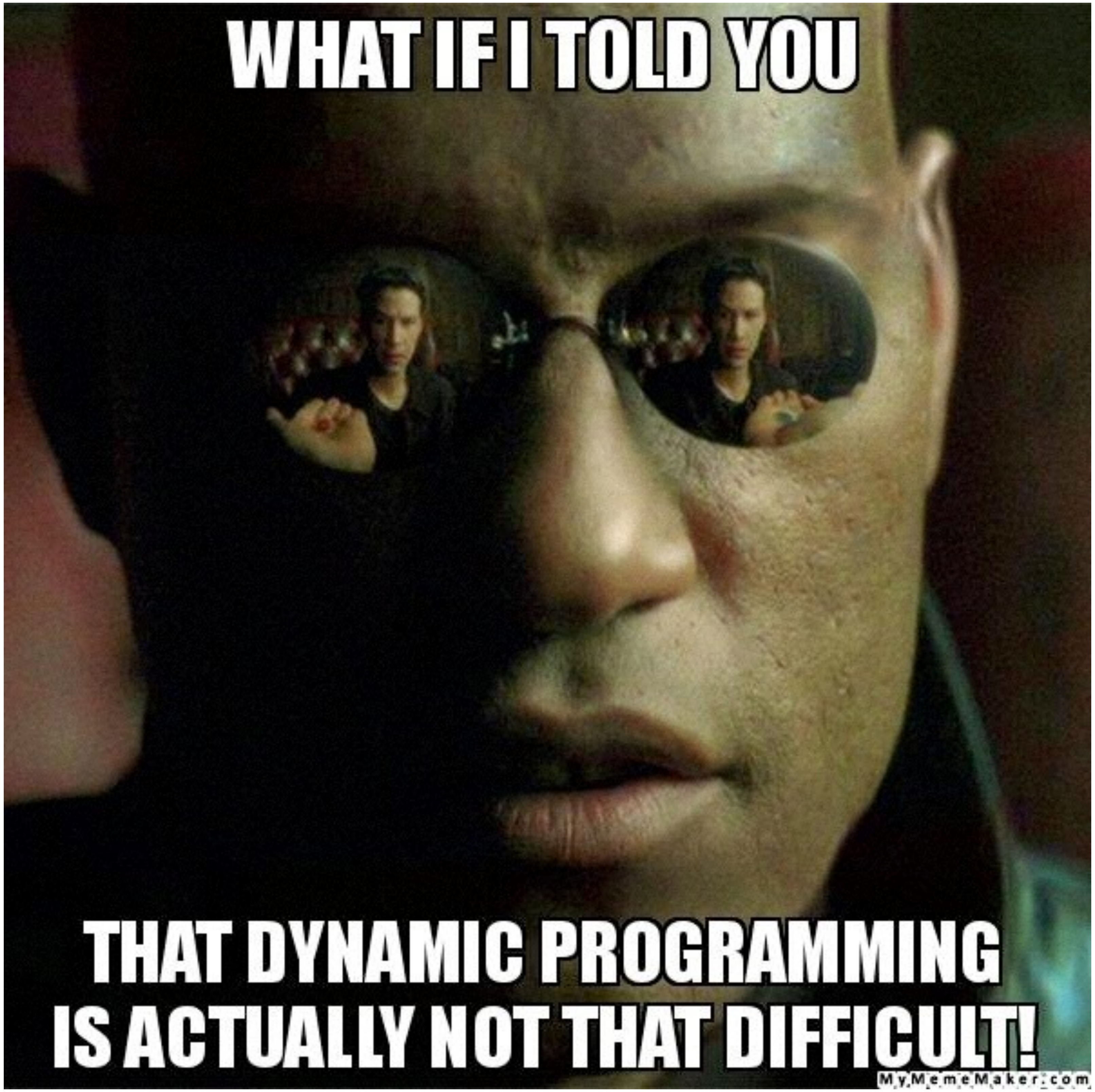
Algorithms Lab

Dynamic Programming


WHAT IF I TOLD YOU
THAT DYNAMIC PROGRAMMING
IS ACTUALLY NOT THAT DIFFICULT!
MyMemeMaker.com

Example problem: A, B two arrays of positive numbers. You can remove *a* last elements from A and *b* last elements from B (both *a* and *b* have to be at least 1), and the cost of such operation is (X - *a*) * (Y - *b*), where X (Y) is the sum of the removed elements from A (B).

