Networks

Types of Network and Robustness

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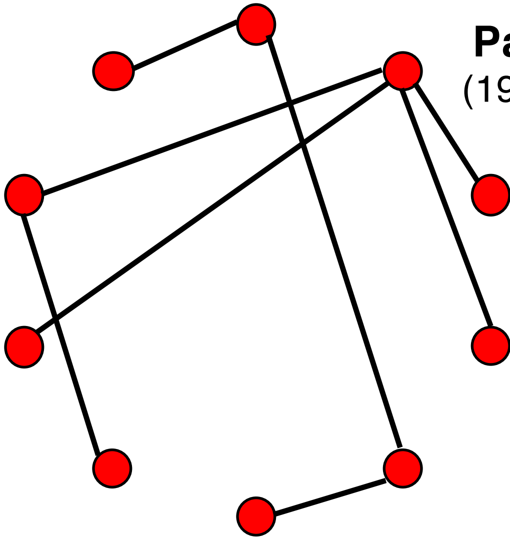
Slides adapted from Lázló Barabási
http://www.barabasilab.com/

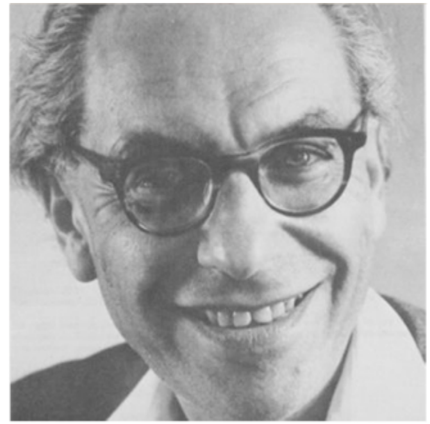
Networks can be characterized by their form

- **Random Networks** – edges are randomly created between nodes
- **Watts Strogatz model** - Small world Networks
- **Scale Free Networks** – a few nodes with many connections

RANDOM (Erdös – Reyni) NETWORK MODEL



Pál Erdös
(1913-1996)



Alfréd Rényi
(1921-1970)

Erdös-Rényi model (1960)

Connect with probability p

p=1/6 N=10

$\langle k \rangle \sim 1.5$

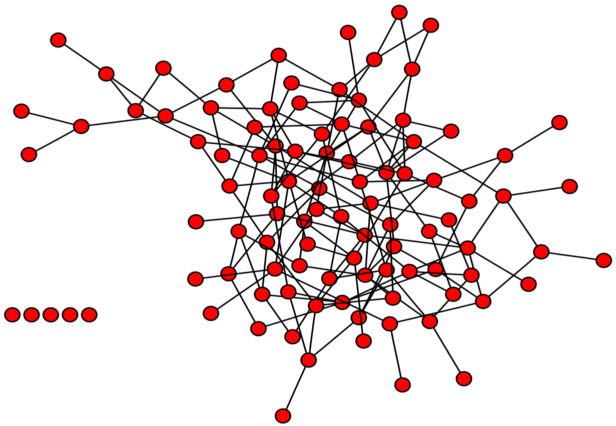
Definition:

A **random graph** is a graph of N labeled nodes where each pair of nodes is connected by a preset probability **p**.

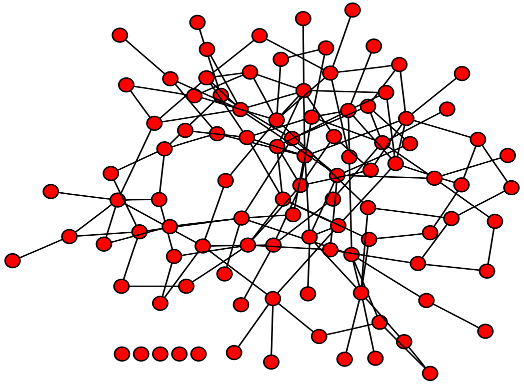
We will call is G(N, p).

RANDOM NETWORK MODEL

p=0.03
N=100



$\mathbf{f_c}$=0.95



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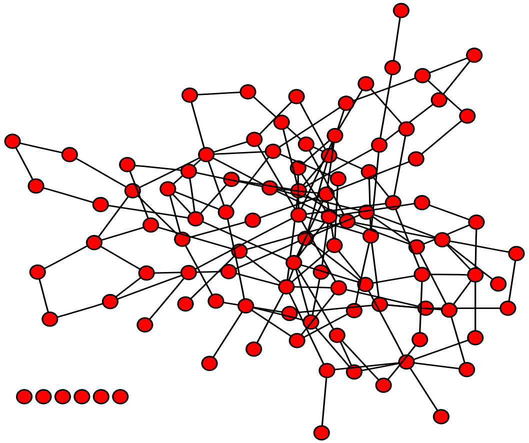
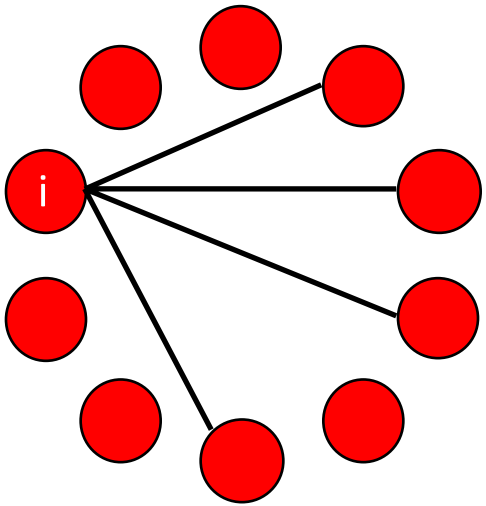


$\mathbf{f_c}$=0.94

Giant Component = largest fully connected sub-graph
$\mathbf{f_c}$= fraction of nodes in giant component

RANDOM NETWORK MODEL



$$p \times p \times p \times p \times (1-p) \times (1-p) \times (1-p) \times (1-p) \times (1-p)$$

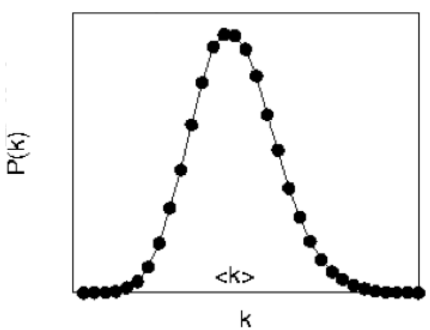
$$=> p^k (1-p)^{(N-1)-k}$$

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

Select k nodes from N-1

probability of having *k* edges

probability of missing N-1-k edges



$$< k >= p(N-1) \qquad \sigma_k^2 = p(1-p)(N-1)$$

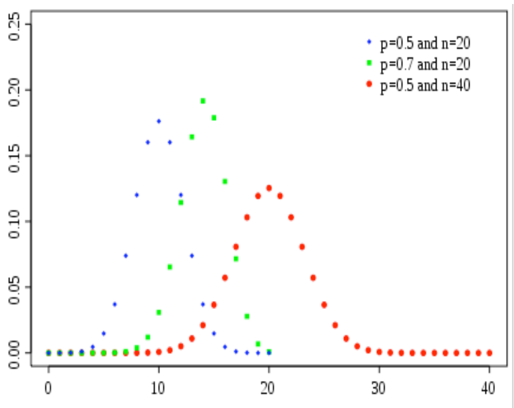
DEGREE DISTRIBUTION OF A RANDOM NETWORK

Exact Result

-binomial distribution-

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

Probability Distribution Function (PDF)



$$< k >= (N-1)p$$

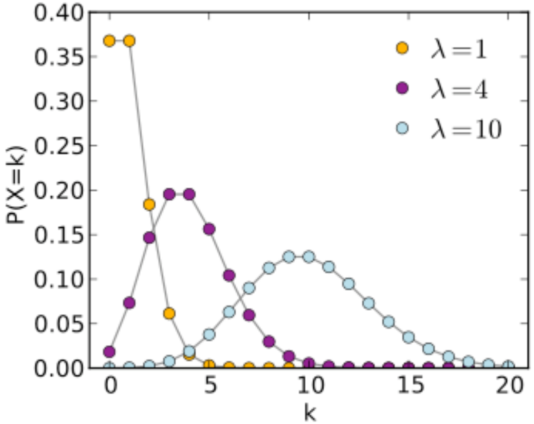
$$< k^2 >= p(1-p)(N-1) + p^2 (N-1)^2$$

$$\sigma_k = (< k^2 > - < k >^2)^{1/2} = [p(1-p)(N-1)]^{1/2}$$

N>>k

-Poisson distribution-

$$P(k) = e^{-<k>} \frac{< k >^k}{k!}$$



$$< k >=< k >$$

$$< k^2 >=< k > (1+ < k >)$$

$$\sigma_k = (< k^2 > - < k >^2)^{1/2} =< k >^{1/2}$$

ARE REAL NETWORKS LIKE RANDOM GRAPHS?

As quantitative data about real networks became available, we can compare their topology with the predictions of random graph theory.

