Nonlinear dynamics

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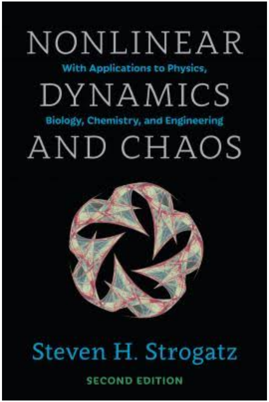
Outline



Reference: S. Strogatz, “Nonlinear dynamics and chaos” (1994)

1. Introduction: what is nonlinear dynamics and what has it got to do with complexity science?
2. Basic terminology: phase space, attractors, stability.
3. Overview of some essentially nonlinear phenomena:
   - Bifurcations and tipping points
   - Multistability
   - Hysteresis
   - Nonlinear oscillations

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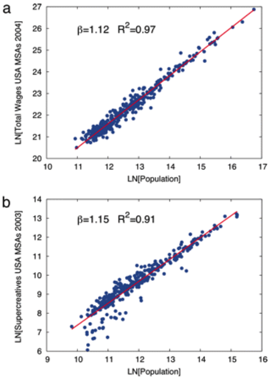
What is nonlinearity?



Superlinear scaling laws in socio-economic data, Bettencourt et al., PNAS (2007)

A nonlinear function is a function which is not linear - the output is not proportional to the input.

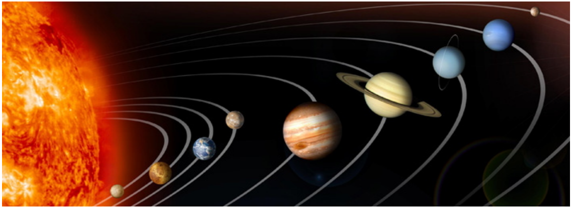
Simplest example:

$$f(x) = x^{\alpha}$$

Linear case is $\alpha = 1$. Linearity is special. Nonlinearity is generic.

Nonlinearity is *not* mystical.

What is dynamics?



Dynamics is a set of rules which define the evolution of a system in terms of its current state.

Discrete dynamics defined in terms of recurrence relations. Example: the quadratic map:

$$z_{n+1} = z_n^2 + c.$$

Continuous dynamics defined in terms of differential equations. Example: Newton's law of motion:

$$m\,\ddot{x} = F(x, \dot{x})$$

Turbulence: Complicated nonlinear equations can have complicated solutions

Two dimensional turbulence in a soap film.

Navier-Stokes eqs for fluid velocity, $\mathbf{v}(\mathbf{x}, t)$, and pressure, $p(\mathbf{x}, t)$:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= -\nabla p + \nu \nabla^2 \mathbf{v} \\ \nabla \cdot \mathbf{v} = 0 & \end{aligned}$$

Turbulence: onset of spatiotemporally disordered flow as $\nu \to 0$.

Complicated: number of unknowns is "infinite" (a function, $\mathbf{v}(\mathbf{x}, t)$).

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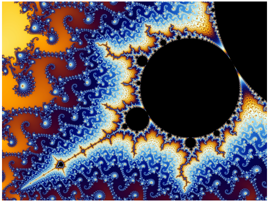
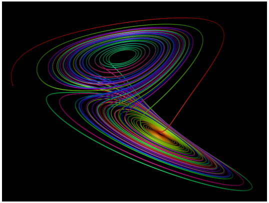
Chaos: Simple nonlinear equations can have complicated solutions



The Mandelbrot set.



Visualisation of the Lorenz attractor.

Quadratic map:

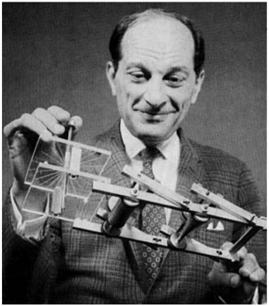
$$z_{n+1} = z_n^2 + c.$$

Lorenz system:

$$\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= x\,(\rho - z) - y \\ \dot{z} &= x\,y - \beta\,z. \end{aligned}$$

1980's revolution: "simple" (low dimensional) nonlinear equations can have very complicated solutions.

