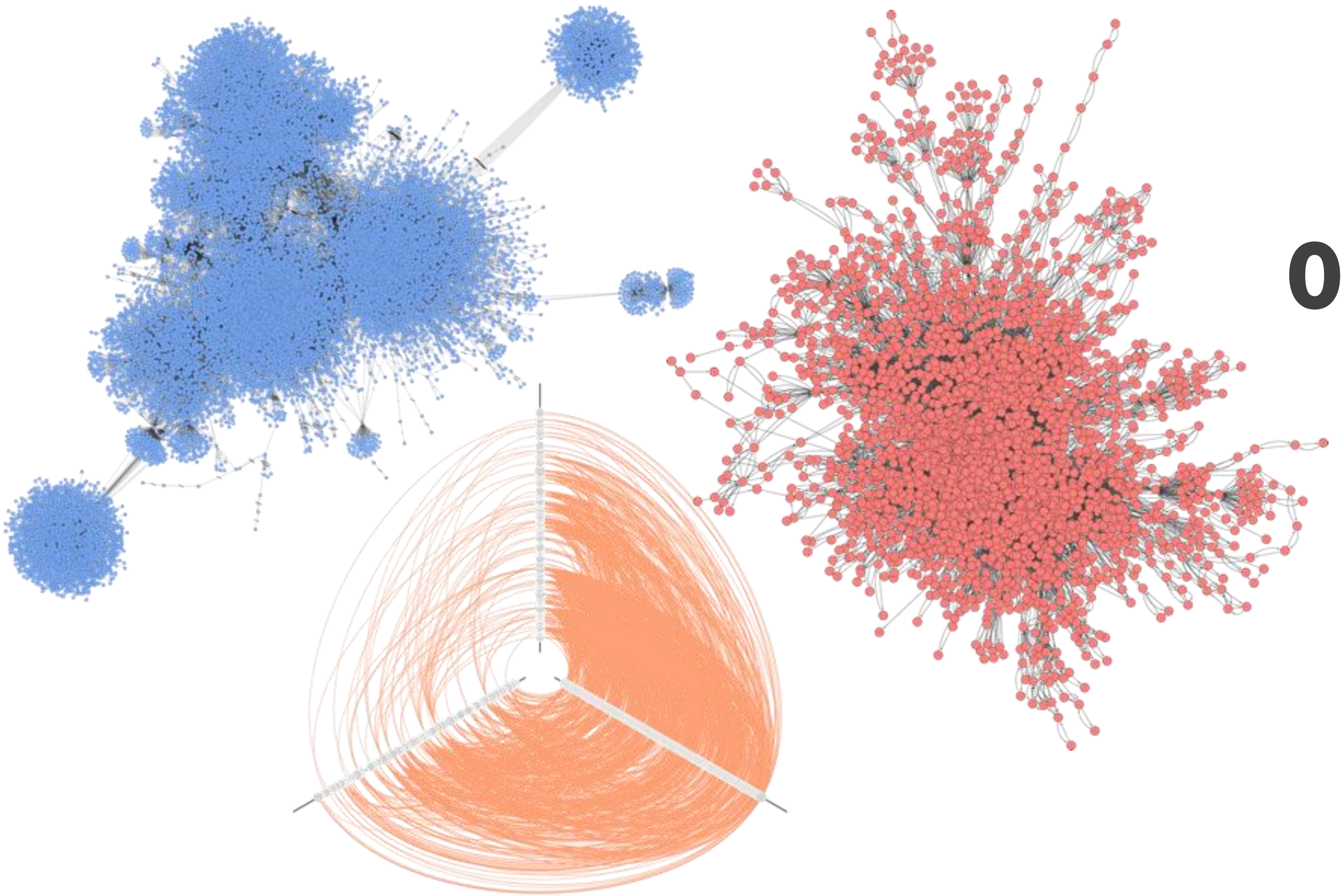
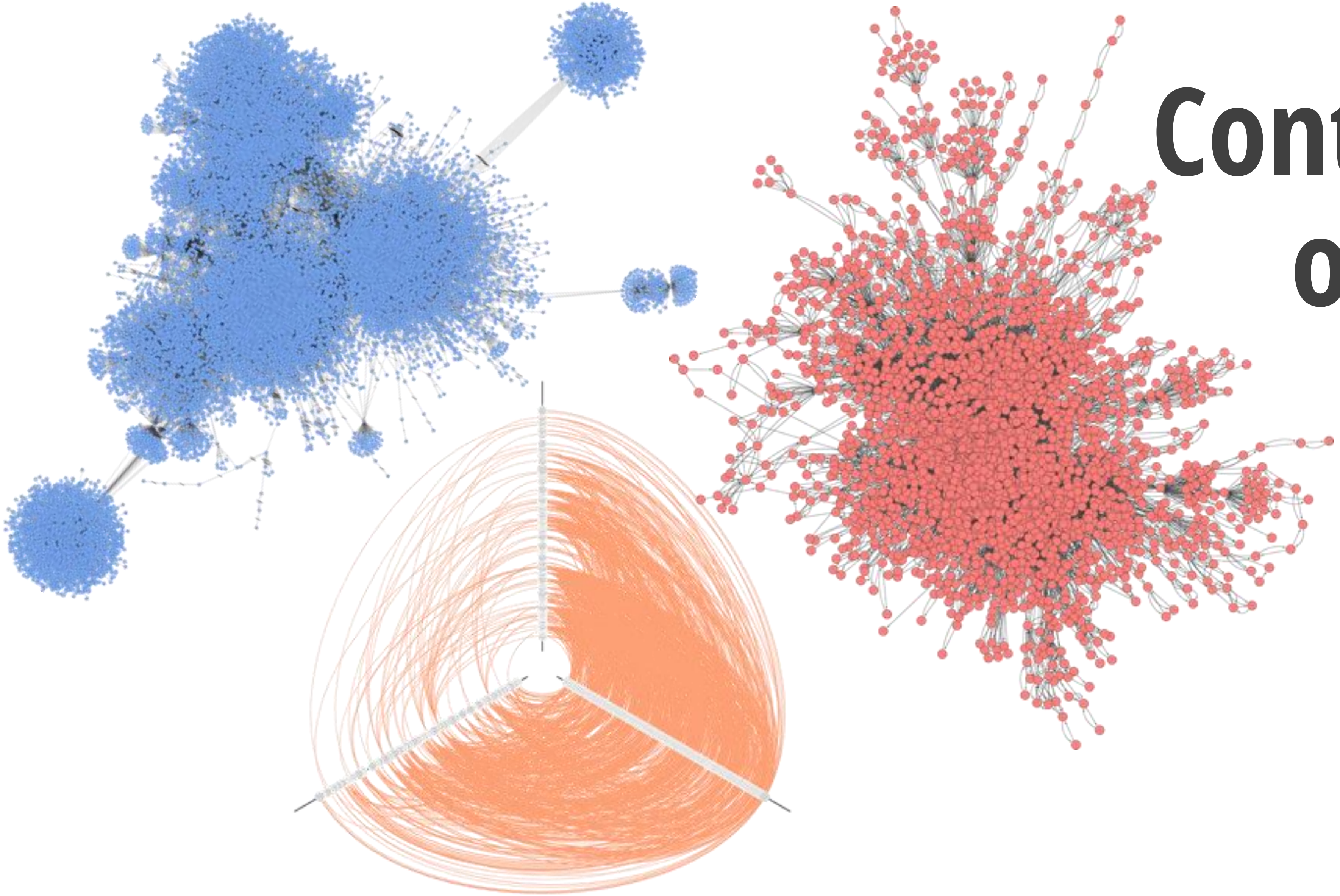


Control of Complex Networks

Justin Ruths

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Established in collaboration with MIT



Controllability of Complex Networks

Justin Ruths

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

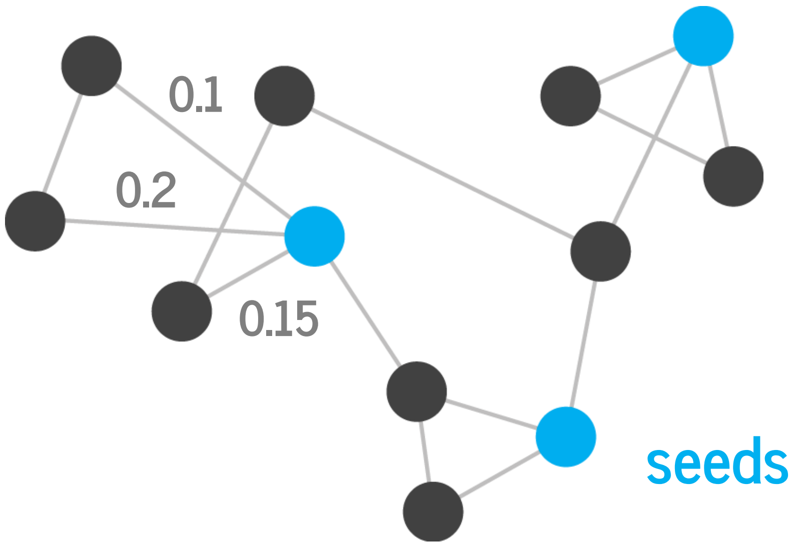
Established in collaboration with MIT

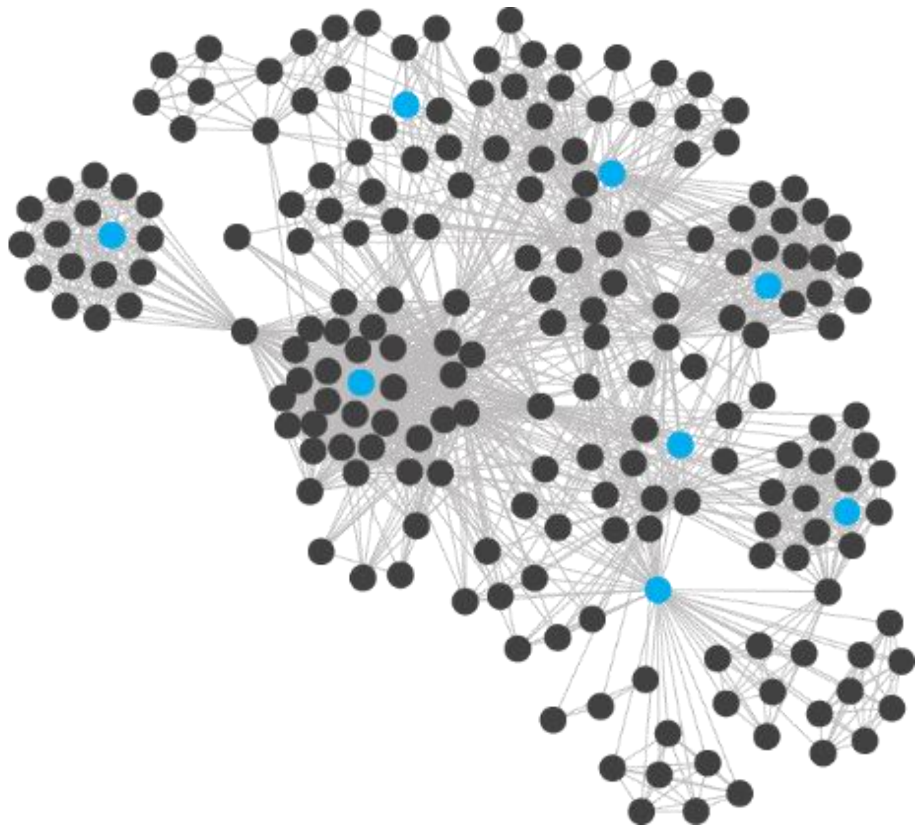
controlling...

