# Contents

# Chapter 1

# 5.1 – Area between two curves

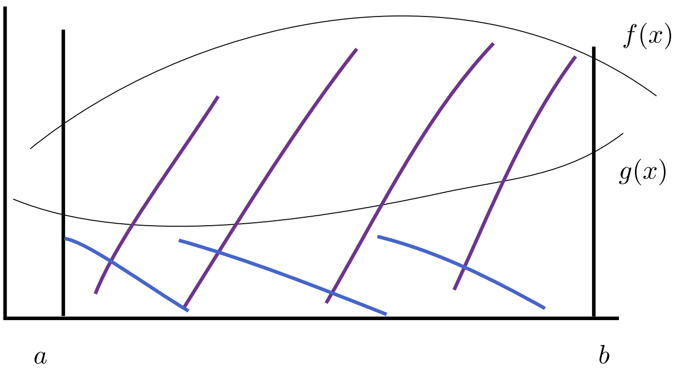


Figure 1.1: area-between-curves-1

Area between $f(x) \quad \& \quad g(x)$.

$$A = \text{Area under } f(x) \text{ - Area under } g(x).$$

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

$$A = \int_a^b \left[f(x) - g(x)\right] dx.$$

**Notitz:**

$$f(x) \geq g(x)$$
$$\forall x \in [a, b].$$
$$(f(x) \text{ is above } g(x)).$$

$$\int_a^c \left[f(x) - 0\right] dx + \int_c^b \left[0 - f(x)\right] dx$$
$$\int_a^c f(x)dx - \int_c^b f(x)dx$$

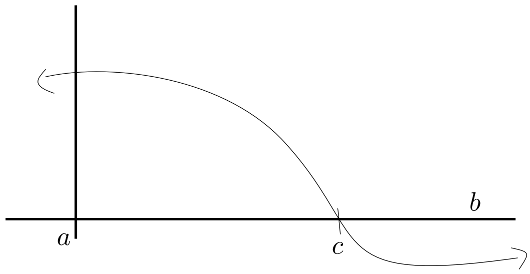


Figure 1.2: are-between-curves-2

Beispiel:

Find the area bounded above by $y = 2x + 5$, and bounded below by $y = x^3$ on $[0, 2]$

$$
\begin{aligned}
A &= \int_0^2 (2x+5) - x^3 \mathrm{dx} \\
&= \int_0^2 2x + 5 = x^3 dx \\
&= x^2 + 5x - \frac{x^4}{4}]_0^2 \\
A &= \left[2^2 + 5 \cdot 2\frac{2^4}{4}\right] - [0] \\
A &= 10
\end{aligned}
$$

Beispiel:

Find area between $y = x^2$ and $y = x + 6$.

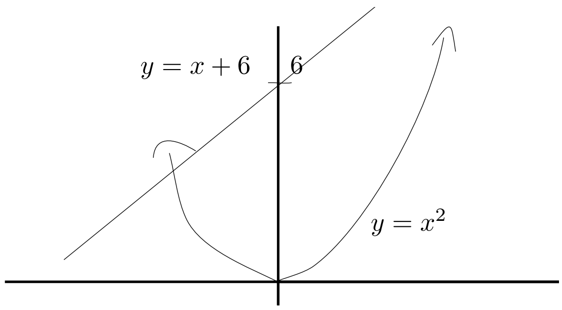


Figure 1.3: area-between-curves-3

Notitz:

**Steps:**

1. Find x-cords of the Intersection of the curves. ( Set $f(x) = g(x)$ )
2. Which function is on the top?
   (Pick one point for each interval)
3. set-up and solve

$$
\begin{aligned}
x + 6 &= x^2 \\
x^2 - x - 6 &= 0 \\
(x-3)\,(x+2)& \\
x - 3 = 0 \qquad & x + 2 = 0 \\
x = 3 \qquad & x = -2
\end{aligned}
$$

Those are the only places $f(x)$ and $g(x)$ are intercepting, so those are the bounds of integration.

