Contents

Chapter 1

(5.1) – Area between two curves

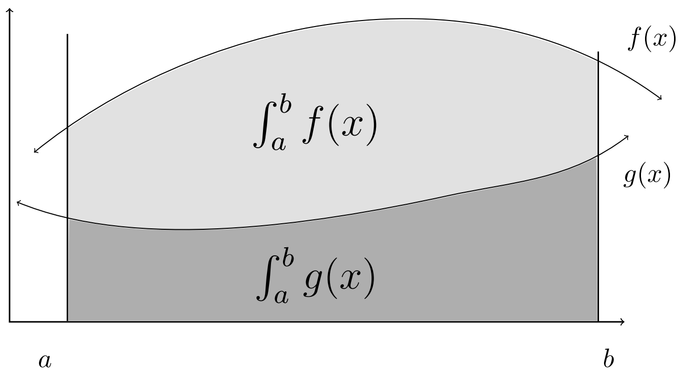

Figure 1.1: area-between-curves-1

Area between $f(x) \quad \& \quad g(x)$.

$$A = \text{Area under } f(x) \text{ - Area under } g(x).$$

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Notitz:

$$f(x) \geq g(x)$$
$$\forall x \in [a, b].$$
$$(f(x) \text{ is above } g(x)).$$

$$\int_a^c [f(x) - 0]\, dx + \int_c^b [0 - f(x)]\, dx$$

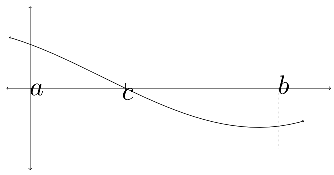

Figure 1.2: are-between-curves-2

$$\int_a^c f(x)dx - \int_c^b f(x)dx$$

Beispiel:

Find the area bounded above by $y = 2x + 5$, and bounded below by $y = x^3$ on $[0, 2]$

$$\begin{aligned} A &= \int_0^2 (2x+5) - x^3 \mathrm{dx} \\ &= \int_0^2 2x + 5 = x^3 dx \\ &= x^2 + 5x - \frac{x^4}{4}]_0^2 \\ A &= \left[2^2 + 5 \cdot 2\frac{2^4}{4}\right] - [0] \\ A &= 10 \end{aligned}$$

Beispiel:

Find area between $y = x^2$ and $y = x + 6$.

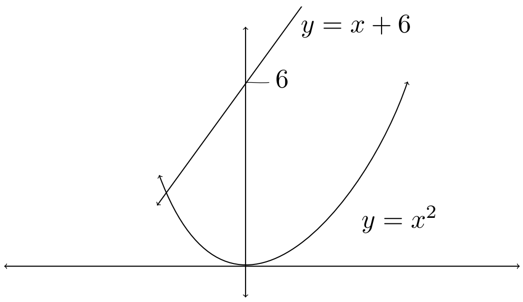

Figure 1.3: area-between-curves-3

Notitz:

Steps:
1. Find x-cords of the Intersection of the curves. (Set $f(x) = g(x)$)
2. Which function is on the top?
   (Pick one point for each interval)
3. set-up and solve

$$
\begin{aligned}
x+6 &= x^2 \\
x^2-x-6 &= 0 \\
(x-3)\,(x+2)& \\
x-3=0 &\qquad x+2=0 \\
x=3 &\qquad x=-2
\end{aligned}
$$

Those are the only places $f(x)$ and $g(x)$ are intercepting, so those are the bounds of integration.

$$A=\int_{-2}^{3}.$$

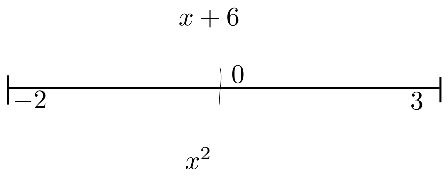

Figure 1.4: figures/area-between-curves-4

$$
\begin{aligned}
A &= \int_{-2}^{3}(x+6)-(x^2)\mathrm{dx} \Rightarrow \int_{-2}^{3} x+6-x^2\mathrm{dx} \\
&= \left[\frac{x^2}{2}+6x-\frac{x^3}{3}\right]_{-2}^{3} = \left[\frac{3^2}{2}+6\cdot 3-\frac{3^3}{3}\right]-\left[\frac{(-2)^2}{2}+6(-2)-\frac{-2^3}{3}\right] \\
&= \left[\frac{9}{2}+18-9\right]-\left[2-12+\frac{8}{3}\right]=\left[\frac{9}{2}+9\right]-\left[-10+\frac{8}{3}\right] \\
&= \frac{27}{2}-\frac{-22}{3} \to \frac{27}{2}+\frac{22}{3}=\frac{125}{6}
\end{aligned}
$$

Beispiel:

Find area bound by $y=x^3$ and $y=x$.

$$
\begin{aligned}
x^3 &= x \\
x^3-x &= 0 \\
x(x^2-1) &= 0 \\
x(x+1)(x-1) &= 0 \\
x &= 0, \quad x=1, \quad x=-1
\end{aligned}
$$

regel

Here we test again which function is where above the other, to know, in which direction to integrate. (always the one on the top minus the one on the bottom, as shown at the start of this chapter.)

