Quantifying causal impact using information theory

To find out where to nudge a dynamical system

Interface

The diminishing role of hubs in dynamical processes on complex networks

Rick Quax[1†], Andrea Apolloni[2,†] and Peter M. A. Sloot[1,3,4]

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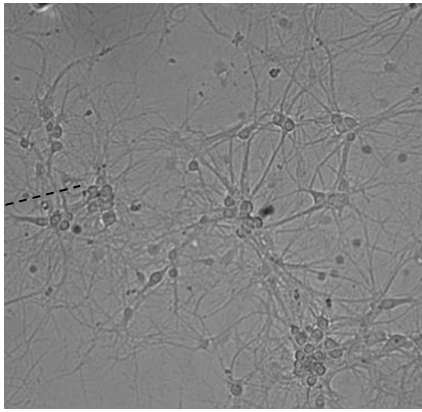
Control theory

- Complicated systems
- Complex systems

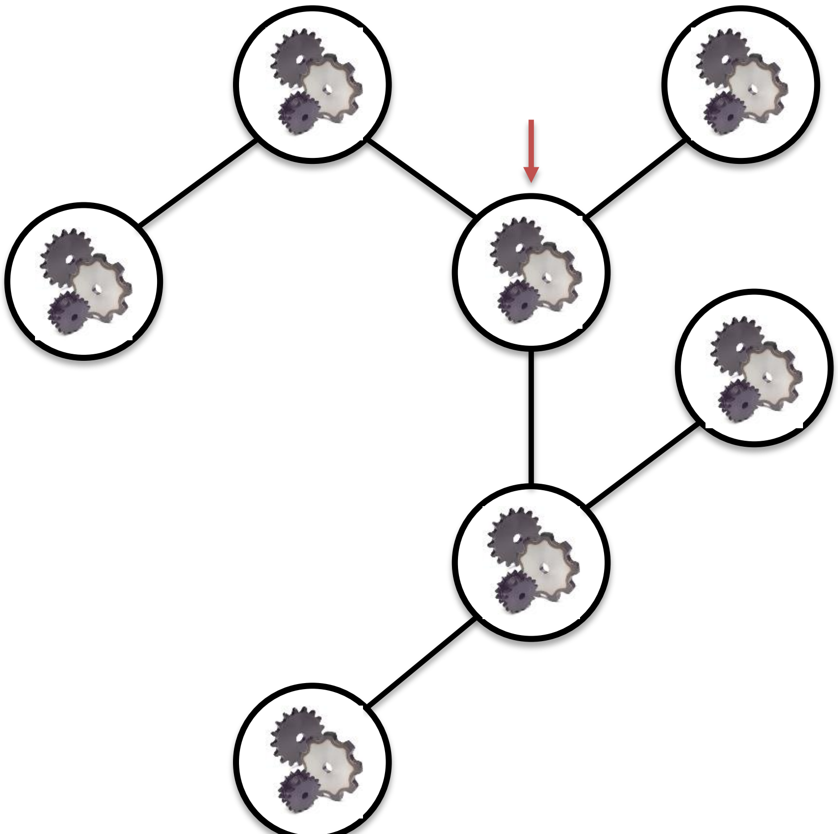


Actual microscope image in vitro

- Self-organization
- Emergent behavior

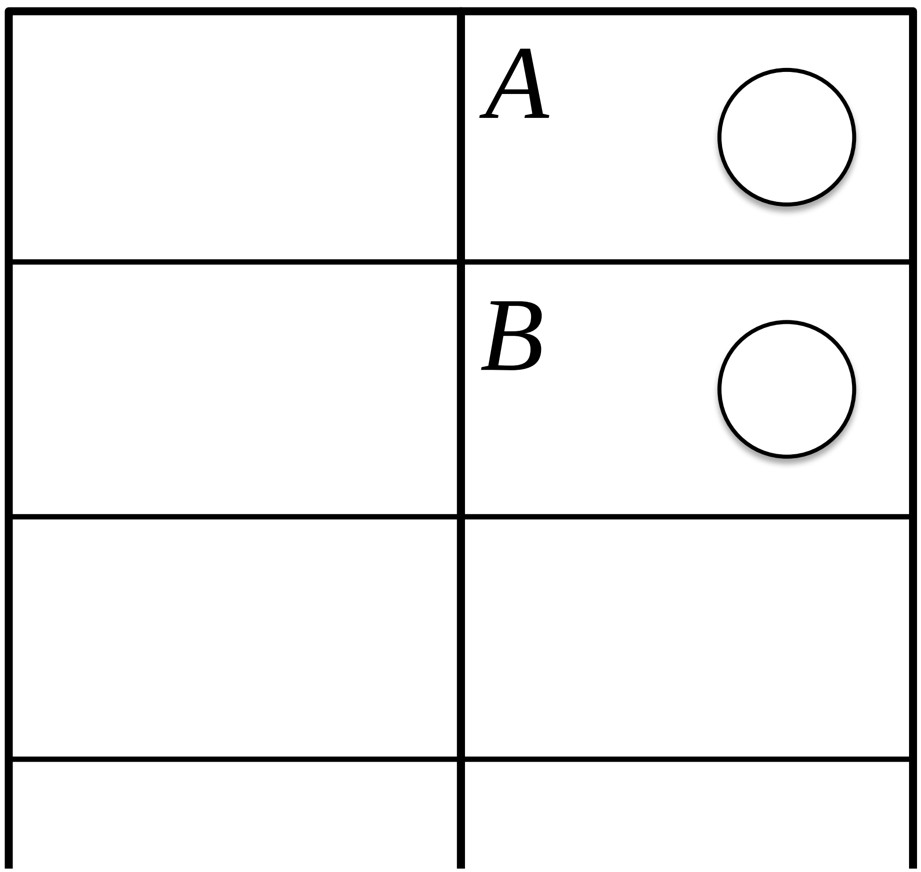
?

How to control a networked system

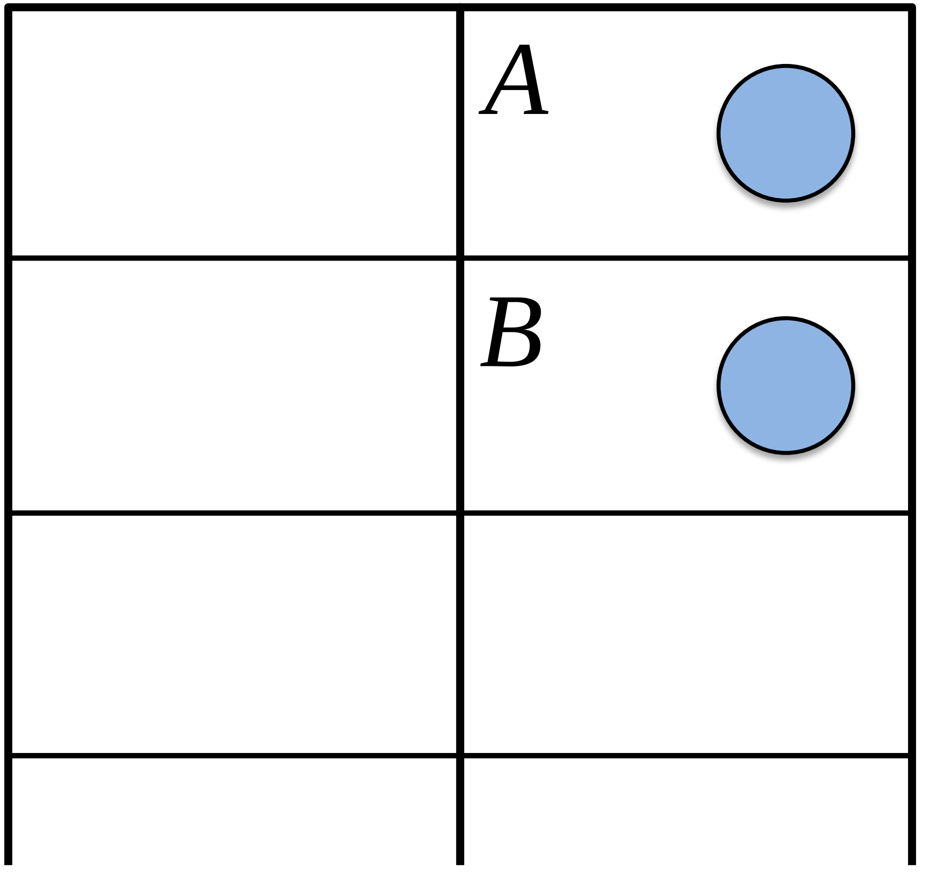


network structure + node dynamics

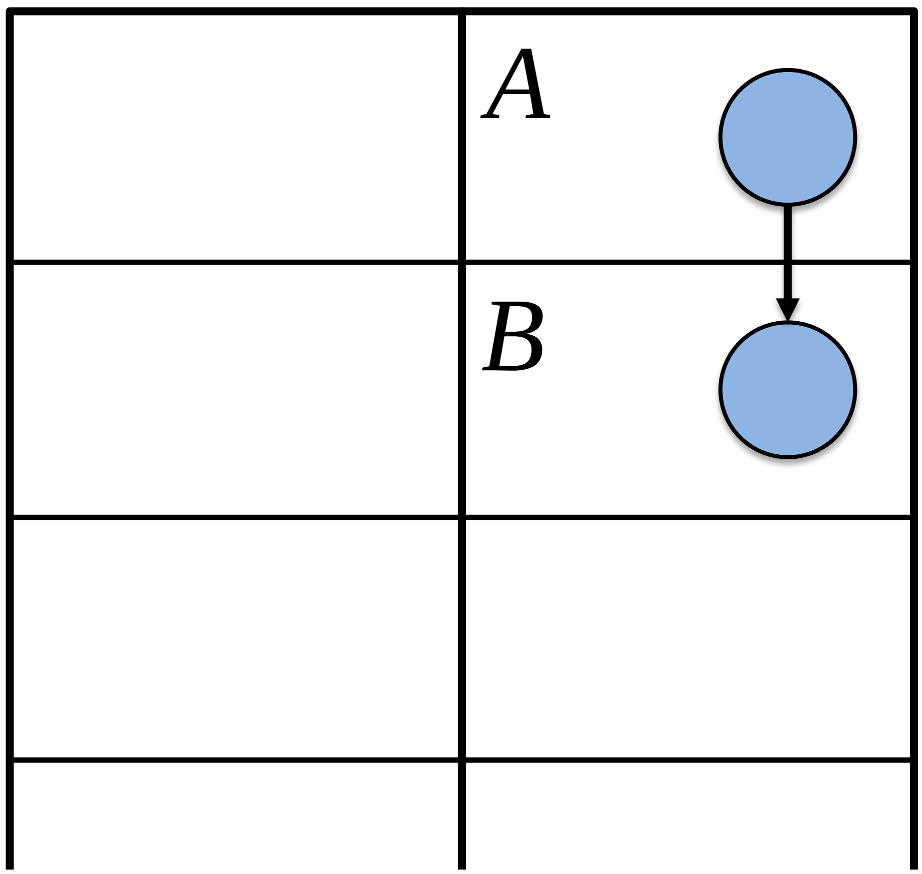
Water flow = causality?



