

# Einführung in L<sup>A</sup>T<sub>E</sub>X

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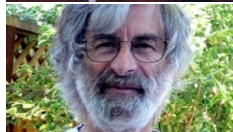
## 1 $\text{\LaTeX}$ ?

## 2 Grundlegender Syntax

## 3 Anderes

# Was ist L<sup>A</sup>T<sub>E</sub>X?

- Knuth hat T<sub>E</sub>Xgemacht
- Lamport hat dann L<sup>A</sup>T<sub>E</sub>Xdaraus gemacht



# Warum L<sup>A</sup>T<sub>E</sub>X?

- Einfacher
- Logisch
- Bibliografie
- macht spaß

# Warum nicht Word?

- Word macht es schwerer Änderungen an großen Dokumenten vorzunehmen.
- Bibliografien werden automatisch gemacht, auch Zitierstil nachträglich änderbar.
- Seitenzahlen, Referenzen, etc. werden automatisch erzeugt.
- *kann man nicht in Vim benutzen.*

# Nutzzwecke

- Ausarbeitungen/Laborberichte
- Präsentationen
- Dokumente
- Lebenslauf
- Bücher



# Paper



Figure 1.3: Example of a Morse function on the torus. At each critical point, the index  $i$ , the number of downward directions is indicated.



Figure 1.4: Two Morse functions on  $S^2$  with a different number of critical points.



Figure 1.5: An example of an embedding where the height function is not Morse.



Figure 1.6: An example of a function that is not Morse.  $f(x, y) = x^2 + y^2$ . Small perturbations of  $f$  are Morse.

## CHAPTER 1. MORSE THEORY

Intuitively, the index of a critical point  $p$  is “the number of downward directions”. Let us give some examples of Morse functions.

**Example 1.3.** Let  $M$  be the torus  $T^2$  embedded in  $\mathbb{R}^3$  as illustrated in Figure 1.3. Then the height function  $h: T^2 \rightarrow \mathbb{R}$  which is the projection on the  $z$ -axis is a Morse function with four critical points. We have a minimum, two saddle points and a maximum, whose indices are 0, 1, 1, 2 respectively.

**Example 1.4.** In Figure 1.2, we have illustrated two embeddings of  $S^2$  in  $\mathbb{R}^3$ , and considering the corresponding height functions, we get two Morse functions  $S^2 \rightarrow \mathbb{R}$ . The first one has only two critical points: a maximum and a minimum. The second one has two maxima, a saddle point and a minimum. Later on we will prove that any manifold admitting a Morse function with only two critical points is homeomorphic to the sphere.

**Nonexample 1.5.** Let  $M = \mathbb{R}^2$  and  $f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto x^2$ . Then all points  $(x, y)$  for  $x = 0$  are critical points of this function. In particular,  $(0, 0)$  is a critical point. As it is impossible to find local coordinates  $(u, v)$  for which  $f$  can be written as  $u^2 + v^2$ , we conclude that  $f$  is not Morse.

**Nonexample 1.6.** Let  $M = \mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto x^3$ . Then  $x = 0$  is a critical point, but  $f$  is not Morse. Note however that if we add a small perturbation to  $f$ , say  $g: x \mapsto x^3 + x$ , then  $g$  would recover  $f$  as Morse. For  $t < 0$ ,  $g$  has two critical points: one of index 1 and one of index 0. If  $t > 0$ ,  $g$  has no critical points.

Note that this last case where  $f$  has no critical points cannot happen if  $M$  is compact. Indeed, any function attains its maximum and minimum on a compact manifold, so we have at least two critical points. On the other hand, the number of critical points is at most finite. This is because of the definition of a Morse function. It implies that critical points are isolated, which on a compact manifold implies that their number is finite. This also immediately rules out the situation we had in the other example, where the set of critical points was a straight line.

## 1.2 Coordinate-free definition

The attentive reader will have noticed that the notion of the index of a critical point could possibly be coordinate dependent and hence ill-defined. In order to show that it is not, we will give an equivalent coordinate-free definition. For this, let us first define the Hessian.