Note that once we have N and <k> for a random network, from it we can derive every measurable property. Indeed, we have:

Average path length:

$$< l_{rand} > \approx \frac{\log N}{\log\langle k\rangle}$$

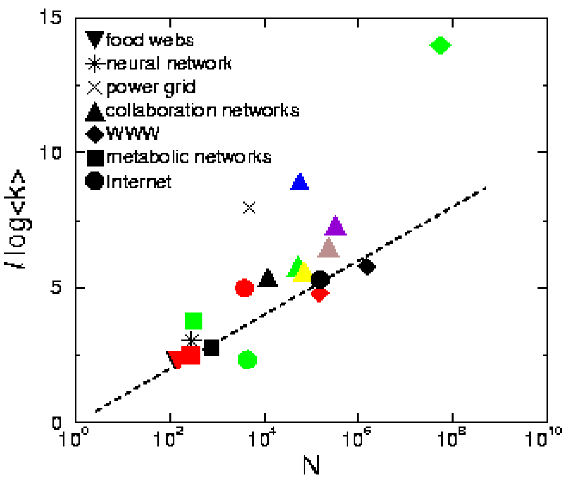
Clustering Coefficient:

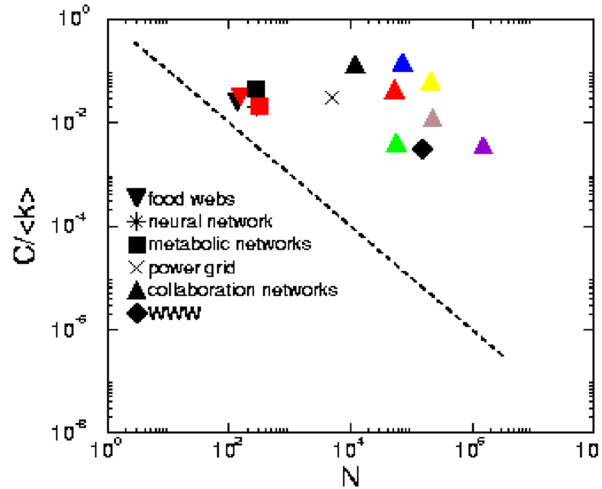
$$C_{rand} = p = \frac{\langle k\rangle}{N}$$

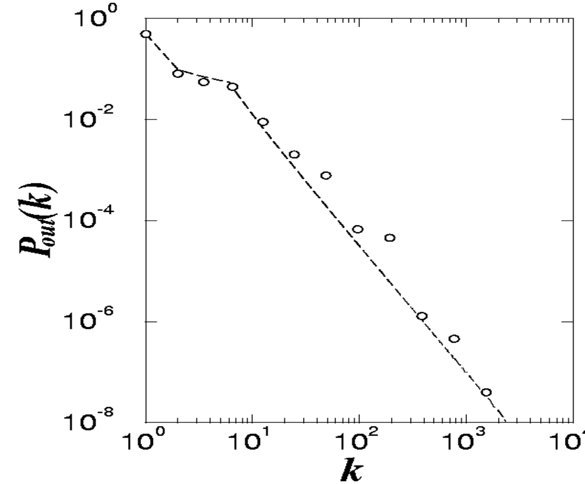
Degree Distribution:

$$P_{rand}(k) \cong C_{N-1}^{k} p^{k} (1-p)^{N-1-k}$$

Empirical findings for real networks







$$l_{data} = \frac{\log N}{\log\langle k\rangle}$$

$$C_{data} \sim const$$

$$P_{data}(k) = \sim k^{-\gamma}$$

$$l_{rand} = \frac{\log N}{\log\langle k\rangle}$$

$$C_{rand} = \frac{\langle k\rangle}{N}$$

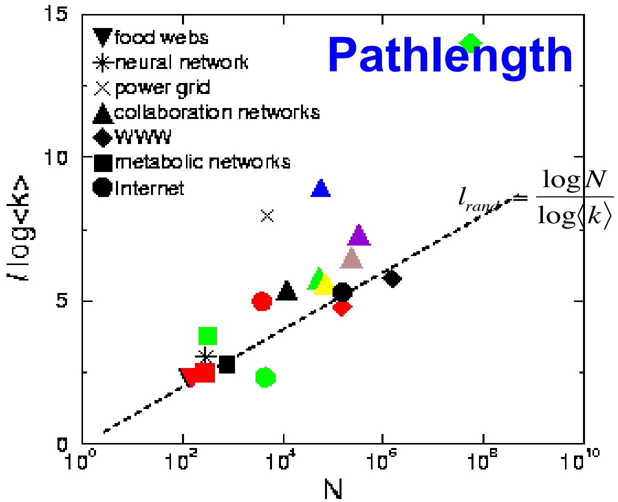
$$P_{rand}(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

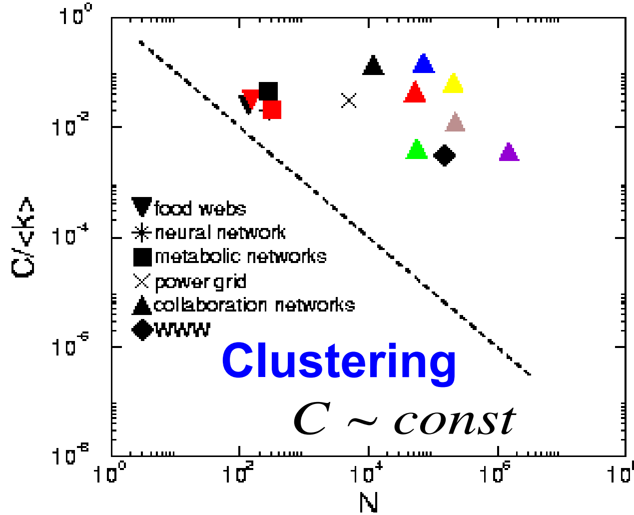
Small World:
distances scale logarithmically with the network size

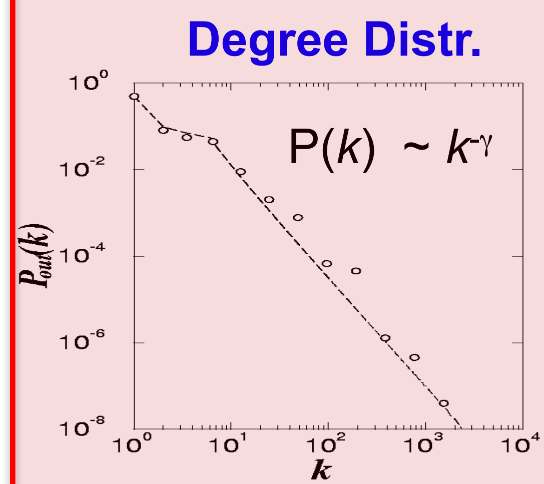
Clustered:
clustering coefficient does not depend on network size.

Scale-free:
The degrees follow a power-laws distribution.

Empirical data for real networks







Regular network	$l \approx N^{1/D}$	👎	$C \sim const$	👍	$P(k)=\delta(k-k_d)$	👎
Erdos-Renyi	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	👍	$C_{rand} = p = \frac{\langle k\rangle}{N}$	👎	$P(k) = e^{-<k>}\frac{<k>^k}{k!}$	👎
Watts-Strogatz	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	👍	$C \sim const$	👍	*Exponential*	👎