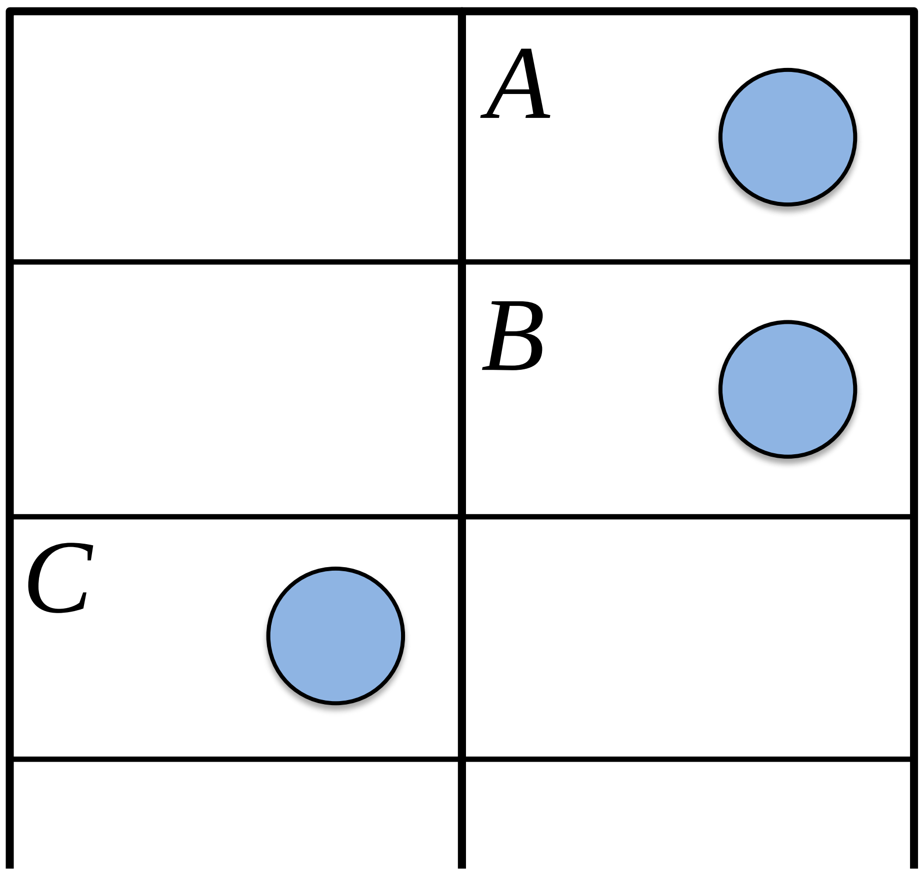
Water flow = causality?



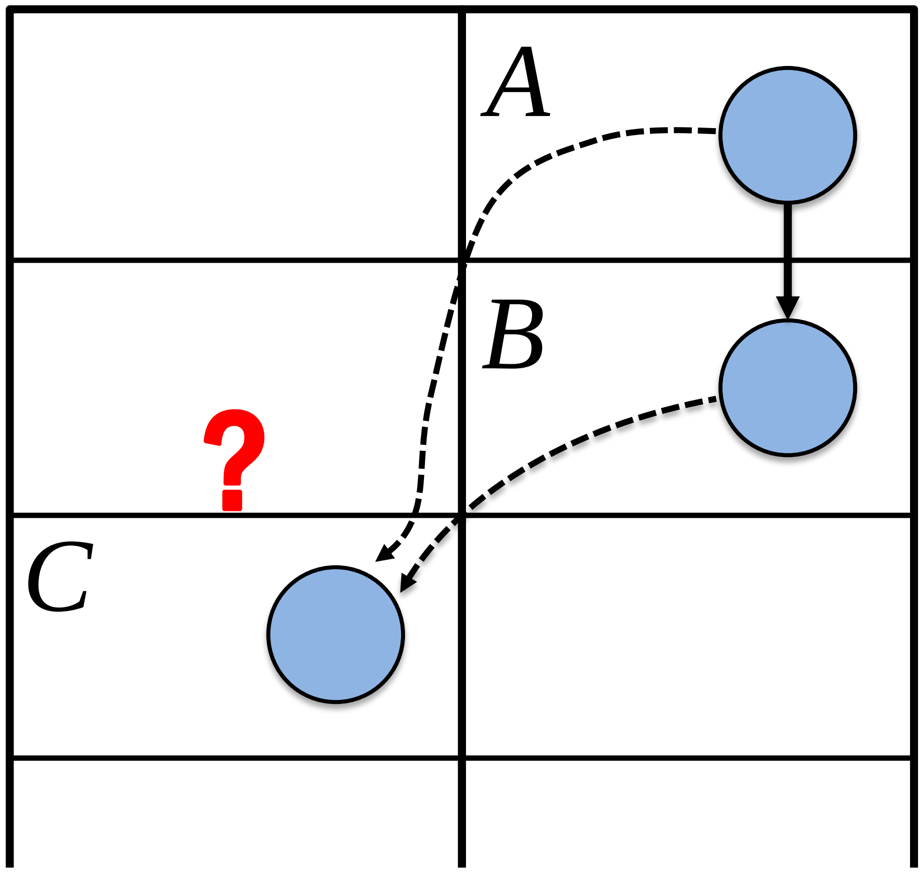
Water flow = causality?



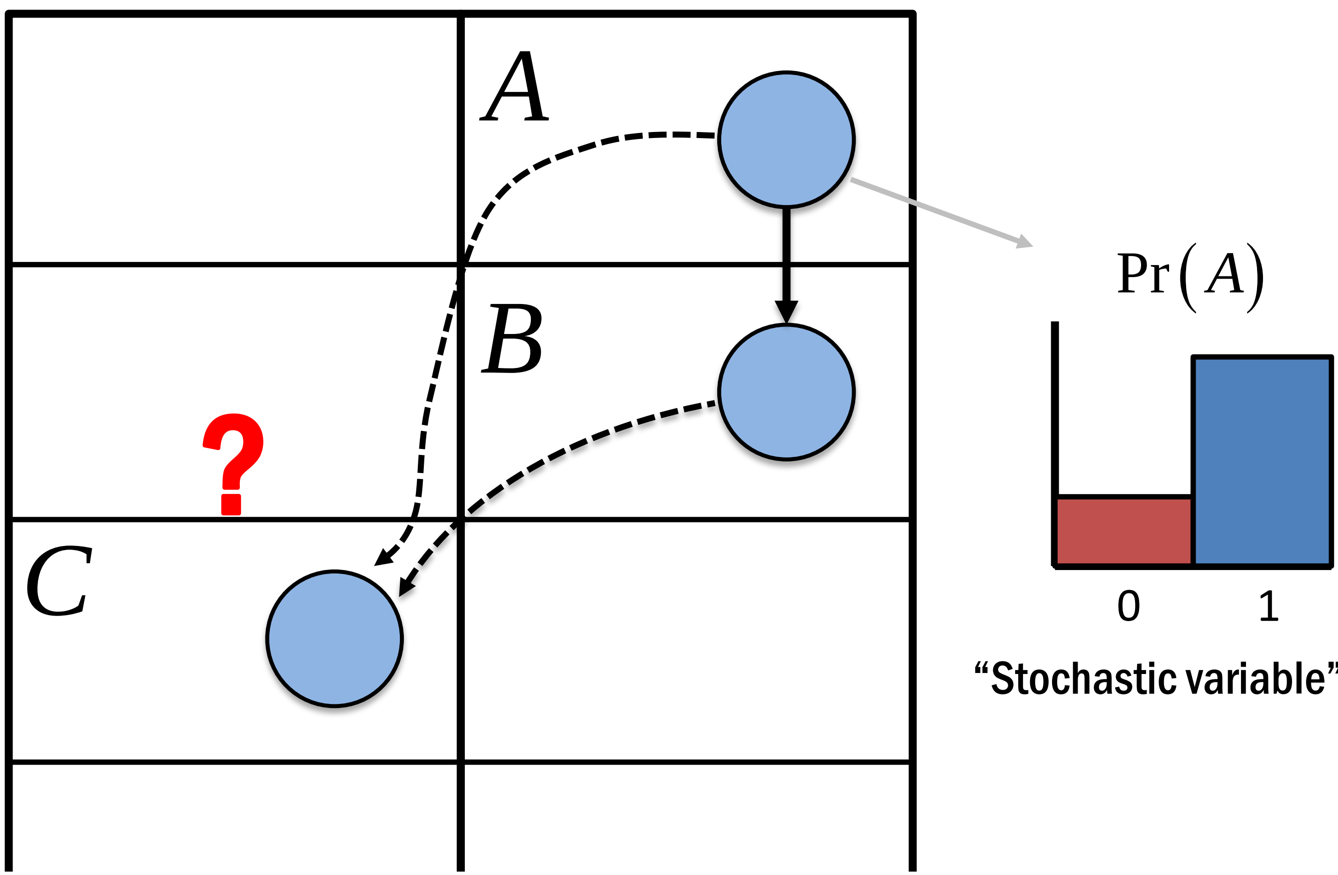
Water flow = causality?



Water flow = causality?

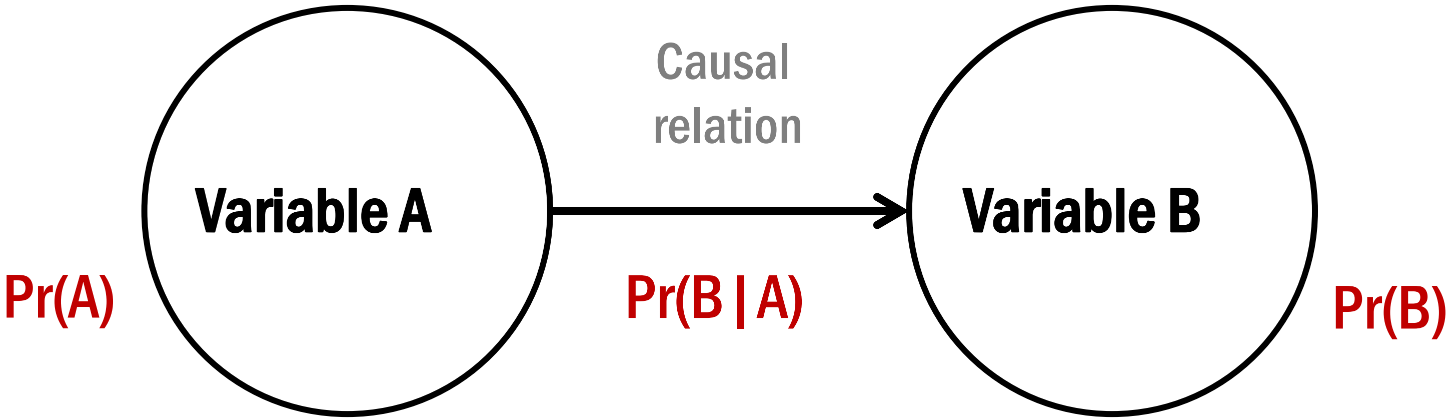


Water flow = causality?