**Definition 1.7 (Hessian).** Let  $M$  be a manifold and  $f: M \rightarrow \mathbb{R}$  a function. Let  $p$  be a critical point of  $f$ . Then we define the Hessian  $H_p$  to be the bilinear form

$$H_p: T_p M \times T_p M \rightarrow \mathbb{R}, \\ (X, Y) \mapsto X(Y)f|_p,$$

where  $V_p$  is a local extension of  $p$  around  $p$ .

## 1.2. COORDINATE-FREE DEFINITION

Because we are only considering the Hessian  $H_p$  at critical points, this is a well defined symmetric bilinear form.<sup>12</sup> In case of a Morse function given locally by  $f(x) = f(x) = x_1^2 + \dots + x_i^2 - x_{i+1}^2 - \dots - x_n^2$ , the Hessian at  $p$  is

$$H_p = 2x_1 \frac{\partial^2 f}{\partial x_1^2} - \dots - 2x_i \frac{\partial^2 f}{\partial x_i^2} + \dots + 2x_n \frac{\partial^2 f}{\partial x_n^2},$$

where  $\partial x_i = dx_i$ . Note in particular that  $H_p$  is non-degenerate and its signature is  $(i, n-i) = (i, n-i)$ , as we have  $i$  negative eigenvalues and  $n-i$  positive eigenvalues. As the signature of a symmetric bilinear form is coordinate independent, this shows that the index of a critical point is as well.

Interestingly, the converse is also true: if  $H_p$  is non-degenerate for all critical points  $p$  of  $f$ , then  $f$  is a Morse function. Many authors take this to be the definition of a Morse function, and then prove the so-called Morse lemma stating that there always exist local coordinates such that  $f$  is given by

$$f(x) = f(p) - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2,$$

which is our definition of a Morse function. With our choice, the Morse lemma takes on the following form:

**Lemma 1.8 (Morse Lemma).** Let  $M$  be a manifold and  $f: M \rightarrow \mathbb{R}$  a smooth function. If for all  $p \in \text{Crit } f$ , the Hessian  $H_p$  is non-degenerate, then  $f$  is Morse.

**Proof.** We follow the proof of Milnor<sup>13</sup>. We may assume that  $M = \mathbb{R}^n$ ,  $p$  is the origin and  $f(p) = 0$ . Then by a version of Taylor’s theorem, we can write

$$f(x) = f(x) = \sum_{i=1}^n \frac{1}{2} x_i^2 + o(|x|^2) \\ = \sum_{i=1}^n \lambda_i x_i^2 + o(|x|^2),$$

where  $g$  are smooth functions. Now, as  $g(x) = o(|x|^2) = 0$ , we can prove this for each  $g$ , giving us the following:

$$f(x) = \sum_{i=1}^n \lambda_i x_i^2 + o(|x|^2).$$

Because the sum is symmetric in  $i$  and  $j$ , we may assume that  $\lambda_1$  is symmetric in  $i$  and  $j$ .<sup>14</sup> Note that

$$\lambda_1(x) = \frac{\partial^2 f}{\partial x_1^2}(x),$$

which is non-degenerate by assumption.

Now we restrict the proof of diagonalization of a non-degenerate quadratic form. We do this by induction. Suppose we have coordinates  $u_1, \dots, u_{n-1}$  a neighborhood of  $0$  such that

$$f = \lambda_1 u_1^2 + \dots + \lambda_{n-1} u_{n-1}^2 + \sum_{i=1}^{n-1} u_i u_n \lambda_i(u_i).$$

<sup>12</sup> The difference between  $H_p(X, Y)$  and  $H_p(Y, X)$  is given by  $H_p(X, Y) - H_p(Y, X) = X(Y)f - Y(X)f = [X, Y]f = 0$ .

The value of  $H_p$  also does not depend on the choice of the normal field. Indeed, suppose  $f$  and  $\tilde{f}$  are two different extensions of  $f$ . Then by symmetry  $H_p = \tilde{H}_p$  on  $T_p$ .