Your **goal** is to **remove all elements from A and B** by repeatedly applying such operation, with the **minimal total cost**!

2 5 10 2 6

7 1 9 4 2

Cost:

Total cost:

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2 5 10 ~~2 6~~

7 1 ~~9 4 2~~

Cost: $(2 + 6 - 2) * (9 + 4 + 2 - 3) = 72$

Total cost: 72

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2 ~~5 10~~ ~~2 6~~

7 ~~1~~ ~~9 4 2~~

Cost: (5 + 10 - 2) * (1 - 1) = 0

Total cost: 72

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~~2~~ ~~5 10~~ ~~2 6~~

~~7~~ ~~1~~ ~~9 4 2~~

Cost: (2 - 1) * (7 - 1) = 6

Total cost: 78

First approach - brute force

Recursively try all possible removals

rec_try(i, j) *// consider only first i elements of*
// A and j elements of B

if (i == 1) return (A[1] - 1) * ($\sum_{t=1}^{j} B[t]$ - j)

if (j == 1) return (B[1] - 1) * ($\sum_{t=1}^{i} A[t]$ - i)

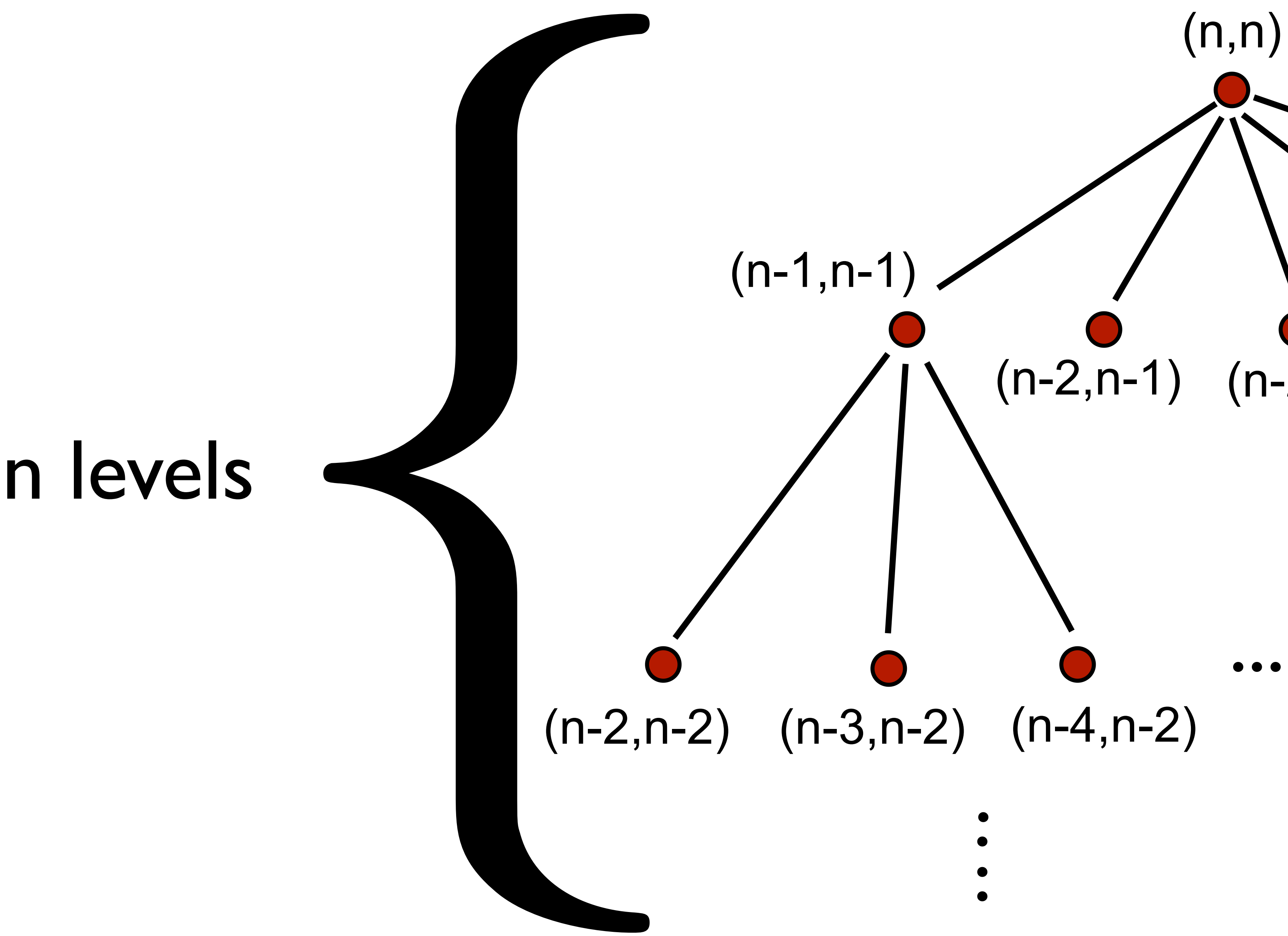
best =($\sum_{t=1}^{i} A[t]$ - i) * ($\sum_{t=1}^{j} B[t]$ - j) *// take all*