- **Extinction:** populations of a species within a food web
- **Epilepsy:** neuron voltages in the brain
- **Disease:** concentration of proteins within a cell
- **Unrest:** sentiment within a population of individuals

influence maximization

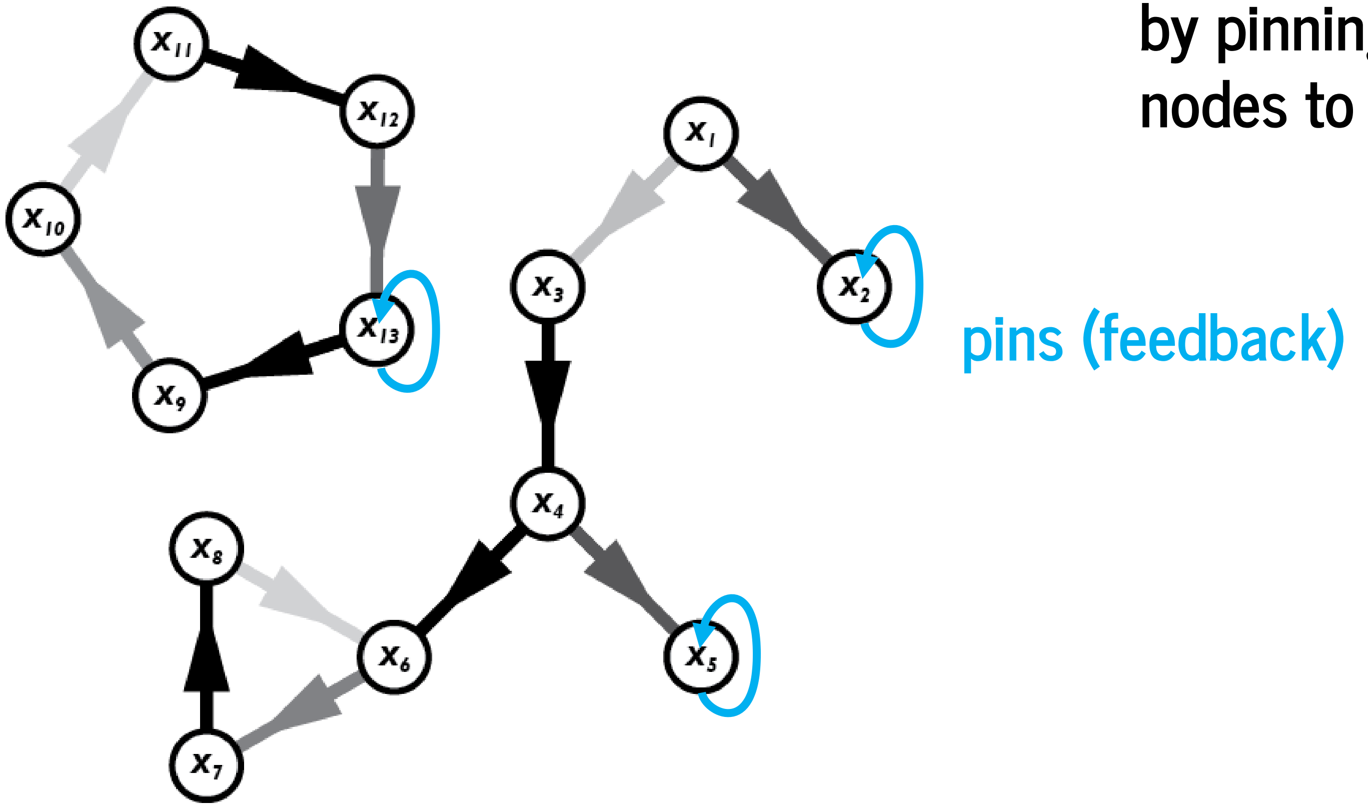
independent cascade models a discrete, stochastic diffusion based on edge weights (probabilities)





Kempe, Kleinberg, Tardos, ACM SIGKDD (2003).

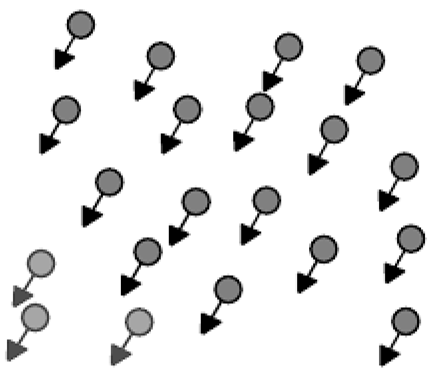
pinning control



by pinning specific nodes: drive all nodes to a reference trajectory

Sorrentino, Bernardo, Garofalo, Chen. Physical Review E, 75, (2007).

multi-agent consensus



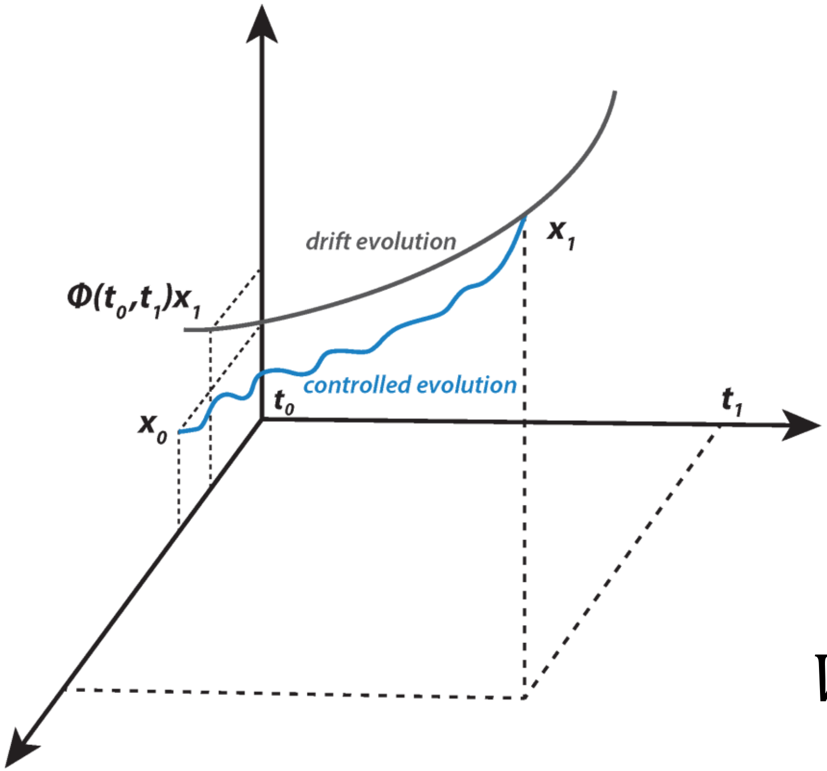
design feedback policies and topologies that lead toward attaining consensus.

$$\dot{x}_i(t) = \sum_{j \in N_i} A_{ij}\big(x_j(t) - x_i(t)\big) \longrightarrow \dot{x}(t) = -Lx(t)$$

$$L = D - A$$

for an undirected connected graph, these dynamics yield asymptotic consensus for all initial states

Olfati-Saber, Fax, Murray (2007).



$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t)$$

$$x_0 - \Phi(t_0,t_1)x_1 \in \mathcal{R}(W(t_0,t_1))$$

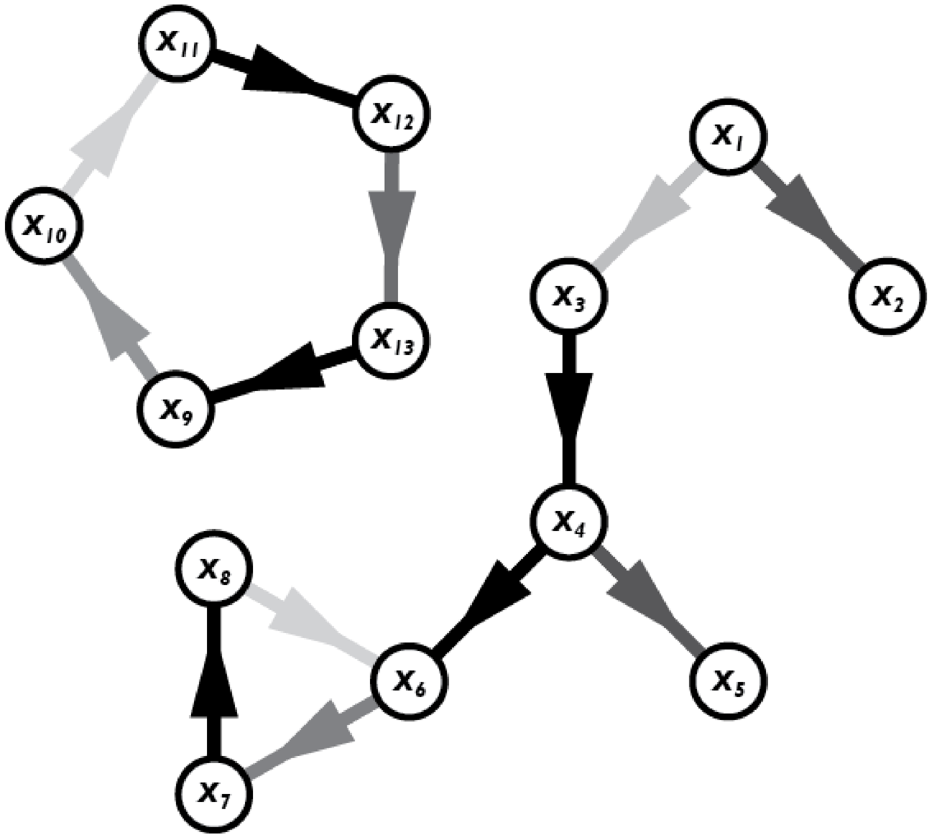
$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,t\,)B(t)B'(t)\Phi'(t_0,t)\,dt$$

controllability is the ability to drive a dynamical system from an arbitrary initial state $x(t_0) = x_0$ to an arbitrary final state, $x(t_1) = x_1$, through the application of a time-varying input $u(t)$.

