dots, X_n$.

## 1.5 The Central Limit Theorem

The sum of many independent or nearly-independent random variables with small variances (relative to the number of RVs being summed) produces bell-shaped distributions.

For example, consider the sum of five dice (Figure 4).

```r
library(dplyr)
dist =
  expand.grid(1:6, 1:6, 1:6, 1:6, 1:6) |>
  rowwise() |>
  mutate(total = sum(c_across(everything()))) |>
  ungroup() |>
  count(total) |>
  mutate(`p(X=x)` = n/sum(n))

library(ggplot2)
```

```
dist |>
  ggplot() +
  aes(x = total, y = `p(X=x)`) +
  geom_col() +
  xlab("sum of dice (x)") +
  ylab("Probability of outcome, Pr(X=x)") +
  expand_limits(y = 0)
```

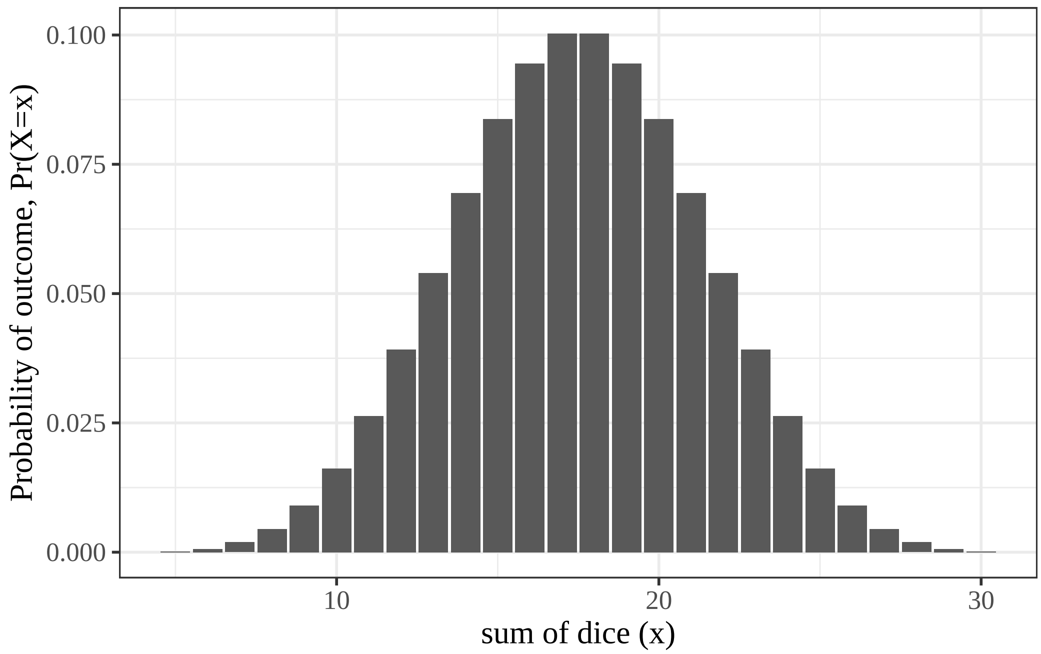



Figure 4: Distribution of the sum of five dice

In comparison, the outcome of just one die is not bell-shaped (Figure 5).

```
library(dplyr)
dist =
  expand.grid(1:6) |>
  rowwise() |>
  mutate(total = sum(c_across(everything()))) |>
  ungroup() |>
  count(total) |>
  mutate(`p(X=x)` = n/sum(n))

library(ggplot2)

dist |>
  ggplot() +
  aes(x = total, y = `p(X=x)`) +
  geom_col() +
  xlab("sum of dice (x)") +
  ylab("Probability of outcome, Pr(X=x)") +
  expand_limits(y = 0)
```