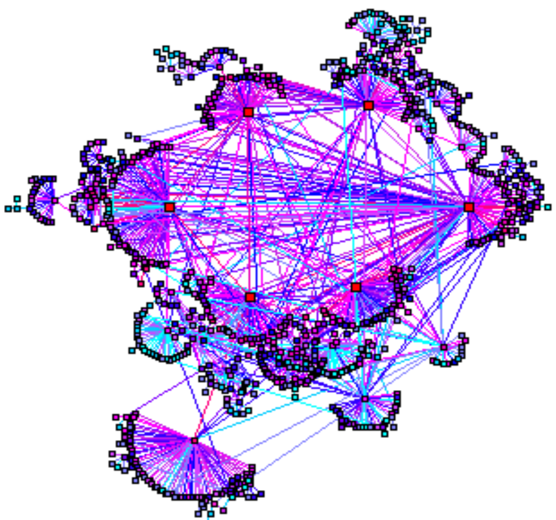
SCALE-FREE
Real Networks

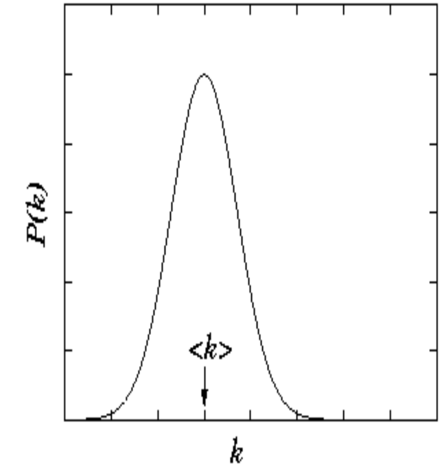
WORLD WIDE WEB

Nodes: **WWW documents**
Links: **URL links**

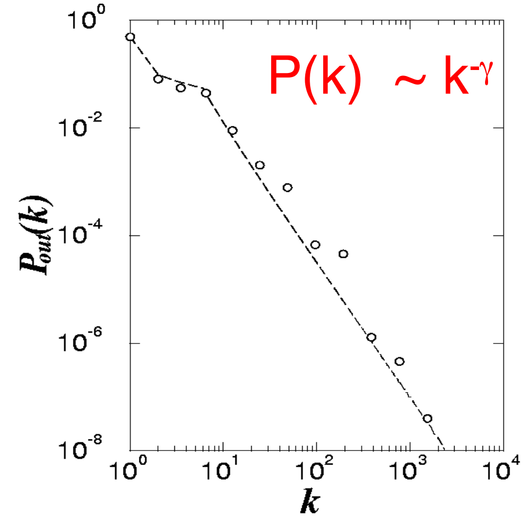
Over 3 billion documents

ROBOT: collects all URL's found in a document and follows them recursively





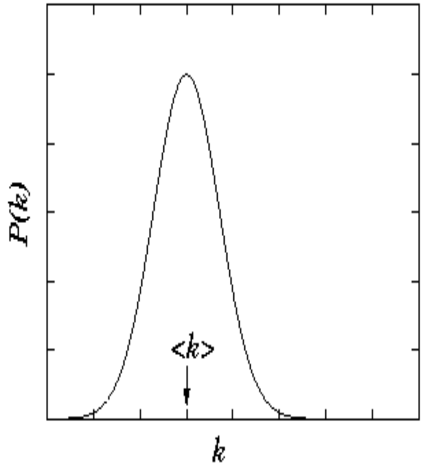
Expected



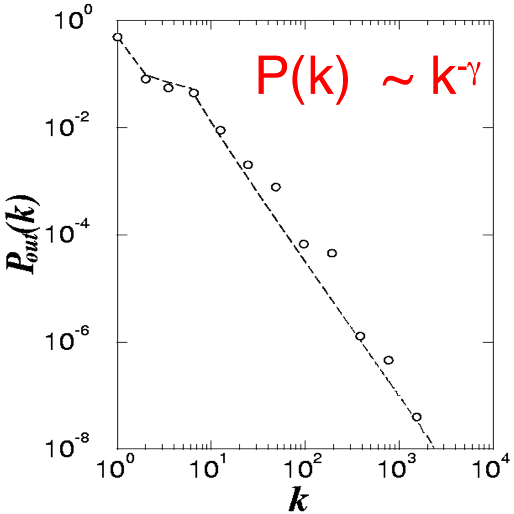
Found

R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

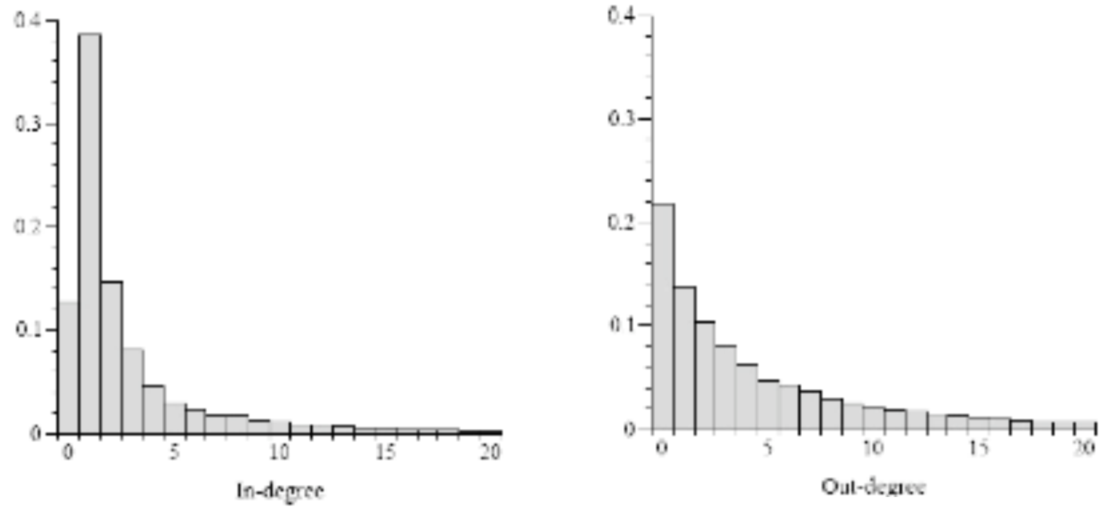
Degree distribution of the WWW



Expected



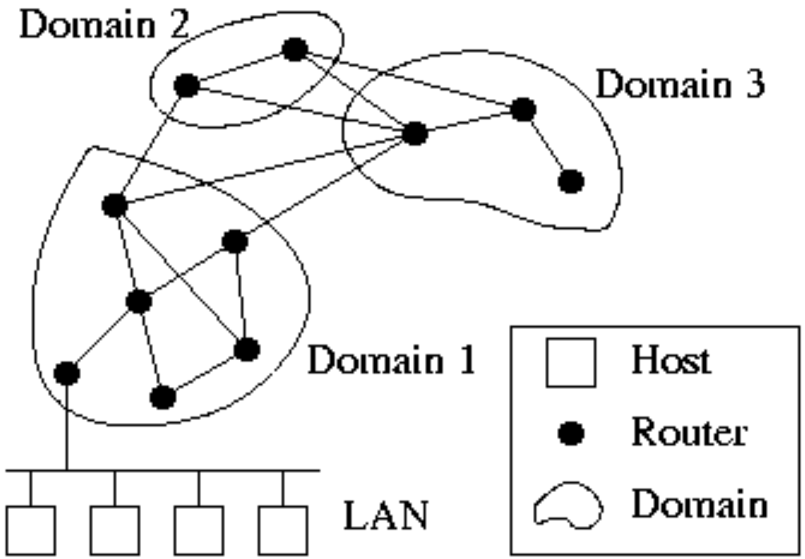
Found

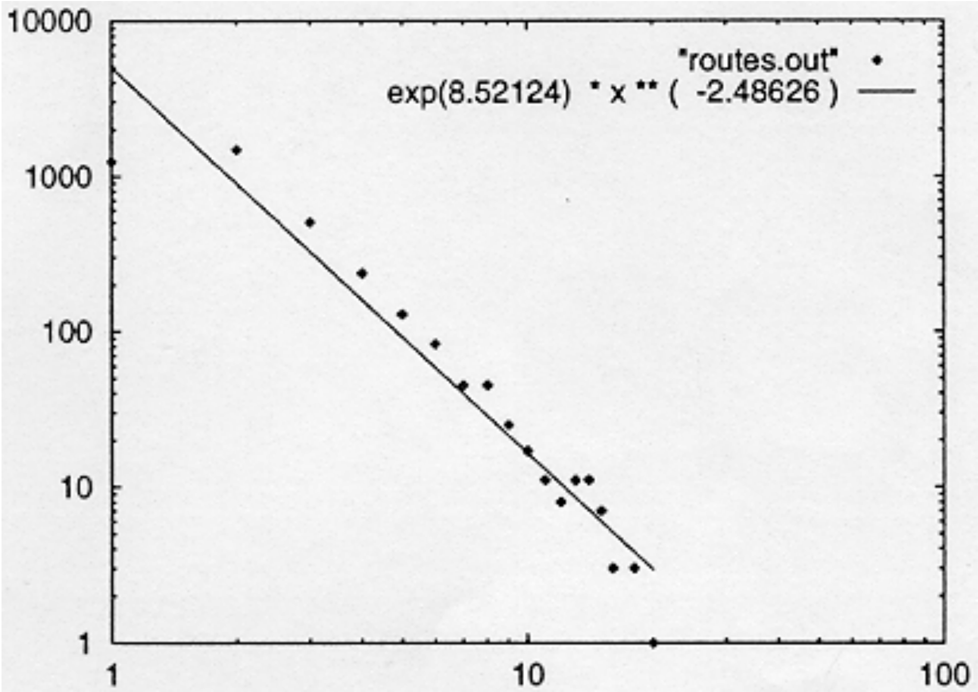


R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

INTERNET BACKBONE

