Complex Algebra

ChatGPT

March 26, 2023

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1 Introduction

Complex numbers are a type of number that extends the real numbers. A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit, which is defined as $\sqrt{-1}$. The real part of a complex number is the coefficient of the real unit, which is a, and the imaginary part is the coefficient of the imaginary unit, which is b. The set of complex numbers is denoted by $\mathbb{C}$.

Complex numbers can be represented graphically on a complex plane, where the horizontal axis represents the real part and the vertical axis represents the imaginary part. The modulus or absolute value of a complex number is the distance from the origin to the point representing the complex number on the complex plane, and is given by $|a + bi| = \sqrt{a^2 + b^2}$. The conjugate of a complex number $a + bi$ is the number $a - bi$.

Complex numbers are used in many areas of mathematics and science, such as in electrical engineering, quantum mechanics, and signal processing. They also have important applications in geometry and in the study of functions of a complex variable.

2 Complex Numbers

Complex numbers are numbers of the form $a+bi$, where a and b are real numbers and i is the imaginary unit, which is defined as $\sqrt{-1}$. Complex numbers can be represented graphically on a complex plane, with the real part of the number plotted on the horizontal axis and the imaginary part plotted on the vertical axis. The modulus or absolute value of a complex number is the distance from the origin to the point representing the complex number on the complex plane, and is given by $|a + bi| = \sqrt{a^2 + b^2}$. The conjugate of a complex number $a + bi$ is the number $a-bi$. The sum, difference, product, and quotient of two complex numbers can be calculated using the rules of algebra.

2.1 Addition and Subtraction

To add or subtract two complex numbers, we add or subtract their real and imaginary parts separately. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are two complex numbers, then their sum is $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$, and their difference is $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$.

2.2 Multiplication

To multiply two complex numbers, we use the distributive property and the fact that $i^2 = -1$. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are two complex numbers, then their product is given by:

$z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2$

Using the fact that $i^2 = -1$, we can simplify this to:

$z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i$

So the real part of the product is the product of the real parts minus the product of the imaginary parts, and the imaginary part of the product is the sum of the product of the real part of one number and the imaginary part of the other, where the imaginary part of the first is multiplied by the real part of the second.

2.3 Division

To divide two complex numbers, we use the fact that we can multiply the numerator and denominator by the conjugate of the denominator, which will eliminate the imaginary part in the denominator. If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ are two complex numbers, then their quotient is given by:

$$\frac{z_1}{z_2} = \frac{a_1 + b_1i}{a_2 + b_2i} = \frac{(a_1 + b_1i)(a_2 - b_2i)}{(a_2 + b_2i)(a_2 - b_2i)}$$

Expanding the numerator and denominator, we get:

$$\frac{z_1}{z_2} = \frac{(a_1a_2 + b_1b_2) + (b_1a_2 - a_1b_2)i}{a_2^2 + b_2^2}$$

So the real part of the quotient is the product of the real parts plus the product of the imaginary parts, divided by the modulus of the denominator, and the imaginary part of the quotient is the product of the real part of the first number and the imaginary part of the second, minus the product of the imaginary part of the first number and the real part of the second, divided by the modulus of the denominator.

3 Polar Form of Complex Numbers

The polar form of a complex number is a way of expressing a complex number using its modulus and argument. If $z = a + bi$ is a complex number, then its modulus $|z|$ is given by $|z| = \sqrt{a^2 + b^2}$ and its argument θ is the angle between the positive real axis and the line connecting the origin and the point representing the complex number on the complex plane. The argument is usually measured in radians and is denoted by θ.

The polar form of the complex number z is then given by:

$z = |z| \cdot (\cos\theta + i\sin\theta)$

This is also known as the exponential form of the complex number. It can also be written as $z = re^{i\theta}$, where $r = |z|$ is the modulus and $e^{i\theta} = \cos\theta + i\sin\theta$ is known as the complex exponential or Euler's formula.

The polar form of a complex number is useful in performing multiplication and division of complex numbers, as well as in finding roots of complex numbers. It also provides a geometric interpretation of complex numbers, where the modulus represents the distance from the origin and the argument represents the angle of rotation.

3.1 Conversion to Polar Form

To convert a complex number from rectangular form to polar form, we need to find its modulus and argument. If $z = a + bi$ is a complex number, then its modulus $|z|$ is given by $|z| = \sqrt{a^2 + b^2}$ and its argument θ is given by $\theta = \arctan\left(\frac{b}{a}\right)$, where arctan is the inverse tangent function.

