NTU-Warwick Winter School on Complexity 2016

Self-Organized Criticality

By

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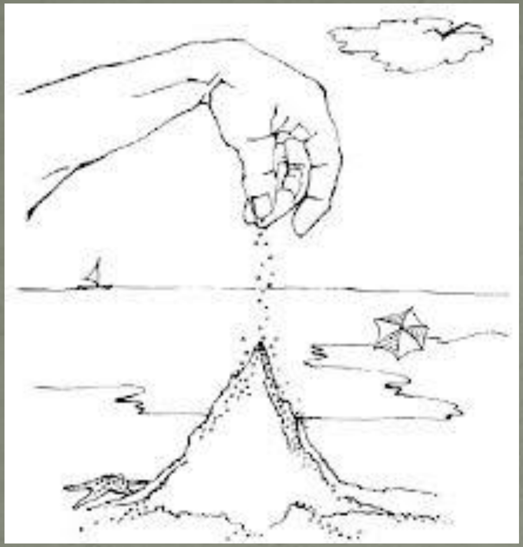
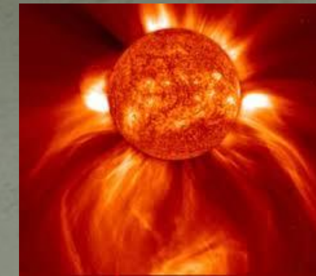


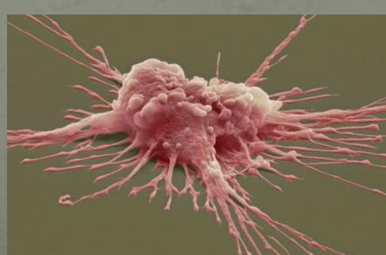
Who could ever calculate the path of a molecule? How do we know that the creation of worlds are not determined by falling grains of sand?

- Victor Hugo, *Les Miserables*

Astronomical Scale



Macroscopic Scale



Mesoscopic Scale

Self-Similarity

Road Network

Leaf Vascular Network

River Network

Lung Tubular Network

Blood Vessel Network

Neural Network

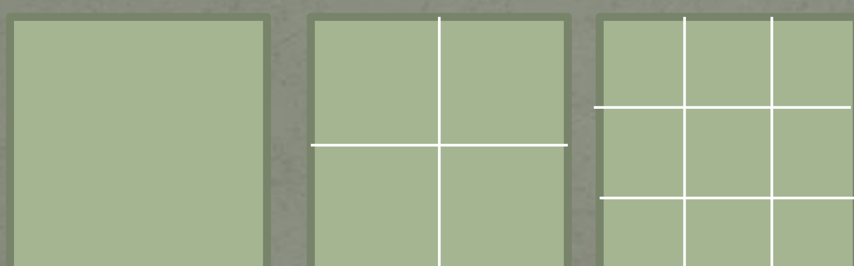
Fractals

m ~ Number of Copies
r ~ Scale Factor
D ~ Fractal Dimension

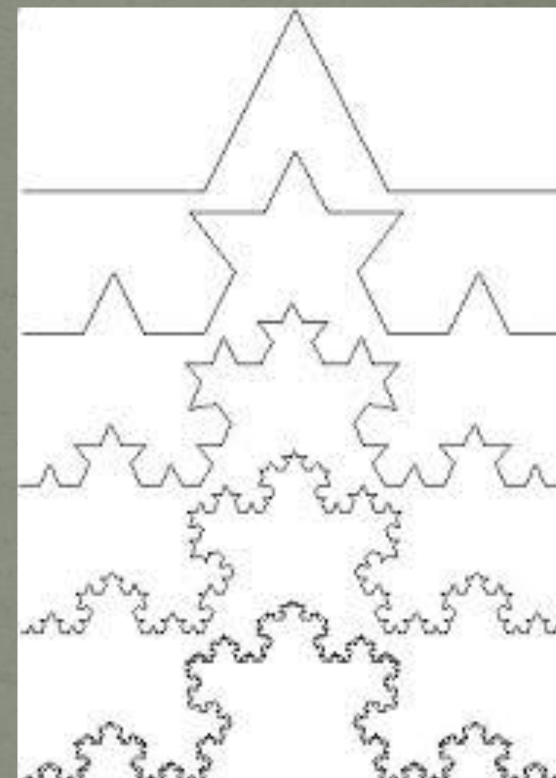
$$m = r^D$$

$$D = \frac{\ln m}{\ln r}$$

$m = 1$ $r = 1$ | $m = 2$ $r = 2$ | $m = 3$ $r = 3$



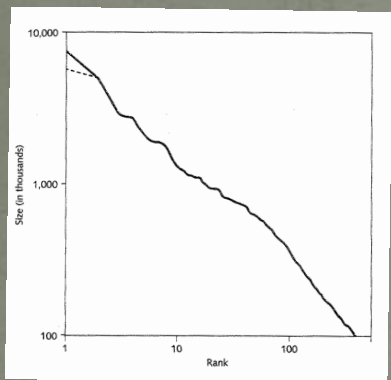
$m = 1$ $r = 1$ | $m = 4$ $r = 2$ | $m = 9$ $r = 3$