Nonlinear dynamics and chaos



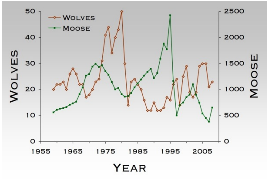
Stan Ulam

"to call the study of chaos non-linear science is like calling zoology the study of non-elephant animals"

- Stanislav Ulam?

A similar sentiment applies to the study of nonlinear dynamics itself.

Nonlinear dynamics and complexity science



Data on predator-prey relationship in a simple food web.

Communicable diseases can be modeled with simple dynamical systems.

Complex systems are high dimensional. Low dimensional dynamical systems are often *models of average properties* of complex systems.

Lotka-Volterra model (ecology):

$$\begin{aligned} \dot{x} &= \alpha x - \beta x y \\ \dot{y} &= -\gamma y + \beta x y. \end{aligned}$$

SIR model (epidemiology):

$$\begin{aligned} \dot{S} &= -\beta\, S\, I \\ \dot{I} &= \beta\, S\, I - \gamma\, I \\ \dot{R} &= \gamma\, I. \end{aligned}$$

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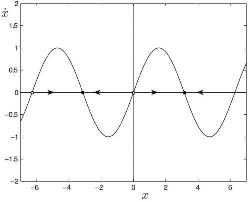
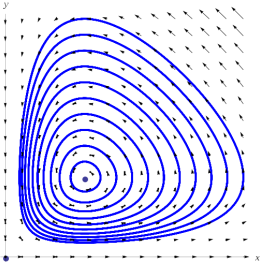
Phase space, trajectories and parameters



Phase space of a one dimensional dynamical system.



Phase space of the Lotka-Volterra model.

One-dimensional case:

$$\dot{x} = sin(\omega\, x)$$

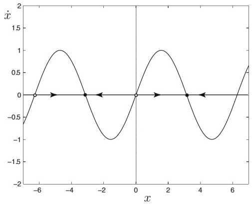
Phase space: line. Trajectories: $x(t)$. Parameters: ω.

Two-dimensional case:

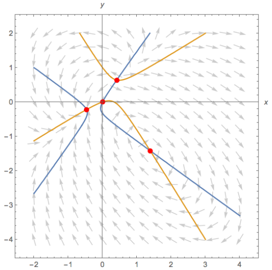
$$\begin{aligned} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= -\gamma y + \beta xy. \end{aligned}$$

Phase space: x-y plane. Trajectories: curves $(x(t), y(t))$. Parameters: α, β and γ.

Fixed points



Fixed points in one dimension.



Nullclines and fixed points in two dimensions.

Fixed points are points in the phase space where the trajectory becomes stationary.

For a one-dimensional system,

$$\dot{x} = f(x),$$

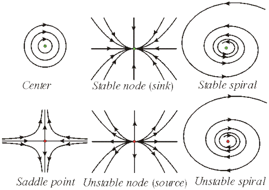
fixed points are found by finding the roots of the equation $f(x) = 0$. For a two-dimensional system,

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

fixed points are found by finding the intersections of the **nullclines** $\dot{x} = f(x, y) = 0$ and $\dot{y} = g(x, y) = 0$.

Stable and unstable fixed points

An unstable equilibrium.



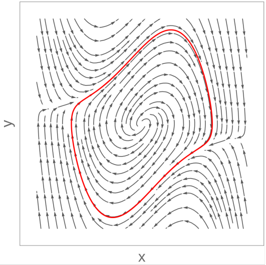
Types of stable and unstable fixed points in a two dimensional dynamical system.

A fixed point, P, is **stable** if trajectories starting close to P approach closer to P.

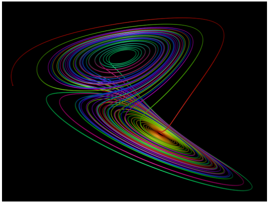
A fixed point, P, is **unstable** if trajectories starting close to P move away from P.

Stability in one dimension is boring. In two dimensions there are multiple types of stable and unstable fixed points.

Invariant sets and attractors



Limit cycle in the phase space of the Van der Pol oscillator.



Strange attractor in the Lorenz system.

A subset, S, of the phase space is called an **invariant set** if trajectories starting in S stay in S forever.

An invariant set, A, is an **attractor** if all trajectories starting near A approach closer to A.