for a = 1 to i - 1
  for b = 1 to j - 1
    cost = ($\sum_{t=i-a+1}^{i} A[i]$ - a) * ($\sum_{t=j-b+1}^{i} B[i]$ - b)
    if (cost + **rec_try**(i - a, j - b) < best)
      best = cost + **rec_try**(i - a, j - b)

return best

First approach - brute force

Recursively try all possible removals



Rough estimate - at least 2^n many calls of try_rec()

First approach - brute force

Recursively try all possible removals

Problem: for some (i,j), we call **try_rec(i,j)** many times

Observation: for a fixed (i,j), **try_rec(i,j)** always returns the same value!

Solution: for each (i,j), store the value which **try_rec(i,j)** returns! Only call **try_rec(i,j)** if this value is not stored yet!

Second approach - brute force

with storing

rec_try(i, j) // consider only first i elements of
// A and j elements of B

if (i == 1) return (A[1] - 1) * ($\sum_{t=1}^{j} B[t]$ - j)

if (j == 1) return (B[1] - 1) * ($\sum_{t=1}^{i} A[t]$ - i)

best =($\sum_{t=1}^{i} A[t]$ - i) * ($\sum_{t=1}^{j} B[t]$ - j) // take all

for a = 1 to i - 1
for b = 1 to j - 1
cost = ($\sum_{t=i-a+1}^{i} A[i]$ - a) * ($\sum_{t=j-b+1}^{i} B[i]$ - b)

if (not stored(i - a, j - b)) rec_try(i-a, j-b)
if (cost **+ stored(i - a, j - b)** < best)
best = cost **+ stored(i - a, j - b)**

store(i,j) <- best

Second approach - brute force with storing

try_rec(i,j) will be executed at most once for each pair (*i*,*j*), $1 \leq i, j \leq n$

two nested loops: one up to *i*, one up to *j*

Total running time: $\mathcal{O}(n^4)$

This was an example of
Dynamic Programming!

Can we improve the running time of the previous algorithm?

Let us observe the cost function more closely

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Cost of taking first 10 elements from array A and 16 from B

$$((\sum_{i=1}^{10} A[i]) - 10) \cdot ((\sum_{i=1}^{16} B[i]) - 16)$$

*In the analysis we remove elements from the beginning and in the code from the end

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Let us observe the cost function more closely

Cost of taking first 10 elements from array A and 16 from B

$$((\sum_{i=1}^{10} A[i]) - 10) \cdot ((\sum_{i=1}^{16} B[i]) - 16)$$

$$= ((\sum_{i=1}^{10} (A[i] - 1)) \cdot ((\sum_{i=1}^{16} (B[i] - 1))$$

*In the analysis we remove elements from the beginning and in the code from the end

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To make analysis simpler

- Decrease every element of both A and B by 1
- Cost of taking a elements from A and b from B is

$$(\sum_{i=1}^{a} A[i]) \cdot (\sum_{i=1}^{b} B[i])$$

*In the analysis we remove elements from the beginning and in the code from the end