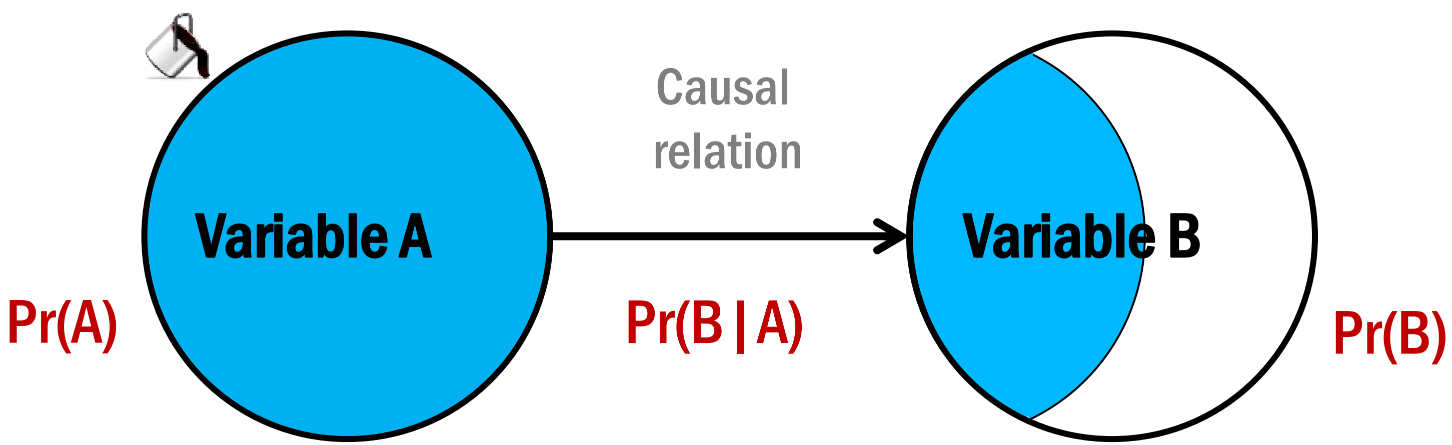
Research question

- Suppose an isolated model A→B where one stochastic variable A influences another B
- P(B | A) encodes the full causality relation



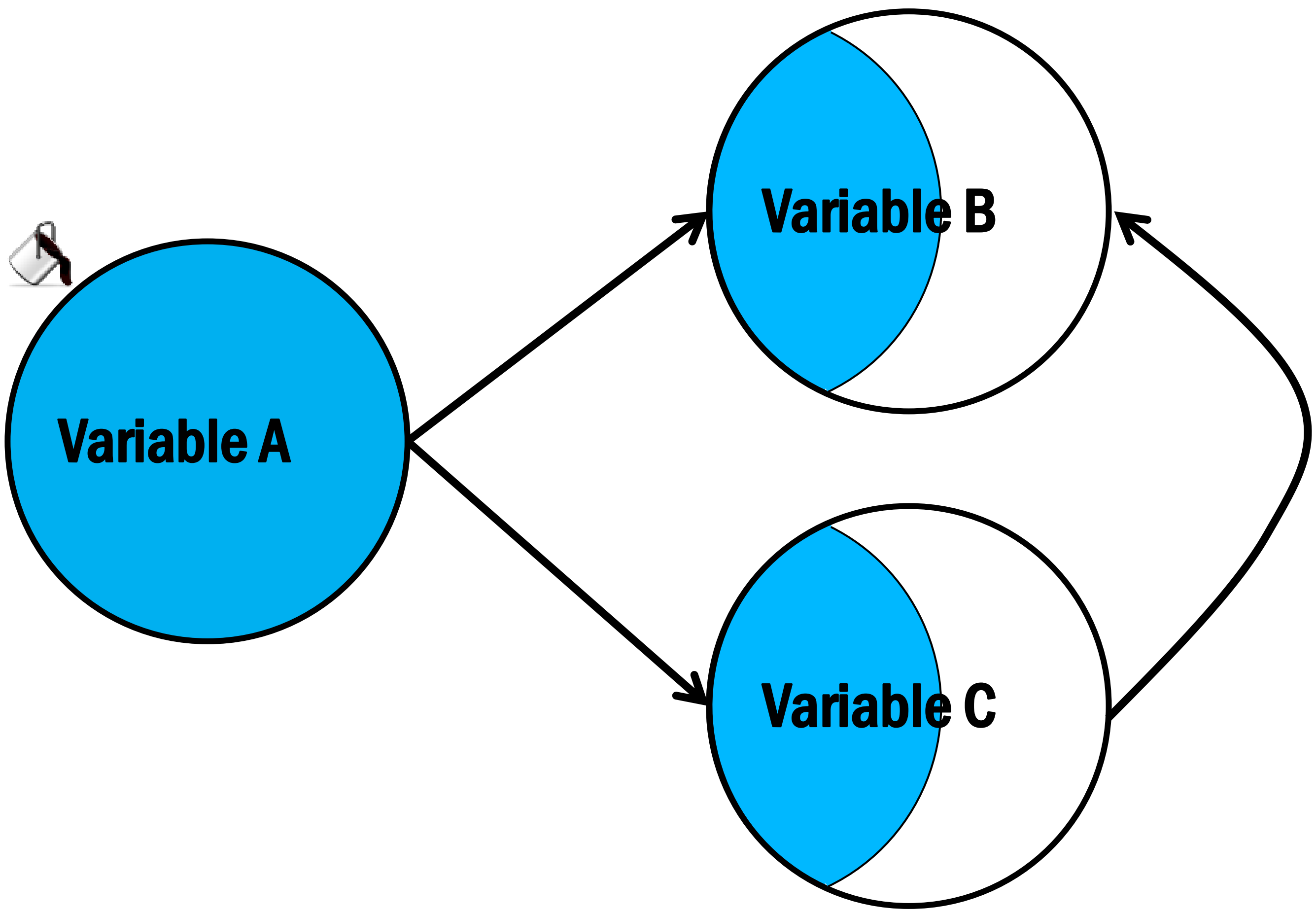
Research question

- Suppose an isolated model A→B where one stochastic variable A influences another B
- P(B | A) encodes the full causality relation
- **Find a mathematical 'indicator' fluid to detect causality**