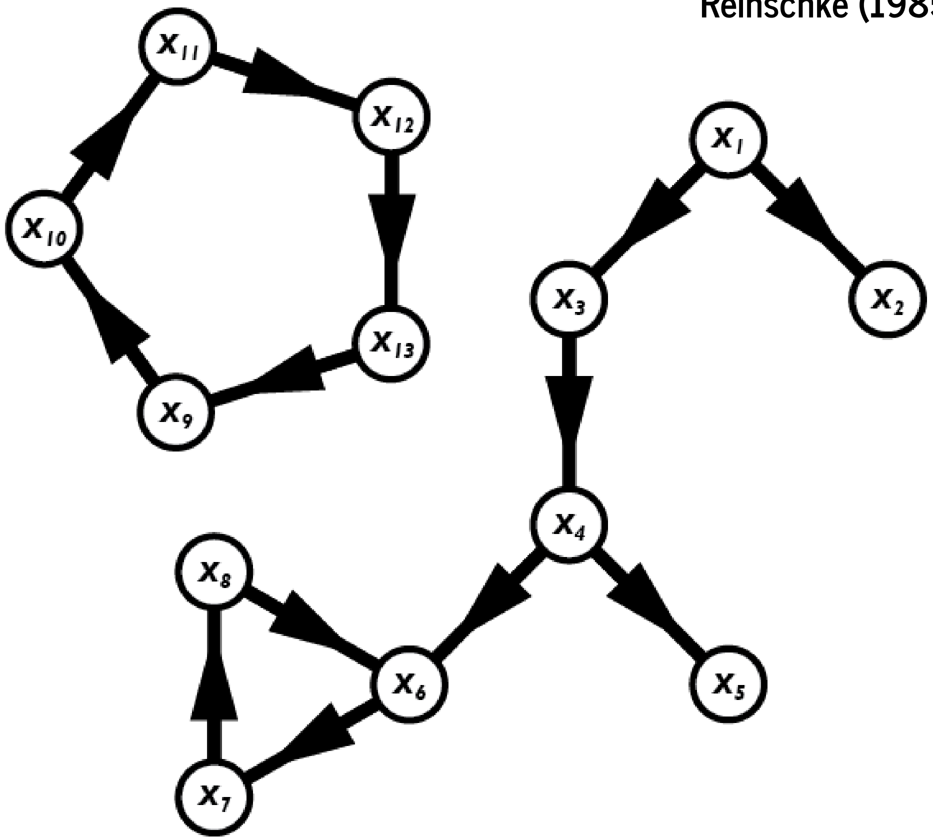
image remade from Roger Brockett's Finite Dimensional Linear Systems

structural control

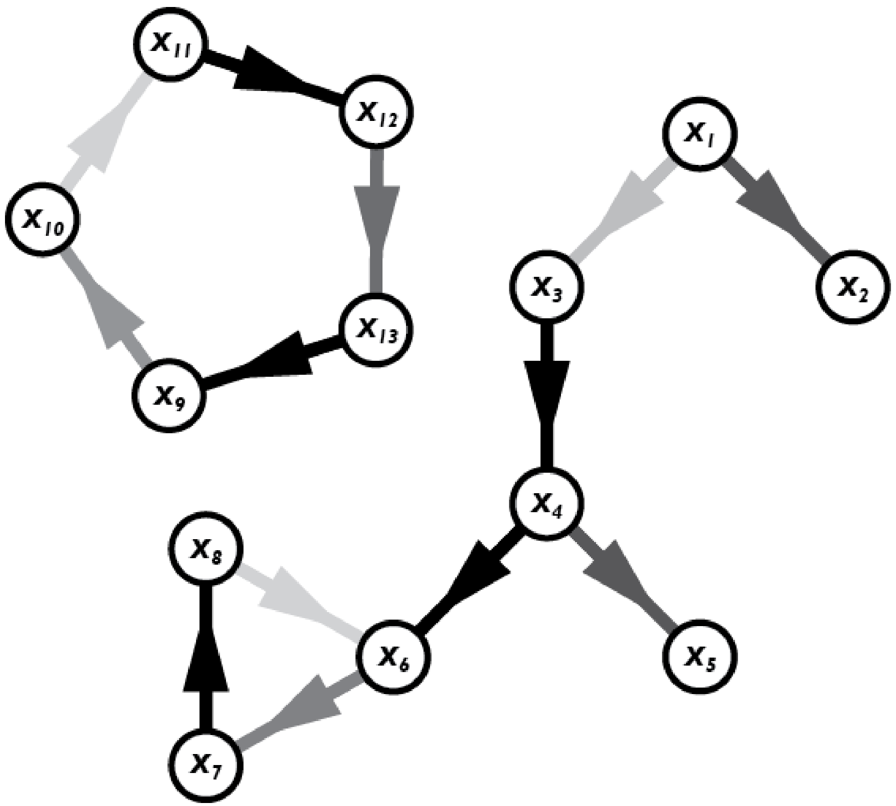
Lin (1974)
Shields & Pearson (1976)
Glover & Silverman (1976)
Schizas & Evans (1981)
Mayeda (1981)
Hosoe (1980)
Reinschke (1985)



weighted

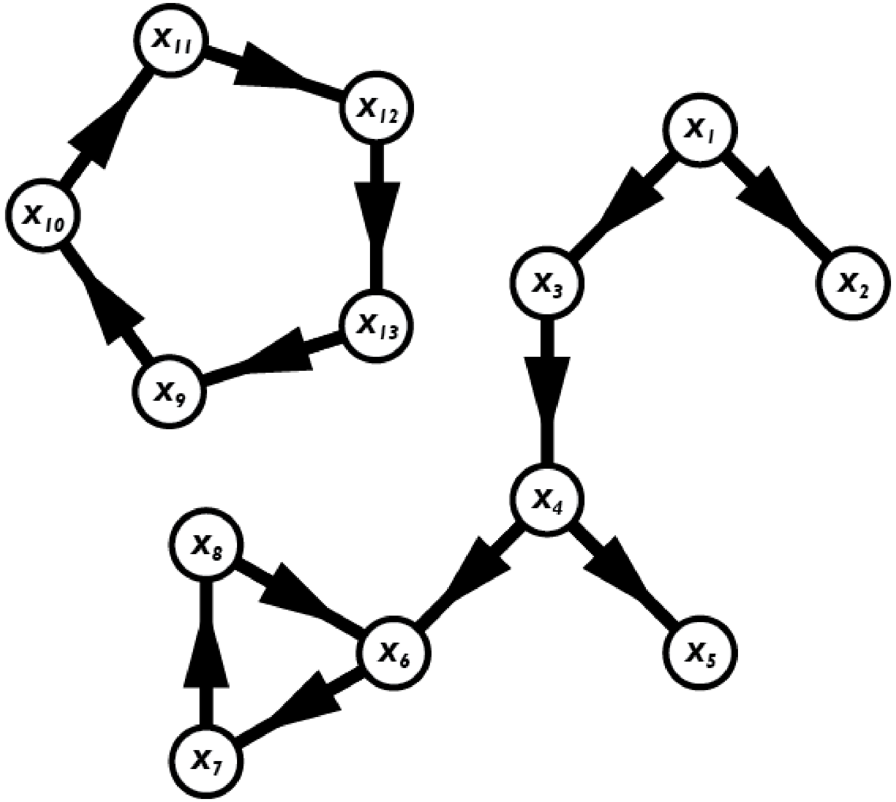


structured



$$\frac{d}{dt}x(t) = Ax(t)$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix}$$



$$\frac{d}{dt}x(t) = \tilde{A}x(t)$$

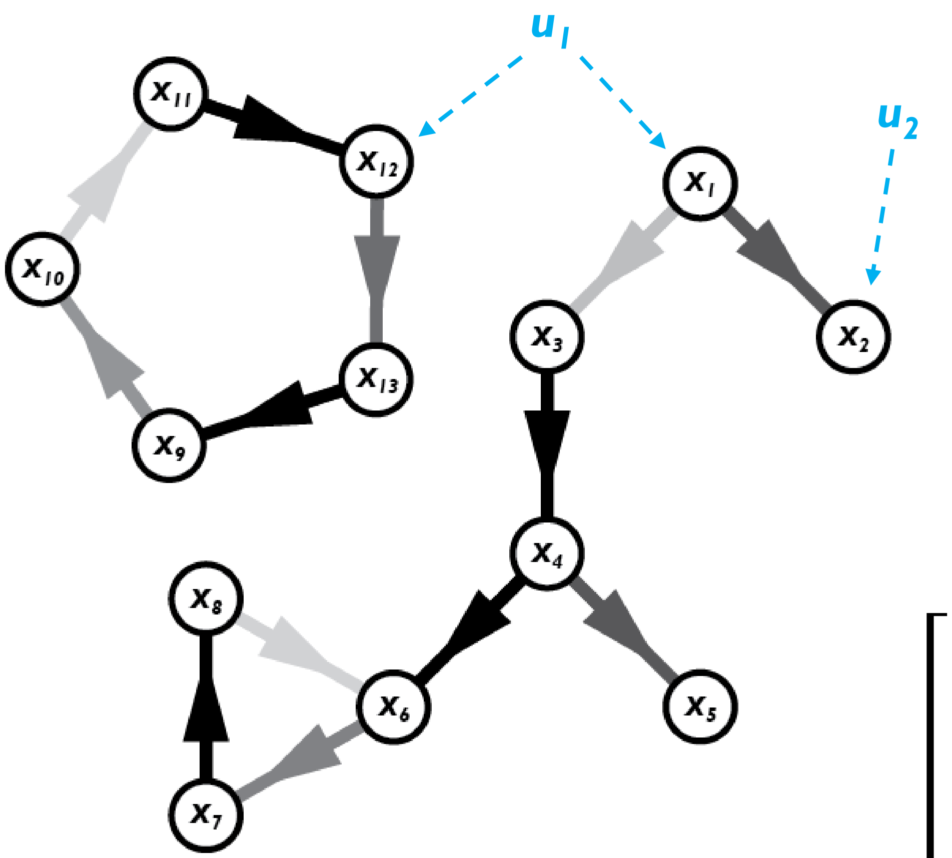
independent parameters

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix}$$

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

- Control typically looks at the problem **given A and B**,
  - Is the system controllable?
  - What is the reachable subspace?

- New problems arising from networks ask **given A**,
  - What is the **B** that guarantees controllability?
  - What are the characteristics of this **B**?