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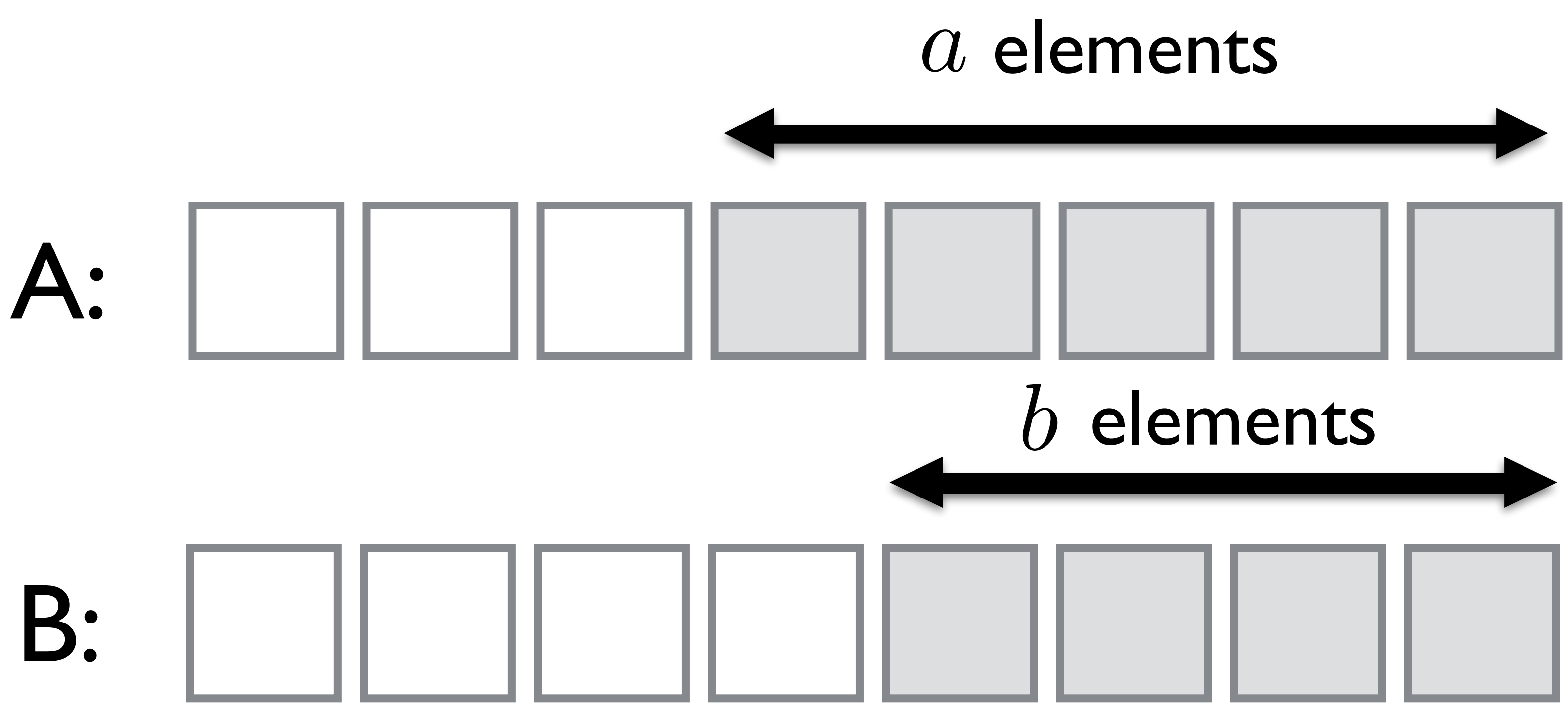

