Formalization of Gilvenko's Theorem in Coq

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1 Motivation

In this project, I use the Coq Proof Assistant to formally prove Gilvenko's Theorem of double negation translation as well as some corollaries. Gilvenko's Theorem states that for all propositions ϕ, ϕ is a theorem of classical propositional logic if and only if $\neg\neg\phi$ is a theorem of intuitionistic propositional logic[2]. This is a well-known result in in the field of proof theory, but is often presented without proof[1]. I found that the result is not intuitively explainable. Why should it necessarily be the case that intuitionistic propositional logic can encode classical proofs via double negation? Why does the absence of the law of excluded middle not affect intuitionistic propositional logic to a greater extent? To resolve these questions, I decided to derive the theorem with formal verification techniques in the Coq Proof Assistant.

2 Formal Proof Outline

The following section gives a short summary of the contents in each file. There is additional documentation in the files themselves.

2.1 PropositionalLogicExpressions.v

In this file, I define an inductive type **Lexp** to represent propositional logic expressions. An element of this type is simply defined as either being an atomic propositional variable, falsum ($\perp$), or a logical connective ($\wedge, \vee, \supset$) joining two subformulas. In this syntax, $\neg\phi$ is just an abbreviation for $\phi \supset \perp$ (which maintains properties of typical classical negation). In addition, various lemmas involving lists are defined in this file for convenience.

2.2 IntuitionisticNaturalDeduction.v

In this file, I define an inductive proposition **NJ** to represent the syntax of intuitionistic natural deduction. I base this definition off Floris van Doorn's formalization of natural deduction[3]. In this approach, I let a list of **Lexp** represent the assumption space Γ, and a single **Lexp** represent the conclusion. I then formalize the typical rules of intuitionistic natural deduction, which are documented fully in this file. I also derive other rules of intuitionistic natural deduction including weakening and the deduction theorem. Floris van Doorn deserves credit for the clever approach to formally proving weakening via a general lemma[3].

2.3 ClassicalNaturalDeduction.v

In this file, I define an inductive proposition **NK** to represent classical natural deduction. **NK** includes (or derives in the case of *ex falso quodlibet*) all of the rules of **NJ**, and adds one classical axiom. I choose a form of the law of double negation elimination for the classical axiom. Additionally, I introduce helpful proofs of weakening and the deduction theorem for this new system. I then prove that the use of alternative classical axioms, specifically the law of excluded middle and Pierce's law (both packaged with *ex falso quodlibet*), would form equivalent systems of **NK**.

2.4 GilvenkosTheorem.v

In this file, I formalize and inductively prove Gilvenko's theorem. The precise statement is located in the theorem titled *DNtranslation*. I then investigate the theoretical consequences of this translation. I formalize the corollary that for all propositions ϕ, $\neg\phi$ is a theorem of classical propositional logic if and only if $\neg\phi$ is a theorem of intuitionistic propositional logic. In other words, the set of all refuted propositions is identical between both systems. I then prove the equiconsistency of **NJ** and **NK**, which subverts the common thought that intuitionistic logic is in some way "safer" than its classical counterpart.

3 Coq Project Set-up

The files should be compiled in the order that they are presented in this document. An appropriate makefile and CoqProject file are provided.

References

[1] Richard Moot and Christian Retore. Classical logic and intuitionistic logic: equivalent formulations in natural deduction, Gödel-Kolmogorov-Glivenko translation, 2016.

[2] Joan Moschovakis. Intuitionistic Logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Fall 2021 edition, 2021.

[3] Floris van Doorn. Propositional Calculus in Coq, 2015.