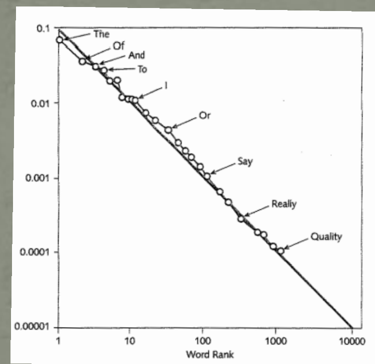


Koch Snowflake

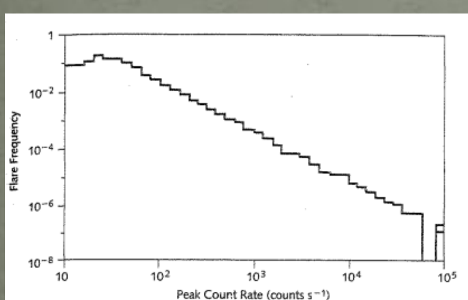
Power Laws

$$P(s) = s^{-\tau}$$

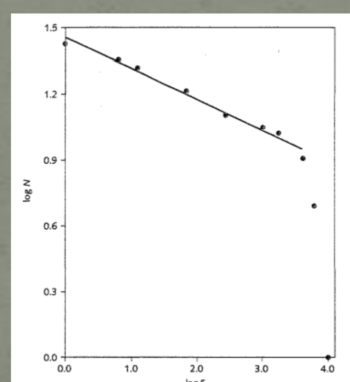




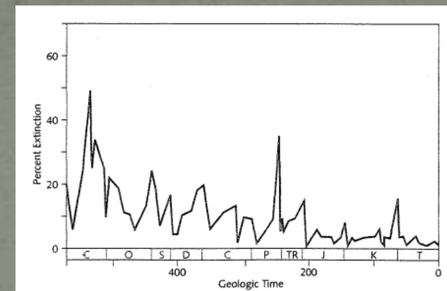
Ranking of Cities Ranking of Words

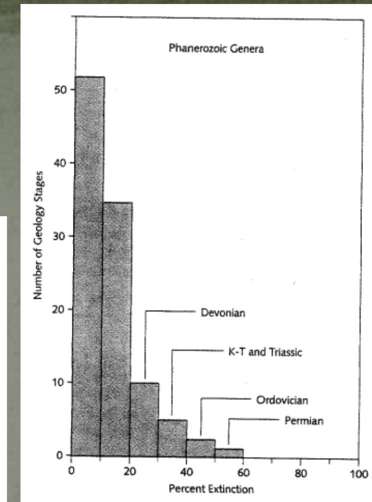


X-Ray Intensities from Solar Flares

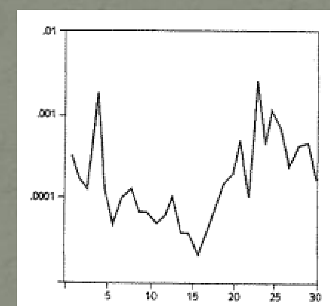


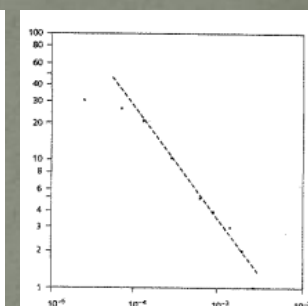
Pulsar Glitches



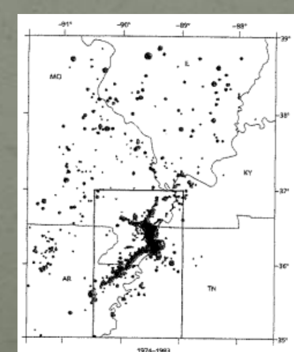


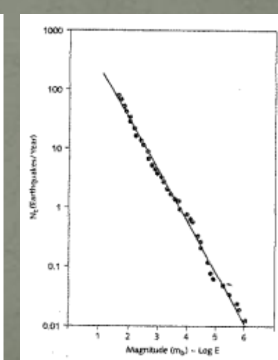
Biological Extinction





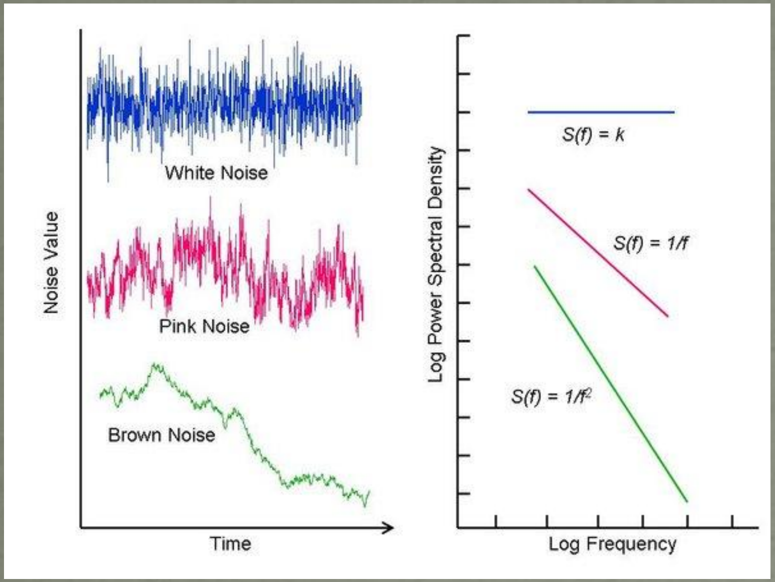
Cotton Price

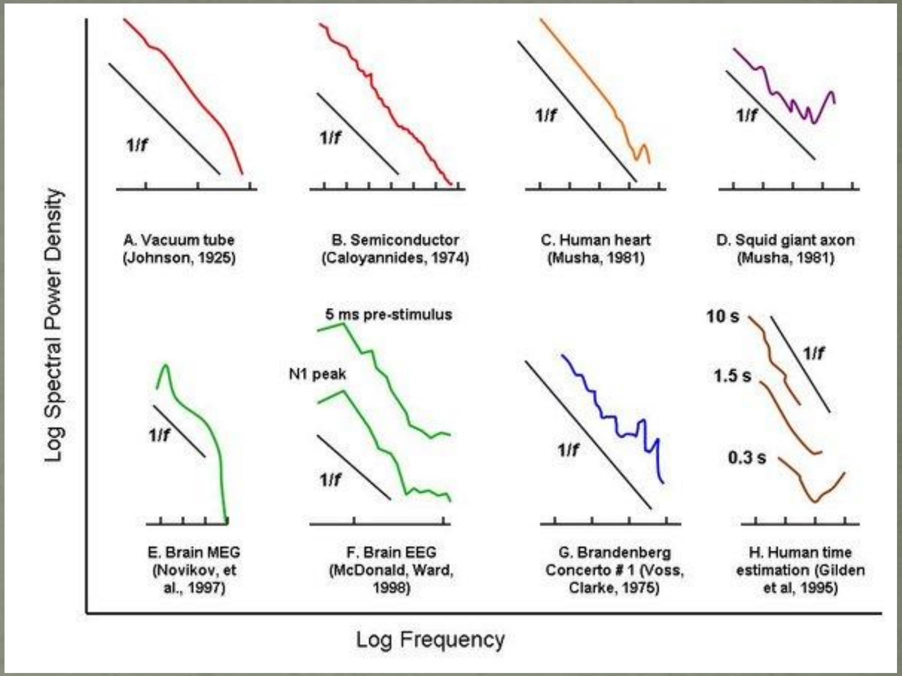




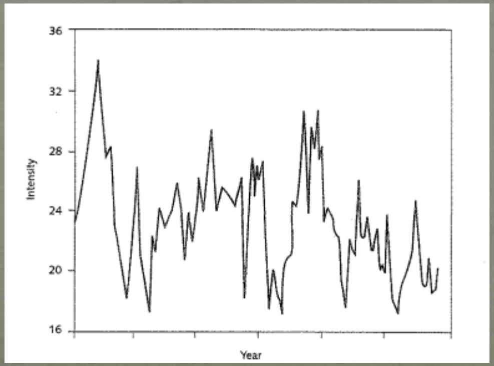
Earthquake Magnitude

1/f Noise

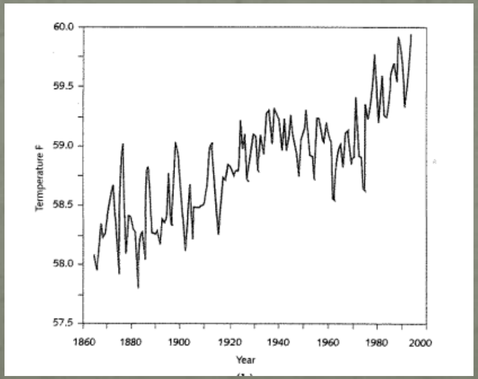




Intensity of Light from Quasar



Global Temperature

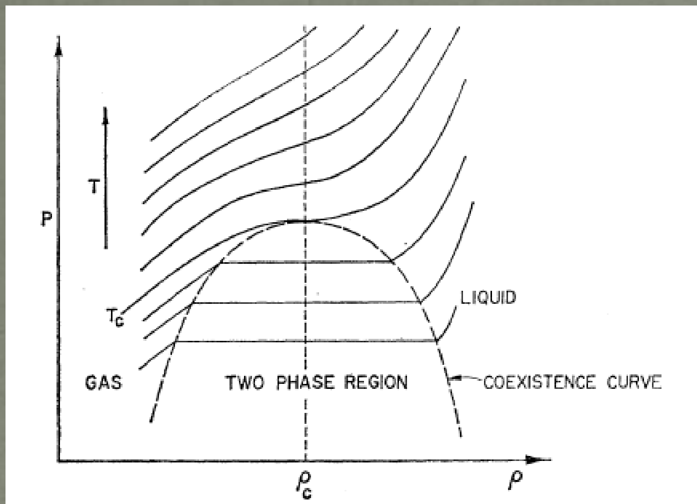


Criticality

Second-Order Phase Transition

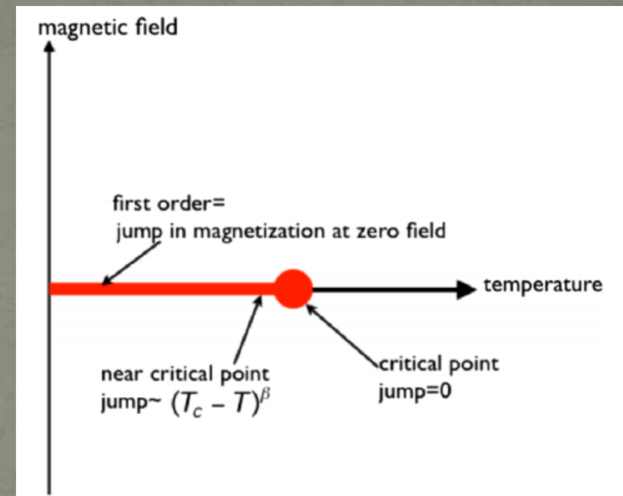
Liquid-gas critical point

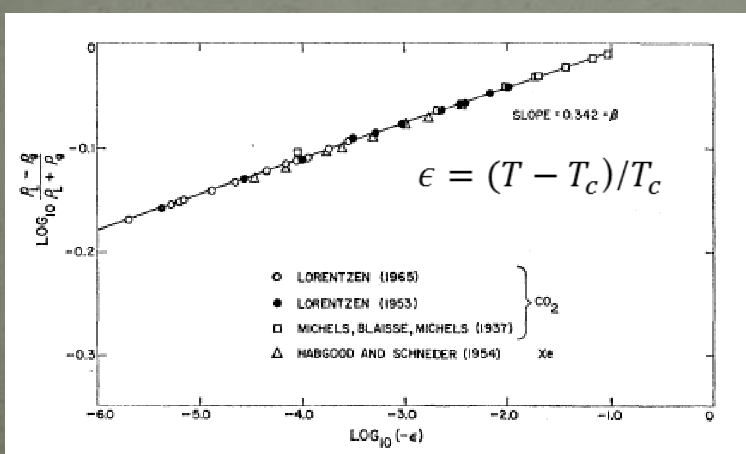
Pressure



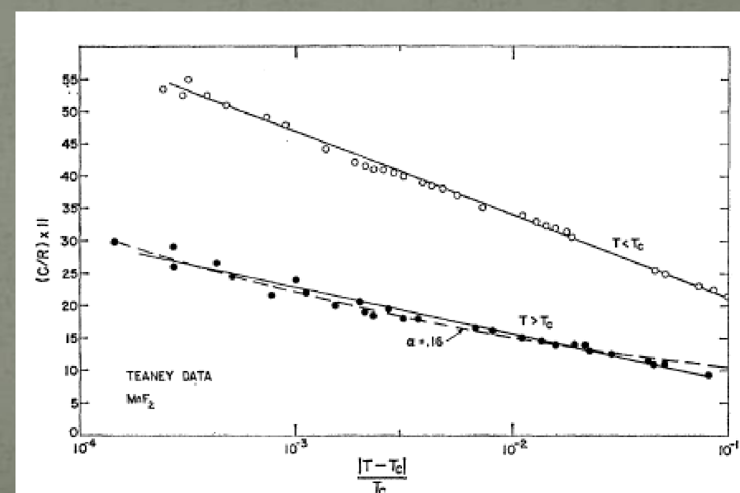
Density - ρ

Ferromagnetism critical point





Data for CO_2 and Xe. Critical index β≈0.34

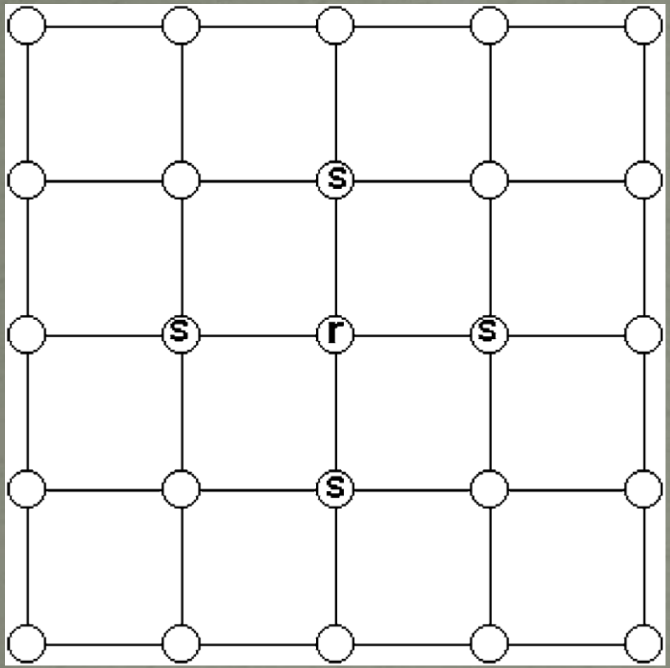


Specific Heat of MnF_2. The power law $C \sim \epsilon^{-0.16}$

Universality

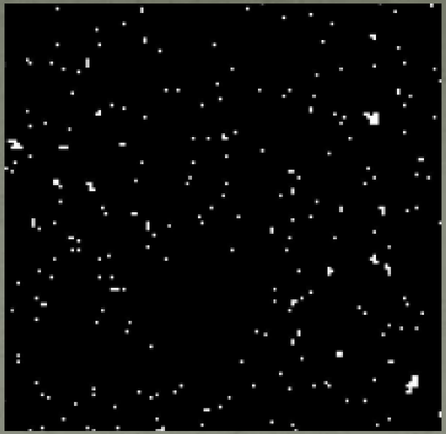