Cost 1: $(A[1] + A[2] + \cdots + A[a]) \cdot (B[1] + B[2] + \cdots + B[b])$

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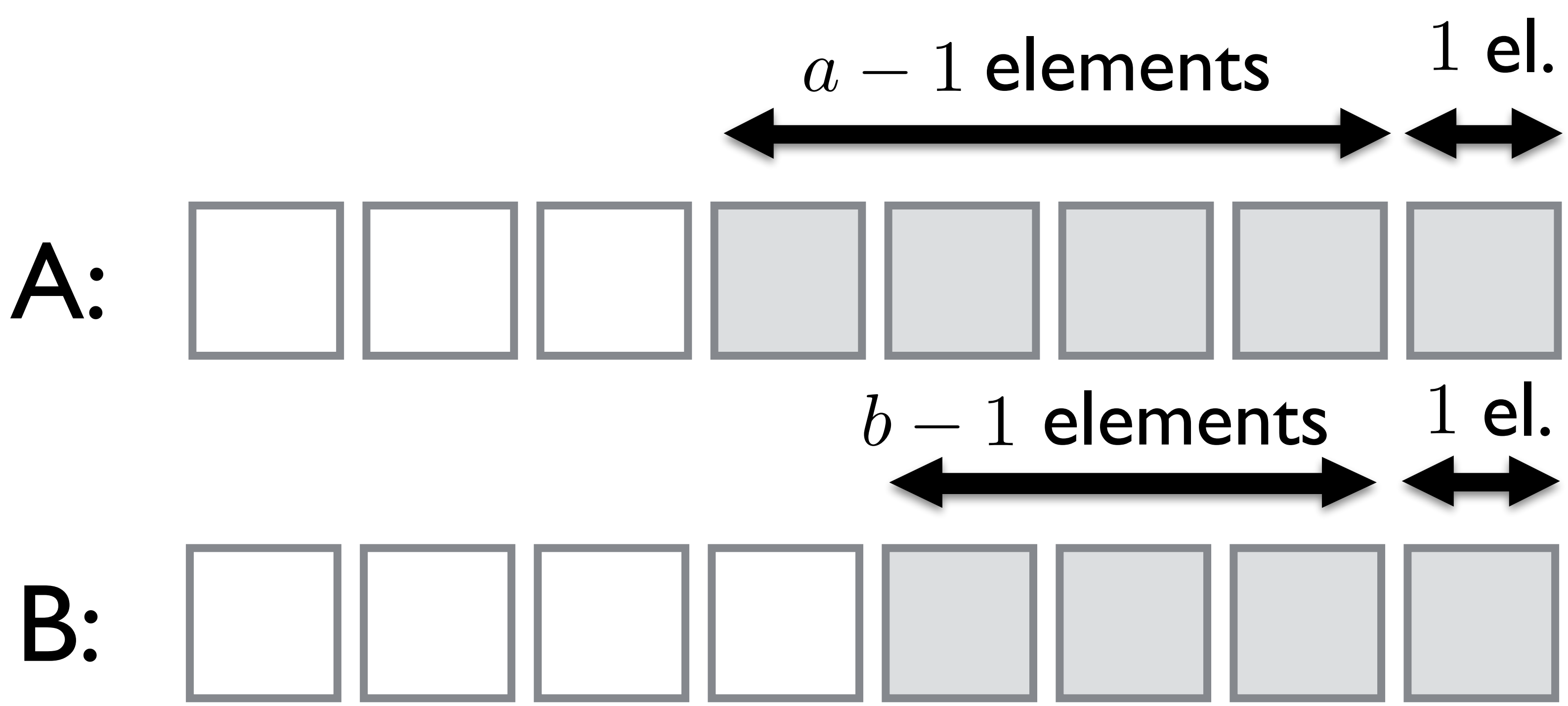