If a is positive, then θ is the angle between the positive real axis and the line connecting the origin and the point representing the complex number on the complex plane. If a is negative, then we need to add π to the argument to get the correct angle. If a is zero and b is positive, then the argument is $\frac{\pi}{2}$, and if a is zero and b is negative, then the argument is $-\frac{\pi}{2}$.

Once we have found the modulus and argument, we can express the complex number in polar form as:

$z = |z| \cdot (\cos\theta + i\sin\theta)$

Alternatively, we can express it in exponential form as:

$z = |z| \cdot e^{i\theta}$

These forms are useful in performing multiplication and division of complex numbers, as well as in finding roots of complex numbers.

4 Euler's Formula

Euler's formula is a mathematical identity that relates the exponential function to the trigonometric functions. It is given by:

$e^{i\theta} = \cos\theta + i\sin\theta$

where i is the imaginary unit and θ is an angle in radians. This formula is also known as the complex exponential function.

Euler's formula is useful in expressing complex numbers in polar form, as well as in solving differential equations and in the study of Fourier series. It provides a connection between the exponential function, which is a fundamental function in calculus and analysis, and the trigonometric functions, which are fundamental in geometry and periodic functions.

5 Complex Conjugate

The complex conjugate of a complex number $z = a + bi$ is the number $\overline{z} = a - bi$. In other words, it is obtained by changing the sign of the imaginary part of the number.

The complex conjugate of a complex number has several important properties. First, the product of a complex number and its conjugate is always a real number, given by:

$$z \cdot \overline{z} = |z|^2 = a^2 + b^2$$

Second, the sum or difference of a complex number and its conjugate is always a real number, given by:

$$z + \overline{z} = 2a$$

$$z - \overline{z} = 2bi$$

Third, a complex number is equal to its conjugate if and only if it is real, that is, if its imaginary part is zero.

The complex conjugate is also useful in finding roots of complex numbers and in performing division of complex numbers, as it allows us to eliminate the imaginary part in the denominator.

5.1 Properties

The complex conjugate of a complex number $z = a+bi$ is the number $\overline{z} = a-bi$, where the sign of the imaginary part is changed. Some properties of the complex conjugate are:

1. The conjugate of a real number is itself. That is, if $z = a$ is a real number, then $\overline{z} = a$.

2. The product of a complex number and its conjugate is a real number. That is, $z \cdot \overline{z} = |z|^2$, where $|z|$ is the modulus of z.

3. The sum and difference of two complex numbers and their conjugates are real and imaginary, respectively. That is, if $z_1 = a_1+b_1i$ and $z_2 = a_2+b_2i$, then $(z_1 + z_2) + (\overline{z_1} + \overline{z_2})$ is real, and $(z_1 - z_2) - (\overline{z_1} - \overline{z_2})$ is imaginary.

4. The conjugate of a product is equal to the product of the conjugates in reverse order. That is, $(z_1 \cdot z_2)^* = \overline{z_2} \cdot \overline{z_1}$.

5. The conjugate of a quotient is equal to the quotient of the conjugates in reverse order. That is, $\left(\frac{z_1}{z_2}\right)^* = \frac{\overline{z_1}}{\overline{z_2}}$, where $z_2 \neq 0$.

5.2 Applications

1. Electrical engineering: Complex numbers are used to represent the amplitude and phase of electrical signals in AC circuits. They also play a key role in the analysis of filters, amplifiers, and control systems.

2. Quantum mechanics: Complex numbers are used to represent the wave function of quantum particles, which describes the probability of finding a particle in a certain state. They are also used to represent the operators that act on these wave functions.

3. Signal processing: Complex numbers are used to represent signals in the frequency domain, which allows for efficient processing of signals using Fourier transforms. They are also used in digital signal processing, where complex signals are manipulated using complex arithmetic operations.

4. Geometry: Complex numbers can be used to represent points in the complex plane, which allows for a geometric interpretation of complex arithmetic operations. They are also used in the study of conformal mappings, which are mappings that preserve angles and shapes.

5. Functions of a complex variable: Complex numbers are used in the study of functions of a complex variable, which are functions that map complex numbers to complex numbers. They are used to define concepts such as analyticity, holomorphy, and singularities, which have important applications in many areas of mathematics and physics.

6 Modulus and Argument

The modulus of a complex number $z = a + bi$ is given by $|z| = \sqrt{a^2 + b^2}$. It represents the distance of z from the origin in the complex plane.

The argument of a complex number $z = a+bi$ is given by $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$. It represents the angle that the line connecting the origin and z makes with the positive real axis in the complex plane. The argument is usually given in radians.

6.1 Modulus

The modulus of a complex number $z = a + bi$ is given by $|z| = \sqrt{a^2 + b^2}$.

6.2 Argument

The argument of a complex number $z = a+bi$ is given by $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$.