Nodes: computers, routers
Links: physical lines

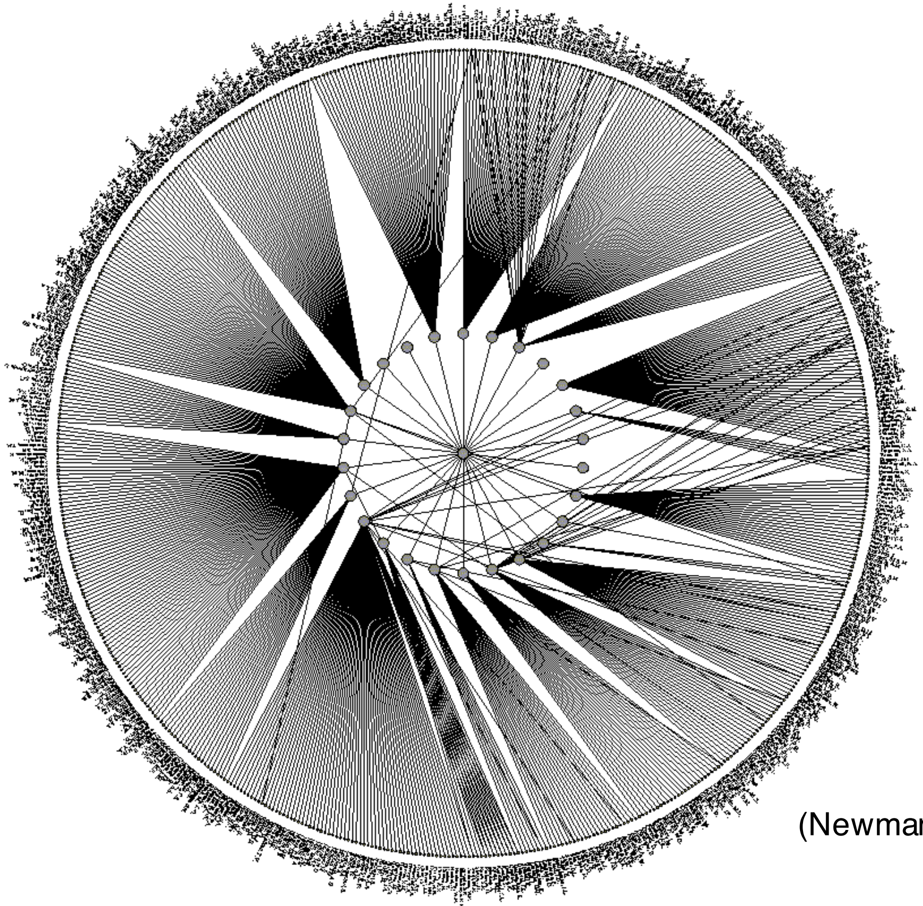


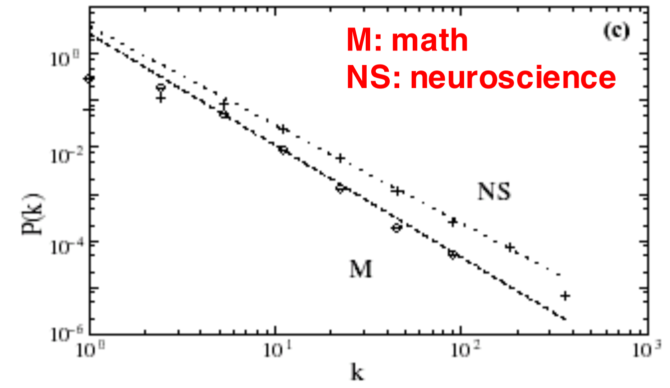


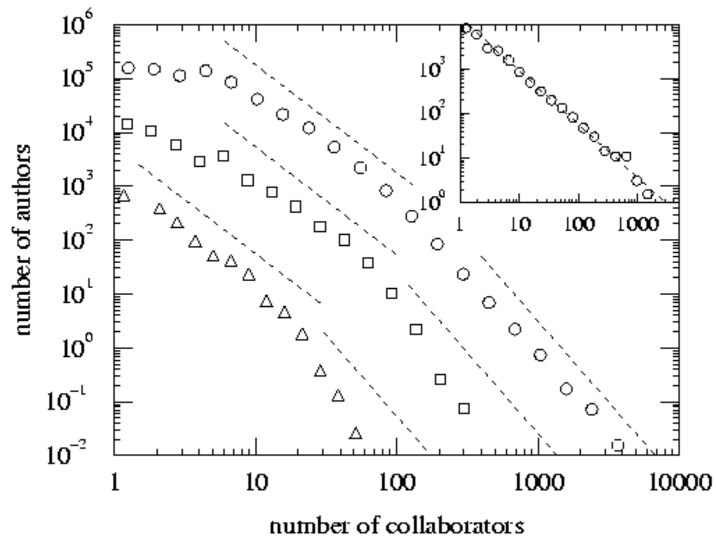
(Faloutsos, Faloutsos and Faloutsos, 1999)

SCIENCE COAUTHORSHIP

Nodes: scientist (authors)
Links: joint publication

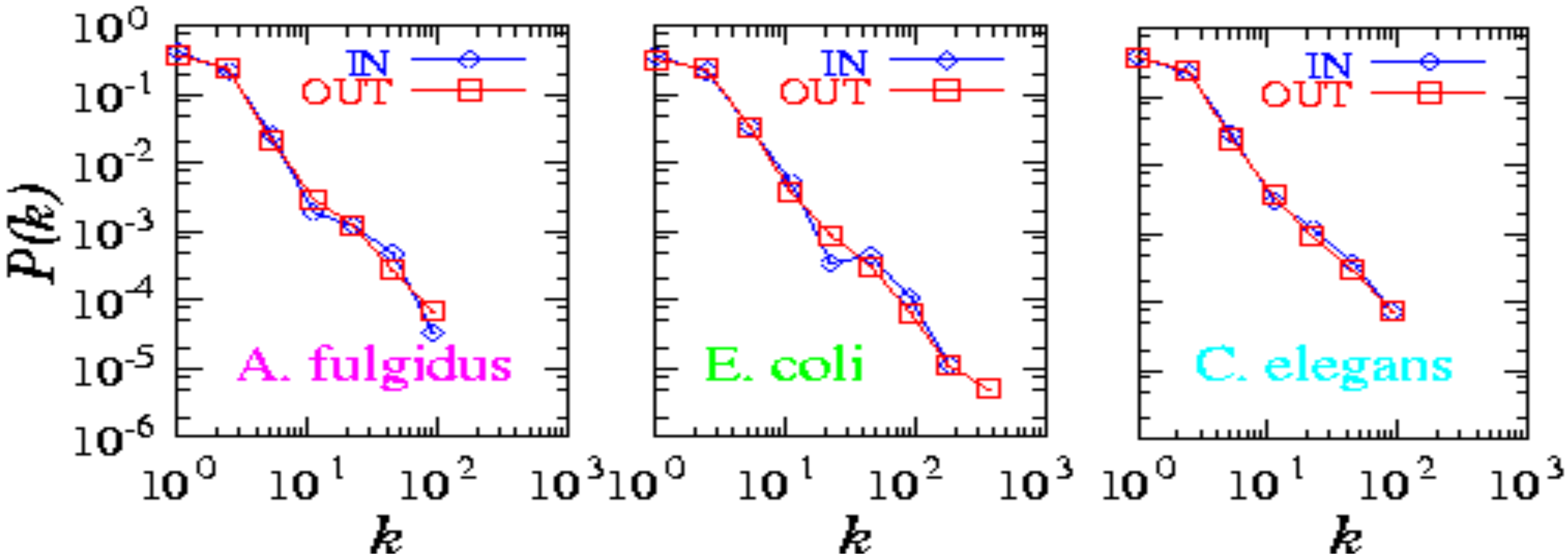






(Newman, 2000, Barabasi et al 2001)

METABOLIC NETWORK



Archaea

Bacteria

Eukaryotes

Organisms from all three domains of life are scale-free!

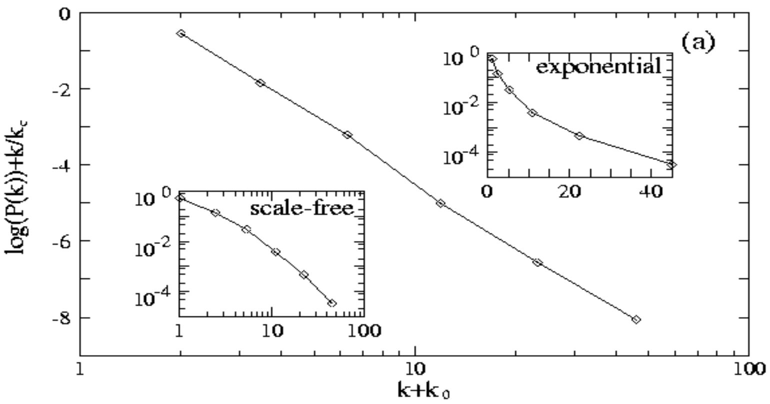
$$P_{in}(k) \approx k^{-2.2}$$

$$P_{out}(k) \approx k^{-2.2}$$

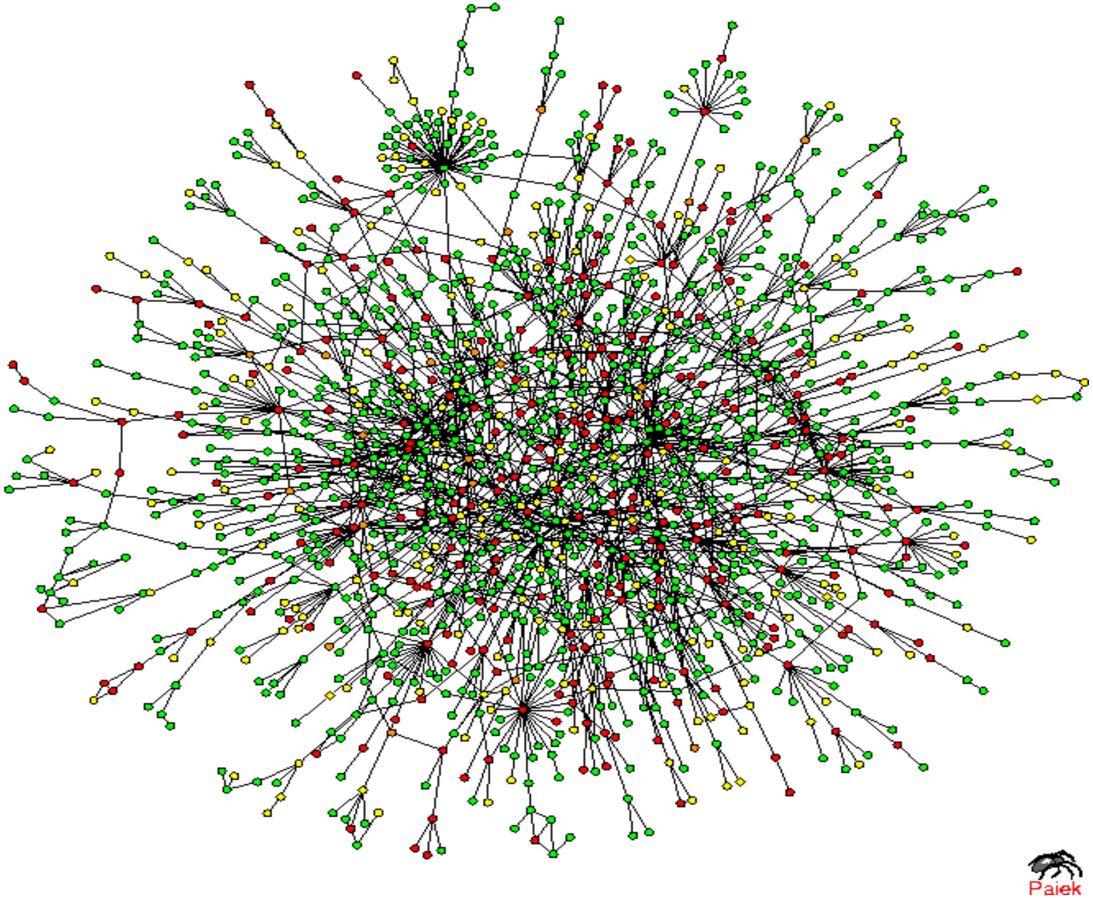
H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.L. Barabasi, *Nature*, 407 651 (2000)