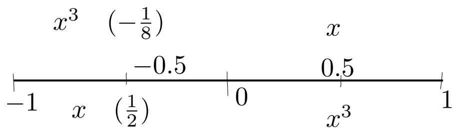

Figure 1.5: figures/area-between-curves-5

Here we add those integrals together, because we want both ares together.

$$
\begin{aligned}
A &= \int_{-1}^{0} x^3 - x \text{ dx} + \int_{0}^{1} x - x^3 \text{ dx} \\
&= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} \\
&= \left[\left(\frac{0}{4} - \frac{0}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)\right] + \left[\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \left(\frac{0}{2} - \frac{0}{4}\right)\right] \\
&= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right] \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{aligned}
$$

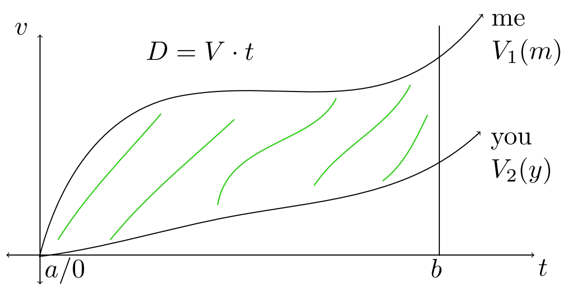

Figure 1.6: figures/area-between-curves-6

In this case, the area represents the distance car 'me' is ahead of car 'you'.

This can then be solved by:

$$A = \int_{0}^{b} V_1(t) - V_2(t) \text{ dt}.$$

Beispiel:

Find the area bound by $x = y^2$ and $y = x - 2$.

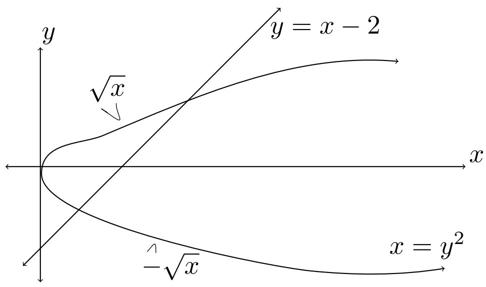

Figure 1.7: figures/area-between-curves-7

$$
\begin{aligned}
x &= y^2 & y &= x - 2 \\
x &= y^2 & x &= y + 2 \\
& y^2 = y + 2 \\
& \downarrow \\
y^2 - y - 2 &= 0 \\
(y-2)(y+1) &= 0 \\
y - 2 &= 0 & y + 1 &= 0 \\
y &= 2 & y &= -1
\end{aligned}
$$

Now we have to decide with what respect to integrate, we can use both, but need to be sure on which on to use.

So we take $x = y^2$, make out of it $\pm\sqrt{x} = \sqrt{y^2}$, out of this we then get $\pm\sqrt{x} = y$.

> **Notitz:**
>
> I got really confused at this point in the lecture, please notice that $y = \pm\sqrt{x}$ and $x = y^2$ are basically the same thing, can recommend to check it out on desmos.

Now we simplify the graph, to make it easier to understand.