Ising Model

A model for ferromagnetic phase transition

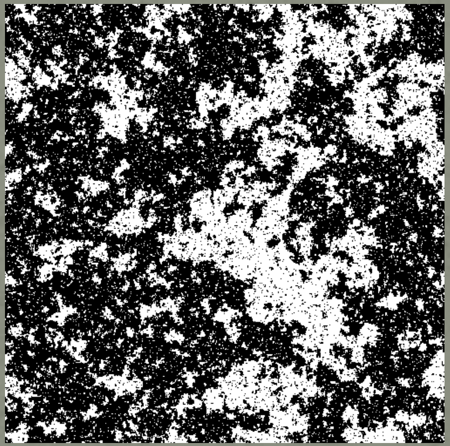


$$H = -J\sum_{i,j}\sigma_i\sigma_j - h\sum_i \sigma_i$$

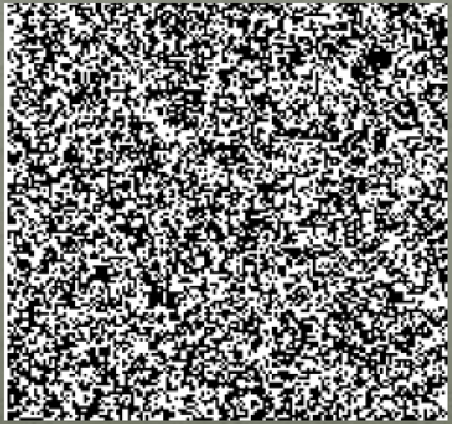
Ising Model Simulation (h = 0)



$T \to 0$



$T \to T_c$



$T \to \infty$

Experimental Comparison

Material	Experimenters	Ref.	T_c (°K)	$\epsilon=\lvert\Delta T\rvert/T_c$ Range for fit	α
			Antiferromagnets		
MnF_2	Teaney	86	67.33 ± 0.01	2×10^{-4}–5×10^{-2}	$\lesssim0.16$
$CoCl_2\cdot6\,H_2O$	Skalyo, Friedberg	84	2.289 ± 0.002	10^{-3}–3×10^{-2} 5×10^{-3}–4×10^{-2}	$\lesssim0.11$
$MnCl_2\cdot4\,H_2O$	Friedberg, Wasscher	89	1.622 ± 0.005	10^{-3}–10^{-1}	
$CuK_2(SO_4)_2\cdot6\,H_2O$	Miedema, Wielinga, Huiskamp	85	0.193 ± 0.001	10^{-3}–2×10^{-2}	$\lesssim0.6$
$CoCs_3Cl_5$			0.52 ± 0.01	4×10^{-3}–2×10^{-2} 4×10^{-3}–5×10^{-2}	$\lesssim0.7$
$RbMnF_3$	Teaney, Moruzzi, Argyle	90	0.83 ± 0.01	2×10^{-4}–5×10^{-2} 2×10^{-3}–2×10^{-2}	$\lesssim0.15$
			Ferromagnets		
Iron	Kraftmakher, Romashina	91	1043.0 ± 1.0	2×10^{-3}–10^{-1} 3×10^{-3}–7×10^{-2}	$\lesssim0.17$
$CuK_2Cl_4\cdot2\,H_2O$	Miedema, Wielinga, Huiskamp	92	0.88 ± 0.01	10^{-3}–10^{-1}	$\lesssim0.10$
Nickel	Kraftmakher	93	627.0	5×10^{-3}–8×10^{-2}	
Value used for scaling law analysis					$\lesssim0.16$
Molecular field theory					0
3-dimensional Ising model					$\lesssim0.2$