TOPOLOGY OF THE PROTEIN NETWORK

Nodes: proteins
Links: physical interactions-binding



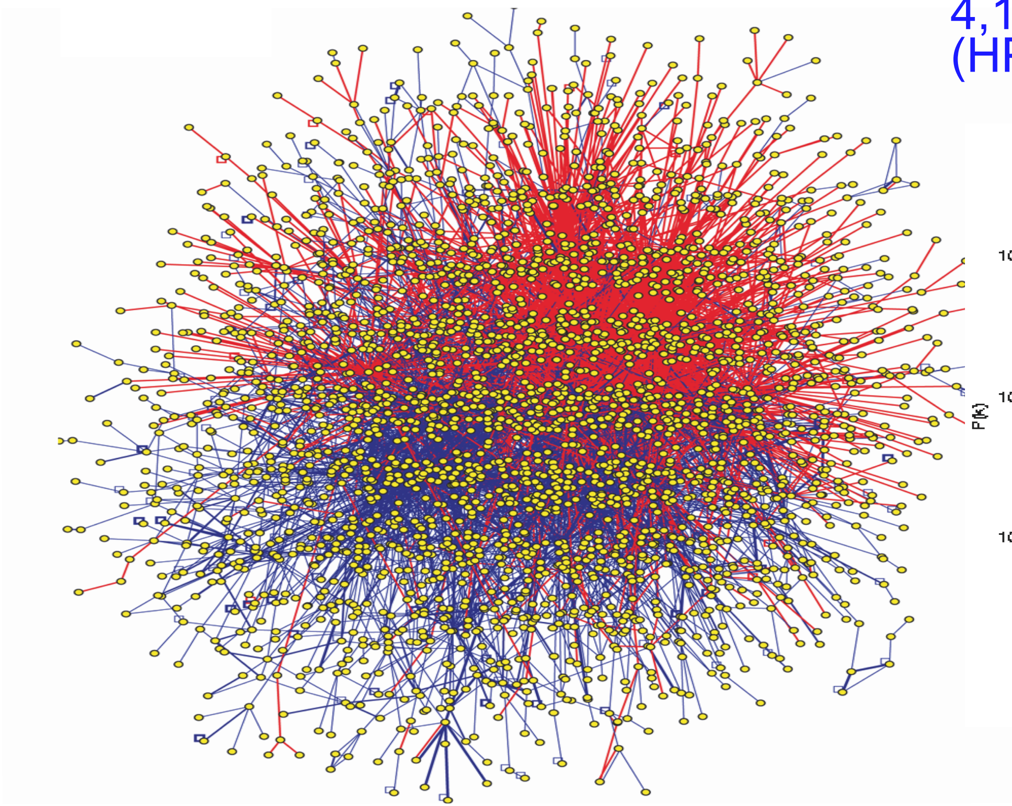
$$P(k) \sim (k + k_0)^{-\gamma} \exp(-\frac{k + k_0}{k_\tau})$$

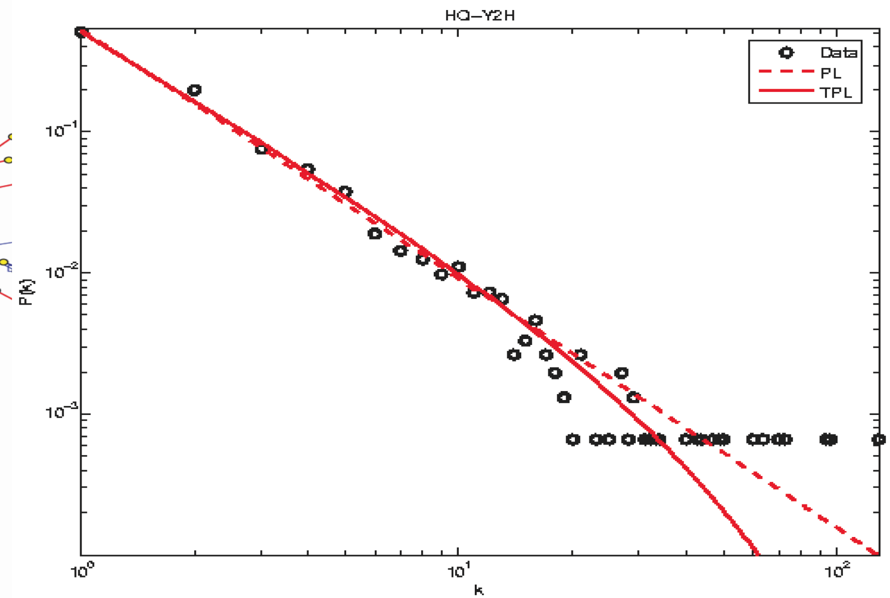


H. Jeong, S.P. Mason, A.-L. Barabasi, Z.N. Oltvai, Nature 411, 41-42 (2001)

HUMAN PROTEIN-PROTEIN INTERACTION NETWORK

2,800 Y2H interactions
4,100 binary LC interactions
(HPRD, MINT, BIND, DIP, MIPS)



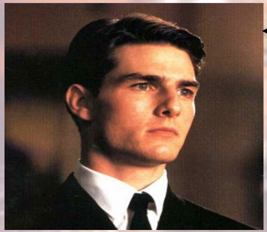


Rual *et al.* Nature 2005; Stelze *et al.* Cell 2005

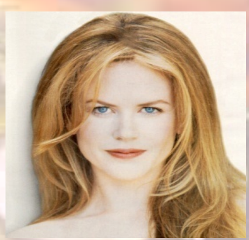
ACTOR NETWORK

Nodes: actors
Links: cast jointly

IMDb Internet Movie Database

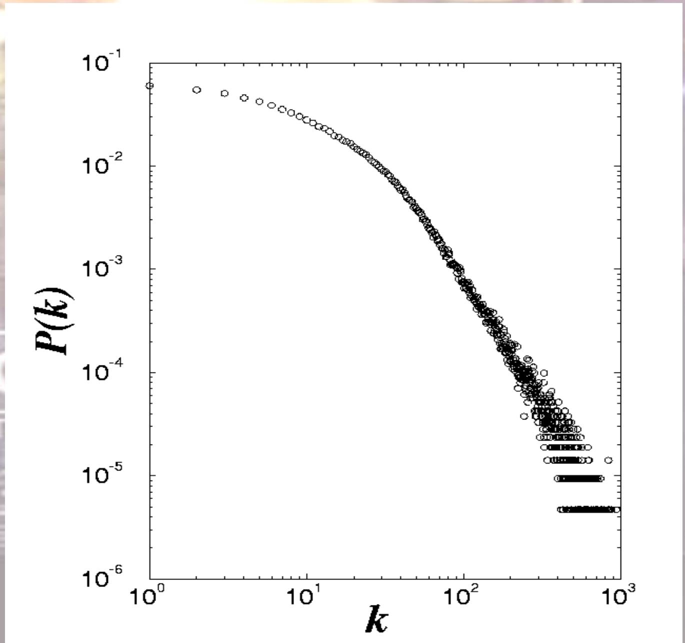


Days of Thunder *(1990)*
Far and Away *(1992)*
Eyes Wide Shut *(1999)*



N = 212,250 actors
$\langle k \rangle = 28.78$

$P(k) \sim k^{-\gamma}$
$\gamma = 2.3$



Many real world networks have a similar architecture:

Scale-free networks

WWW, Internet (routers and domains), electronic circuits, computer software, movie actors, coauthorship networks, sexual web, instant messaging, email web, citations, phone calls, metabolic, protein interaction, protein domains, brain function web, linguistic networks, comic book characters, international trade, bank system, encryption trust net, energy landscapes, earthquakes, astrophysical network…

SCALE-FREE MODEL

(BA model)

BA MODEL: Preferential Attachment

(A) Growth: At each timestep we add a new node with m($\leq m_0$) links that connects the new node to m nodes already in the network.

(B) Preferential attachment: The probability $\Pi(k_i)$ that a link of the new node connects to node i depends on the degree k_i of node i as:

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

we need both!