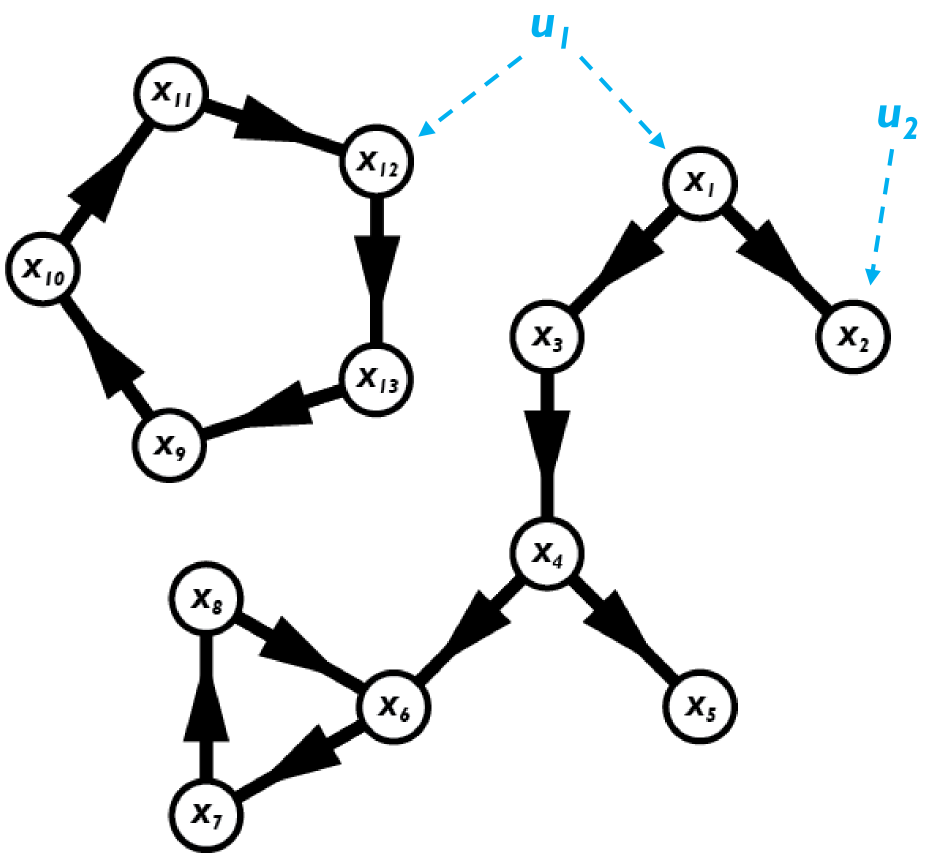


$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

$$rank[W(t_0, t_1)] \\ = rank[B, AB, A^2B, \dots, A^{n-1}B]$$

Kalman rank condition

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix}$$



$$\frac{d}{dt}x(t) = \tilde{A}x(t) + \tilde{B}u(t)$$

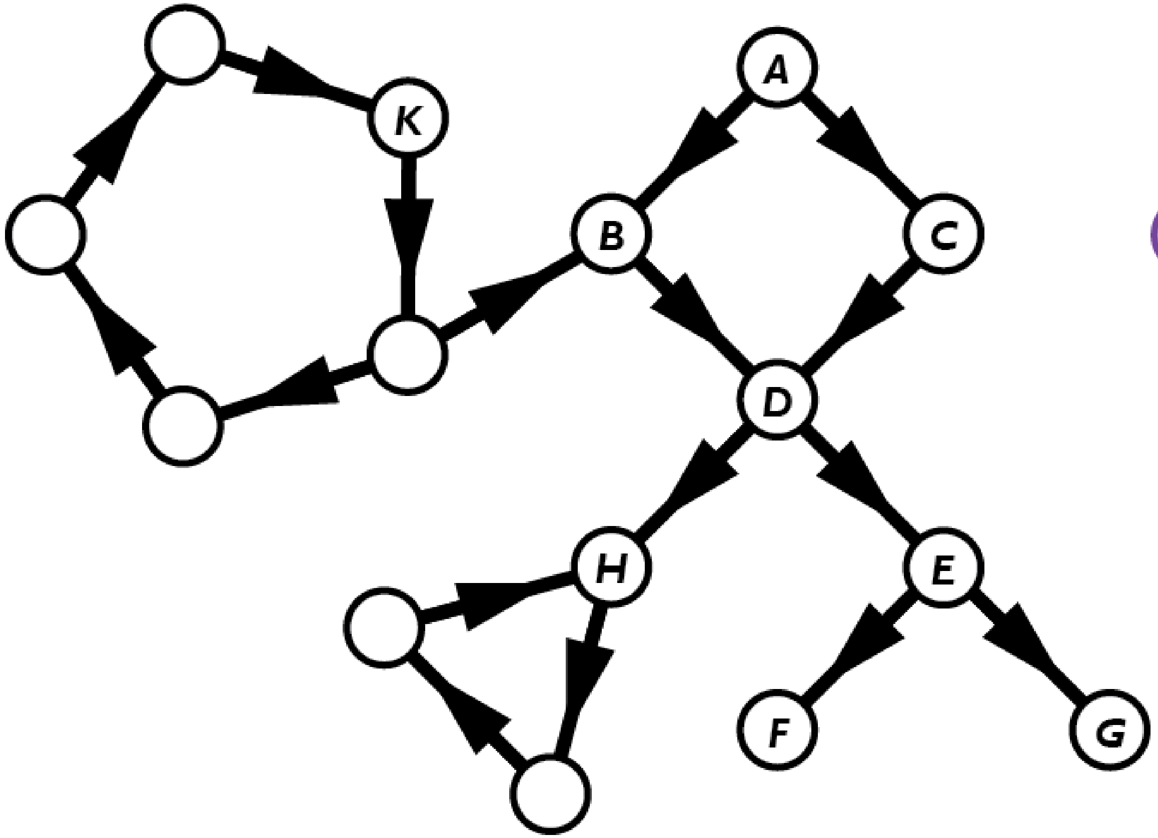
$$\begin{aligned} rank[W(t_0, t_1)] \\ = grank[\tilde{A}, \tilde{B}] \end{aligned}$$

except for a set of measure zero

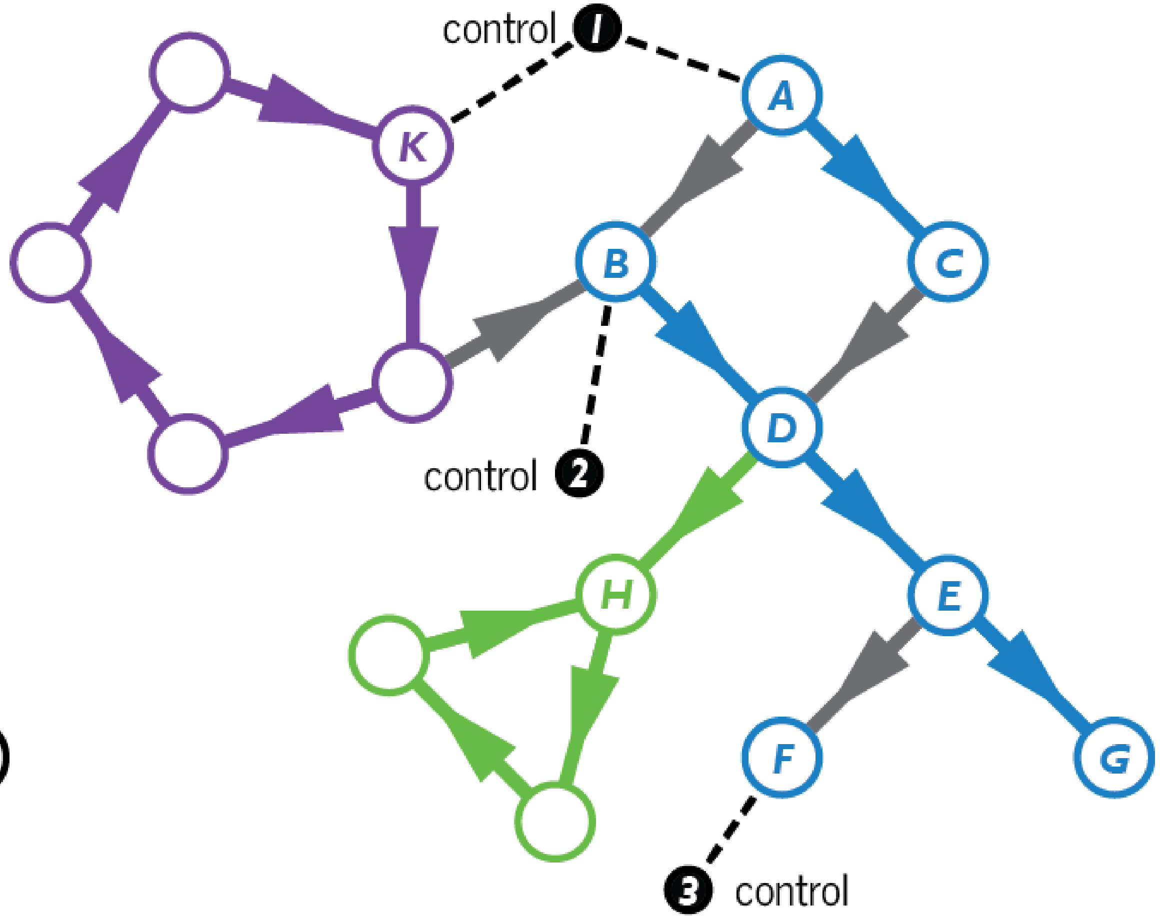
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{13} \end{bmatrix}$$

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

- The system (A,B) is structurally controllable if and only if:
  - the system is irreducible and the generic rank of [A B] = n;
  - there exists a vertex disjoint union of cacti that covers all state vertices;
  - every state vertex is the end of a U-rooted path and there exists a disjoint union of a U-rooted path family and a cycle family that covers all state vertices.



directed network



control structure (cacti)

reducibility: there are nodes not reachable from the controls (i.e., union of cacti do not cover all nodes)

$$P^T A P = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

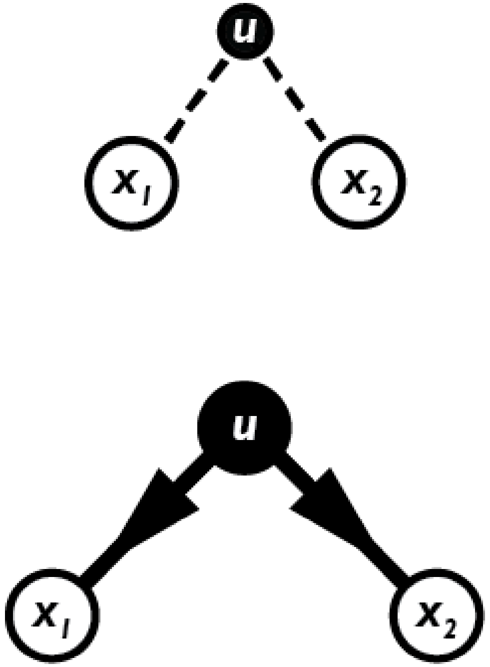
$$P^T B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}$$

$$\Rightarrow \quad \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u(t)$$

g-rank of [A B] = n: there are sufficient independent controls (i.e., vertex disjoint cacti)

generic properties hold for almost all values (i.e., they do not hold for a measure-zero set)