$X(Y)f = Y(X)f = X(Y)\tilde{f}$ . This also shows linearity of the second covariant derivative.

<sup>13</sup> John Milnor, *Morse Theory* (AMS-50), Vol. 16, Princeton university press, 2010, p. 6.

<sup>14</sup> If  $\lambda_1$  is not symmetric, we can replace it by  $\lambda_{1,0} = \frac{1}{2}(\lambda_1 + \lambda_1^T)$ . Then  $\lambda_{1,0}$  is symmetric and we still have  $\sum_{i=1}^n \lambda_{1,0} x_i^2 + o(|x|^2)$ .

# Abbildung: Auszug einer Masterarbeit über Morse Theory





## Toeplitz Matrix

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \dots & \dots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \dots & \dots & a_2 & a_1 & a_0 \end{bmatrix}.$$

# Physik Beispiel

Sequential Quantum Circuits as Maps between Gapped Phases.[2]

$$\frac{1}{|G|} \sum_g X_i^g \rightarrow \sum_h T_{i-1}^h T_i^h, \quad i = 2, \dots, N,$$

$$\frac{1}{|G|} \sum_g X_1^g \rightarrow \frac{1}{|G|} \sum_{h,h'} e^{-\frac{2\pi i}{|G|}(h'-h)g} T_1^h T_N^{h'} \prod_{i=1}^N X_i^g,$$

$$\sum_h T_i^h T_{i+1}^h \rightarrow \frac{1}{|G|} \sum_g X_i^g, \quad i = 2, \dots, N$$

$$\sum_h T_1^h T_2^h \rightarrow \frac{1}{|G|} \sum_g X_1^g \prod_{i=1}^N X_i^g.$$

# Wie man es benutzt

- Arch-basiert: `pacman -S texlive-basic`
- Debian-basiert: `apt-get install texlive-full`
- MacOS: MiKTeX
- Windows: MacTeX

# Wie Ich es benutze

1 2 3 4 5 6 7 8 9 Desktop | br:45 Stats-1 | End in 2:02 hours, after this Calculus 1 after a break of 45 min / 43% | v 40% | 100% | 100% | 17-06 09:14

32 CHAPTER 5. DISCRETE PROBABILITIES

5.2 Histogram from Probability Distribution.

Horizontal	Values of random variable
Vertical	Probability
Example	
Weighted die	
$x$	$P(x)$
1	.05
2	.15
3	.35
4	.30
5	.10
6	.05

Notice

$0 \leq P(x) \leq 1$   
 $\sum P(x) = 1$

Figure 5.1: Prob. distribution of the weighted die

5.2.1 Mean, Variance, Standard Deviation

Mean:

$$\mu = \frac{\sum x \cdot f}{N}$$

$$\mu = \sum \left[ x \cdot \frac{f}{N} \right]$$

$$\mu = \sum [x \cdot P(x)] \leftarrow \text{mean, expected value}$$

```

31 Values are unusual if they lie outside of:
32 \begin{itemize}
33   \item $\mu + 2\sigma$
34   \item $\mu - 2\sigma$
35 \end{itemize}
36
37 \begin{figure}[ht]
38   \centering
39   \includegraphics[width=0.6\textwidth]{lec_05_01}
40   \caption{Prob. distribution of the weighted die}
41   \label{fig:05-01}
42 \end{figure}
43
44 Here we took the dice example, and 1.08 as 5.72 mark our range of $\pm$ two
45 standard deviations, whatever's outside is considered unusual. So we get 1 \&
46 to be unusual.
47
48 If $P(A) \leq .05$ ``A'' is considered unusual.
49
50 \text{Flip a coin 1000 times} \{
51   P(\text{exactly 501 heads})=0.0252 \leq .05 \implies \text{unusual}
52 \}
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54 \{
55   P(\text{501 or more heads})=0.487 \implies \text{usual}
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