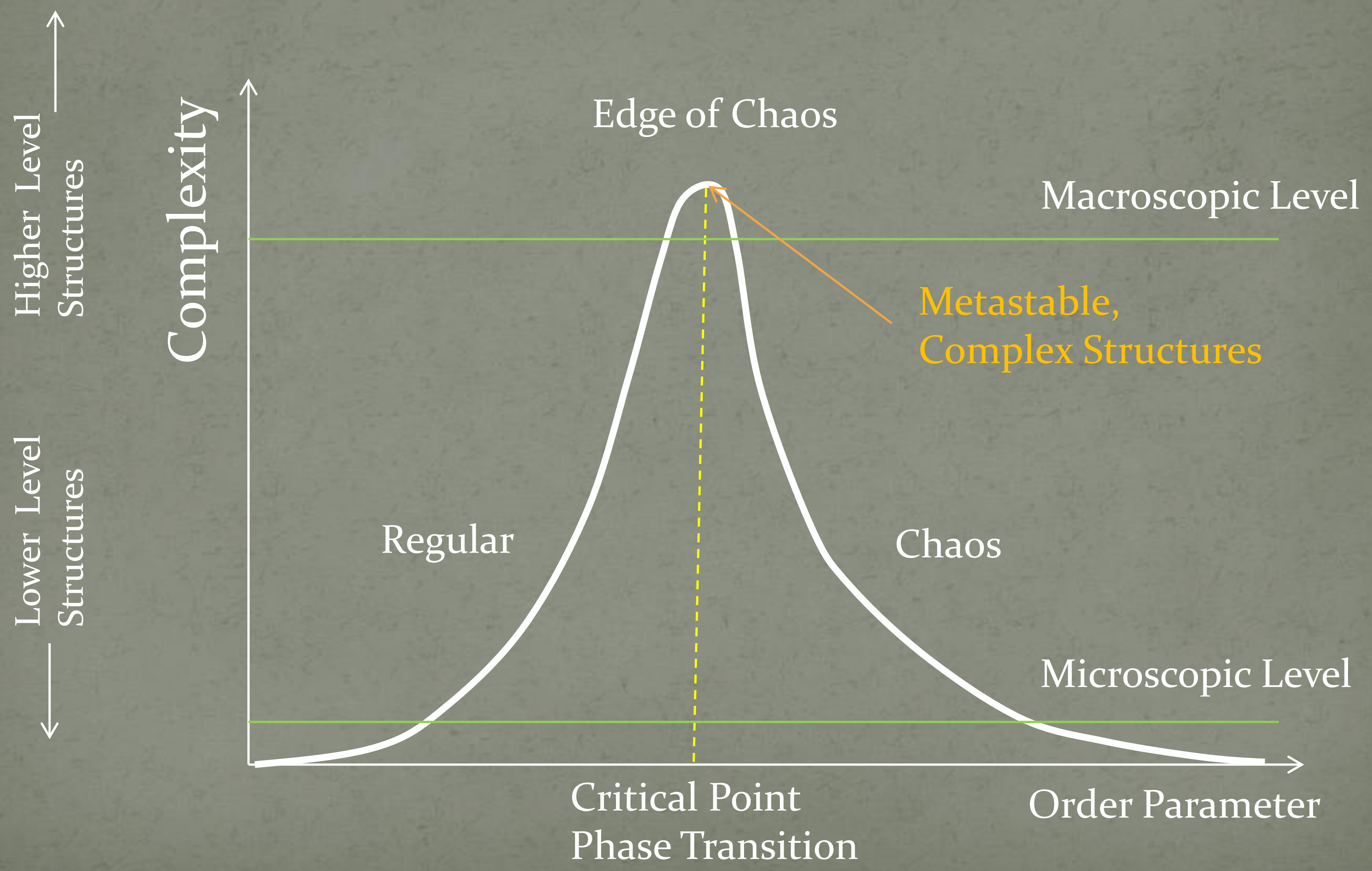
Specific Heat

Experimental Comparison

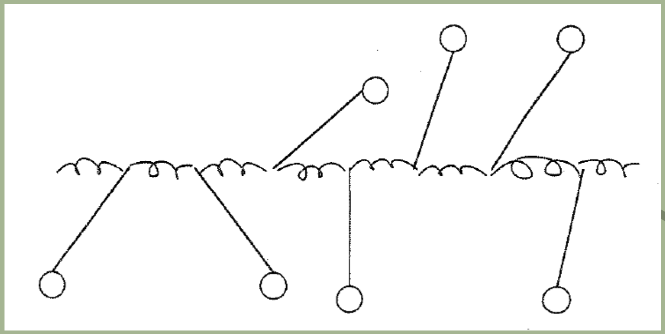
Material	Experimenters	Ref.	Method	T_c (°K)	$\epsilon = \lvert \Delta T \rvert / T_c$ Range for fit	β
			Antiferromagnets			
MnF_2	Heller, Benedek	51	NMR on F^{19}	67.336±0.003	8×10^{-5}–2×10^{-2}	0.335±0.01
$CuCl_2 \cdot 2H_2O$	Poulis, Hardeman	76	NMR, Protons	4.337±0.003	5×10^{-4}–10^{-2} 10^{-2}–10^{-1}	0.18±0.07 0.29±0.03
$CoCl_2 \cdot 6H_2O$	Sawatzky, Bloom	52	NMR, Protons	2.275	10^{-2}–10^{-1}	0.15±0.05
	Van der Lugt, Poulis	77	NMR, Protons	2.275	5×10^{-2}–2×10^{-1}	0.23±0.02
$KMnF_3$	Cooper, Nathans	69	Neutron scattering	88.06±0.02	10^{-2}–10^{-1}	0.33
			Ferromagnets			
Iron	Preston, Hanna, Heberle	71	Mössbauer Fe^{57}	1042.0±0.3	2×10^{-3}–10^{-1}	0.34±0.02
	Potter	78	Magnetocaloric effect	1035.0±2.0	4×10^{-3}–2×10^{-1}	0.36±0.08
Nickel	Howard, Dunlap, Dash	29	Mössbauer Fe^{57}	629.4	5×10^{-4}–10^{-2} 10^{-2}–1.6×10^{-1}	0.51±0.04 0.33±0.03
EuS	Heller, Benedek	79	NMR, Eu^{153}	16.50±0.03	10^{-2}–10^{-1}	0.33±0.015
$YFeO_3$	Gorodetsky, Shtrikman, Treves	30	Vibrating sample magnetometer	643	2×10^{-4}–3×10^{-3}	0.55±0.04
	Eibschutz, Shtrikman, Treves	80	Mössbauer Fe^{57}	640	10^{-2}–3×10^{-1}	0.354±0.005
$CrBr_3$	Senturia, Benedek	81	NMR, Br^{79}, Br^{81}	32.56±0.015	7×10^{-3}–5×10^{-2}	0.365±0.015
Value used for scaling law analysis						0.33±0.03
Molecular field theory						0.5
3-dimensional Ising model						0.313±0.004

Spontaneous Magnetization

Complexity

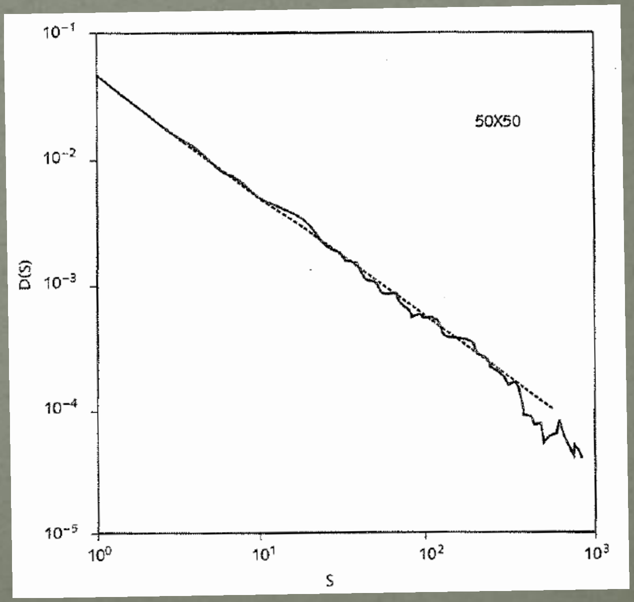

Higher Level Structures
Lower Level Structures
Complexity
Edge of Chaos
Macroscopic Level
Metastable, Complex Structures
Regular
Chaos
Microscopic Level
Critical Point
Phase Transition
Order Parameter

Coupled Pendulums

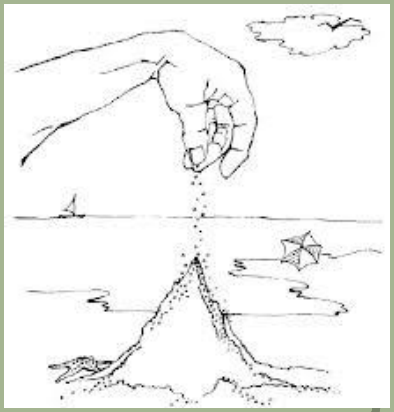


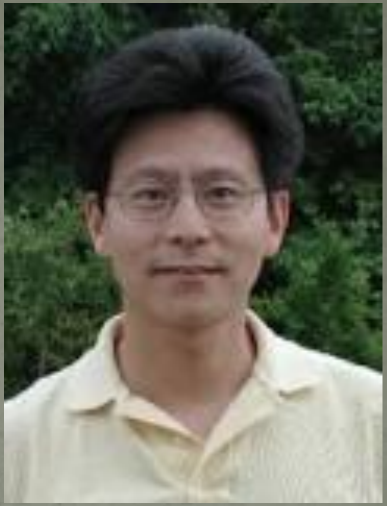
Per Bak

Pioneer in the physics of complex systems



Sandpiles





Chao Tang

Kurt Wiesenfeld

Discoverers of Self-Organized Criticality

Per Bak

BTW Sandpile Model

Example of SOC: sandpile model on 2D square lattice

$$(\text{active})\ z(x,y) \rightarrow z(x,y) - 4\ (\text{topple})$$

$$z(x \pm 1, y) \rightarrow z(x \pm, y) + 1$$

$$z(x, y \pm 1) \rightarrow z(\mathrm{x}, y \pm 1) + 1$$

1	1	0	2	3	0
0	3	2	1	1	3
2	2	1	3	3	1
2	0	0	2	0	1
1	1	3	2	2	1
3	2	1	1	0	2

1	1	0	2	3	0
0	3	2	**2**	1	3
2	2	**2**	**0**	4	1
2	0	0	3	0	1
1	1	3	2	2	1
3	2	1	1	0	2

1	1	0	2	3	0
0	3	2	**2**	**2**	3
2	2	**2**	**1**	**0**	**2**
2	0	0	**3**	**1**	1
1	1	3	2	2	1
3	2	1	1	0	2