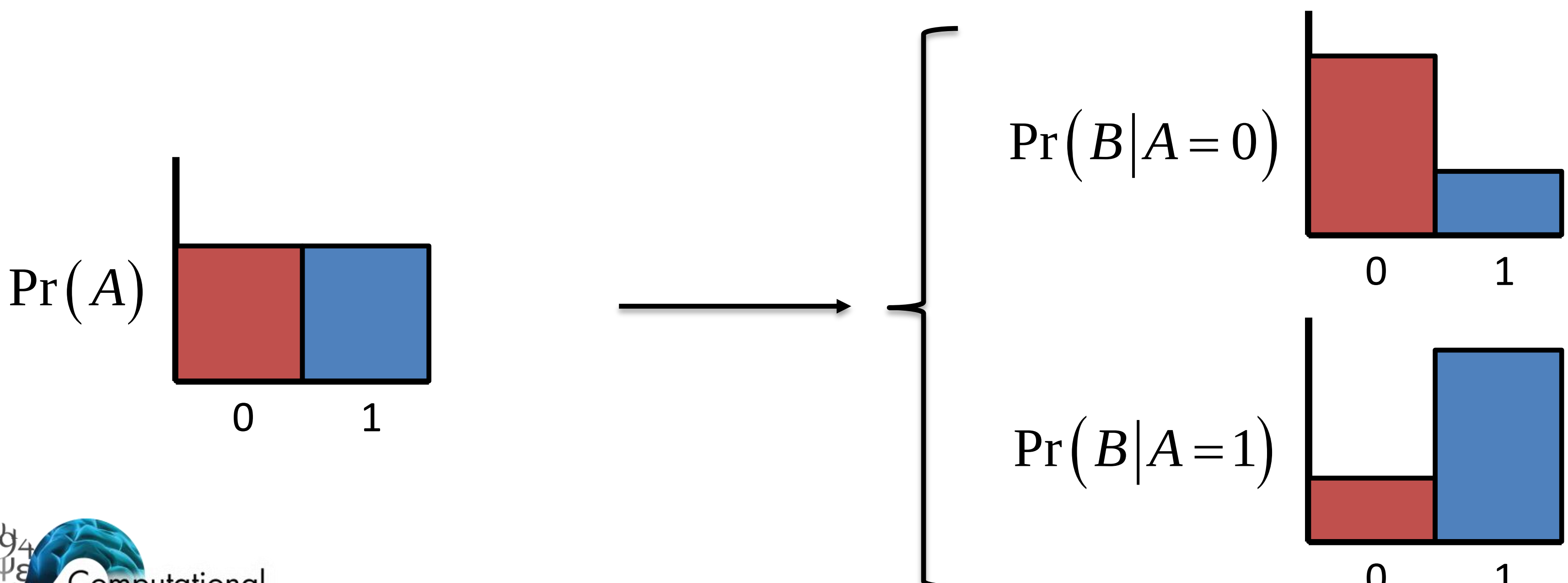


Major problem

- Non-causal **correlations!**

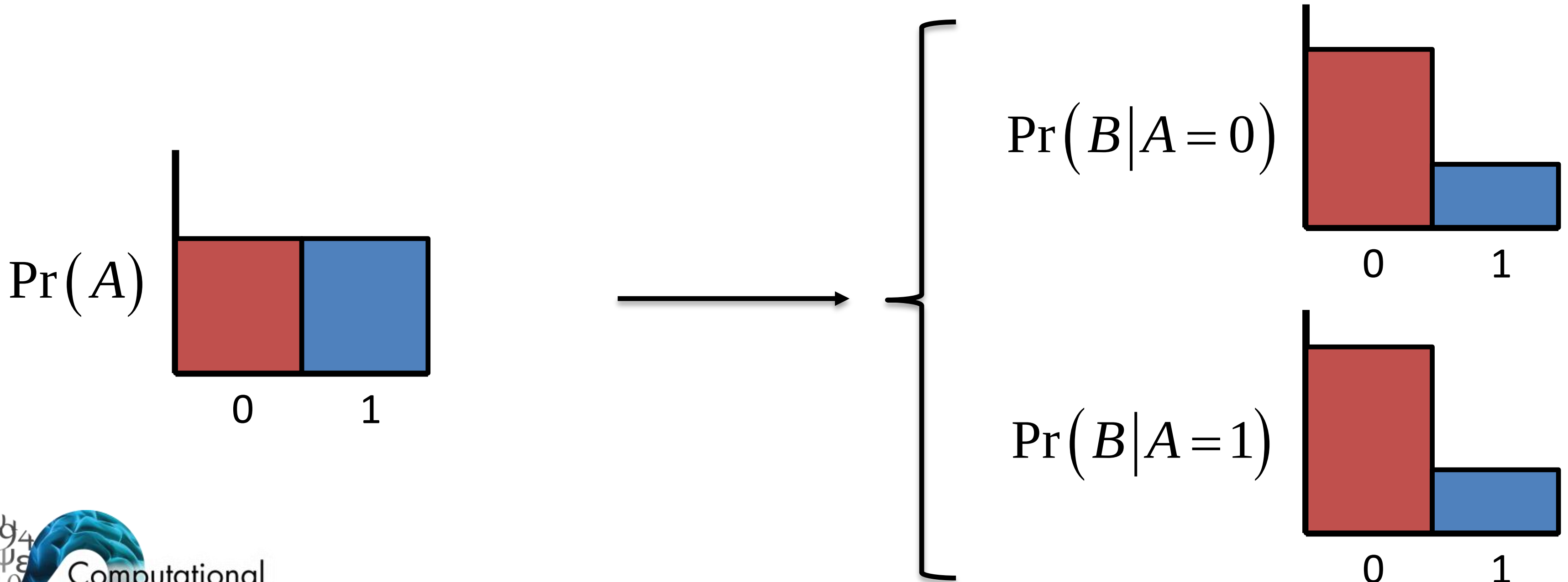


Probabilistic causal relation



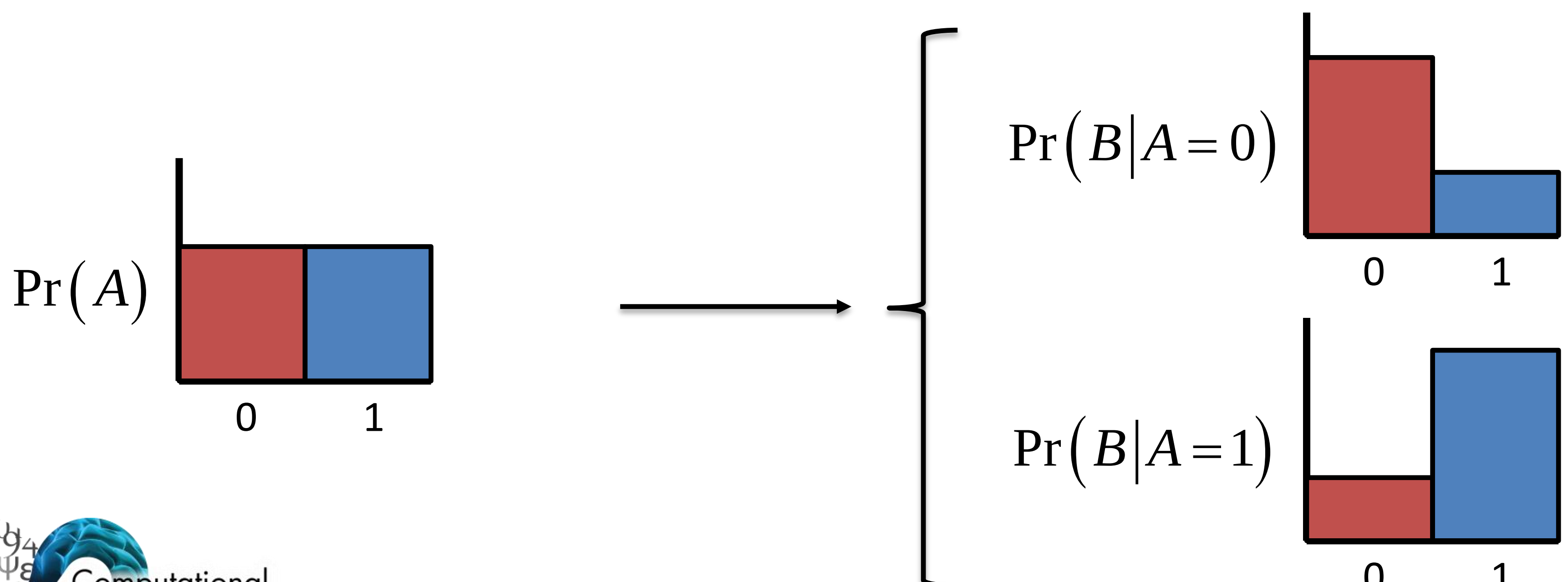
Probabilistic causal relation

Zero causal effect



Probabilistic causal relation

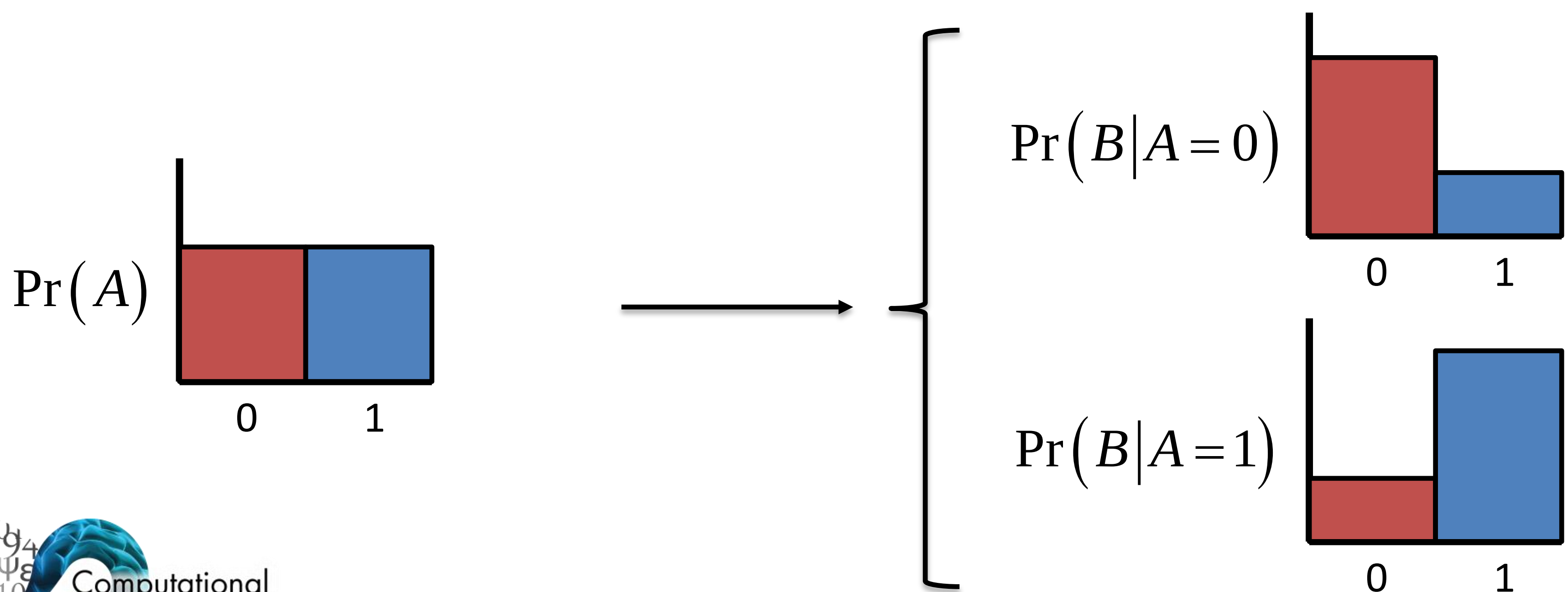
Non-zero causal effect



Probabilistic causal relation

Kullback-Leibler divergence

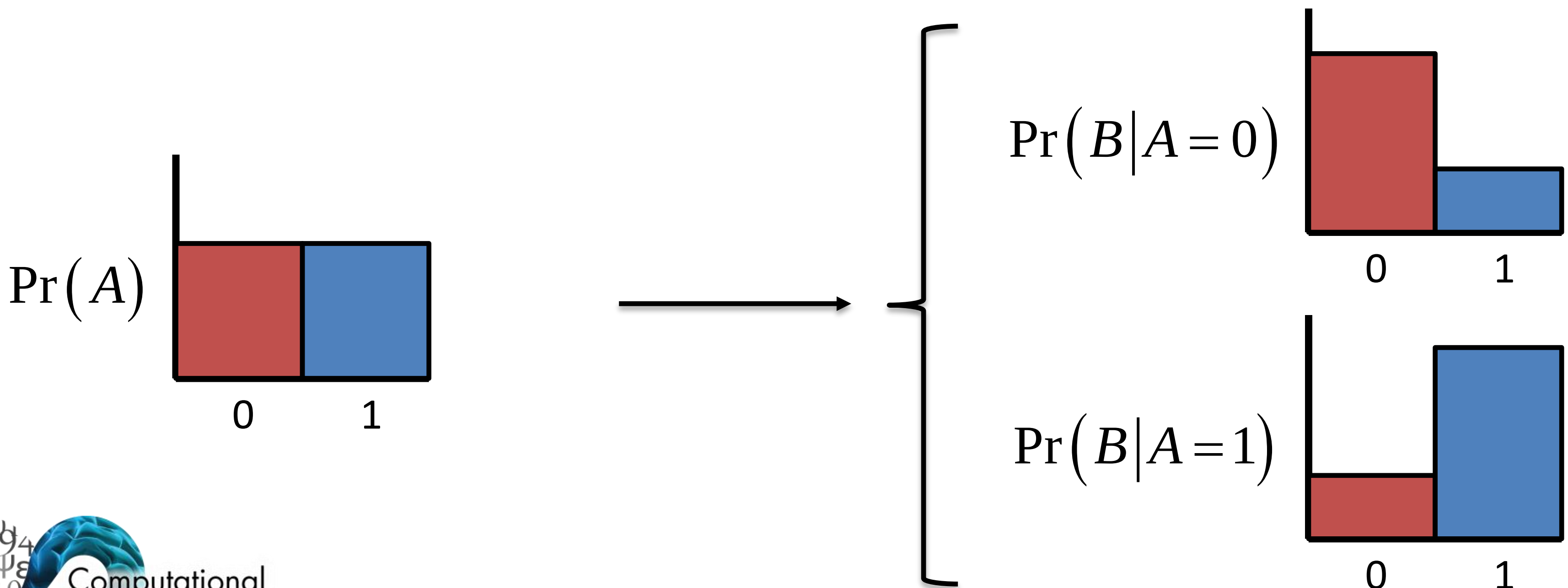
$$\mathrm{E}_A\left[D_{\mathrm{KL}}\left(\Pr(B \mid A=a) \,\|\, \Pr(B)\right)\right]$$