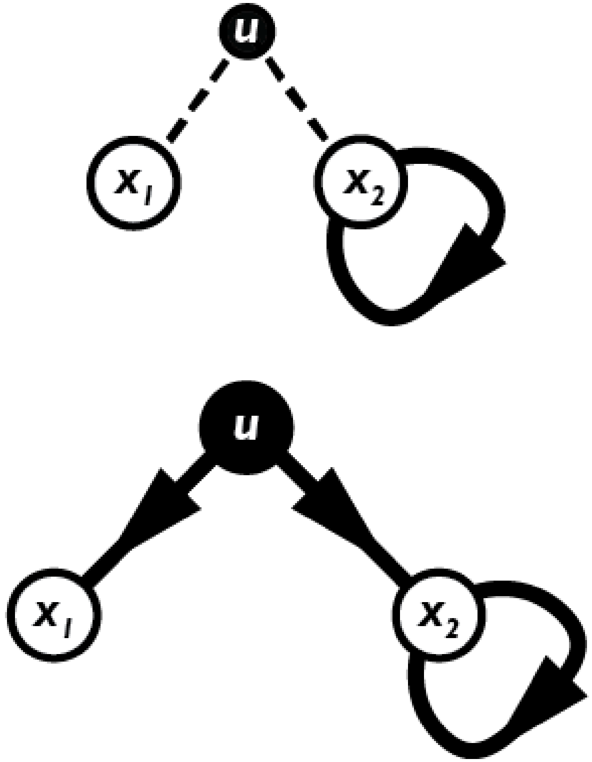
irreducible
g-rank < n



$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} * \\ * \end{pmatrix}$$

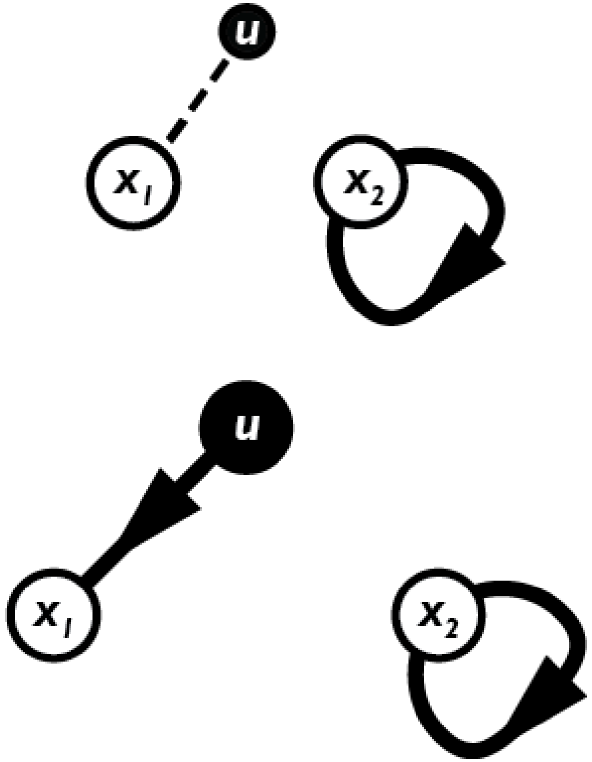
irreducible
g-rank = n



$$A = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

$$B = \begin{pmatrix} * \\ * \end{pmatrix}$$

reducible
g-rank = n



$$A = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

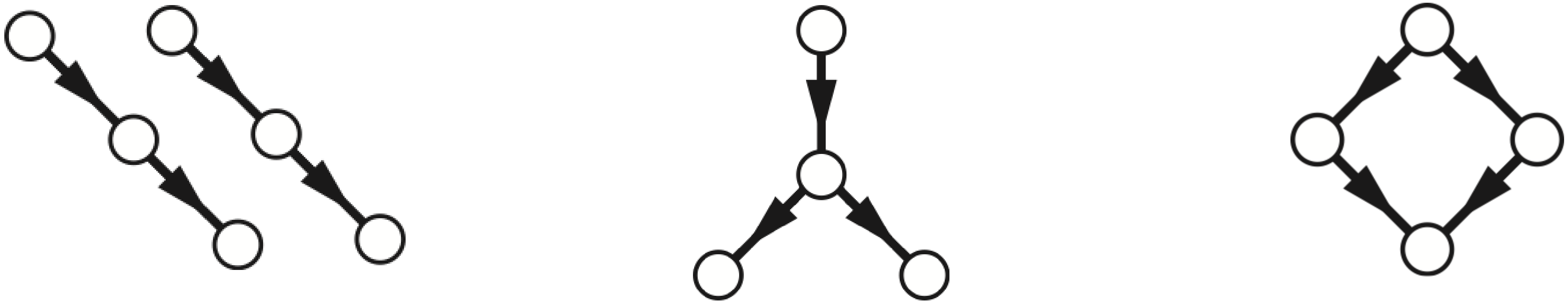
$$B = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