Sandpile Dynamics

… sandpile model on 2D square lattice

1	1	0	2	3	0
0	3	2	2	2	3
2	2	2	1	0	2
2	0	0	3	1	1
1	1	3	4	2	1
3	2	1	1	0	2

1	1	0	2	3	0
0	3	2	2	2	3
2	2	2	1	0	2
2	0	1	4	1	1
1	2	0	1	3	1
3	2	2	2	0	2

1	1	0	2	3	0
0	3	2	2	2	3
2	2	2	2	0	2
2	0	1	0	2	1
1	1	4	1	3	1
3	2	1	2	0	2

1	1	0	2	3	0
0	3	2	2	2	3
2	2	2	1	0	2
2	0	0	4	1	1
1	1	4	0	3	1
3	2	1	2	0	2

1	1	0	2	3	0
0	3	2	2	2	3
2	2	2	2	0	2
2	0	2	0	2	1
1	2	0	2	3	1
3	2	2	2	0	2

1	1	0	2	3	0
0	3	2	2	2	3
2	2	2	2	0	2
2	0	2	0	2	1
1	2	0	2	3	1
3	2	2	2	0	2

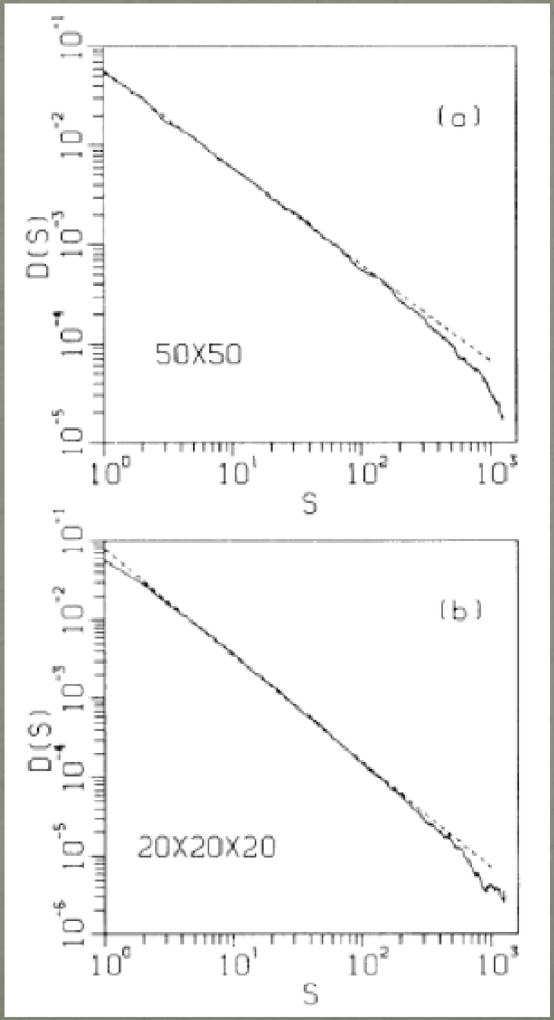
Power Law Distributions

Distribution of Cluster Size

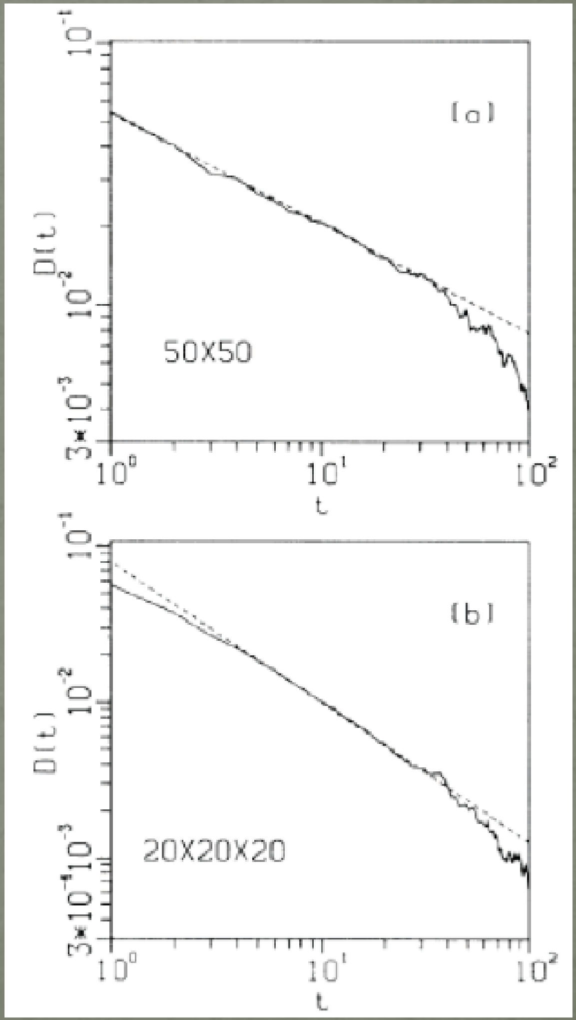
Distribution of Lifetime

2-Dimensional

3-Dimensional



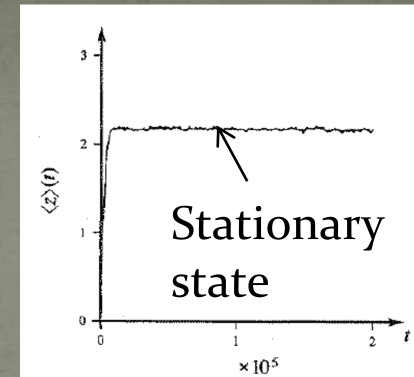
Cluster Size

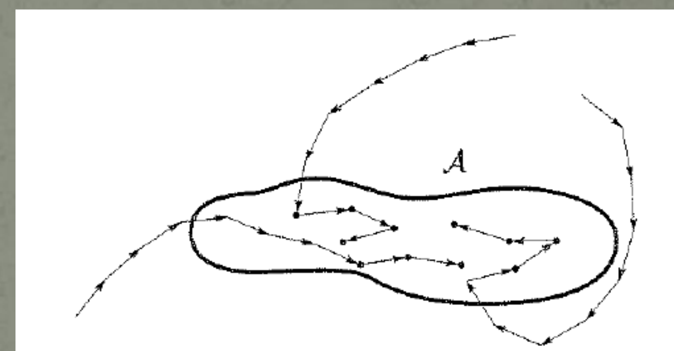


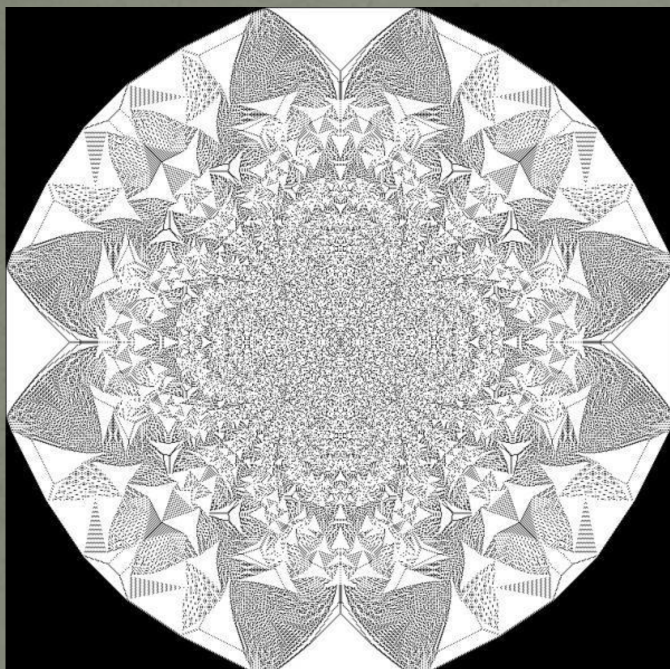
Lifetime

Self-Organized Criticality

Attractor for
Metastable
Configurations

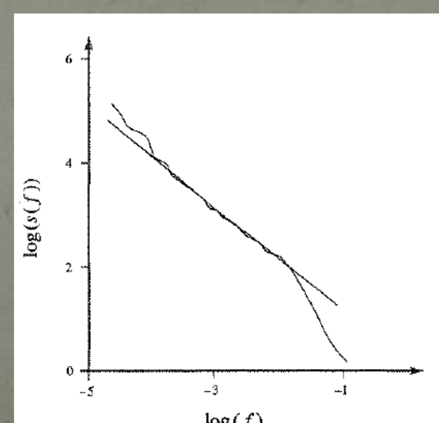






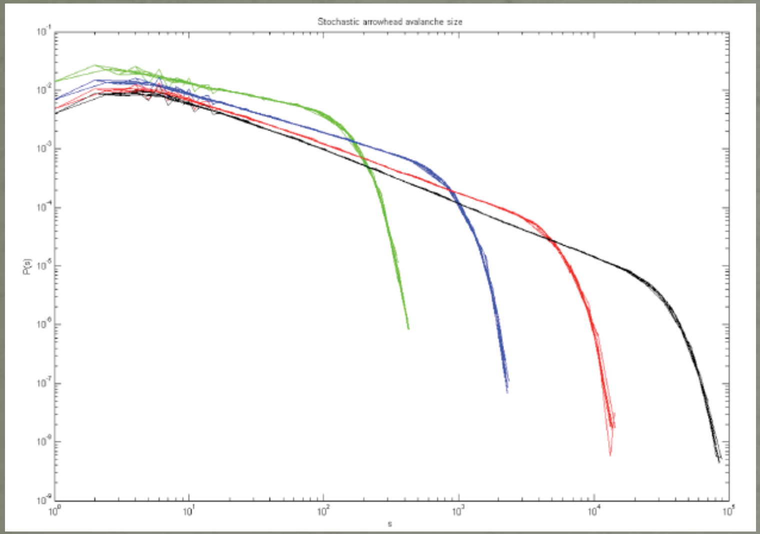
Source: Netherlands Organization for
Scientific Research