6.3 Properties

1. The modulus of a complex number is always non-negative: $|z| \geq 0$.

2. The modulus of a complex number is zero if and only if the complex number itself is zero: $|z| = 0 \iff z = 0$.

3. The product of two complex numbers is equal to the product of their moduli: $|z_1 z_2| = |z_1| \cdot |z_2|$.

4. The quotient of two complex numbers is equal to the quotient of their moduli: $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$.

5. The argument of a complex number is not unique, since the tangent function has a period of π. In other words, if θ is an argument of z, then so is $\theta + k\pi$, where k is any integer.

6. If z is a non-zero complex number, then it can be written in polar form as $z = |z|(\cos\theta + i\sin\theta)$, where θ is an argument of z.

7. The argument of a product of two complex numbers is equal to the sum of their arguments: $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, provided the arguments are taken in the same range.

8. The argument of a quotient of two complex numbers is equal to the difference of their arguments: $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$, provided the arguments are taken in the same range and $z_2 \neq 0$.

7 Complex Functions

A complex function is defined as $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ is a complex number, and $u(x, y)$ and $v(x, y)$ are real-valued functions of x and y.

7.1 Definitions

1. Holomorphic function: A complex function is said to be holomorphic or analytic in a region of the complex plane if it is differentiable at every point in that region.

2. Meromorphic function: A complex function is said to be meromorphic in a region of the complex plane if it is analytic except for a finite number of isolated singularities.

3. Entire function: A complex function is said to be entire if it is analytic everywhere in the complex plane.

4. Rational function: A complex function is said to be rational if it can be expressed as the ratio of two polynomials.

5. Periodic function: A complex function is said to be periodic if it satisfies the condition $f(z + T) = f(z)$ for some complex number T and all z in the domain of f.

7.2 Complex Derivatives

The complex derivative of a complex function $f(z)$ is defined as:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where Δz is a complex number approaching 0. If this limit exists, then the function is said to be complex differentiable at z, and $f'(z)$ is the complex derivative of $f(z)$ at z.

The Cauchy-Riemann equations provide a useful tool for determining when a function is complex differentiable. If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$, then the partial derivatives of u and v with respect to x and y must satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If these equations hold at a point, then $f(z)$ is complex differentiable at that point, and its complex derivative is given by:

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

8 Complex Roots

A complex number w is said to be a complex root of a polynomial $p(z)$ if $p(w) = 0$. The fundamental theorem of algebra states that every non-constant polynomial with complex coefficients has at least one complex root.

If $p(z)$ is a polynomial of degree n, then it can be factored as:

$$p(z) = c(z - w_1)(z - w_2)\cdots(z - w_n)$$

where c is a constant, and $w_1, w_2, \ldots, w_n$ are the complex roots of $p(z)$. Note that a complex root may appear multiple times in the factorization if it has multiplicity greater than 1.

To find the complex roots of a polynomial, we can use various methods such as the quadratic formula for quadratic polynomials, or the cubic formula and quartic formula for cubic and quartic polynomials, respectively. However, for polynomials of higher degree, there is no general formula that can express the roots in terms of radicals.

Instead, we can use numerical methods such as Newton's method or the Durand-Kerner method to approximate the roots of a polynomial. These methods involve starting with an initial guess for a root, and then iteratively refining the guess until it converges to a root of the polynomial.

8.1 Square Roots

Given a complex number z, a square root of z is a complex number w such that $w^2 = z$. Note that there are two possible square roots of a non-zero complex number z, namely w and $-w$.

To find the square roots of a complex number z, we can use the following formula:

$$w = \pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

where $z = r\operatorname{cis}\theta$ is the polar form of z, and $\operatorname{cis}\theta = \cos\theta + i\sin\theta$ is the complex exponential function. Here, we take the positive square root if we want the principal square root of z.

Alternatively, we can use the quadratic formula to find the square roots of a complex number $z = a + bi$:

$$w = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i \ \mathrm{sgn}(b)\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$$

where $\mathrm{sgn}(b)$ is the sign function, which is equal to 1 if b is positive, and -1 if b is negative.

Note that if z is a non-zero complex number, then its square roots are always distinct, unless z is a non-zero real number.

8.2 Cube Roots

Given a complex number z, a cube root of z is a complex number w such that $w^3 = z$. Note that there are three possible cube roots of a non-zero complex number z.

To find the cube roots of a complex number z, we can use the following formula:

$$w_k = \sqrt[3]{|z|}\left(\cos\frac{\mathrm{Arg}(z)+2\pi k}{3} + i\sin\frac{\mathrm{Arg}(z)+2\pi k}{3}\right)$$

where $|z|$ is the modulus of z, $\mathrm{Arg}(z)$ is the principal argument of z (i.e., the argument in the interval $(-\pi, \pi]$), and $k = 0, 1, 2$.