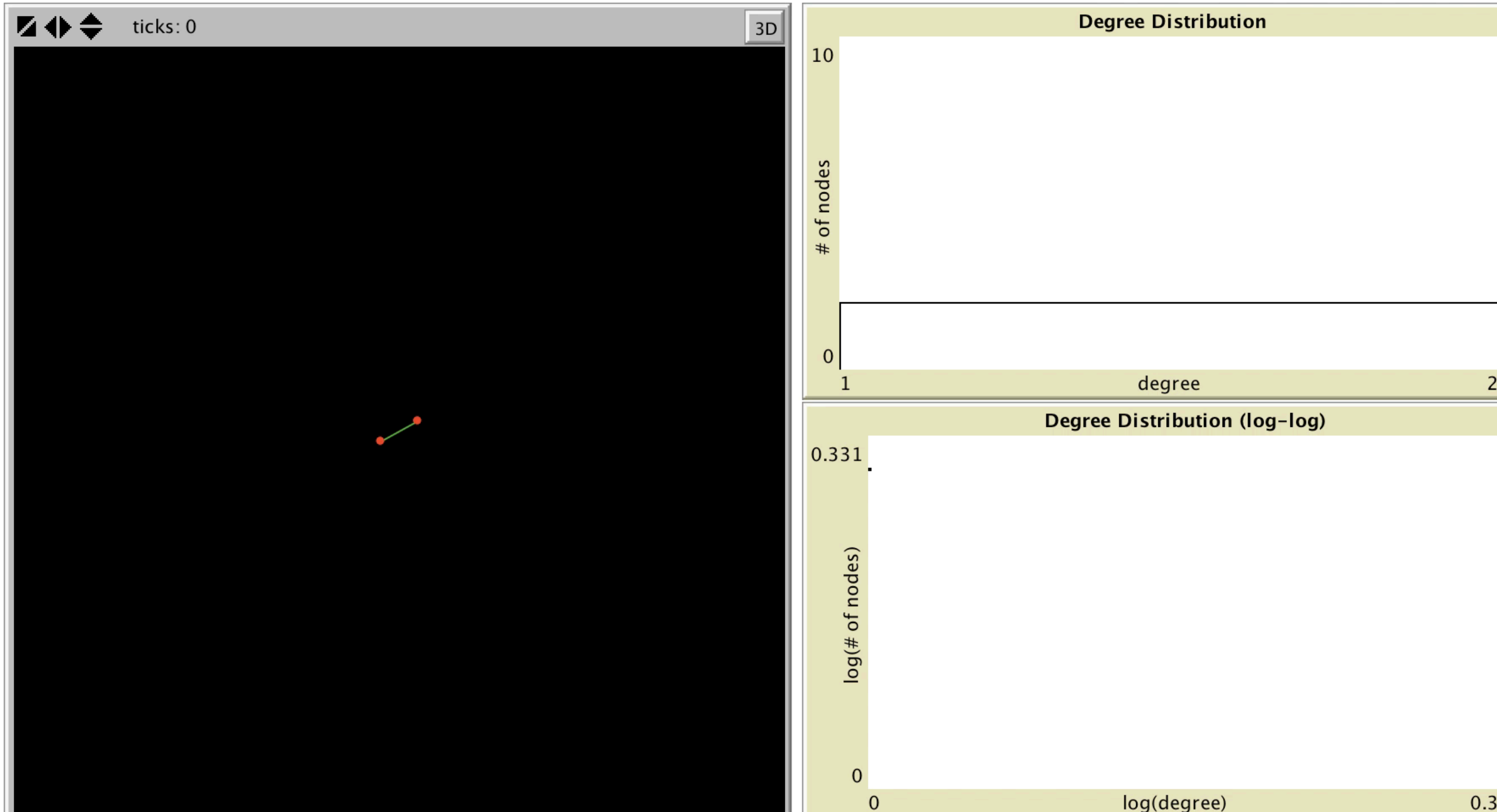
After t timesteps the Barabási-Albert model generates a network with $N = t + m_0$ nodes and $m_0 + mt$ links

BA MODEL

(1) Networks continuously expand by the addition of new nodes

(2) New node has m links added

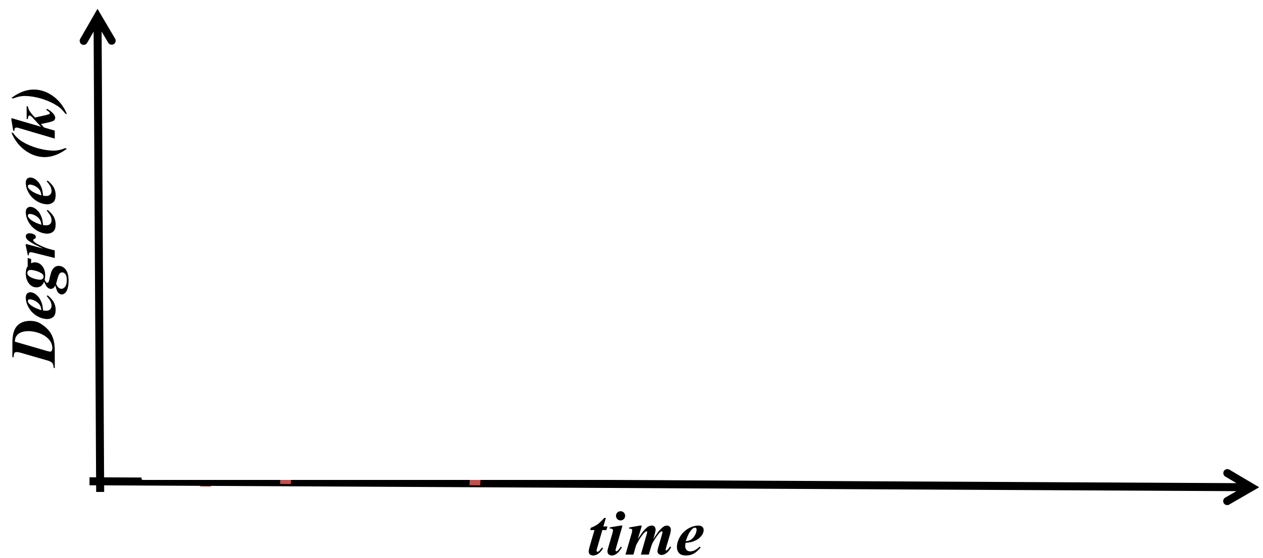
Barabási & Albert, *Science* **286,** 509 (1999)



m=1

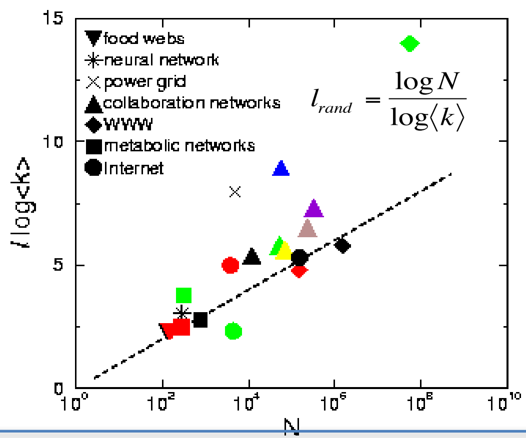
Fitness Model: Can Latecomers Make It?

SF model: $k(t) \sim t^{1/2}$ **(first mover advantage)**

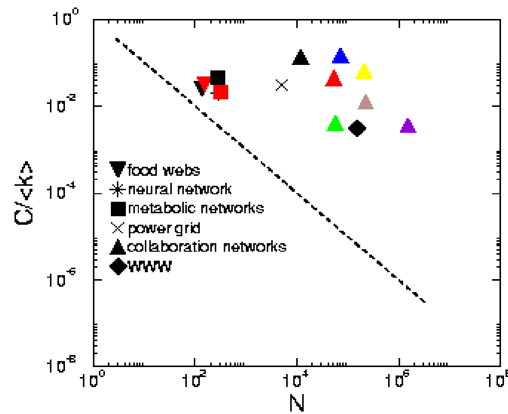


EMPIRICAL DATA FOR REAL NETWORKS

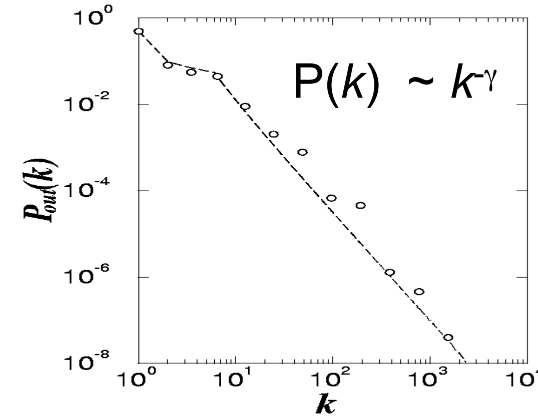
Pathlenght



Clustering



Degree Distr.