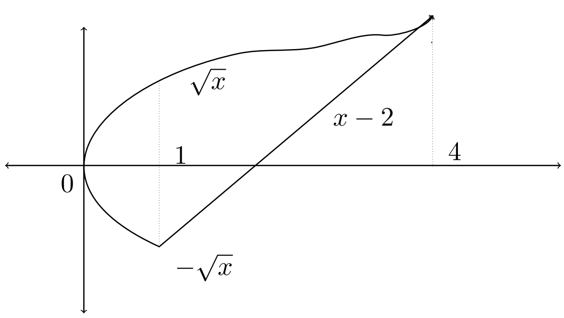

Figure 1.8: figures/area-between-curves-8

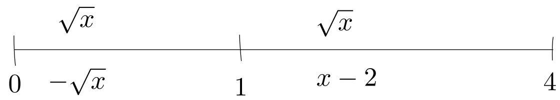

Figure 1.9: figures/area-between-curves-9

$$
\begin{aligned}
A &= \int_0^1 \sqrt{x} - (-\sqrt{x}) \text{ dx} + \int_1^4 \sqrt{x} - (x-2) \text{ dx} \\
&= \int_0^1 2\sqrt{x} \text{ dx} + \int_1^4 \sqrt{x} - x + 2 \text{ dx} \\
&= 2\int_0^1 x^{\frac{1}{2}} \text{ dx} + \int_1^4 x^{\frac{1}{2}} - x + 2 \text{ dx} \\
&= \left[2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]_1^4 \\
&= \left[\frac{4x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + \left[\frac{2x^{\frac{3}{1}}}{3} - \frac{x^2}{2} + 2x\right]_1^4 \\
&\downarrow \quad \text{evaluating} \quad \downarrow \\
&= \left(\frac{4 \cdot 1 \cdot \frac{3}{1}}{3} - 0\right) + \left[\left(\frac{2(4)^{\frac{3}{1}}}{3} - \frac{4^2}{2} + 2 \cdot 4\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)\right] \\
&= \frac{4}{3} + \left[\left(\frac{16}{3} - 8 + 8\right) - \frac{13}{6}\right] \\
&= \frac{4}{3} + \frac{16}{3} - \frac{13}{6} = \frac{27}{6} \\
&= \frac{9}{2}
\end{aligned}
$$

Now doing it with the other axis, in respect to 'y'.

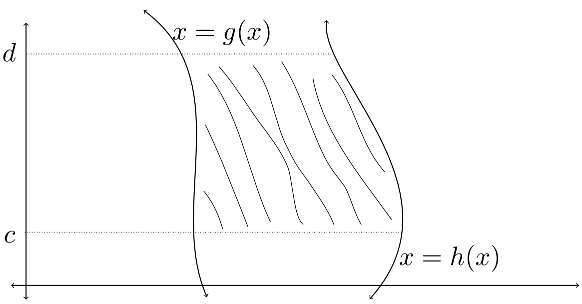

Figure 1.10: figures/area-between-curves-10

$$A = \int_c^d h(y) - g(y)\mathrm{dy}.$$

(If $h(y) \geq g(y)$ for [c,d].]

Now doing it with the prior example:

$$\begin{aligned} x = y^2 \qquad & y = x - 2 \\ y^2 =& y + 2 \\ y^2 - y - 2 = 0 \quad & \rightarrow \quad y = 2, -1 \end{aligned}$$

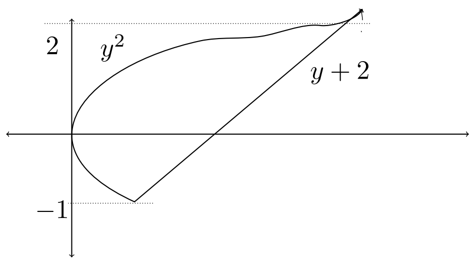

Figure 1.11: Area Between Curves 11

$$\begin{aligned} A &= \int_{-1}^{2} y + 2 - y^2 \; dy \\ &= \left[\frac{y^2}{2} + 2y = \frac{y^3}{3}\right]_{-1}^{2} \\ &= \frac{9}{2} \end{aligned}$$

Chapter 2

(5.2) – Volume of solids by disks and washers method

2.1 Volume of solids by slicing

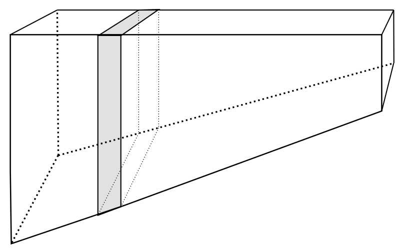

Figure 2.1: figures/disks-1

Cut into thin slabs. Then use Summations to set up an integration.

To do this, find area of cross section.

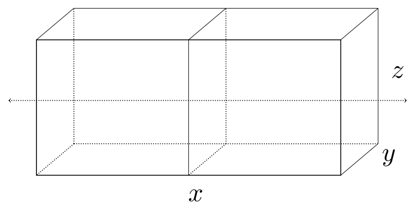

Figure 2.2: figures/disks-2

Here we can see that is on an axis or some like that:

$$V = (y \cdot z) \cdot x.$$

Where $y \cdot x$ is the surface area of the cross seaction and x the length.

Now we can find the volume of any solid that is bound my planes $\perp$ to x-axis. At points 'a' and 'b'.