Alternatively, we can use the following formula to find the principal cube root of a complex number $z = a + bi$:

$$w = \sqrt[3]{\frac{\sqrt{a^2+b^2}+a}{2}} + i\frac{\mathrm{sgn}(b)\sqrt[3]{\left|\frac{\sqrt{a^2+b^2}-a}{2}\right|}}{\sqrt{2}}$$

where $\mathrm{sgn}(b)$ is the sign function, which is equal to 1 if b is positive, and -1 if b is negative.

Note that if z is a non-zero complex number, then its cube roots are always distinct, unless z is a non-zero real number.

9 Applications of Complex Numbers

9.1 Electrical Engineering

Complex numbers have many applications in electrical engineering, including:

1. AC circuit analysis: In AC circuit analysis, complex numbers are used to represent the amplitudes and phases of sinusoidal signals. This allows us to analyze circuits with resistors, capacitors, and inductors, and determine their behavior under different operating conditions.

2. Impedance and admittance: Complex numbers are used to represent the impedance and admittance of circuits, which are important parameters in AC circuit analysis. Impedance is the complex-valued counterpart of resistance, and admittance is the complex-valued counterpart of conductance.

3. Phasor analysis: Phasor analysis is a technique used in AC circuit analysis to simplify the analysis of circuits with sinusoidal signals. It involves representing sinusoidal signals as complex numbers, and performing circuit analysis using complex arithmetic.

4. Control systems: Complex numbers are used in control systems to represent the frequency response of a system, which is a measure of how the system responds to sinusoidal inputs at different frequencies. The frequency response is represented using a complex function called the transfer function.

5. Signal processing: Complex numbers are used in signal processing to represent signals in the frequency domain, using a complex function called the Fourier transform. This allows us to analyze signals in terms of their frequency components, and perform filtering operations to remove unwanted frequencies.

9.2 Signal Processing

Signal processing is a field of study that deals with the analysis, modification, and synthesis of signals such as sound, images, and biological signals. Complex algebra plays a crucial role in signal processing applications. Here are some examples:

1. Fourier analysis: The Fourier transform is a mathematical tool used in signal processing to transform a signal from the time domain to the frequency domain. It is defined as follows:

$$\mathcal{F}(f(t))(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

   where $f(t)$ is the signal in the time domain, $\mathcal{F}(f(t))(\omega)$ is the Fourier transform of the signal in the frequency domain, and ω is the angular frequency. The Fourier transform is used for applications such as filtering, compression, and feature extraction.

2. Digital signal processing: Digital signal processing (DSP) is the use of digital processing techniques to manipulate signals. Complex algebra is used in DSP for operations such as filtering, convolution, and correlation. For example, the discrete Fourier transform (DFT) is a digital version of the Fourier transform, which is used extensively in DSP.

3. Image processing: Image processing is the analysis and manipulation of digital images. Complex algebra is used in image processing for operations such as filtering, convolution, and Fourier analysis. For example, the 2D Fourier transform is used to analyze the frequency content of an image.

In conclusion, complex algebra plays a critical role in signal processing applications, from Fourier analysis to digital signal processing and image processing.

9.3 Quantum Mechanics

Quantum mechanics is a fundamental theory of physics that describes the behavior of matter and energy at the atomic and subatomic level. Complex algebra plays a central role in the mathematical formalism of quantum mechanics. Here are some examples of applications of complex algebra in quantum mechanics:

1. Wave functions: The wave function is a fundamental concept in quantum mechanics that describes the quantum state of a particle. It is a complex-valued function that satisfies the Schrödinger equation. The probability density of finding a particle at a certain location is given by the squared modulus of the wave function. Complex algebra is used to manipulate wave functions, compute probabilities, and solve the Schrödinger equation.

2. Operators: Operators are mathematical objects that represent physical observables in quantum mechanics. They act on wave functions and return another wave function or a scalar value. Operators can be represented as matrices, and complex algebra is used to manipulate them, compute eigenvalues and eigenvectors, and perform operations such as normalization and projection.

3. Quantum states: Quantum states are described by complex vectors in a Hilbert space. They can be represented using bra-ket notation, where the bra represents the complex conjugate of a vector and the ket represents the vector itself. Complex algebra is used to manipulate quantum states, perform measurements, and compute probabilities.

In conclusion, complex algebra is an essential tool for the mathematical formalism of quantum mechanics. It is used to manipulate wave functions, operators, and quantum states, compute probabilities, and solve the Schrödinger equation.