$$A = \int_{-2}^{3} .$$

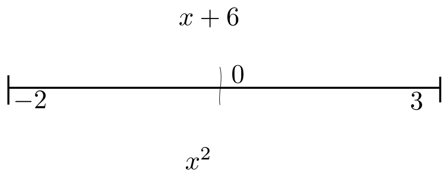


Figure 1.4: figures/area-between-curves-4

$$
\begin{aligned}
A &= \int_{-2}^{3} (x+6) - (x^2)\mathrm{dx} \Rightarrow \int_{-2}^{3} x + 6 - x^2 \mathrm{dx} \\
&= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} = \left[\frac{3^2}{2} + 6 \cdot 3 - \frac{3^3}{3}\right] - \left[\frac{(-2)^2}{2} + 6(-2) - \frac{-2^3}{3}\right] \\
&= \left[\frac{9}{2} + 18 - 9\right] - \left[2 - 12 + \frac{8}{3}\right] = \left[\frac{9}{2} + 9\right] - \left[-10 + \frac{8}{3}\right] \\
&= \frac{27}{2} - \frac{-22}{3} \rightarrow \frac{27}{2} + \frac{22}{3} = \frac{125}{6}
\end{aligned}
$$

Beispiel:

Find area bound by $y = x^3$ and $y = x$.

$$
\begin{aligned}
x^3 &= x \\
x^3 - x &= 0 \\
x(x^2 - 1) &= 0 \\
x(x+1)(x-1) &= 0 \\
x = 0,& \quad x = 1, \quad x = -1
\end{aligned}
$$

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Here we test again which function is where above the other, to know, in which direction to integrate. (always the one on the top minus the one on the bottom, as shown at the start of this chapter.)

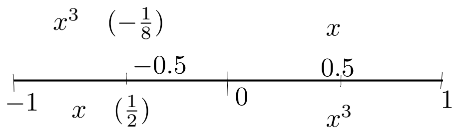


Figure 1.5: figures/area-between-curves-5

Here we add those integrals together, because we want both ares together.

$$
\begin{aligned}
A &= \int_{-1}^{0} x^3 - x \text{ dx} + \int_{0}^{1} x - x^3 \text{ dx} \\
&= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} \\
&= \left[\left(\frac{0}{4} - \frac{0}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)\right] + \left[\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \left(\frac{0}{2} - \frac{0}{4}\right)\right] \\
&= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right] \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{aligned}
$$

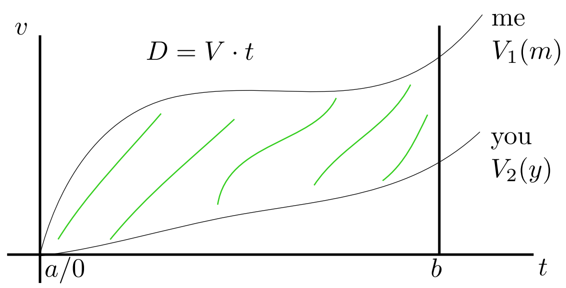


Figure 1.6: figures/area-between-curves-6

In this case, the area represents the distance car 'me' is ahead of car 'you'.

This can then be solved by:

$$A = \int_{0}^{b} V_1(t) - V_2(t) \text{ dt}.$$

**Beispiel:**

Find the area bound by $x = y^2$ and $y = x - 2$.

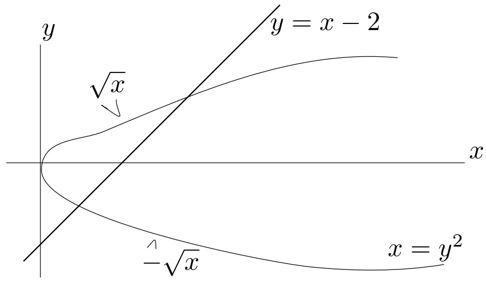


Figure 1.7: figures/area-between-curves-7

$$
\begin{aligned}
x = y^2 \qquad & y = x - 2 \\
x = y^2 \qquad & x = y + 2 \\
y^2 &= y + 2 \\
&\downarrow \\
y^2 - y - 2 &= 0 \\
(y-2)(y+1) &= 0 \\
y - 2 = 0 \qquad & y + 1 = 0 \\
y = 2 \qquad & y = -1
\end{aligned}
$$

Now we have to decide with what respect to integrate, we can use both, but need to be sure on which on to use.

So we take $x = y^2$, make out of it $\pm\sqrt{x} = \sqrt{y^2}$, out of this we then get $\pm\sqrt{x} = y$.

> **Notitz:**
>
> I got really confused at this point in the lecture, please notice that $y = \pm\sqrt{x}$ and $x = y^2$ are basically the same thing, can recommend to check it out on desmos.