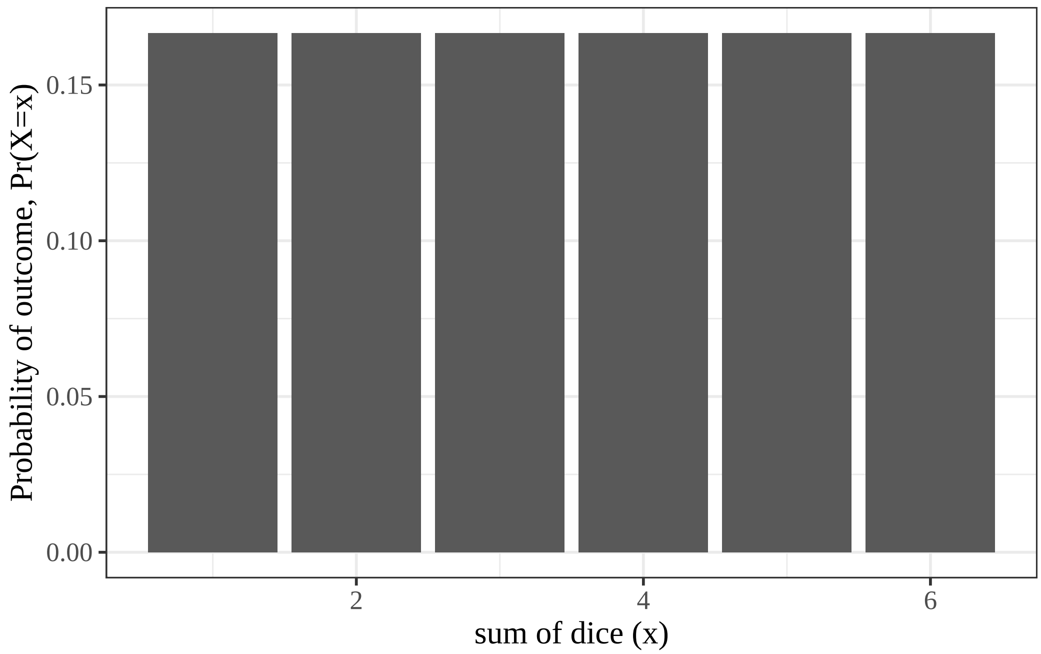



Figure 5: Distribution of the outcome of one die

What distribution does a single die have?

Answer: discrete uniform on 1:6.

## 1.6 Additional resources

- Miller (2017)

Dobson, Annette J, and Adrian G Barnett. 2018. *An Introduction to Generalized Linear Models*. 4th ed. CRC press. https://doi.org/10.1201/9781315182780.

Kalbfleisch, John D, and Ross L Prentice. 2011. *The Statistical Analysis of Failure Time Data*. John Wiley & Sons.

Klein, John P, and Melvin L Moeschberger. 2003. *Survival Analysis: Techniques for Censored and Truncated Data*. Vol. 1230. Springer. https://link.springer.com/book/10.1007/b97377.

Kleinbaum, David G, and Mitchel Klein. 2012. *Survival Analysis: A Self-Learning Text*. 3rd ed. Springer. https://link.springer.com/book/10.1007/978-1-4419-6646-9.

Miller, Steven J. 2017. *The Probability Lifesaver : All the Tools You Need to Understand Chance*. A Princeton Lifesaver Study Guide. Princeton: Princeton University Press. https://press.princeton.edu/books/hardcover/9780691149547/the-probability-lifesaver.

Rothman, Kenneth J., Timothy L. Lash, Tyler J. VanderWeele, and Sebastien Haneuse. 2021. *Modern Epidemiology*. Fourth edition. Philadelphia: Wolters Kluwer.

Vittinghoff, Eric, David V Glidden, Stephen C Shiboski, and Charles E McCulloch. 2012. *Regression Methods in Biostatistics: Linear, Logistic, Survival, and Repeated Measures Models*. 2nd ed. Springer. https://doi.org/10.1007/978-1-4614-1353-0.