A fixed point is an invariant set but there are more interesting higher dimensional examples:

- A closed curve in the phase space is called a **limit cycle**.
- Fractal invariant sets occuring in chaotic dynamical systems are called **strange attractors**.

Bifurcations

Existence, location and stability of invariant sets depend on parameters of the dynamical system.
A bifurcation is a change in the qualitative structure of a dynamical system as parameters are varied.

- **Local bifurcation:** a parameter change causes the stability of a fixed point to change.
- **Global bifurcation:** a parameter change causes a "larger" invariant set, such as a limit cylce, collide with a fixed point.

We will look at some examples of local bifurcations.

Types of bifurcations 1: Saddle-node (or fold) bifurcation and tipping points

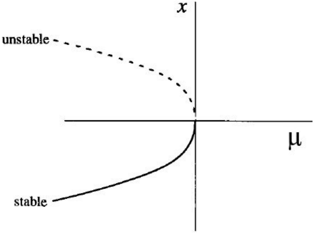
Model example:

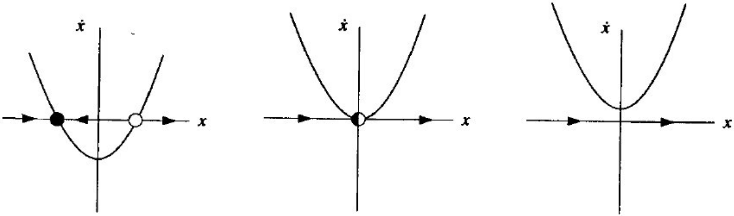
$$\dot{x} = \mu + x^2$$

Fixed points:

$$\begin{aligned} \mu + x^2 &= 0 \\ \Rightarrow x &= \pm\sqrt{-\mu} \end{aligned}$$

Bifurcation diagram:





(a) $\mu < 0$ (b) $\mu = 0$ (c) $\mu > 0$.

Tipping point at $\mu = 0$.

Types of bifurcations 2: (Supercritical) pitchfork bifurcation and multistability

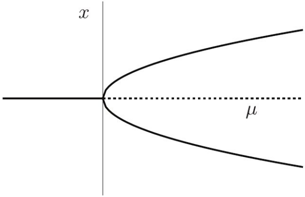
Model example:

$$\dot{x} = \mu x - x^3$$

Fixed points:

$$\begin{aligned} \mu x - x^3 &= 0 \\ \Rightarrow x\left(\mu - x^2\right) &= 0 \\ \Rightarrow x &= 0 \;\text{ or }\; x = \pm\sqrt{\mu} \end{aligned}$$

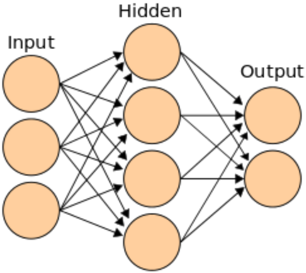
Bifurcation diagram:



For $\mu > 0$ observe the phenomenon of **multistability**: there is more than one stable attractor.
Where you end up depends on where you start.

Practical example: Multistability in visual perception

The Spinning Dancer by Nobuyuki Kaya-hara



A simple neural network.

Our visual system uses nonlinear dynamical systems defined on neural networks for cognition. Objects are fixed points of these networks.

Types of bifurcations 3: (Subcritical) pitchfork bifurcation

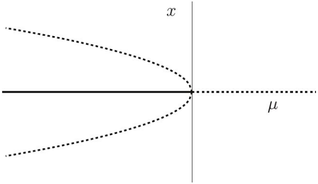
Model example:

$$\dot{x} = \mu x + x^3$$

Fixed points:

$$\begin{aligned} \mu x + x^3 &= 0 \\ \Rightarrow x\left(\mu + x^2\right) &= 0 \\ \Rightarrow x &= 0 \ \text{ or } \ x = \pm\sqrt{-\mu} \end{aligned}$$

Bifurcation diagram:



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Types of bifurcations 3: Stabilised subcritical pitchfork bifurcation and hysteresis

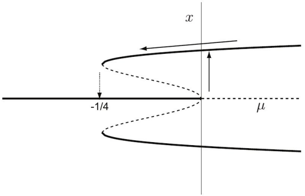
Model example:

$$\dot{x} = \mu x + x^3 - x^5$$

Fixed points:

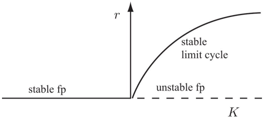
$$\begin{aligned} 0 &= x\,(x^4 - x^2 - \mu) \\ \Rightarrow x &= 0 \\ x &= \pm\sqrt{\frac{1}{2}\left(1+\sqrt{1+4\mu}\right)} \\ x &= \pm\sqrt{\frac{1}{2}\left(1-\sqrt{1+4\mu}\right)} \end{aligned}$$

Bifurcation diagram:

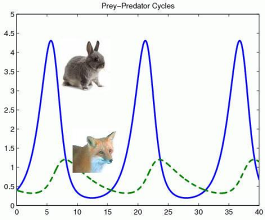


Hysteresis: when μ crosses 0 from below, system jumps to one of the two distant stable points but does not jump back when μ re-crosses zero from above.

Types of bifurcations 4: Hopf bifurcation and nonlinear oscillations



Bifurcation diagram.



Nonlinear oscillations in the Lotka-Volterra model.

Model example (in polar coordinates, (r, θ)):

$$\dot{r} = r\,(K - r^2), \quad \dot{\theta} = 1.$$

As K passes through 0 from below, $r = 0$ fixed point becomes unstable and creates a limit cycle of radius $\sqrt{K}$ for $K > 0$.

Nonlinear oscillations:

- Frequency-dependent amplitude.
- Harmonic generation.
- Global bifurcations.

Conclusions and summary

- Nonlinearity is generic.
- Low dimensional dynamical systems appear as models of average or macroscopic properties of complex systems.
- Dynamical systems exhibit a range of interesting and inherently nonlinear phenomena. We looked at a few: bifurcations, tipping points, multistability, hysteresis, nonlinear oscillations.
- There is much more to nonlinear dynamics than chaos!

Further reading:

- S. Strogatz, “Nonlinear dynamics and chaos” (1994)

Mathematics for Real-World Systems Centre for Doctoral Training

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MathSys is a collaboration of three interdisciplinary research centres, each with their own excellent records of world-leading research involving application of mathematics to real-world problems: the Centre for Complexity Science; Warwick Systems Biology; and Warwick Infectious Disease Epidemiology Research (WIDER) centre.

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This has the potential to shift the current state of the art in the training of cohorts of new researchers to combine cutting-edge mathematical skills with the ability to understand and model real-world systems "

Dame Julie Moore,
University Hospitals, Birmingham

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- Fluctuation driven phenomena in non-equilibrium statistical mechanics
  21 - 25 September 2015
  Orgainsers: Colm Connaughton, Oleg Zaboronski, Eric Vanden Eijnden, R. Rajesh
- Mathematics of kinetically constrained dynamics and metastability
  4 - 8 January 2016
  Organisers: Paul Chleboun, Stafen Grosskinsky, Fabio Martinelli
- Random matrix theory and strongly correlated systems
  21 - 24 March 2016
  Organisers: Oleg Zaboronski, Roger Tribe, Yan Fyodorov, Gernot Akemann, Neil O'Connell
- Real world risks and extremes: correlation and quantification (this meeting will be held at WBS London in the Shard)
  8 April 2016
  Organisers: Nick Watkins, Joanna Faure-Walker, Ariel Liebman, Colm Connaughton, Trevor Maynard
- Fluctuation-driven phenomena in biological systems
  18 - 22 April 2016
  Organisers: Stefan Grosskinsky, Thomas House, Anton Zilman
- Extreme events in the Earth and planetary sciences
  4 - 8 July 2016
  Organisers: Freddy Bouchet, Colm Connaughton, Ira Didenkulova, Alexandra Tzella
- Statistics of extreme and singular events in spatially extended systems
  11 - 15 July 2016
  Organisers: Sergey Nazarenko, Jason Laurie, Colm Connaughton
- Ergodicity breaking and anomalous dynamics
  10-12 August 2016
  Organisers: Colm Connaughton, Nick Moloney, Yuzuru Sato and Nick Watkins

http://www2.warwick.ac.uk/fac/sci/maths/research/events/2015-16/symposium/