Fractals

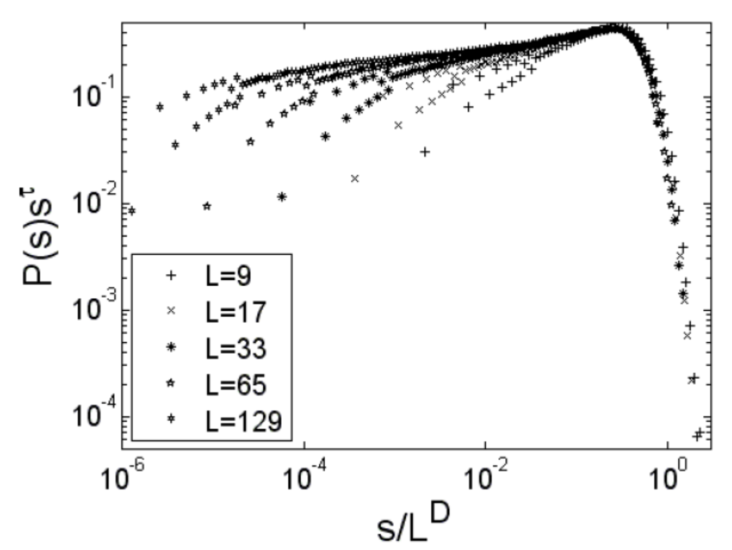


1/f behavior

Self-Organized Criticality



Data Collapse



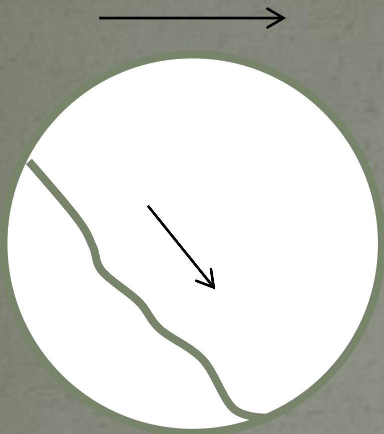
Finite Size Scaling

$$\boxed{P(s) = a_s s^{-\tau} G_s\left(\frac{s}{b_s L^D}\right)}$$

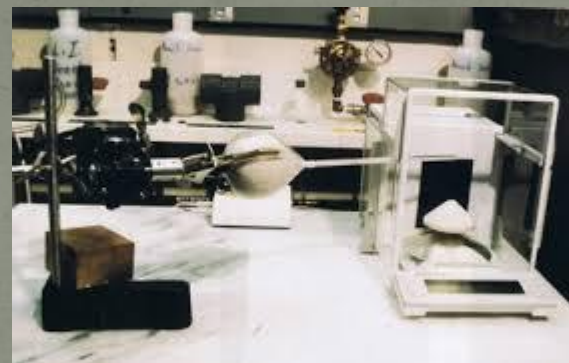
SOC Features

- Slow Drive/Fast Relaxation
- Open/Dissipative
- Threshold/Instability
- Contingent/History
- Avalanche/Fluctuations

Experimental Verifications

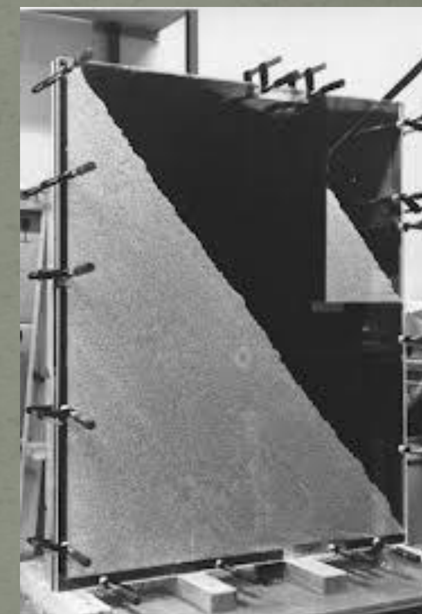


Rotating Drum Experiment

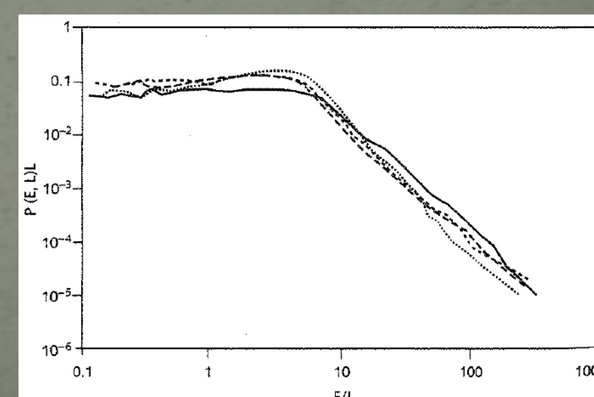


IBM Experiment

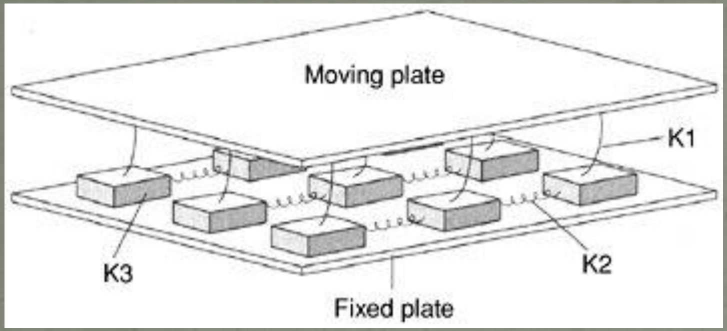
Norwegian Rice Pile

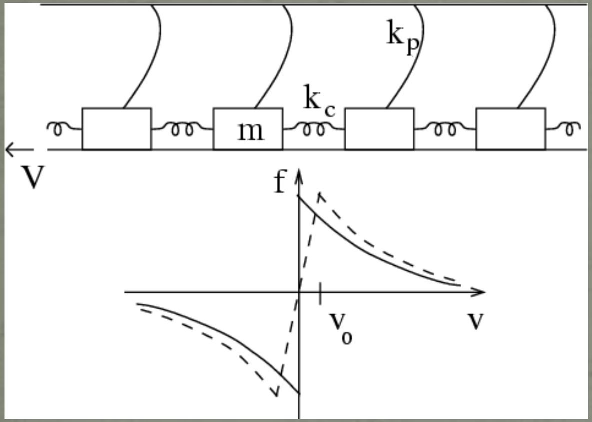


University of Michigan Experiment



Earthquakes





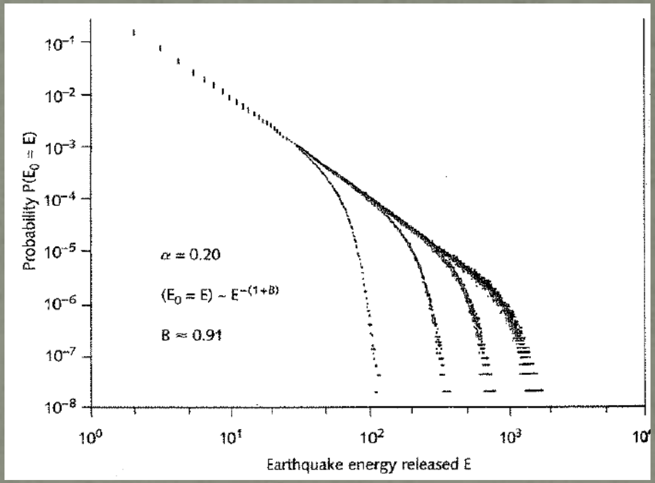
Burridge-Knopoff Block-Spring Model

OFC Model
Non-conservative SOC Model

Homogeneous driving

$$E_i \to E_i + \varepsilon$$

$$E_i \geq E_c \Rightarrow \begin{cases} E_i \to 0, \\ E_{nn} \to E_{nn} + \alpha E_i \end{cases}$$



Gutenberg-Richter Law

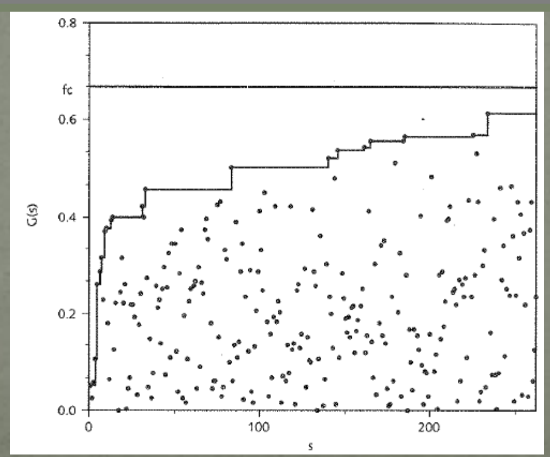
The Earth Crust has self-organized to a critical state.