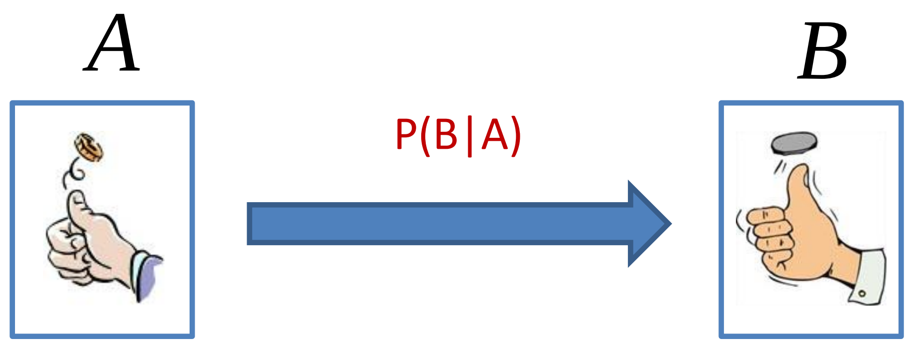
Probabilistic causal relation

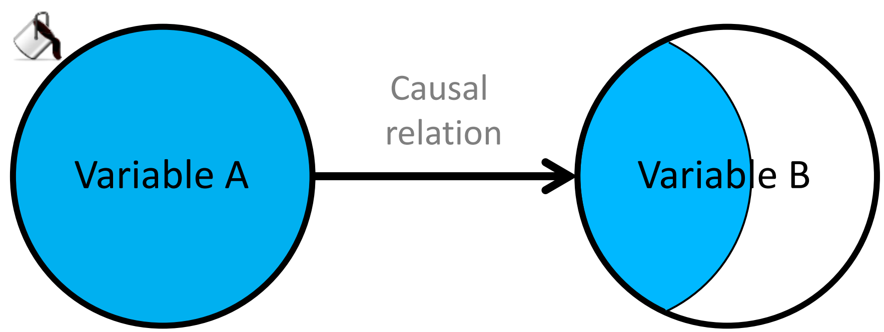
Kullback-Leibler divergence → Mutual information

$$E_A\left[D_{\mathrm{KL}}\left(\Pr(B \mid A=a) \,\|\, \Pr(B)\right)\right] \quad = \quad I(A:B)$$



Causality → information flow





$I(A:A)$ $I(A:B)$

“Entropy” **“Mutual information”**

Entropy of a coin flip

$A =$ 

$$\begin{cases} 0 & \Pr(A=0)=0.5 \\ 1 & \Pr(A=1)=0.5 \end{cases}$$

$\Rightarrow$ Carries **1 bit** of information

Entropy of a coin flip

$A =$  $\begin{cases} 0 \\ 1 \end{cases}$ $\Pr(A=0)=0.5$, $\Pr(A=1)=0.5$ → **Carries 1 bit of information**

$A =$  $\begin{cases} 0 \\ 1 \end{cases}$ $\Pr(A=0)=0$, $\Pr(A=1)=1$ → **Carries 0 bits of information**

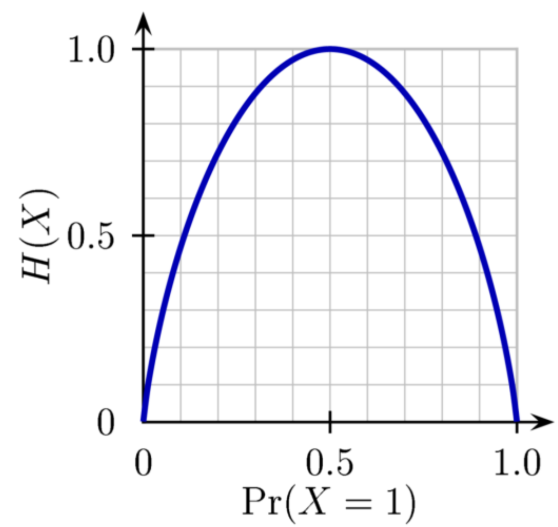
Entropy of a coin flip

$A =$  $\begin{cases} 0 & \Pr(A=0)=0.5 \\ 1 & \Pr(A=1)=0.5 \end{cases}$ → **Carries 1 bit of information**

$A =$  $\begin{cases} 0 & \Pr(A=0)=0 \\ 1 & \Pr(A=1)=1 \end{cases}$ → **Carries 0 bits of information**

In general:

$$H(A) = \sum_{a \in \{0,1\}} \Pr(A=a) \cdot \log_2 \frac{1}{p(A=a)}$$



Mutual information between coins

A

Transform

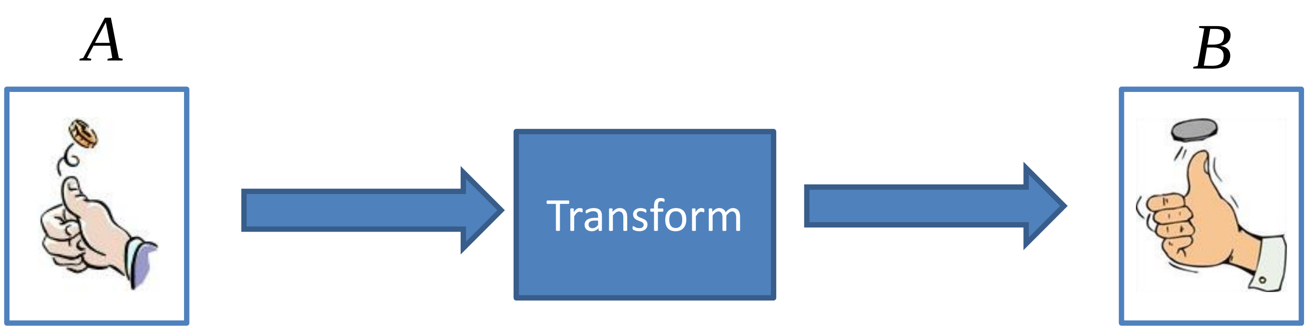
B

$$\begin{cases} \Pr(A=0)=0.5 \\ \Pr(A=1)=0.5 \end{cases}$$

$$\Pr(B \mid A)$$

$$\Pr(A \mid B) \quad ?$$

Mutual information between coins



$$\begin{cases} \Pr(A=0)=0.5 \\ \Pr(A=1)=0.5 \end{cases}$$

$$\Pr(B \mid A)$$

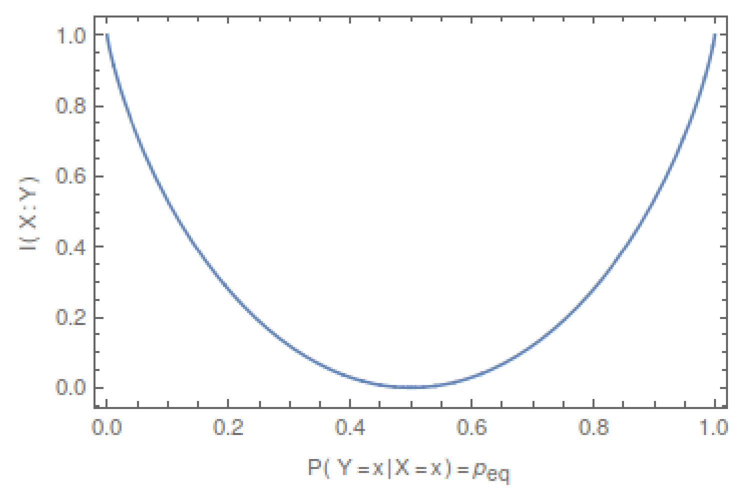
$$\Pr(A \mid B) \quad ?$$

$$I(X:Y)=\sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)p(y)}$$

$\Pr(B=x \mid A=x)=1$ → **1 bit transferred**

$\Pr(B=x \mid A=x)=1/2$ → **0 bits transferred**

$\Pr(B=x \mid A=x)=p_{\text{eq}}$ →



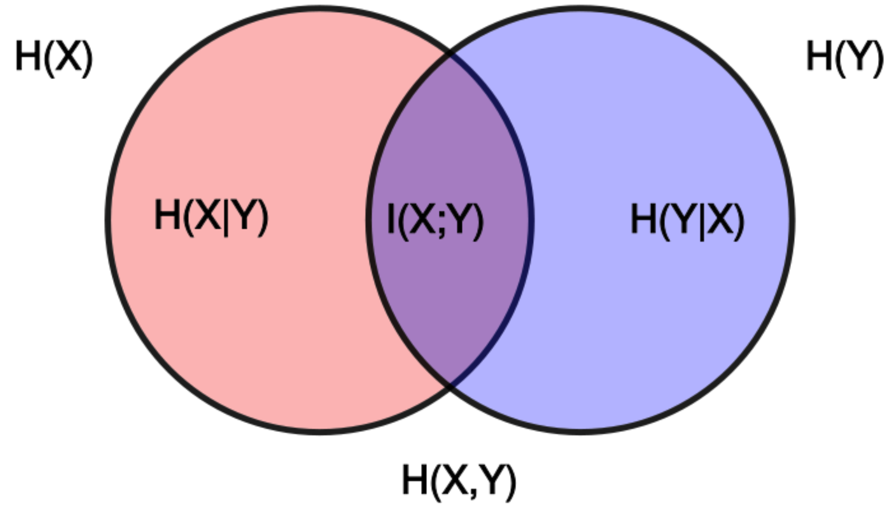
Summary of information theory

$$H(X) = -\sum_{X=x} p(x) \log p(x)$$

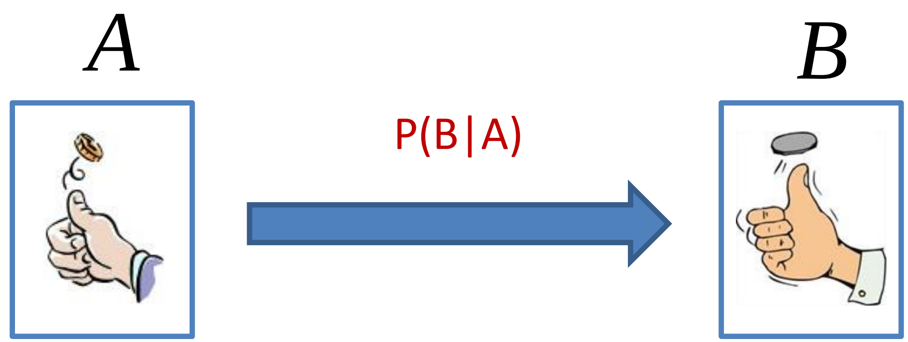
"Entropy"

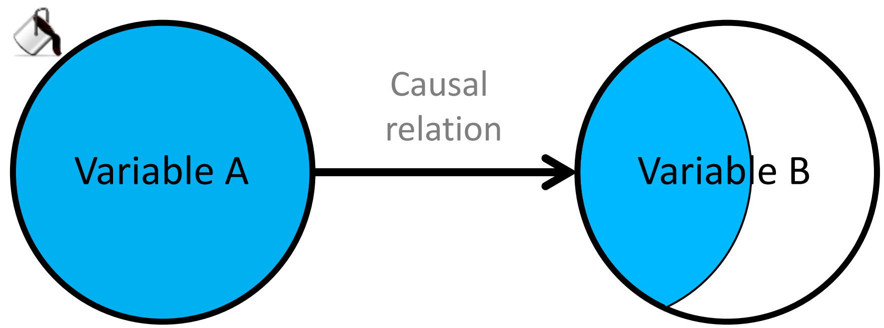
$$I(X:Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)},$$
$$= H(X) - H(X|Y).$$

