Einführung in LaTeX

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Was ist LaTeX?

- Erfinder
- warum er es erfand

Warum LaTeX?

- Einfacher
- Logisch
- Bibliografie
- macht spaß

Nutzzwecke

- Ausarbeitungen/Laborberichte
- Präsentationen
- Dokumente
- Bücher

Berichte

- Laborberichte

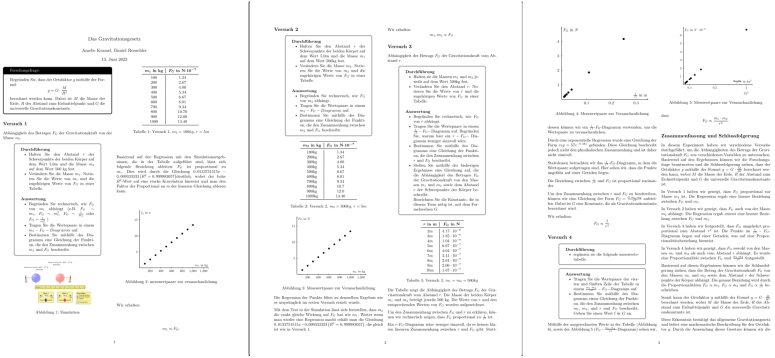
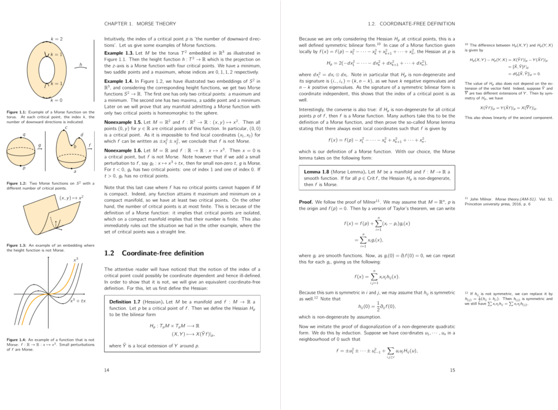


Abbildung: Laborprotokoll Gravitationsgesetz

Paper



CHAPTER 1. MORSE THEORY

Intuitively, the index of a critical point p is 'the number of downward directions'. Let us give some examples of Morse functions.

Example 1.3. Let M be the torus T^2 embedded in $\mathbb{R}^3$ as illustrated in Figure 1.1. Then the height function $h : T^2 \to \mathbb{R}$ which is the projection on the z-axis is a Morse function with four critical points. We have a minimum, two saddle points and a maximum, whose indices are $0, 1, 1, 2$ respectively.

Example 1.4. In Figure 1.2, we have illustrated two embeddings of S^2 in $\mathbb{R}^3$, and considering the corresponding height functions, we get two Morse functions $S^2 \to \mathbb{R}$. The first one has only two critical points: a maximum and a minimum. The second one has two maxima, a saddle point and a minimum. Later on we will prove that any manifold admitting a Morse function with only two critical points is homeomorphic to the sphere.

Nonexample 1.5. Let $M = \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x^2$. Then all points $(0, y)$ for $y \in \mathbb{R}$ are critical points of this function. In particular, $(0, 0)$ is a critical point. As it is impossible to find local coordinates (x_1, x_2) for which f can be written as $\pm x_1^2 \pm x_2^2$, we conclude that f is not Morse.

Nonexample 1.6. Let $M = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$. Then $x = 0$ is a critical point, but f is not Morse. Note however that if we add a small perturbation to f, say $g_t : x \mapsto x^3 + tx$, then for small non-zero t, g is Morse. For $t < 0$, g_t has two critical points: one of index 1 and one of index 0. If $t > 0$, g_t has no critical points.

Note that this last case where f has no critical points cannot happen if M is compact. Indeed, any function attains it maximum and minimum on a compact manifold, so we have at least two critical points. On the other hand, the number of critical points is at most finite. This is because of the definition of a Morse function: it implies that critical points are isolated, which on a compact manifold implies that their number is finite. This also immediately rules out the situation we had in the other example, where the set of critical points was a straight line.

Figure 1.1: Example of a Morse function on the torus. At each critical point, the index k, the number of downward directions is indicated.

Figure 1.2: Two Morse functions on S^2 with a different number of critical points.

Figure 1.3: An example of an embedding where the height function is not Morse.

Figure 1.4: An example of a function that is not Morse: $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$. Small perturbations of f are Morse.

1.2 Coordinate-free definition

The attentive reader will have noticed that the notion of the index of a critical point could possibly be coordinate dependent and hence ill-defined. In order to show that it is not, we will give an equivalent coordinate-free definition. For this, let us first define the Hessian:

Definition 1.7 (Hessian)**.** Let M be a manifold and $f : M \to \mathbb{R}$ a function. Let p be a critical point of f. Then we define the Hessian H_p to be the bilinear form

$$\begin{aligned} H_p : T_pM \times T_pM &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto X(\tilde{Y} f)|_p, \end{aligned}$$

where $\tilde{Y}$ is a local extension of Y around p.