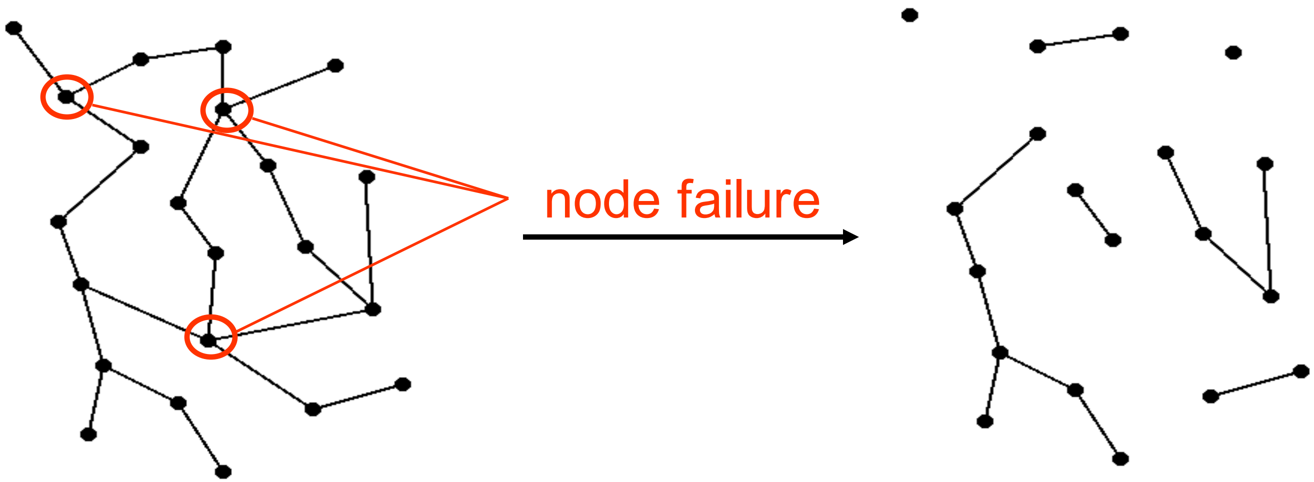


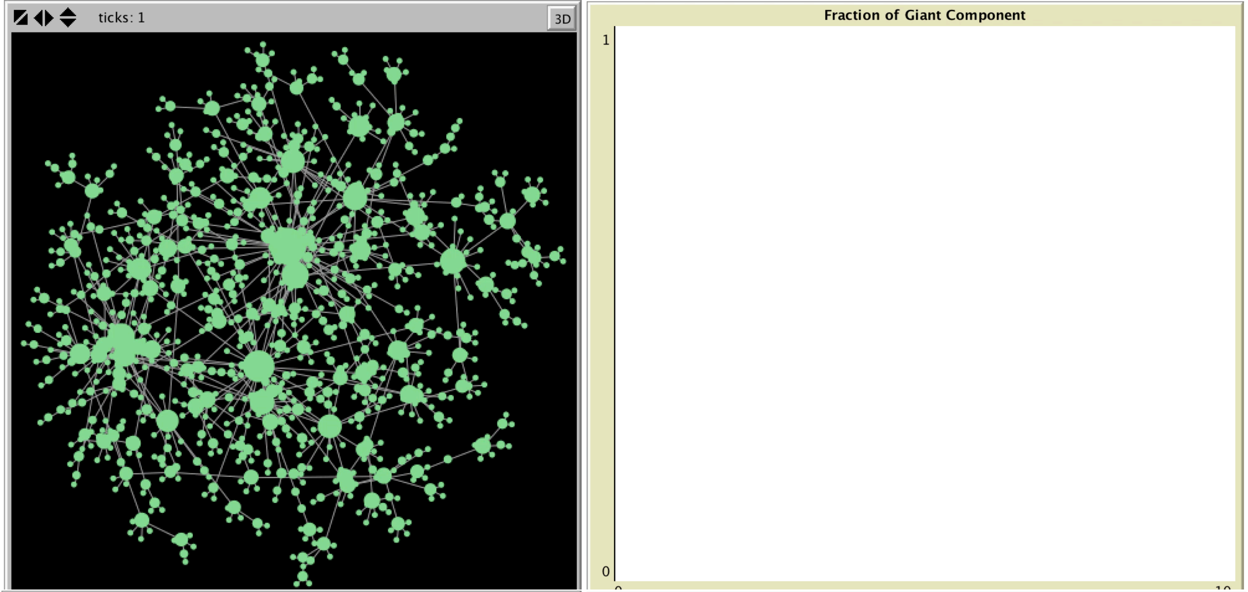
Regular network	$l \approx N^{1/D}$	$C \sim const$	$P(k)=\delta(k-k_d)$
Erdos-Renyi	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	$C_{rand} = p = \frac{\langle k\rangle}{N}$	$P(k) = e^{-\langle k\rangle}\frac{\langle k\rangle^k}{k!}$
Watts-Strogatz	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	$C \sim const$	*Exponential*
Barabasi-Albert	$l \approx \frac{\ln N}{\ln\ln N}$	$C \sim \frac{(\ln N)^2}{N}$	$P(k) \sim k^{-\gamma}$

Network Science

Robustness

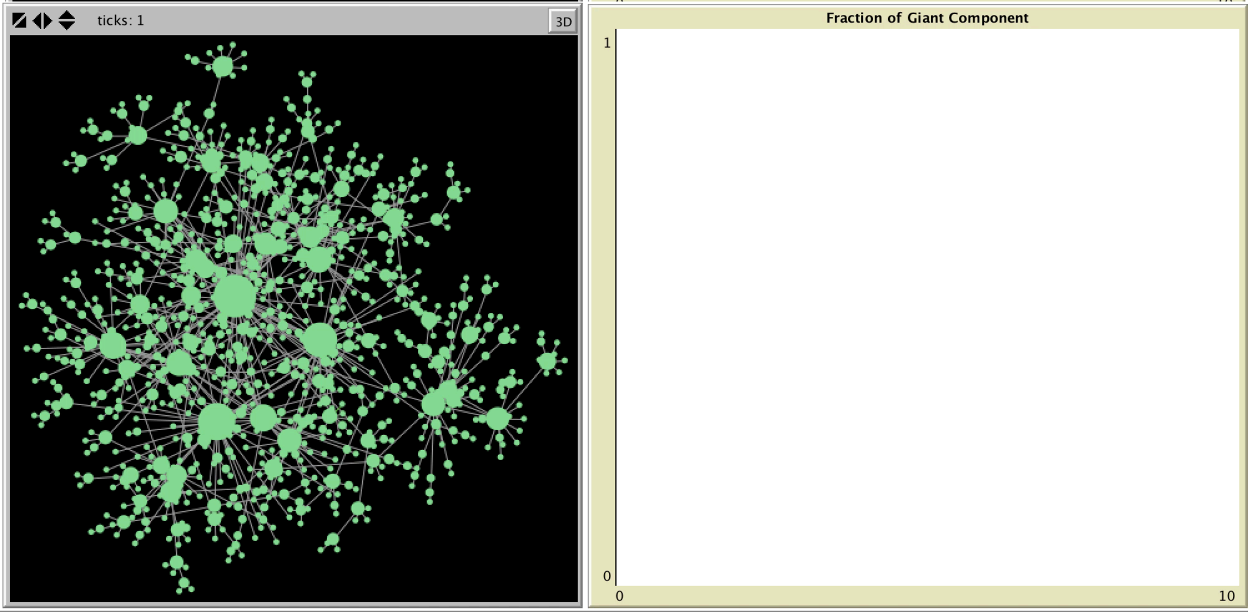
Could the network structure contribute to robustness?