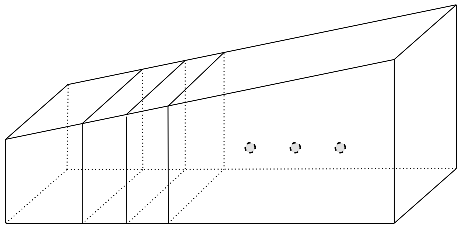

Figure 2.3: figures/disks-3

Cut into slabs with a width of Δx.

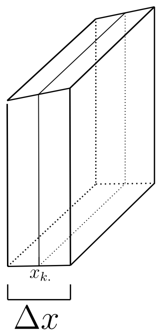

Figure 2.4: figures/disks-4

Pick arbitrary point '$x_{k.}$', on each sub-interval. Find Cross-sectional Area at '$x_{k.}$'.

$$V_k = A(x_{k.}) \cdot \Delta x$$

Where $A(x_{k.}$ is the cross-sectional area and Δx the length.

$$V = \sum_{k=1}^{n} A(x_{k.} \cdot \Delta x \quad \text{(Approx V)}$$

Because we don't want to have an approximation:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_{k.})\Delta x$$

$$V = \int_a^b A(x)\ \mathrm{dx}$$

$A(x)$ = cross sectional area over [a,b]

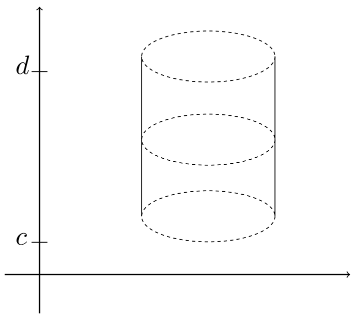

Figure 2.5: figures/disks-5

$$V = \int_c^d A(y)dy$$

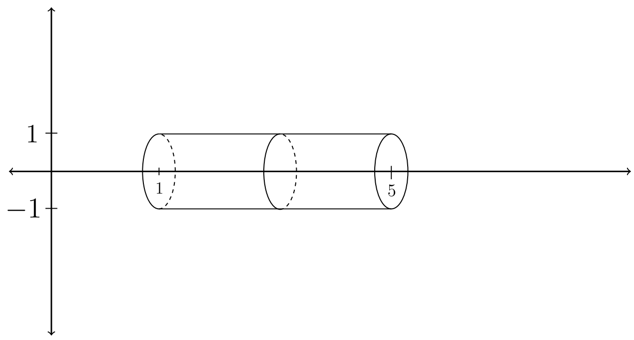

Figure 2.6: figures/disks-6

Some things to the picture

$$A = \pi \cdot r$$
$$A(x) = \pi(1) = \pi$$
$$V = \int_1^5 \pi dx$$

To get the volume we can either take the Integral.

$$V = \pi x]_1^5 = \pi \cdot 5 - \pi \cdot 1 = 4\pi.$$

Or in this case also do it without calculus, but given that it's just a simple task, and tasks wont stay simple, we did it with calculus.

2.2 Solid of revolution

First some examples, the following are all still do-able with just some algebra, but the point to get across is, it can be done with every functino there is, at least most of em, e.g. one could do it with x^2 in a definite integral, by just doing

$$\pi \cdot \int_{-1}^{1} x^2 dx.$$

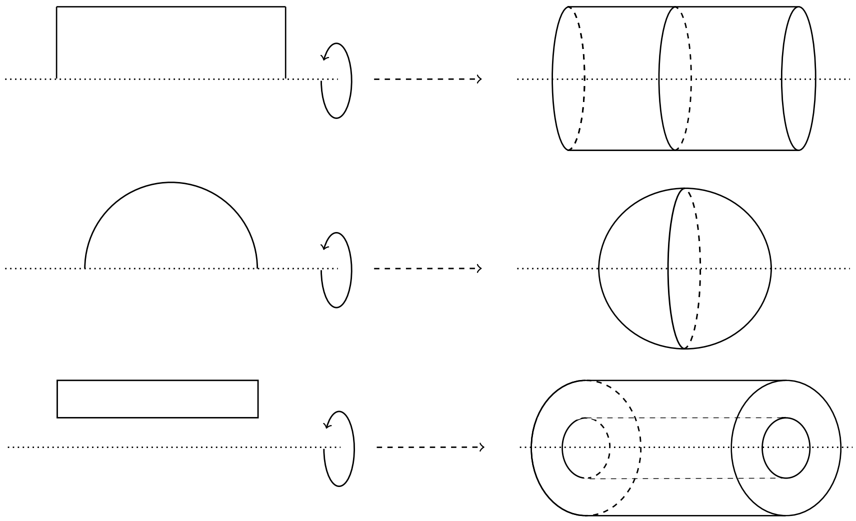

Figure 2.7: figures/disks-7

Notitz:

paused at 47:00