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1.2. COORDINATE-FREE DEFINITION

Because we are only considering the Hessian H_p at critical points, this is a well defined symmetric bilinear form.[10] In case of a Morse function given locally by $f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$, the Hessian at p is

$$H_p = 2(-dx_1^2 - \cdots - dx_k^2 + dx_{k+1}^2 + \cdots + dx_n^2),$$

where $dx_i^2 = dx_i \otimes dx_i$. Note in particular that H_p is non-degenerate and its signature is $(i_-, i_+) = (k, n-k)$, as we have k negative eigenvalues and $n - k$ positive eigenvalues. As the signature of a symmetric bilinear form is coordinate independent, this shows that the index of a critical point is as well.

Interestingly, the converse is also true: if H_p is non-degenerate for all critical points p of f, then f is a Morse function. Many authors take this to be the definition of a Morse function, and then prove the so-called Morse lemma stating that there always exist local coordinates such that f is given by

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2,$$

which is our definition of a Morse function. With our choice, the Morse lemma takes on the following form:

Lemma 1.8 (Morse Lemma)**.** Let M be a manifold and $f : M \to \mathbb{R}$ a smooth function. If for all $p \in \operatorname{Crit} f$, the Hessian H_p is non-degenerate, then f is Morse.

Proof. We follow the proof of Milnor[11]. We may assume that $M = \mathbb{R}^n$, p is the origin and $f(p) = 0$. Then by a version of Taylor's theorem, we can write

$$\begin{aligned} f(x) &= f(p) + \sum_{i=1}^{n} (x_i - p_i) g_i(x) \\ &= \sum_{i=1}^{n} x_i g_i(x), \end{aligned}$$

where g_i are smooth functions. Now, as $g_i(0) = \partial_i f(0) = 0$, we can repeat this for each g_i, giving us the following:

$$f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x).$$

Because this sum is symmetric in i and j, we may assume that h_{ij} is symmetric as well.[12] Note that

$$h_{ij}(0) = \frac{1}{2} \partial_{ij} f(0),$$

which is non-degenerate by assumption.

Now we imitate the proof of diagonalization of a non-degenerate quadratic form. We do this by induction. Suppose we have coordinates $u_1, \cdots, u_n$ in a neighbourhood of 0 such that

$$f = \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u),$$

[10] The difference between $H_p(X, Y)$ and $H_p(Y, X)$ is given by

$$\begin{aligned} H_p(X, Y) - H_p(Y, X) &= X(\tilde{Y} f)|_p - Y(\tilde{X} f)|_p \\ &= [\tilde{X}, \tilde{Y}] f|_p \\ &= df_p[\tilde{X}, \tilde{Y}]|_p = 0. \end{aligned}$$

The value of H_p also does not depend on the extension of the vector field. Indeed, suppose $\tilde{Y}$ and $\overline{Y}$ are two different extensions of Y. Then by symmetry of H_p, we have

$$X(\tilde{Y} f)|_p = Y(\tilde{X} f)|_p = X(\overline{Y} f)|_p.$$

This also shows linearity of the second component.

[11] John Milnor. *Morse theory.(AM-51)*. Vol. 51. Princeton university press, 2016, p. 6

[12] If h_{ij} is not symmetric, we can replace it by $h_{(ij)} = \frac{1}{2}(h_{ij} + h_{ji})$. Then $h_{(ij)}$ is symmetric and we still have $\sum x_i x_j h_{ij} = \sum x_i x_j h_{(ij)}$.

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Abbildung: Auszug einer Masterarbeit über Morse Theory

Beispiel 1

Beispiele:

- Irgendwas mit Euler [1]

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{k=1}^{m}\frac{\partial^2}{\partial y_k^2}.$$

- Analysis Aufgabe:

$$\lim_{x\to\int_0^\infty \sqrt{t}e^{-t}dt}\left(\left(\sum_{n=0}^{\infty}\frac{x^{4n_4}}{(2n+1)(4n+3)(4n+4)}\right)''\right).$$

Toeplitz Matrix

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}.$$

Physik Beispiel

Sequential Quantum Circuits as Maps between Gapped Phases.[2]

$$\frac{1}{|G|}\sum_g X_i^g \to \sum_h T_{i-1}^h T_i^h, \quad i = 2, \dots, N,$$

$$\frac{1}{|G|}\sum_g X_1^g \to \frac{1}{|G|}\sum_{h,h'} e^{-\frac{2\pi i}{|G|}(h'-h)g} T_1^h T_N^{h'} \prod_{i=1}^N X_i^g,$$

$$\sum_h T_i^h T_{i+1}^h \to \frac{1}{|G|}\sum_g X_i^g, i = 2, \dots, N$$

$$\sum_h T_1^h T_2^h \to \frac{1}{|G|}\sum_g X_1^g \prod_{i=1}^N X_i^g.$$

Literatur

[1] Marcin Baranek u. a. *On the randomized Euler algorithm under inexact information*. 2023. arXiv: 2307.04718 [math.NA].

[2] Xie Chen u. a. *Sequential Quantum Circuits as Maps between Gapped Phases*. 2023. arXiv: 2307.01267 [cond-mat.str-el].