Now we simplify the graph, to make it easier to understand.

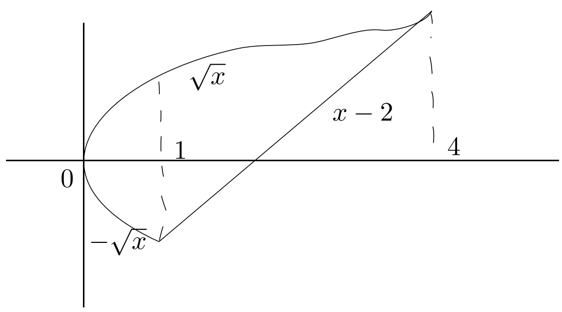


Figure 1.8: figures/area-between-curves-8

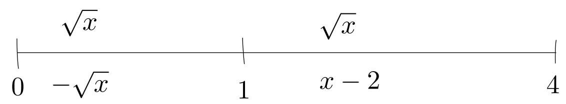


Figure 1.9: figures/area-between-curves-9

$$
\begin{aligned}
A &= \int_0^1 \sqrt{x} - (-\sqrt{x}) \text{ dx} + \int_1^4 \sqrt{x} - (x-2) \text{ dx} \\
&= \int_0^1 2\sqrt{x} \text{ dx} + \int_1^4 \sqrt{x} - x + 2 \text{ dx} \\
&= 2\int_0^1 x^{\frac{1}{2}} \text{ dx} + \int_1^4 x^{\frac{1}{2}} - x + 2 \text{ dx} \\
&= \left[ 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 + \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} + 2x \right]_1^4 \\
&= \left[ \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 + \left[ \frac{2x^{\frac{3}{1}}}{3} - \frac{x^2}{2} + 2x \right]_1^4 \\
&\downarrow \quad \text{evaluating} \quad \downarrow \\
&= \left( \frac{4 \cdot 1 \cdot \frac{3}{1}}{3} - 0 \right) + \left[ \left( \frac{2(4)^{\frac{3}{1}}}{3} - \frac{4^2}{2} + 2 \cdot 4 \right) - \left( \frac{2}{3} - \frac{1}{2} + 2 \right) \right] \\
&= \frac{4}{3} + \left[ \left( \frac{16}{3} - 8 + 8 \right) - \frac{13}{6} \right] \\
&= \frac{4}{3} + \frac{16}{3} - \frac{13}{6} = \frac{27}{6} \\
&= \frac{9}{2}
\end{aligned}
$$

Now doing it with the other axis, in respect to 'y'.

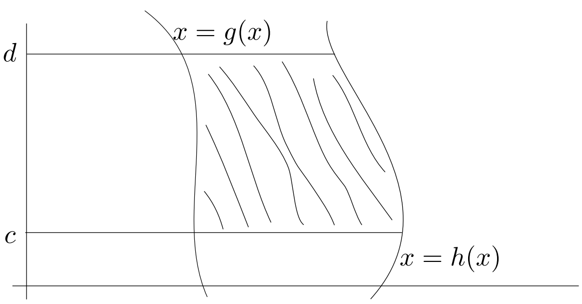


Figure 1.10: figures/area-between-curves-10

$$A = \int_c^d h(y) - g(y)\text{dy}.$$

(If $h(y) \geq g(y)$ for [c,d].)

Now doing it with the prior example:

$$\begin{aligned} x = y^2 \qquad & y = x - 2 \\ y^2 =& y + 2 \\ y^2 - y - 2 = 0 \quad & \rightarrow \quad y = 2, -1 \end{aligned}$$

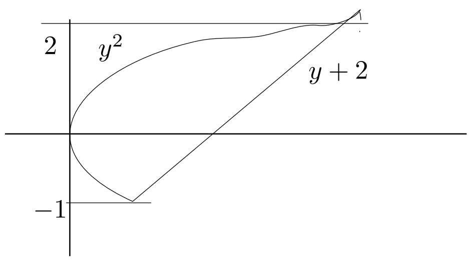


Figure 1.11: Area Between Curves 11

$$\begin{aligned} A &= \int_{-1}^{2} y + 2 - y^2 \; dy \\ &= \left[\frac{y^2}{2} + 2y = \frac{y^3}{3}\right]_{-1}^{2} \\ &= \frac{9}{2} \end{aligned}$$

# Chapter 2

# 5.2 – Volume of solids by disks and washers method