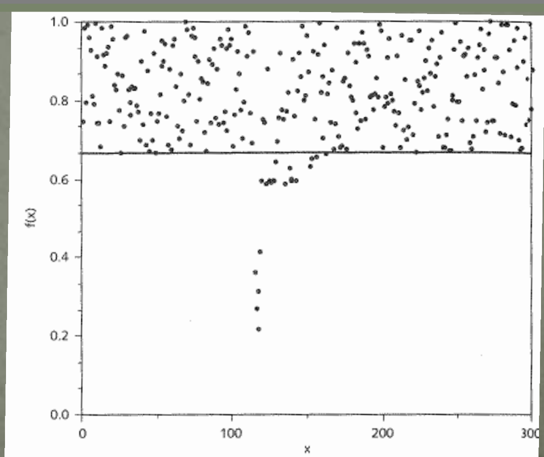
Biological Evolution

Bak-Sneppen Model : Random numbers between 0 and 1 are arranged in a circle. At each time step, the lowest number, and the number at its two neighbors, are each replaced by new random numbers.

$$f(t) = f_c - A\left(\frac{t}{N}\right)^{-1/(\gamma-1)}$$

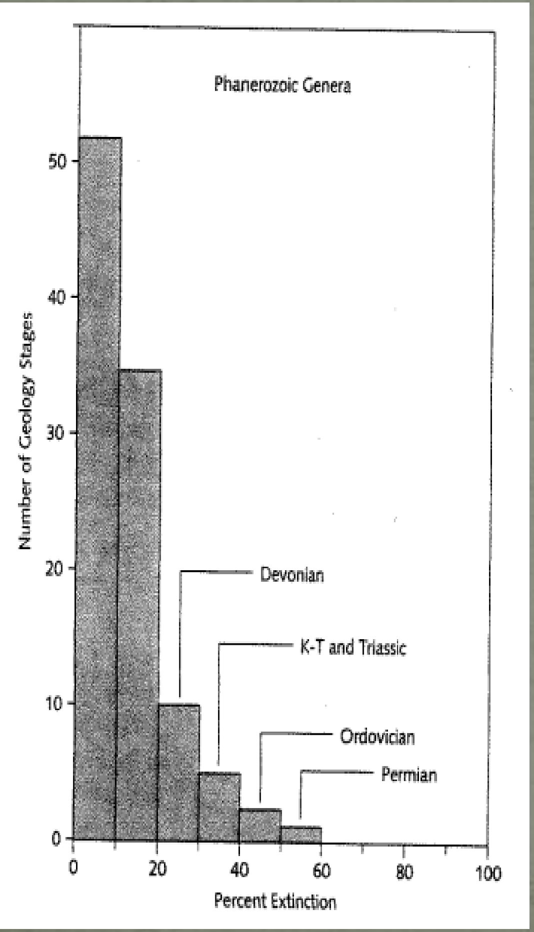
Self-Organization

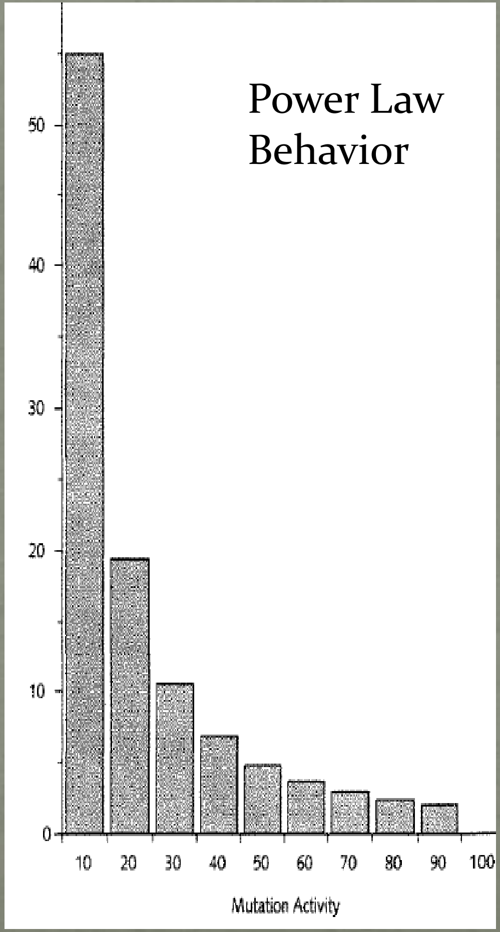


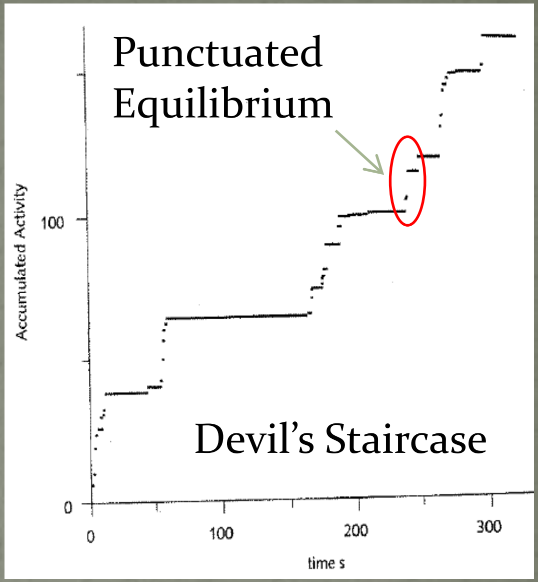


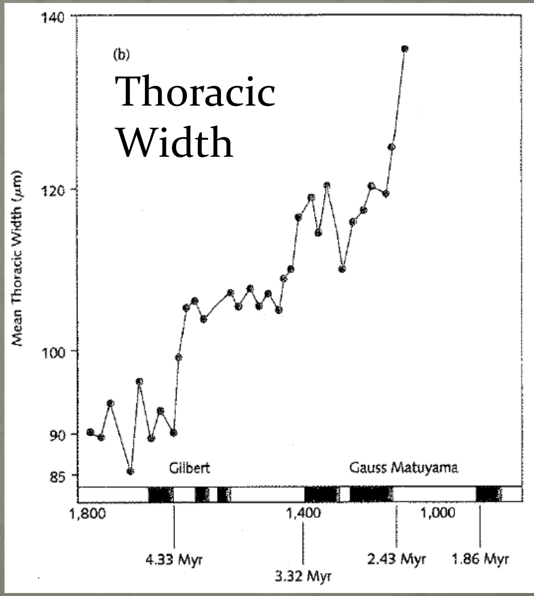
Punctuated Equilibrium

- Cambrian Explosion
- Dinosaur Extinction



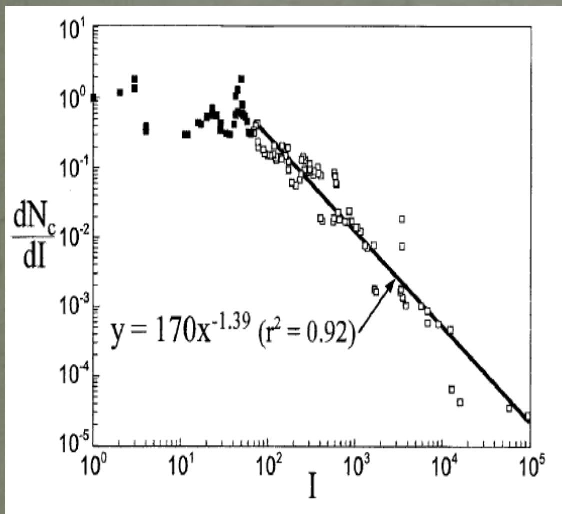




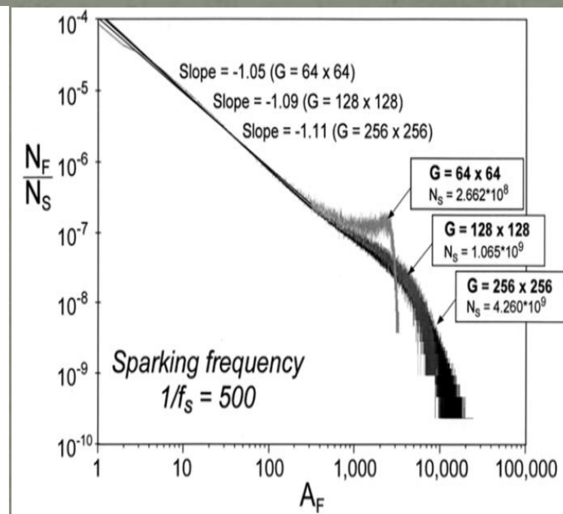


Riots

Richardson's Power Law,
Statistics of Deadly Quarrels



Wars



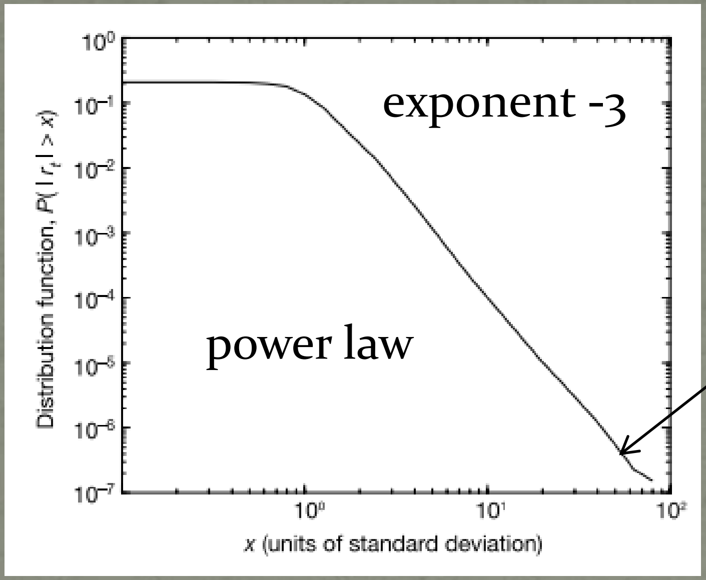
Forest Fire

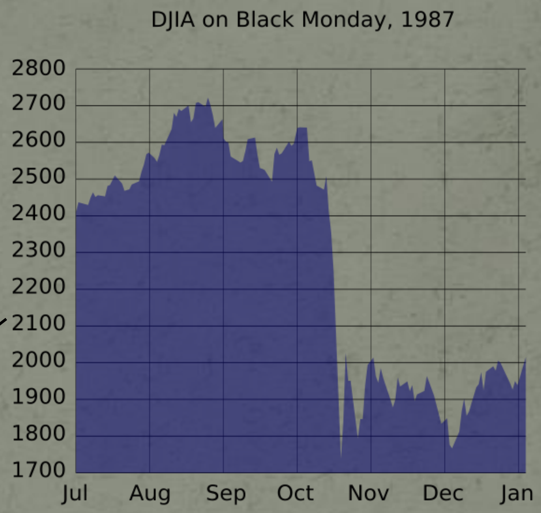
Little India Riots

Forest Fire Model

- A cell with burning tree turns into an empty cell
- A tree will burn if at least one neighbor is burning
- A tree ignites with probability f even if no neighbor is burning
- A tree appears in an empty cell with probability p