"Mutual information"



Causality → information flow





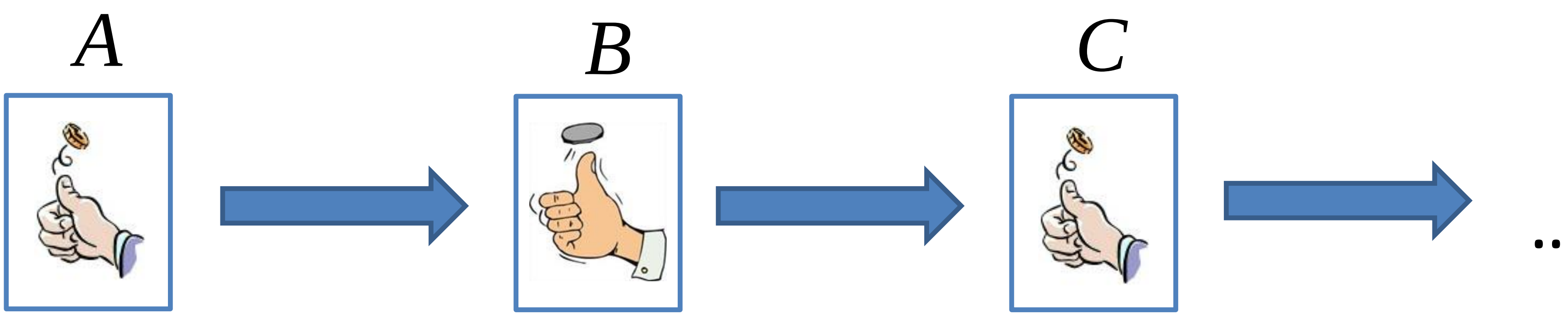
$$H(A) = I(A:A)$$

$$I(A:B)$$

"Entropy"

"Mutual information"

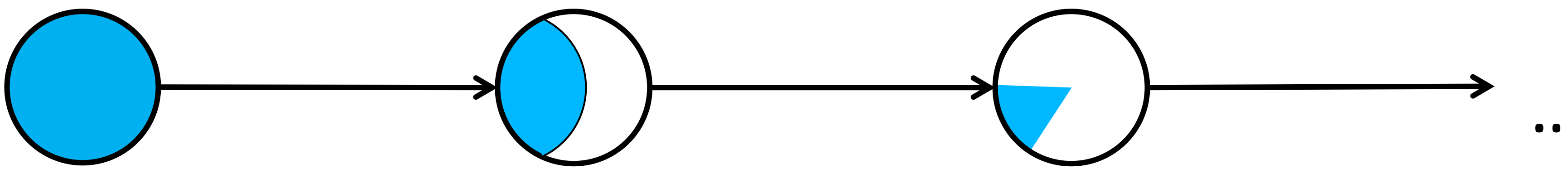
Chain of interactions



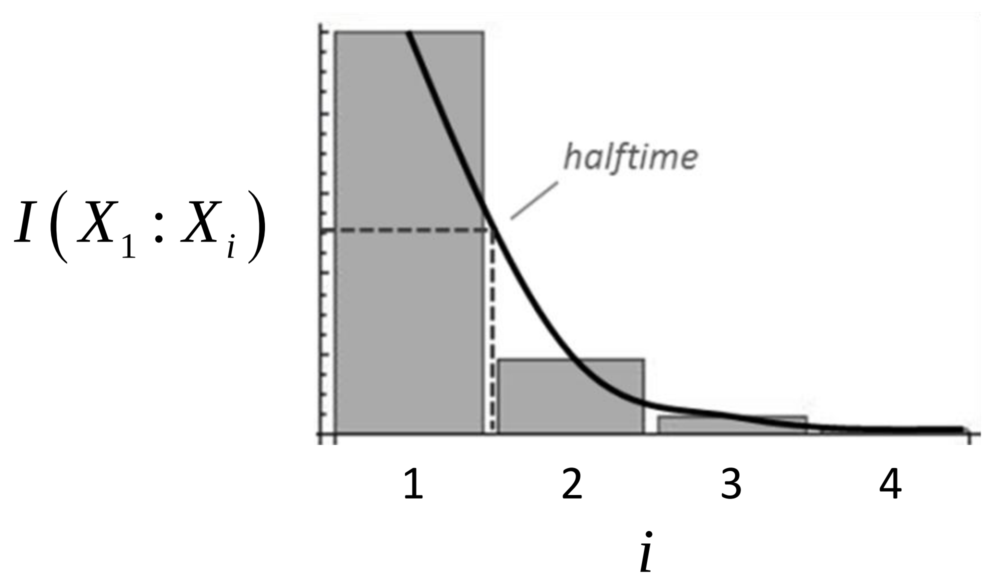
$$\Pr(A) \longrightarrow \Pr(B|A) \longrightarrow \Pr(C|B) \longrightarrow \ldots$$

$$\begin{cases} 1/2 & A=0 \\ 1/2 & A=1 \end{cases} \longrightarrow \begin{cases} \alpha & B=A \\ 1-\alpha & B\neq A \end{cases} \longrightarrow \begin{cases} \alpha & C=B \\ 1-\alpha & C\neq B \end{cases} \longrightarrow \ldots$$

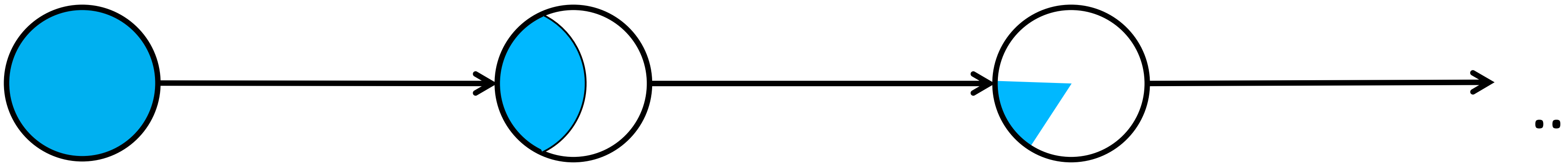
Information dissipation length



$$I(X_1 : X_1) \longrightarrow I(X_1 : X_2) \longrightarrow I(X_1 : X_3) \longrightarrow \ldots$$



Information dissipation length



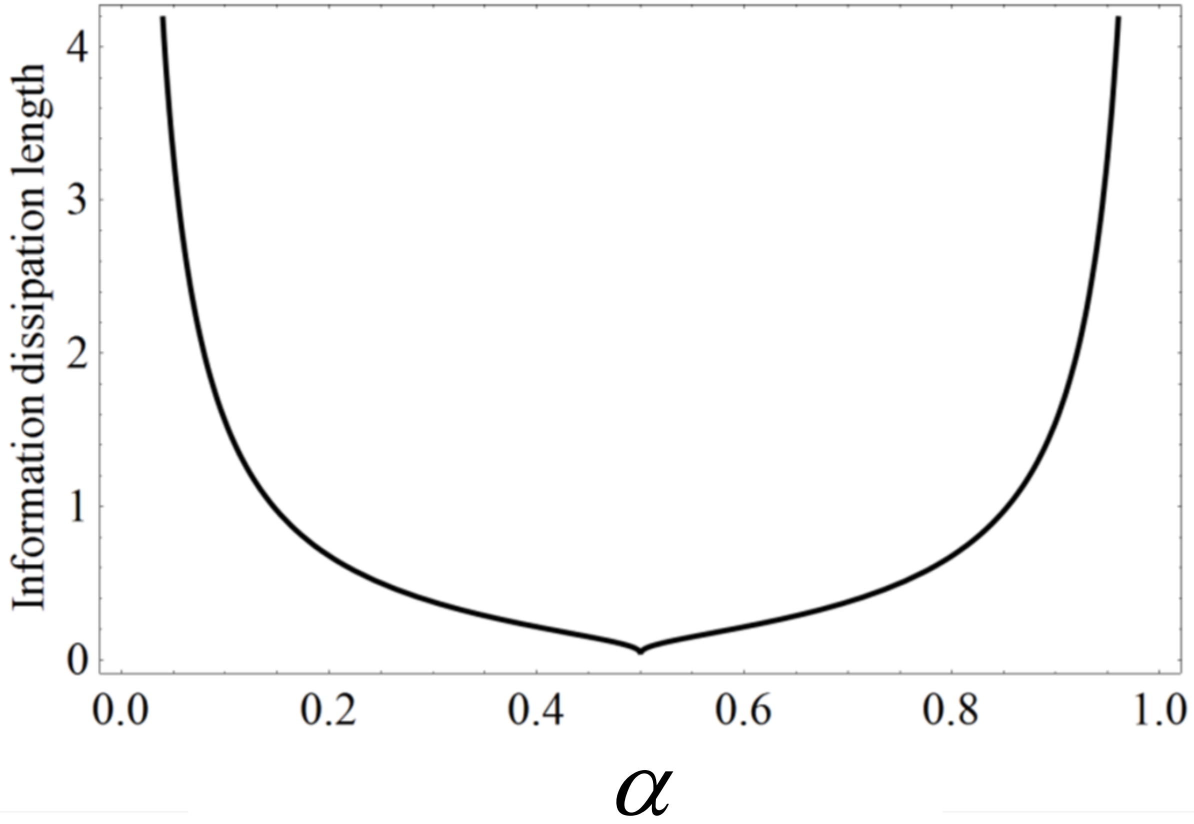
$$f \equiv \lim_{i\to\infty} \frac{I(X_1 : X_{i+1})}{I(X_1 : X_i)} = (2\alpha - 1)^2$$

$$I(X_1 : X_1) = 1 \longrightarrow I(X_1 : X_2) = f \longrightarrow I(X_1 : X_3) = f^2 \longrightarrow \quad \ldots$$