dilation

$$|T(S)| < |S|$$

set of nodes with edges pointing to S

a subset of nodes

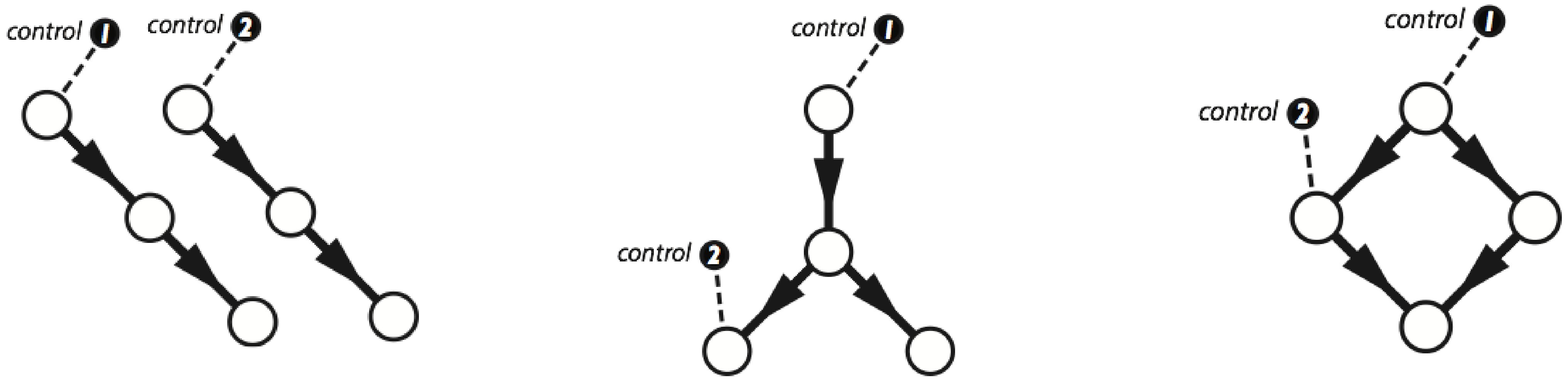


dilation

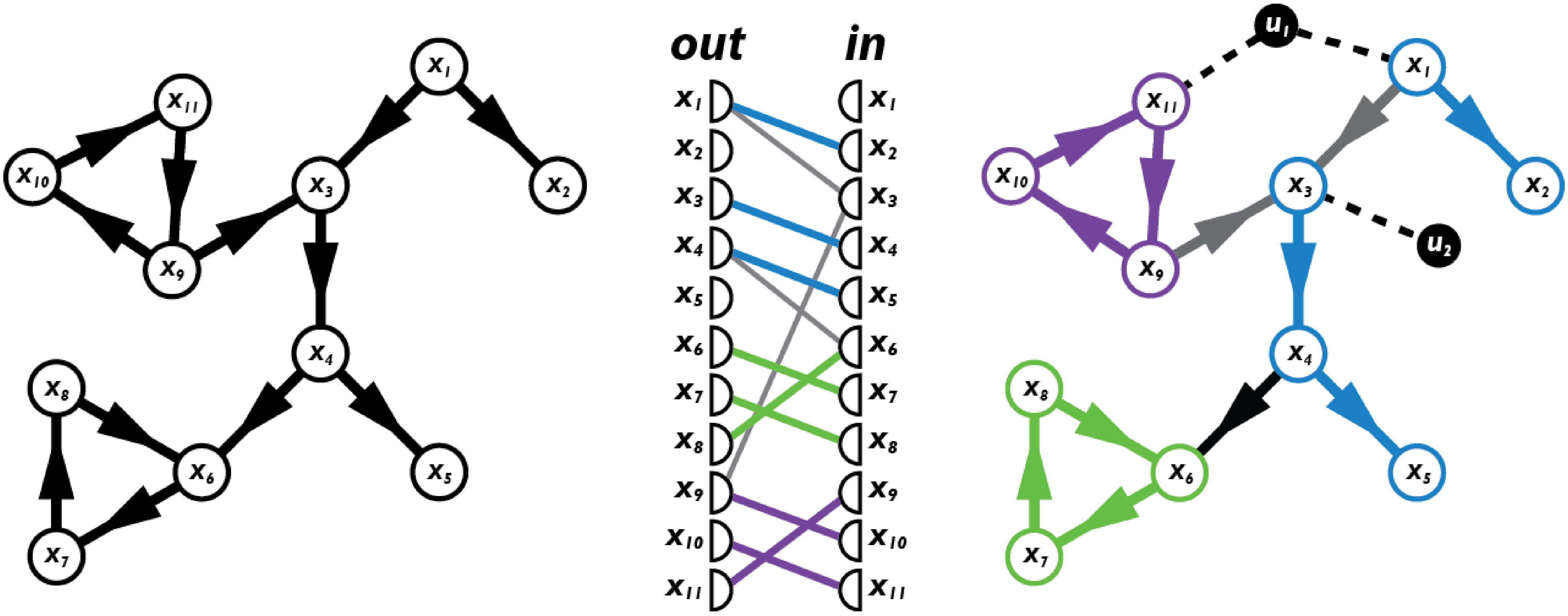
$$|T(S)| < |S|$$

set of nodes with edges pointing to S

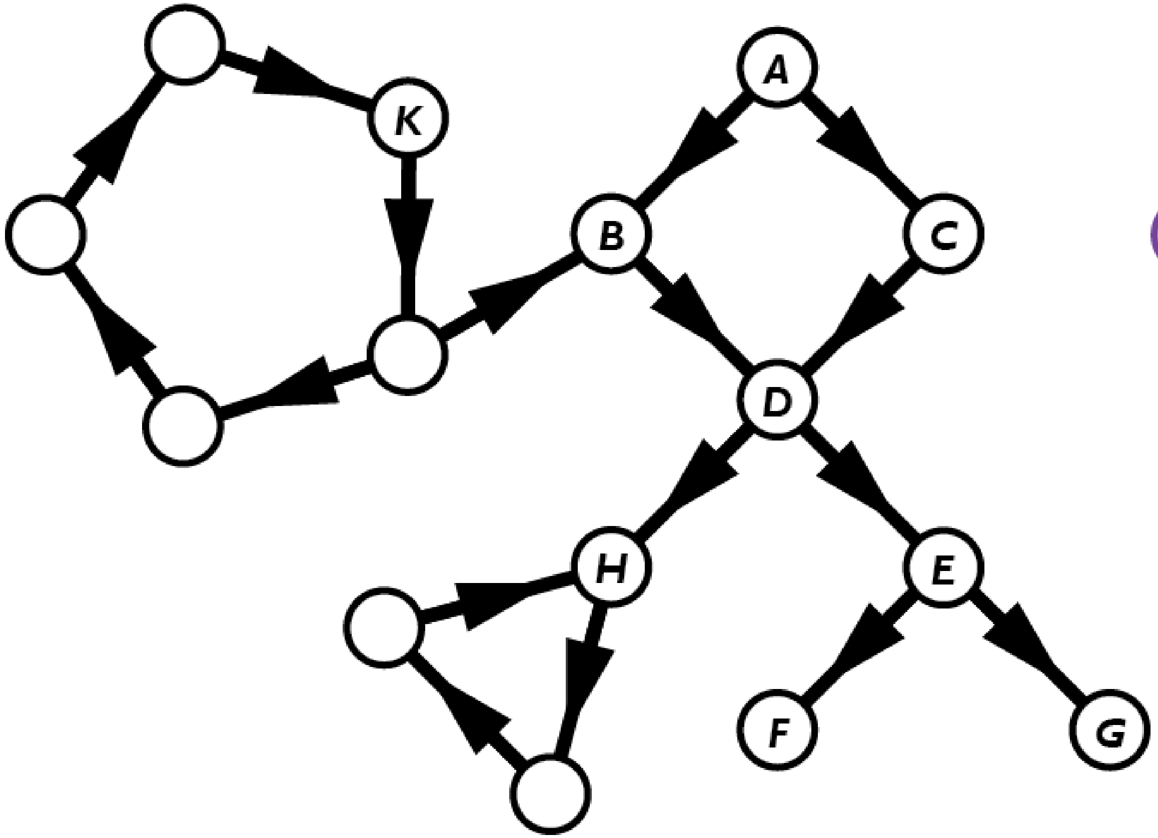
a subset of nodes



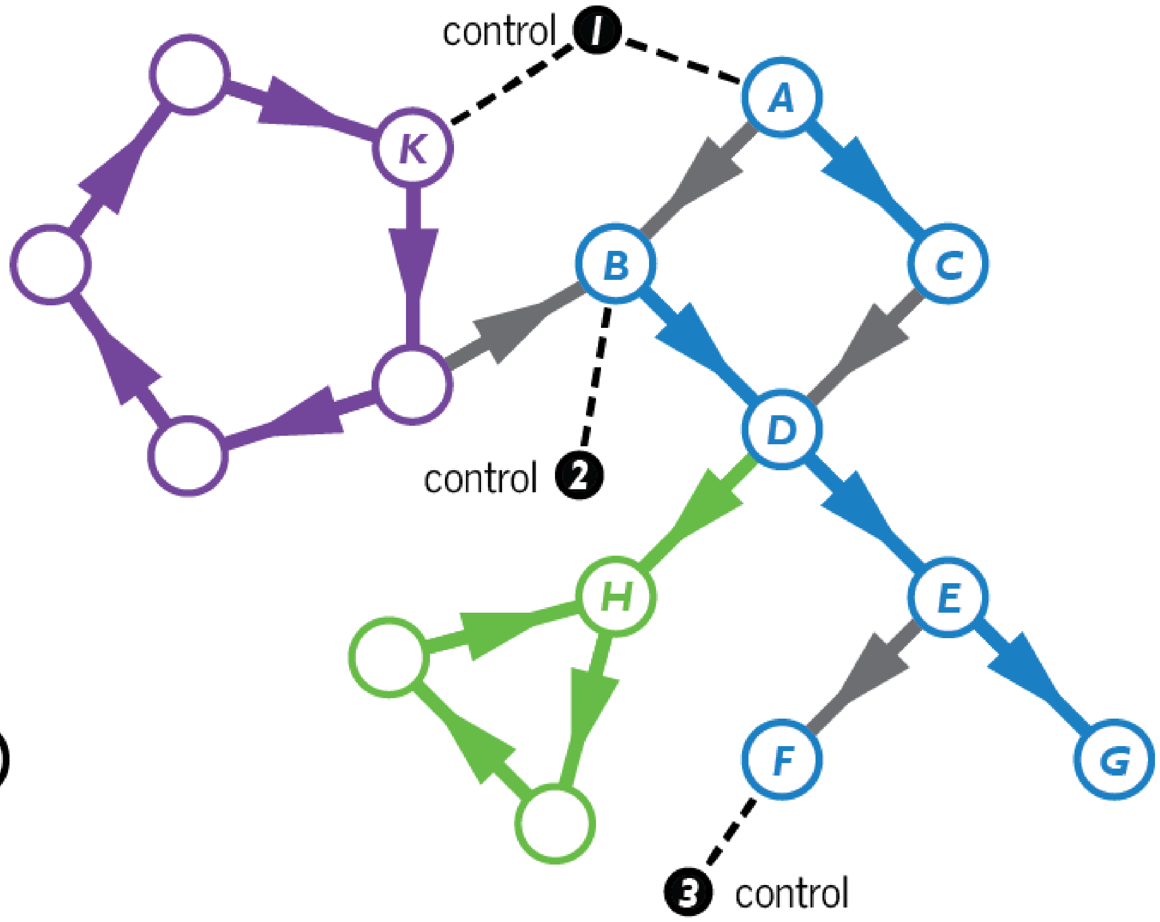
maximum matching



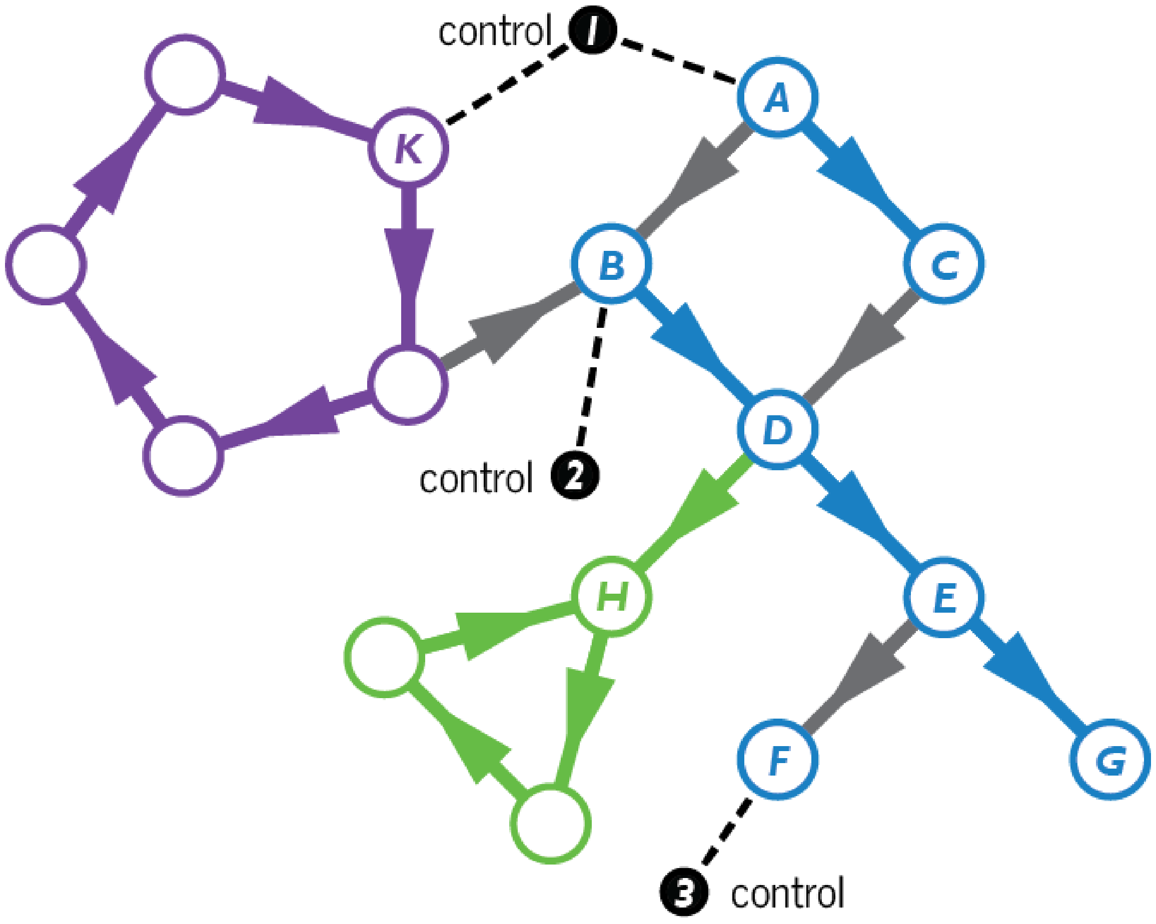
Murota, Matrices and Matroids for Systems Analysis, 2000



directed network



control structure (cacti)



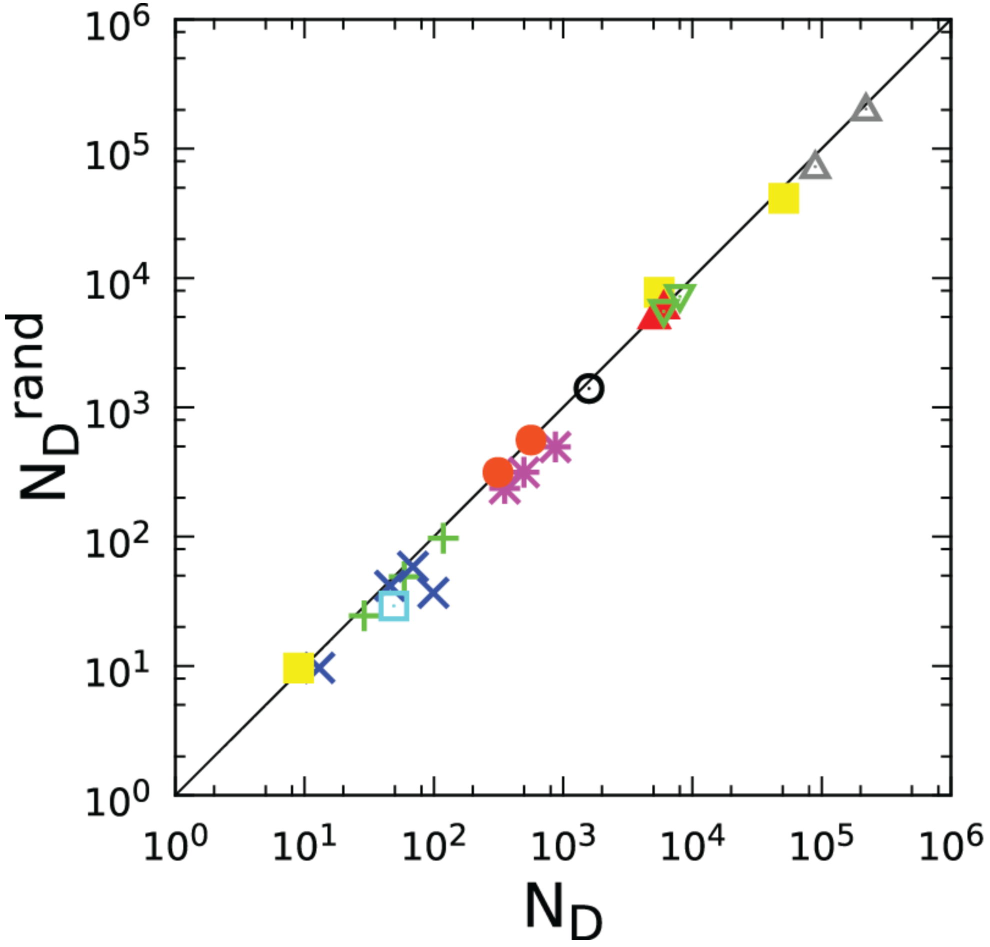
stems

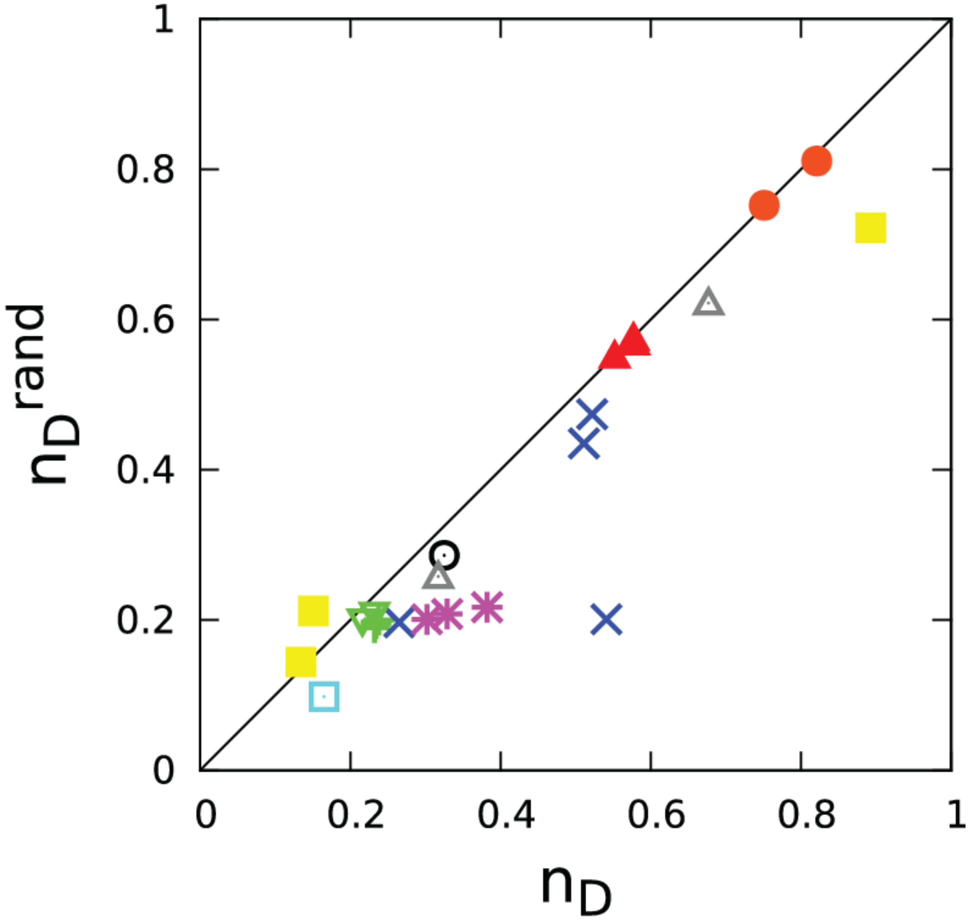
buds via distinguished edges

cycles must become buds

control structure (cacti)