Cost 1: $(A[1]+A[2]+\cdots+A[a])\cdot(B[1]+B[2]+\cdots+B[b])$

Cost 2: $A[1]B[1]+(A[2]+\cdots+A[a])(B[2]+\cdots+B[b])$

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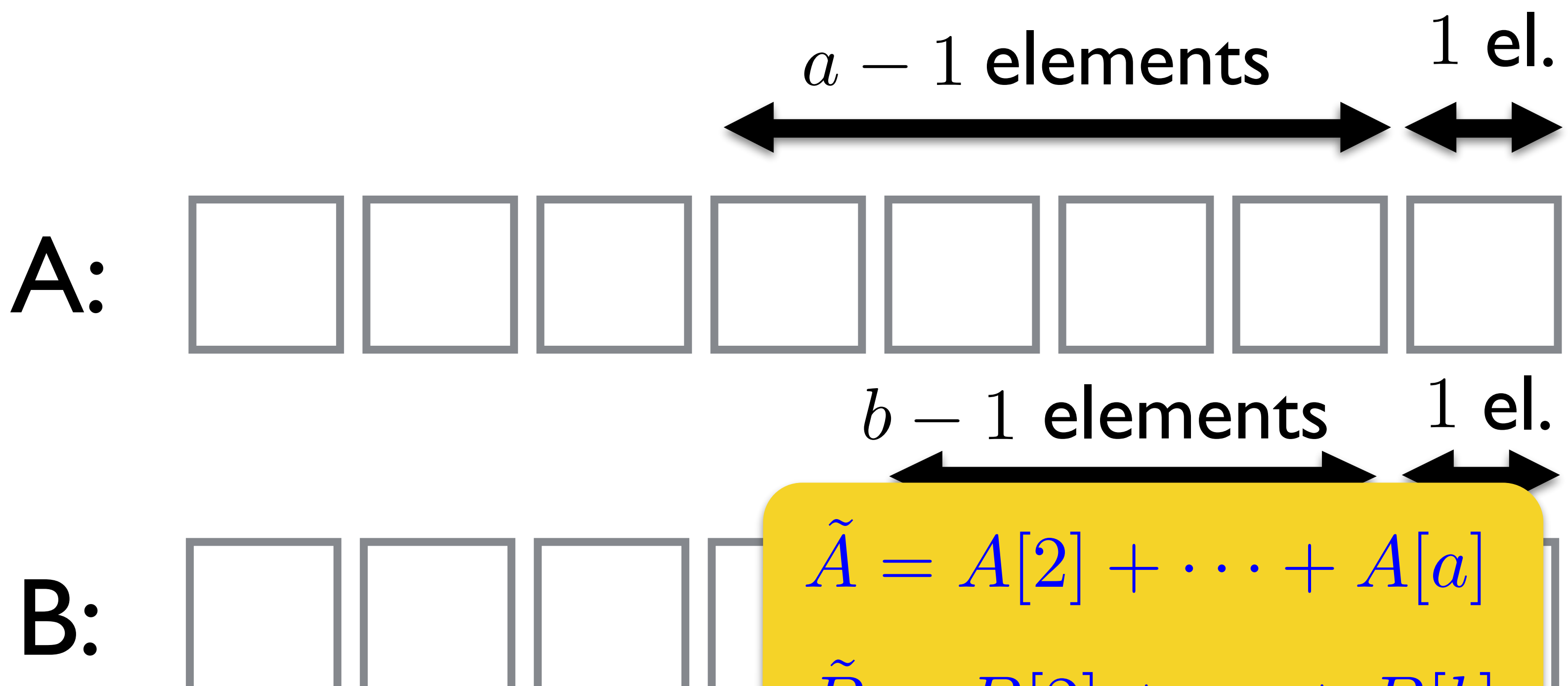

$\tilde{A} = A[2] + \cdots + A[a]$

$\tilde{B} = B[2] + \cdots + B[b]$

Cost 1: $(A[1] + A[2] + \cdots + A[a]) \cdot (B[1] + B[2] + \cdots + B[b])$

Cost 2: $A[1]B[1] + (A[2] + \cdots + A[a])(B[2] + \cdots + B[b])$

Can we improve the running time of the previous algorithm?