When is $f^n \stackrel{?}{=} \frac{1}{2}$

Information dissipation length

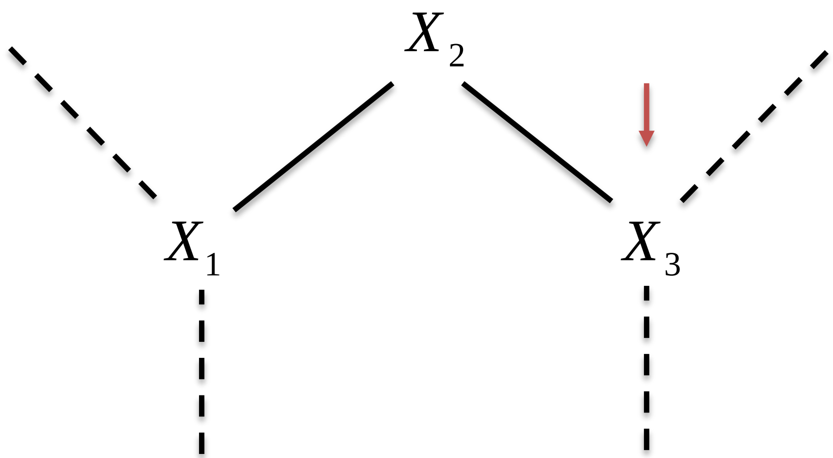
$$\log_f \frac{1}{2}$$



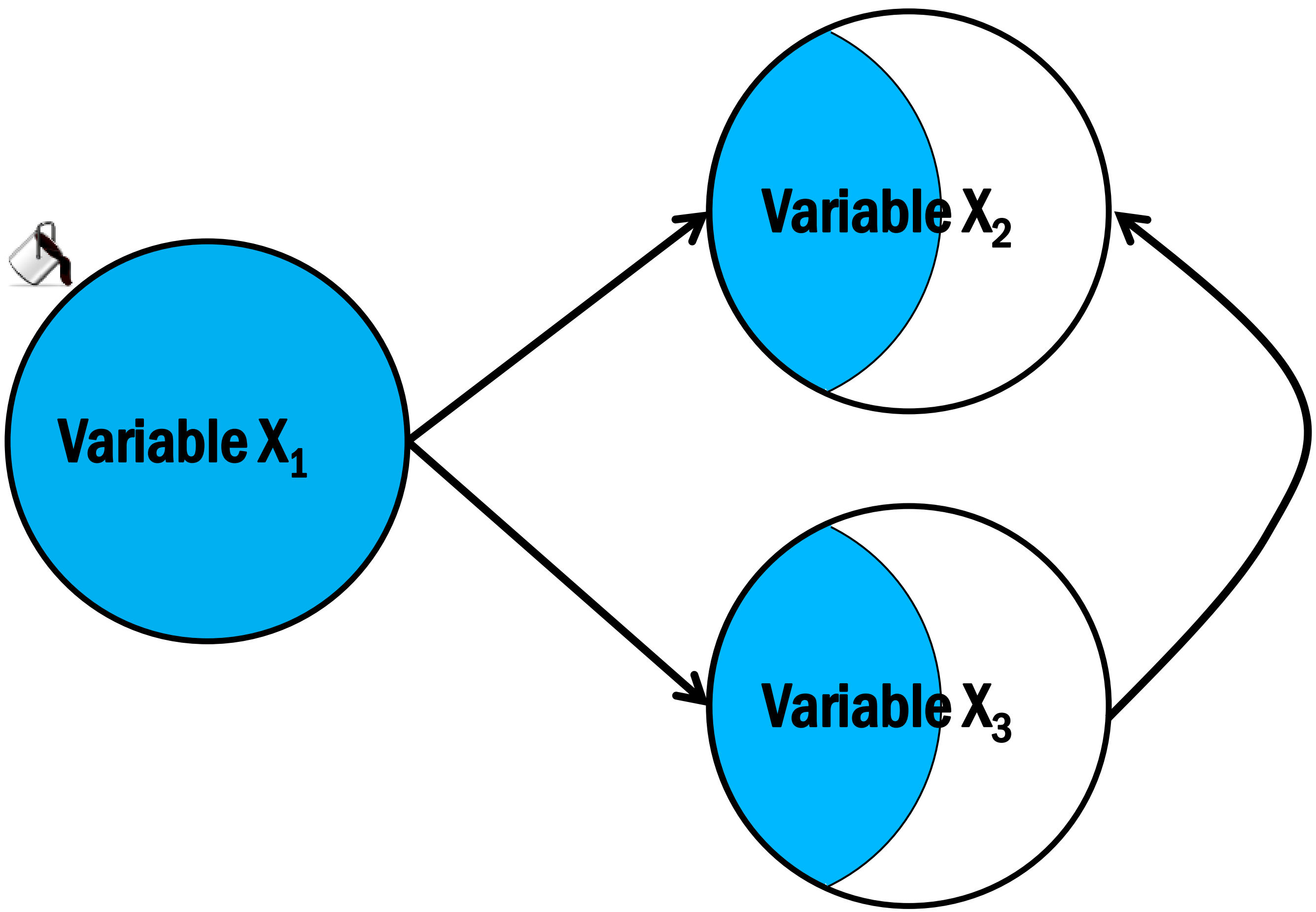
Information dissipation…

NOW TO NETWORKED SYSTEMS

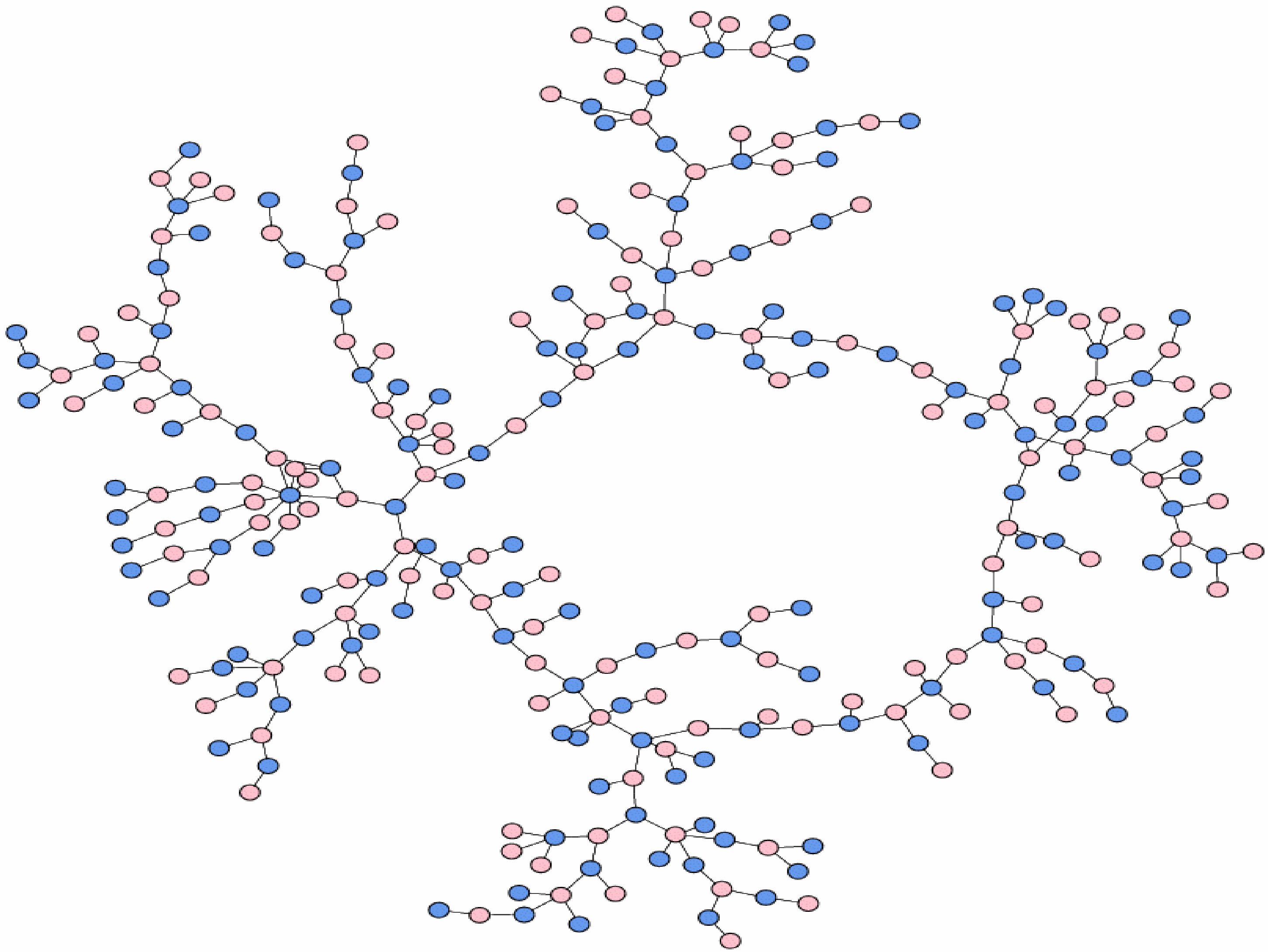
Network of variables



Beware!



No short loops



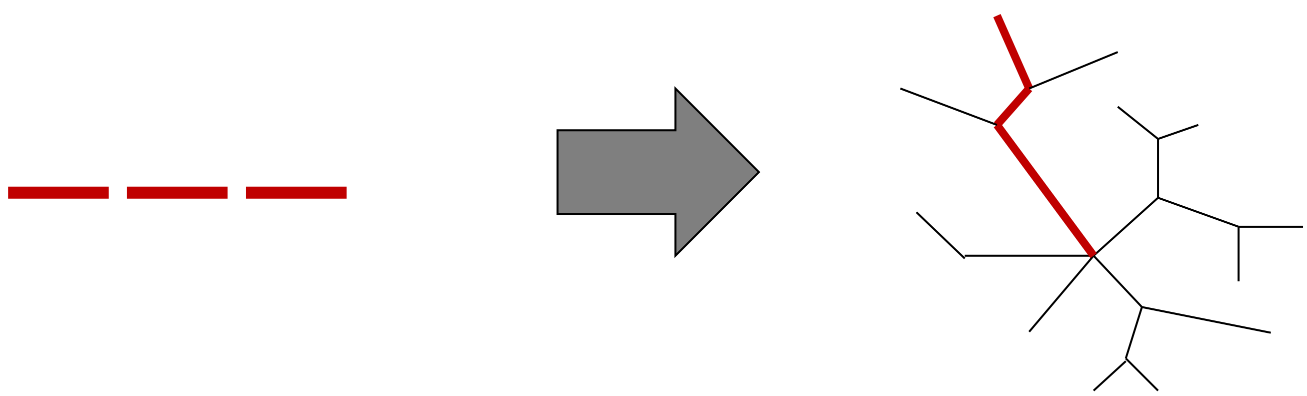
Interface

The diminishing role of hubs in dynamical processes on complex networks

Rick Quax[1,†], Andrea Apolloni[2,†] and Peter M. A. Sloot[1,3,4]

Network

- Very large networks
  - Locally tree-like (i.e., no short loops)
    - E.g., large and
      no community-structure / modularity
    - **Any degree distribution** can be chosen