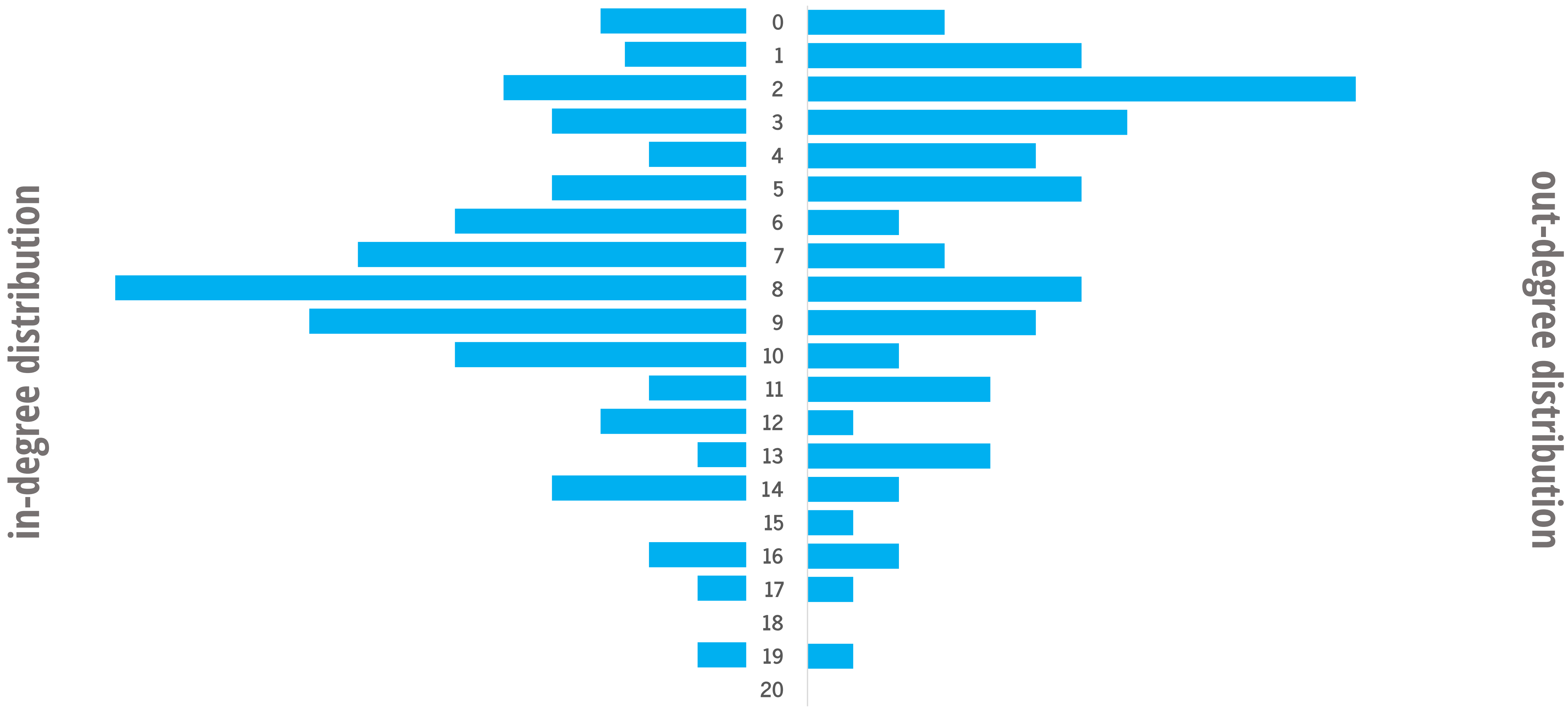
correlation with degree distribution





Liu, Slotine, Barabasi (2011)

Posfai, Liu, Slotine, Barabasi (2013)


in-degree distribution
0
1
2
3
4
5
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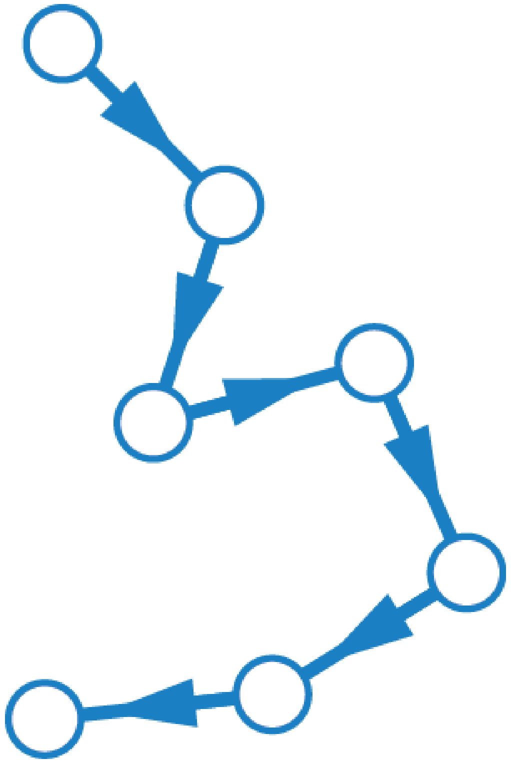
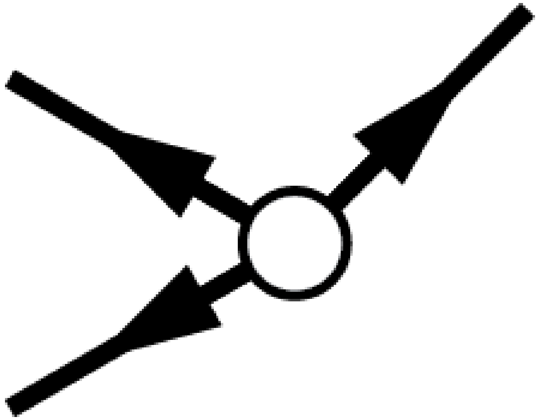
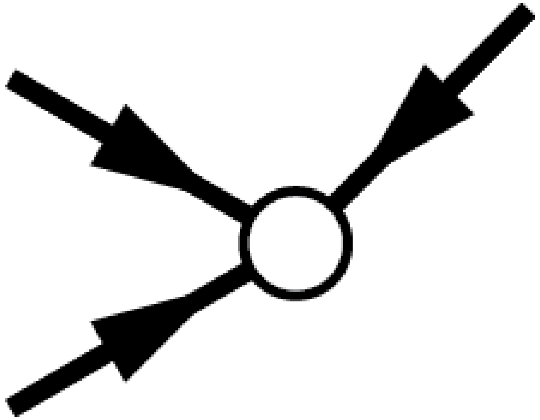
11
12
13
14
15
16
17
18
19
20
out-degree distribution