Stock Market Crashes





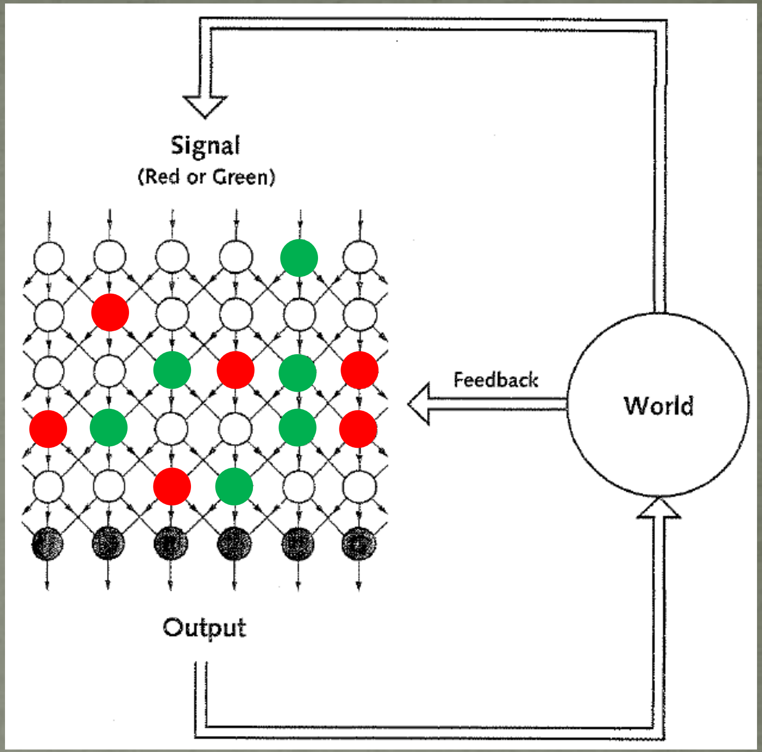
Instabilities

- Imitation
- Herding
- Cooperativity
- Feedbacks

Self-Organizes → to Criticality → Speculative Bubbles → Crashes

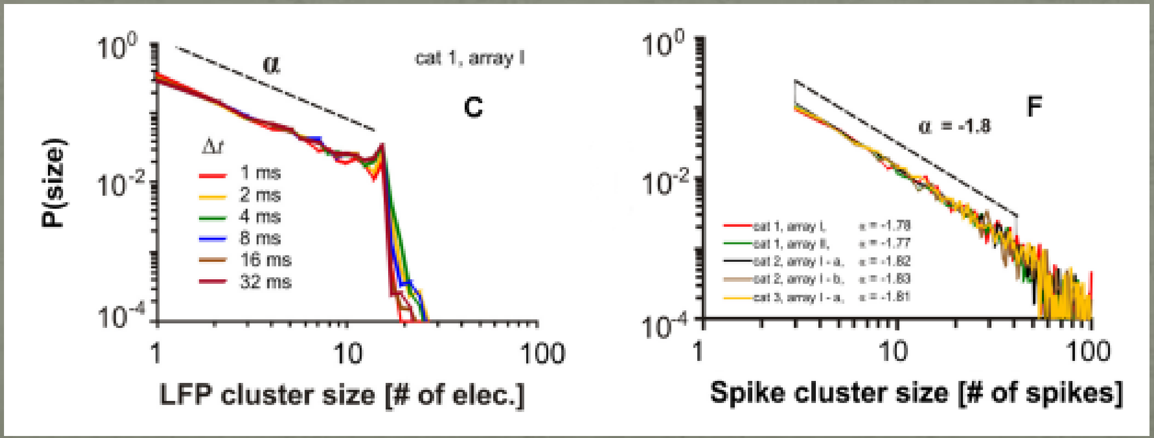
The Brain

Observation
Other thoughts } THOUGHTS ~ small or large avalanche

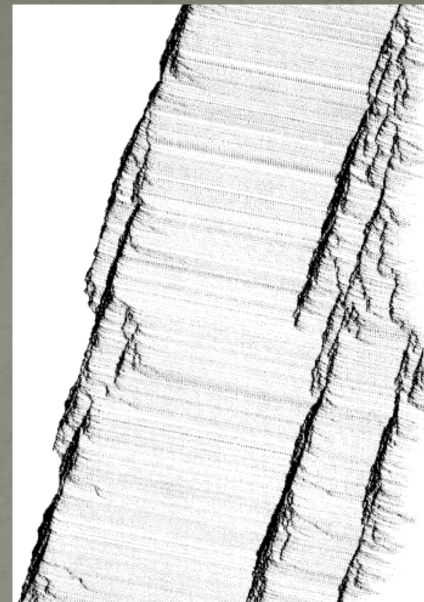


Brain Self Organizes into a Critical State

- Subcritical ~ access limited information
- Supercritical ~ too noisy



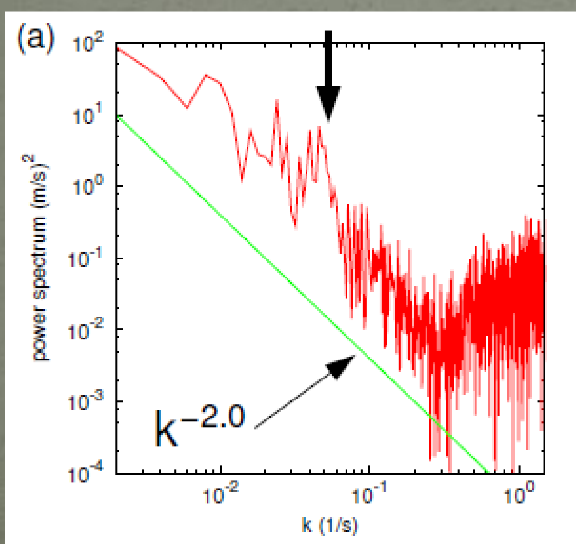
Traffic Jams



The critical state, with jams of all sizes, is the most efficient state, that can be reached dynamically.

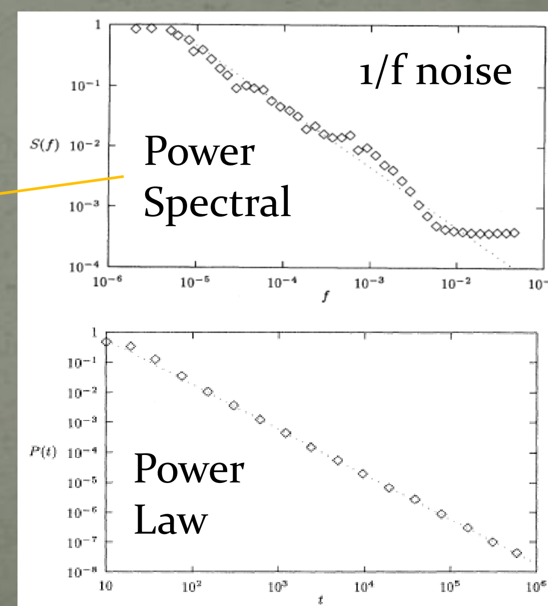
Subcritical ~ free flow (under-utilization)

Supercritical ~ jammed (over-utilization)



From time series of number of vehicles at fixed location

Lifetime distribution from emergent jam



A Relook at SOC

- What is distinct about SOC?
  - Slowly driven, interaction dominated threshold system.
  - Self-organization versus tuning of parameters
  - Robustness of critical behavior
- Is there a theory of SOC systems?
  - Mean field theory
  - Exact solution in terms of operators for Abelian sandpile
  - Langevin equations
  - Dynamically driven renormalization group
- Has SOC taught us anything new about the world?
  - The importance of fluctuations
- Is there anyway predictive power in SOC?
  - Fluctuations have prevented us from predicting SOC systems in detail.
  - Understanding of mechanisms can provide insights into possible measures
    1. Having small or medium size fire/ Releasing social tensions in small or medium groups
    2. Create friction in the system ~ Cooling measures, e.g. Stock market, Property market.

SOC – Where do we go from here?

- Inconclusive experimental evidence on the possible causal relationship between the emergent power laws and the underlying self-organized critical state
  - Variable selection
  - Gibrat's law – growth process by importance measure
  - Coherent noise model (non-critical steady state)
  - Highly optimized tolerance (non-critical self-organizing state)
- Are the empirical distributions of complex systems exactly power law?
  - Pareto, log-normal, log-Cauchy distributions look similar in log-log plot
  - Heavy tailed distributions
- Dragon Kings

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