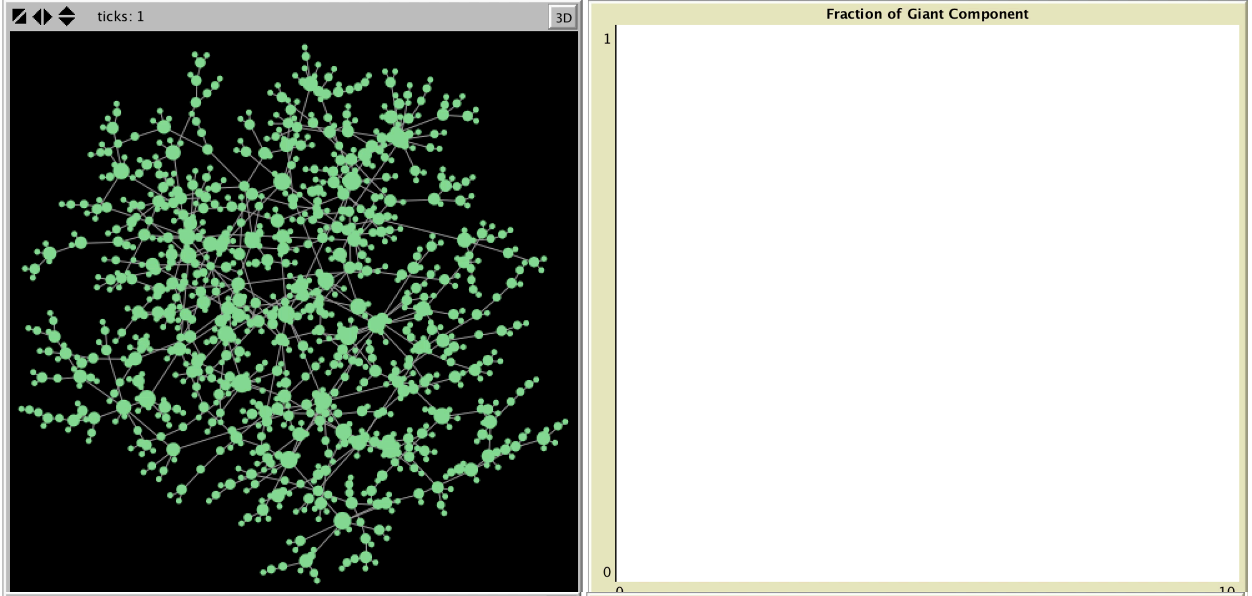
Attack

Barabási-Albert



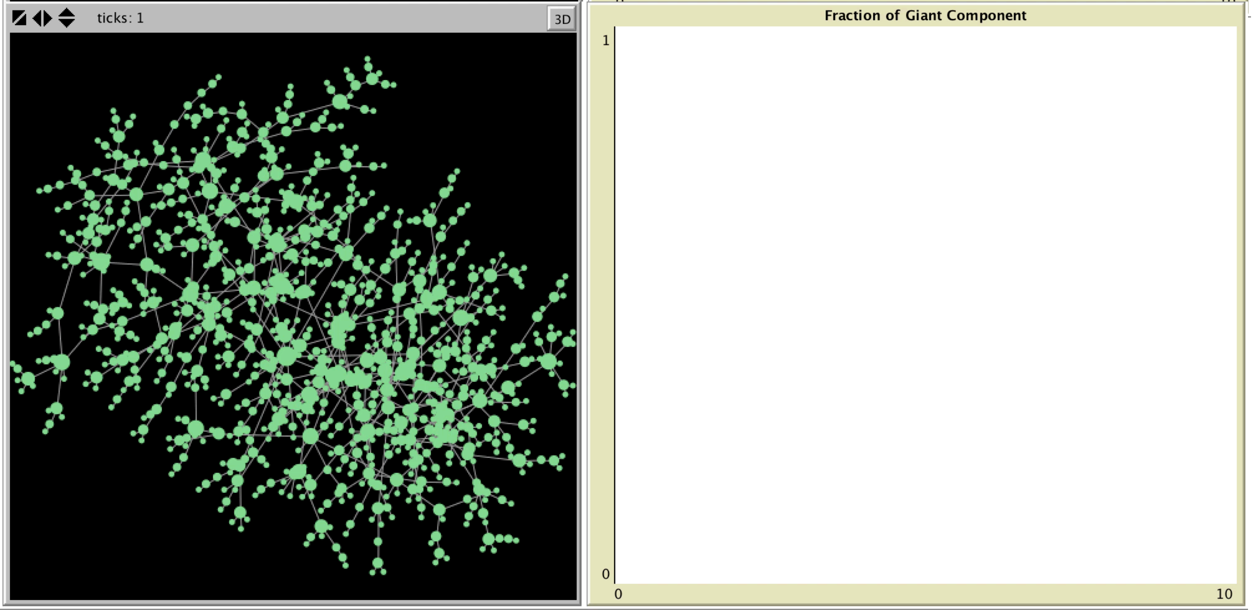
Failure

Barabási-Albert



Attack

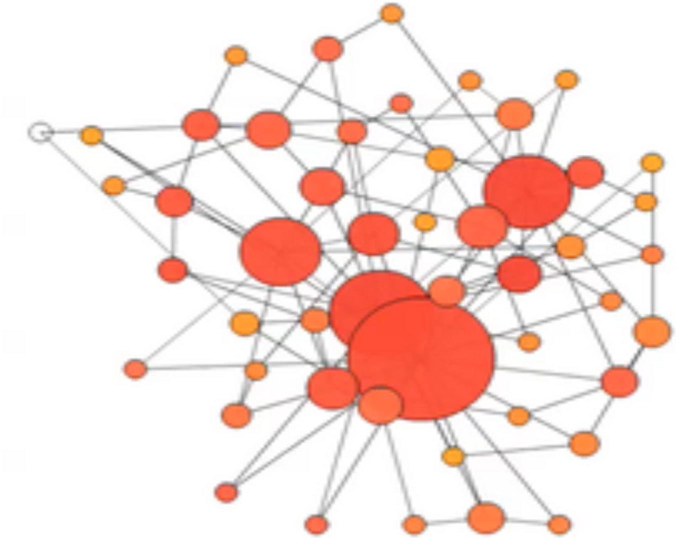
Erdös-Reyni

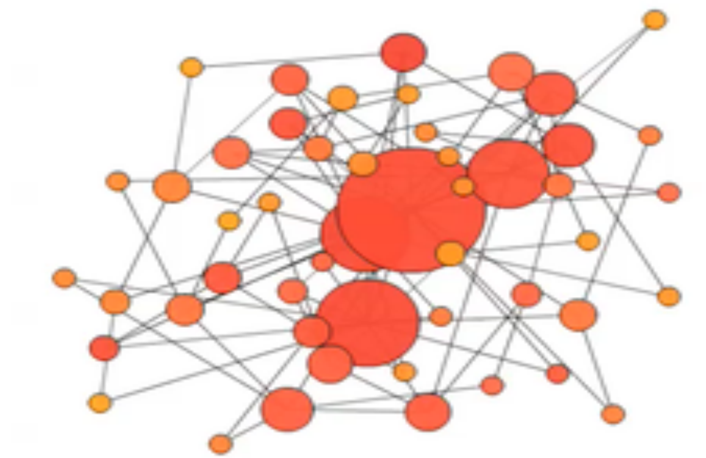


Failure

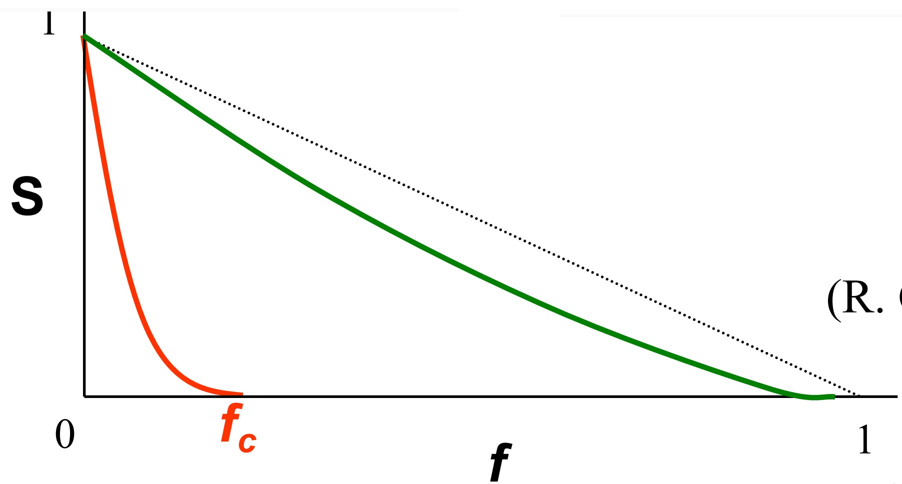
Erdös-Reyni

Achilles' Heel of scale-free networks





Attacks



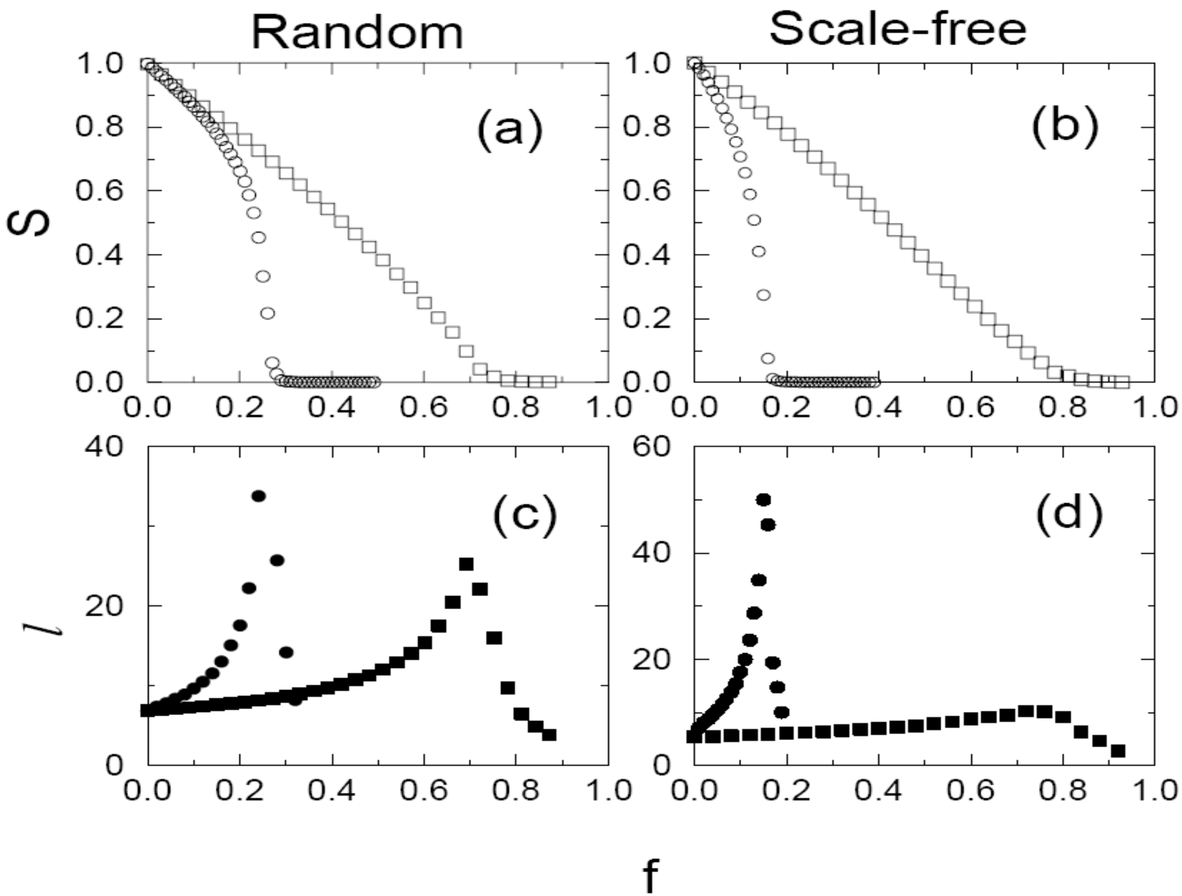
Failures

$\gamma \leq 3 : f_c = 1$

(R. Cohen et al PRL, 2000)

Albert, Jeong, Barabási, *Nature* **406** 378 (2000)

Scale-free networks are more error tolerant, but also more vulnerable to attacks



- squares: random failure
- circles: targeted attack

Failures: little effect on the integrity of the network.
Attacks: fast breakdown