Network

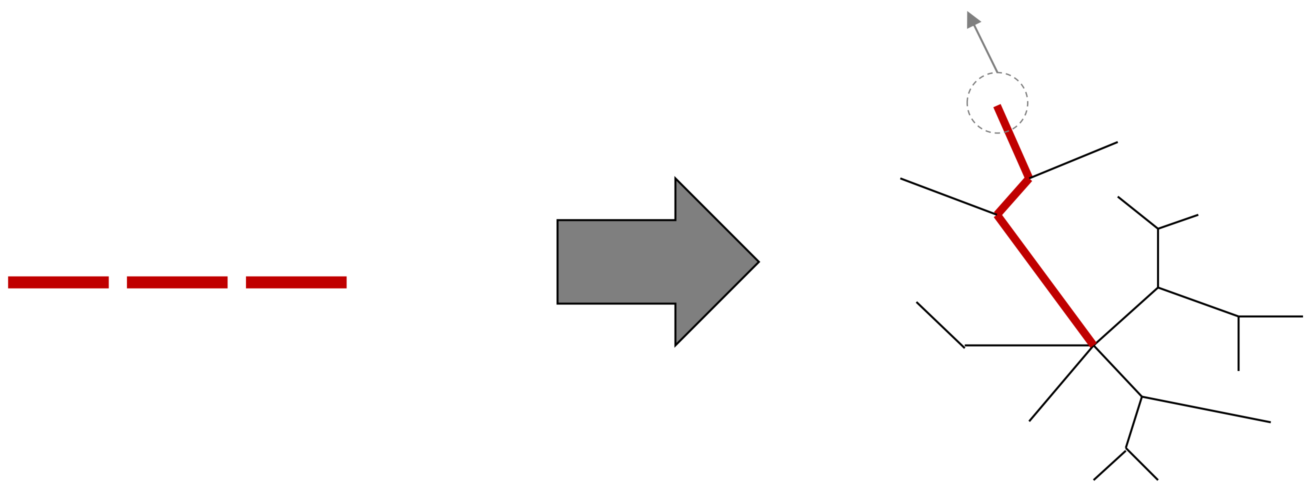
The diminishing role of hubs in dynamical processes on complex networks

Rick Quax[1†], Andrea Apolloni[2,†] and Peter M. A. Sloot[1,3,4]

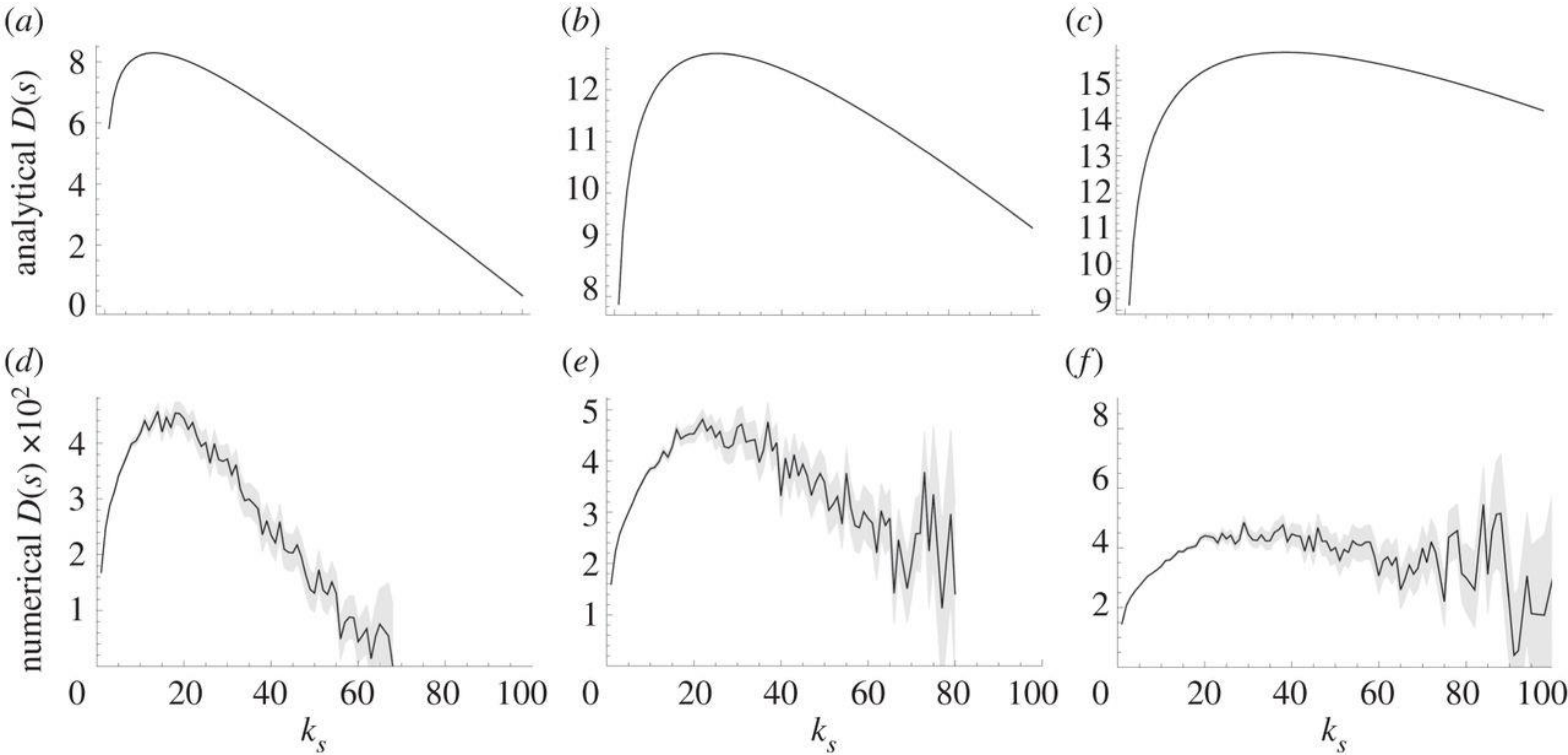
- Network structure
  - Locally tree-like (i.e., no short loops)
    - E.g., large and
      no community-structure / modularity
    - **Any degree distribution** can be chosen

Generalized energy function

$$p(s_i^{t+1} = x \mid s_j^t, \ldots) \propto \exp \sum_j -E(x, s_j^t)$$



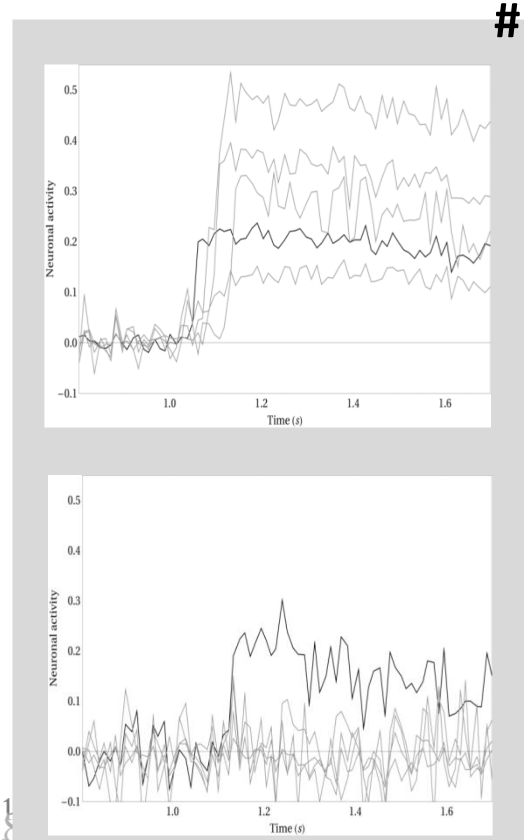
Not the *influentials* but the *man in the street* drives change



Qualitative evidence from experiments

Network of neurons cultured in a Petri dish

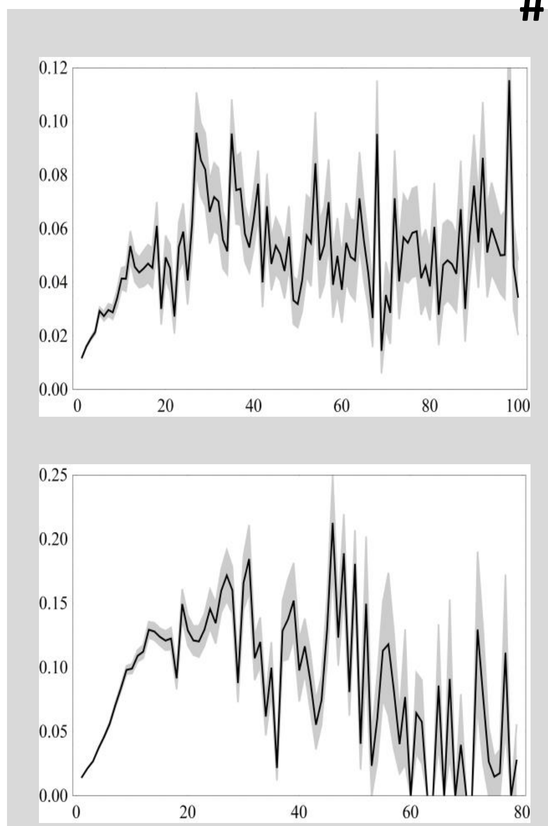
#1



Ivenshitz & Segal (2010)

Social network of word-of-mouth marketing

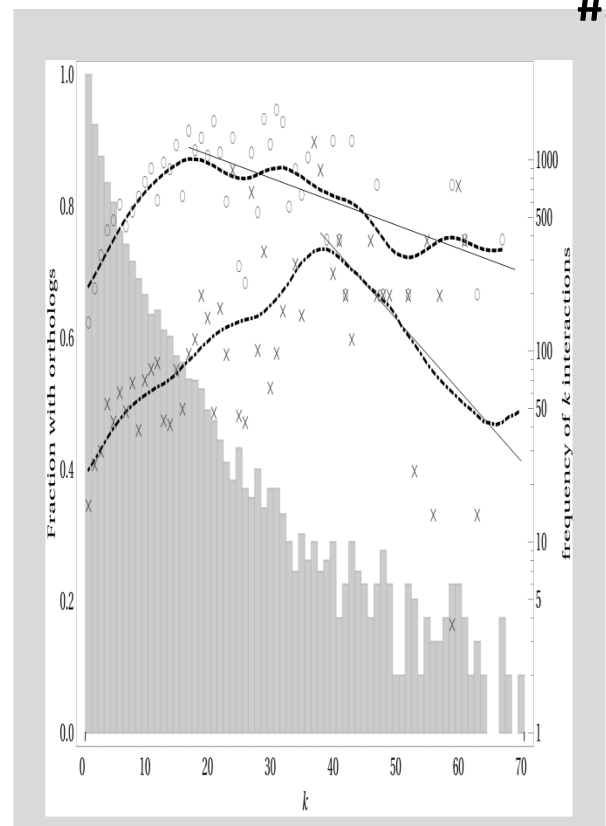
#2



Leskovec et al. (2007)

Gene regulation network

#3



Brown & Jurisica (2007)

Conclusion

- The strength of a causal relation can be quantified using information theory
- The major hurdle to take (or avoid) is correlation
- Locally tree-like networks avoid the hurdle
- The man-in-the-street drives the system behavior for a particular class of dynamics (not the hubs)

Software: https://bitbucket.org/rquax/jointpdf

Quiz

1. Does a small nudge always result in a small effect?
2. If $H(A)=0$ can A then still have causal influence on B?