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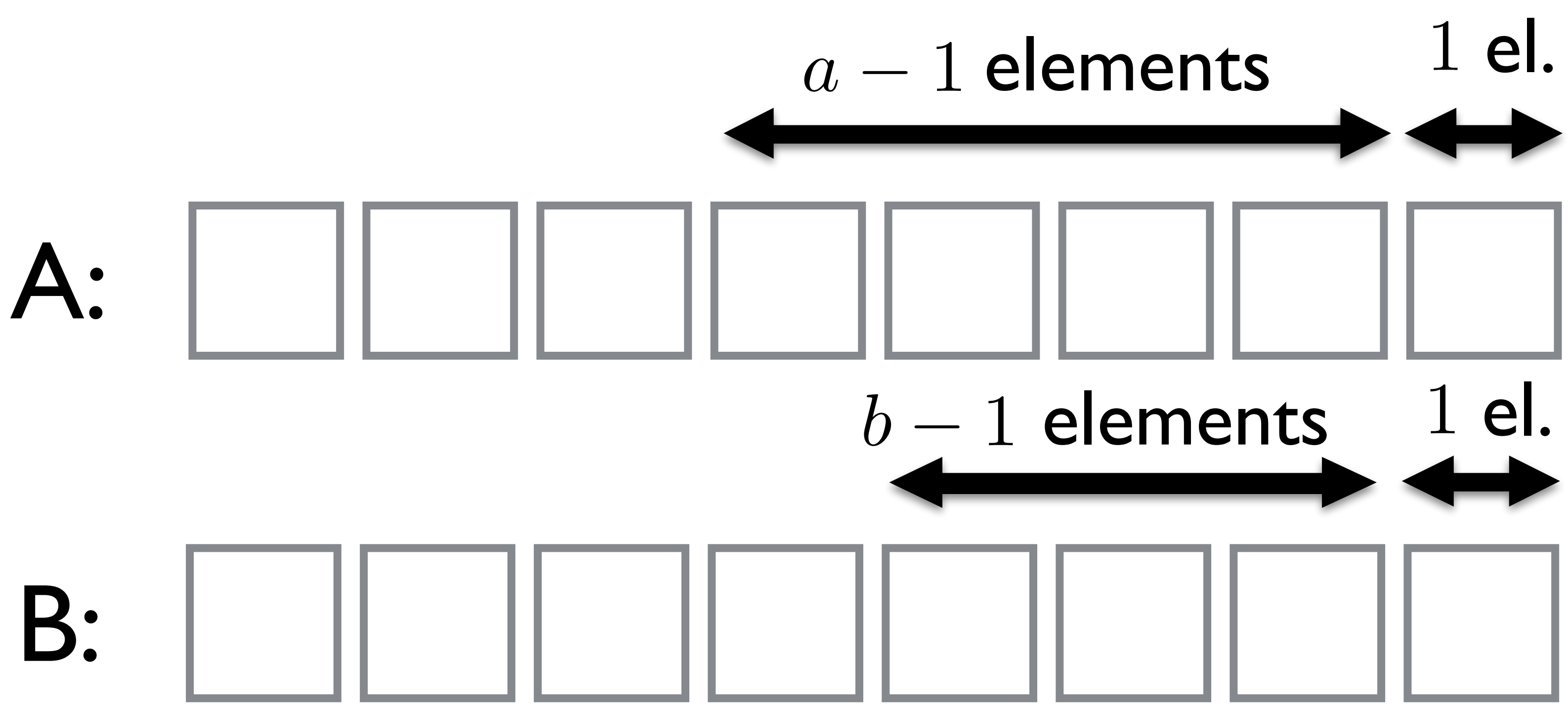

Cost 1: $(A[1]+\tilde{A})(B[1]+\tilde{B})$

Cost 2: $A[1]B[1]+\tilde{A}\tilde{B}$

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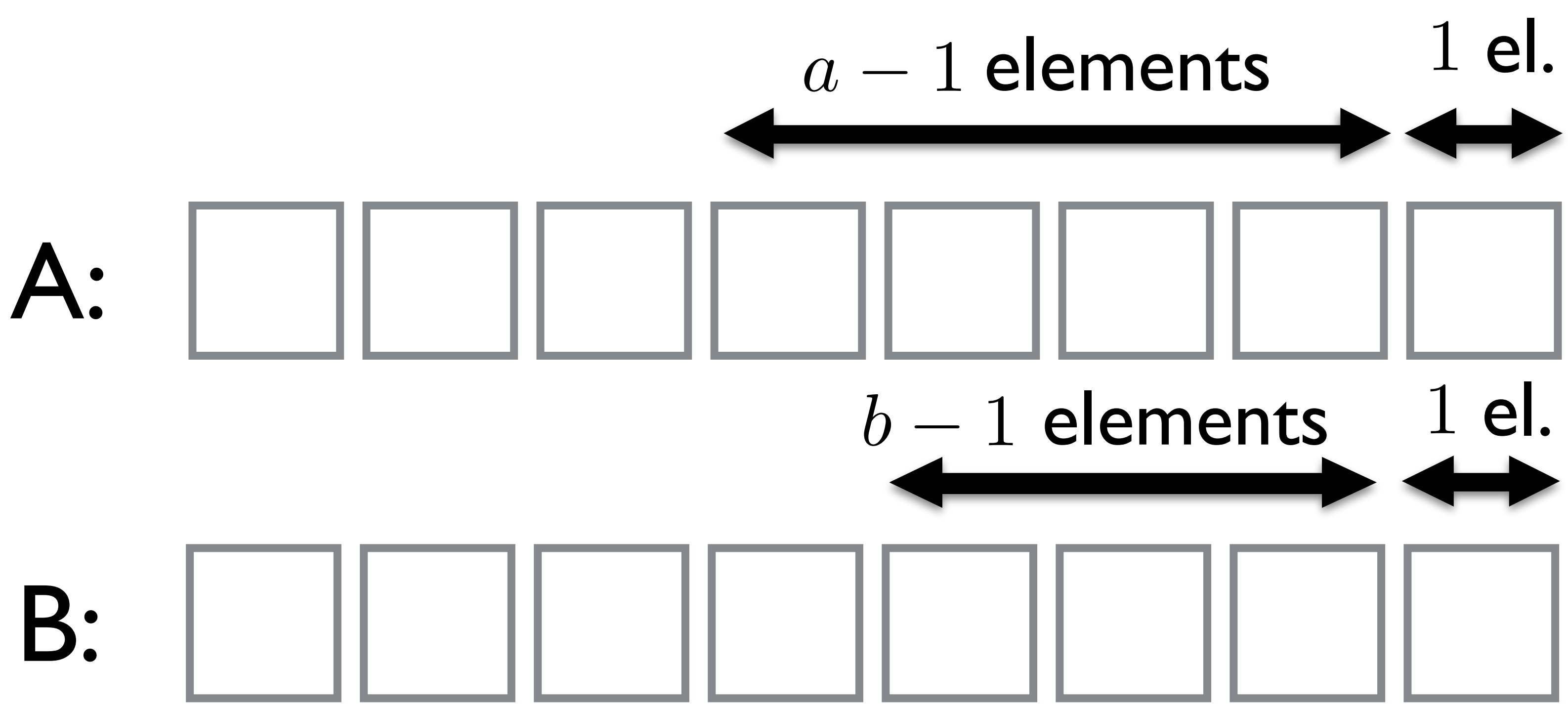
Let us observe the cost function more closely



Cost 1: $(A[1]+\tilde{A})(B[1]+\tilde{B}) = A[1]B[1] + \tilde{A}\tilde{B} + A[1]\tilde{B} + \tilde{A}B[1]$

Cost 2: $A[1]B[1] + \tilde{A}\tilde{B}$

Can we improve the running time of the previous algorithm?

Let us observe the cost function more closely

Conclusion?

Always take one element from each array. Right?

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Conclusion?

Always take one element from each array. Right? **No!**

Previous analysis only works when $a > 1$ and $b > 1$

Therefore: there exists an optimal strategy such that each removal takes either **exactly one element from A or exactly one element from B**!!!

Third approach

```
rec_try(i, j)  // consider only first i elements of
               // A and j elements of B

   if (i == 1) return (A[1] - 1) * ( Σ_{t=1}^{j} B[t]  - j)

   if (j == 1) return (B[1] - 1) * ( Σ_{t=1}^{i} A[t] - i)
   best =( Σ_{t=1}^{i} A[t] - i) * ( Σ_{t=1}^{j} B[t] - j) // take all

   for a = 1 to i - 1 // take one element from B
       cost = ( Σ_{t=i-a+1}^{i} A[i] - a) * (B[j] - 1)

       if (not stored(i - a, j - 1)) rec_try(i-a, j - 1)
       if (cost + stored(i - a, j - 1) < best)
          best = cost + stored(i - a, j - 1)

   for b = 1 to j - 1
       similar ...

   store(i,j) <- best
```

$$\text{cost} = \left(\sum_{t=i-a+1}^{i} A[i] - a\right) * (B[j]-1)$$

Third approach

try_rec(i,j) will be executed at most once
for each pair (*i*,*j*), $1 \leq i, j \leq n$

two **non-nested** loops: one up to *i*, one up to *j*

Total running time: $\mathcal{O}(n^3)$