source

N_s

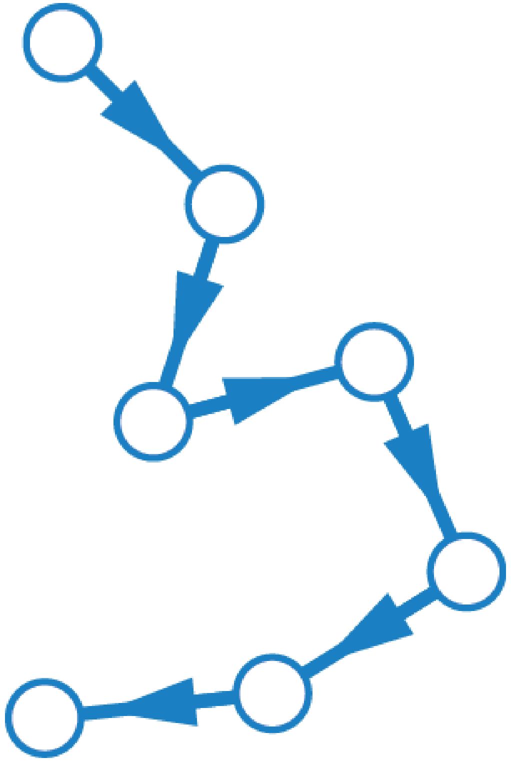


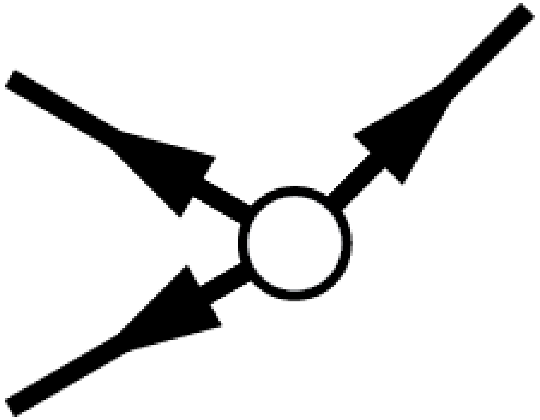
sink

N_t

number of stems

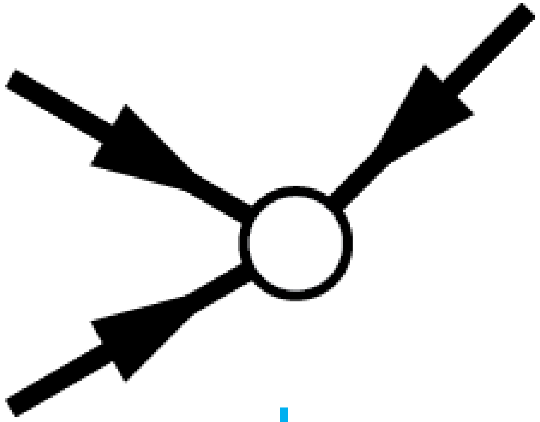
$\max(N_s, N_t)$





source

N_s



surplus

sink

$N_e = \max(0, N_t - N_s)$

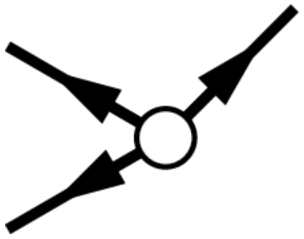
number of stems

$N_s + N_e$

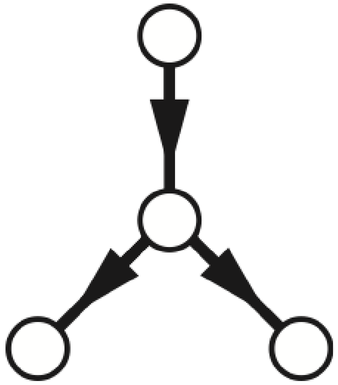
number of controls

$$N_c = N_s + N_e + N_i$$

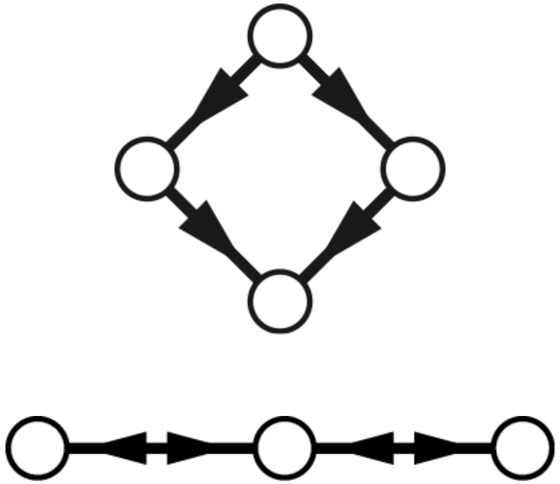
sources

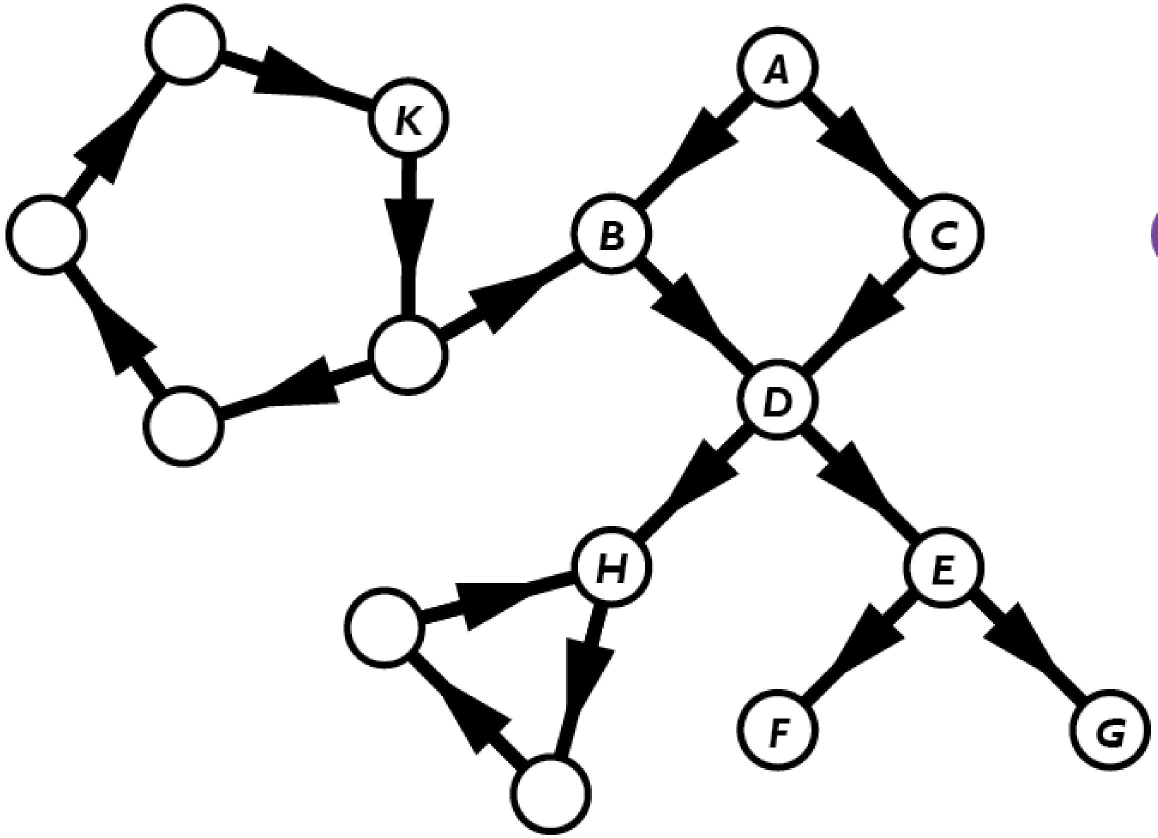


external dilations
(surplus sinks)

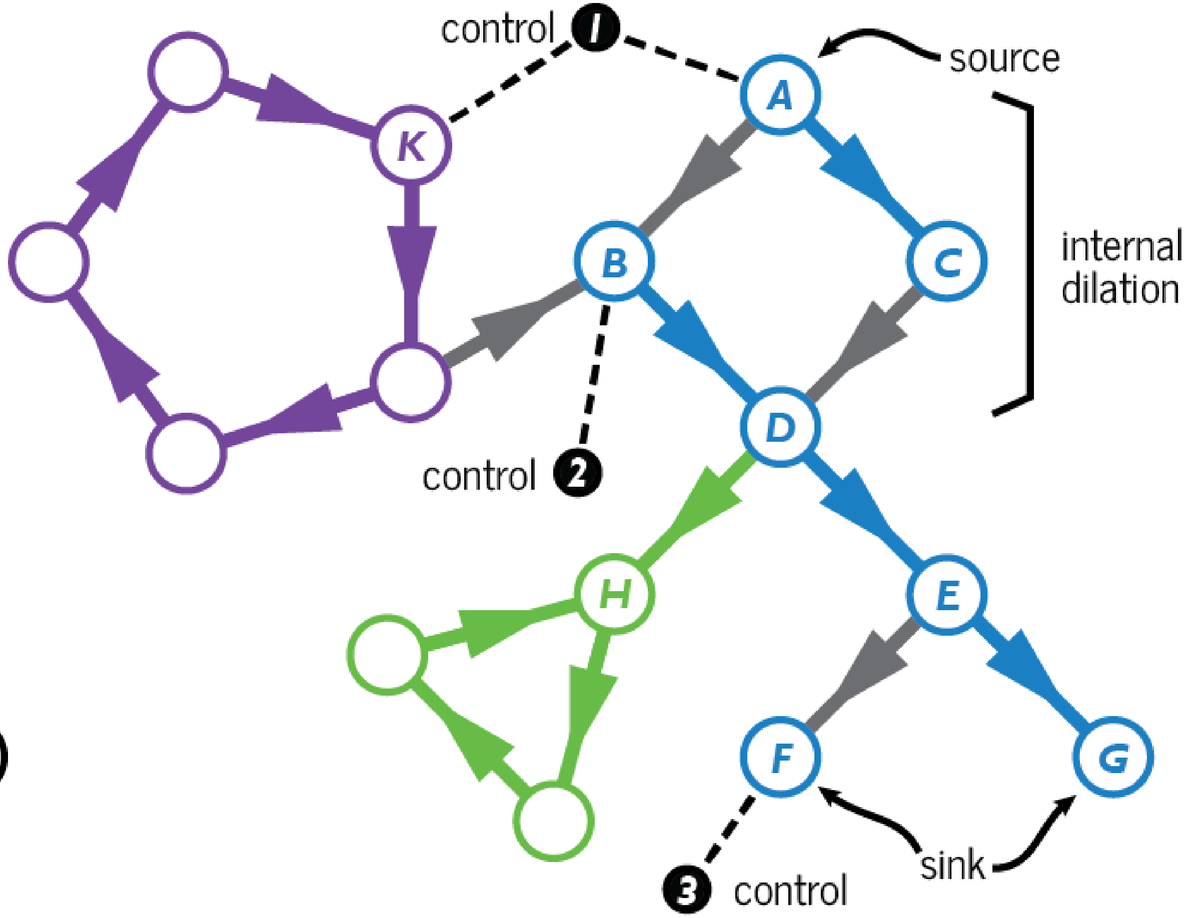


internal dilations





directed network



control structure (cacti)

- Erdos-Renyi

  p is the probability that any two nodes have and edge between them

- Barabasi-Albert

  with each new node m edges are added such that the probability to connect to a node is proportional to its degree

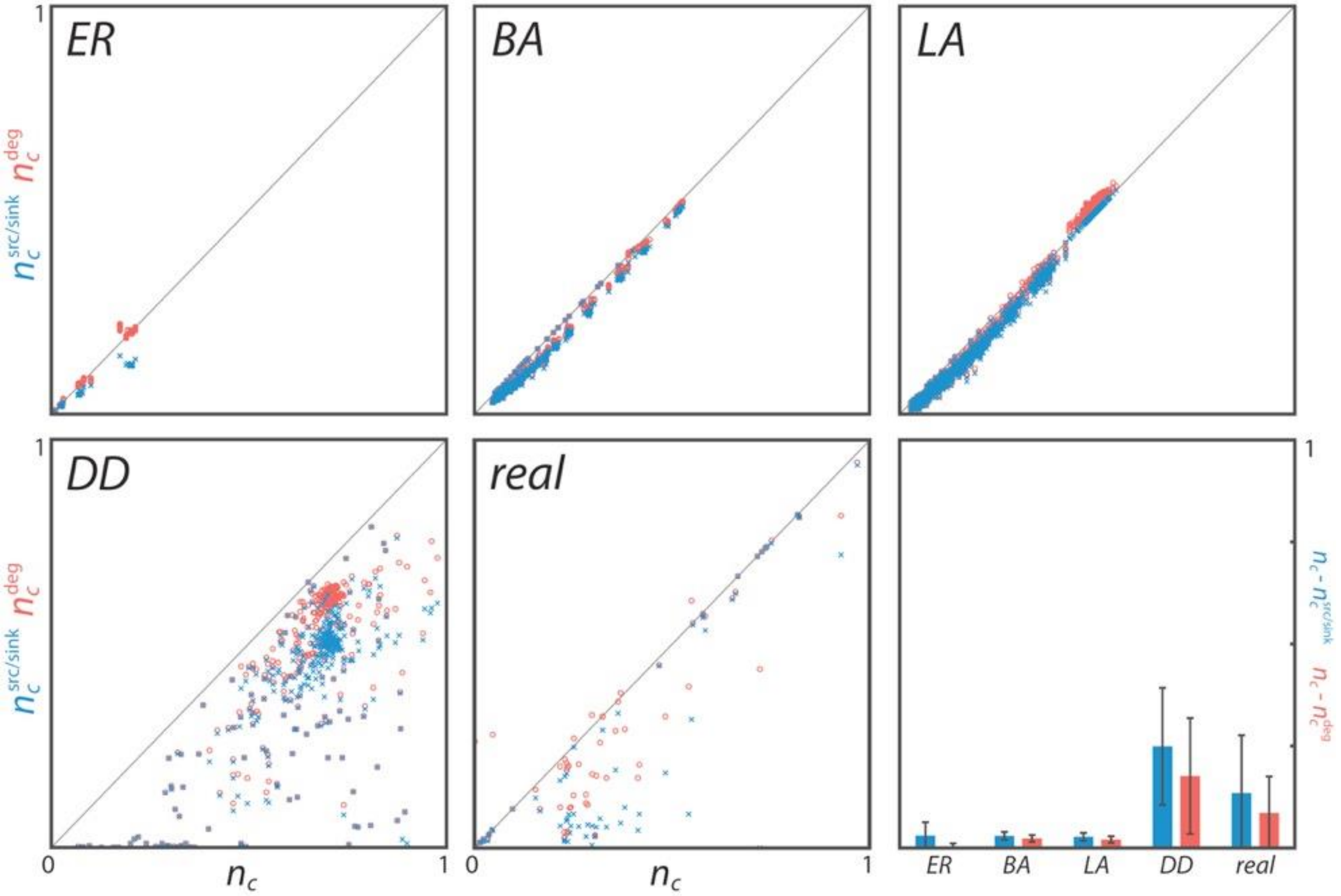
- Local Attachment

  with each new node, m edges are added. r are added randomly (the parents) and m-r are added to the outbound neighbors of the parents.

- Duplication-Divergence

  duplicate a randomly chosen node and retain each edge with probability s.

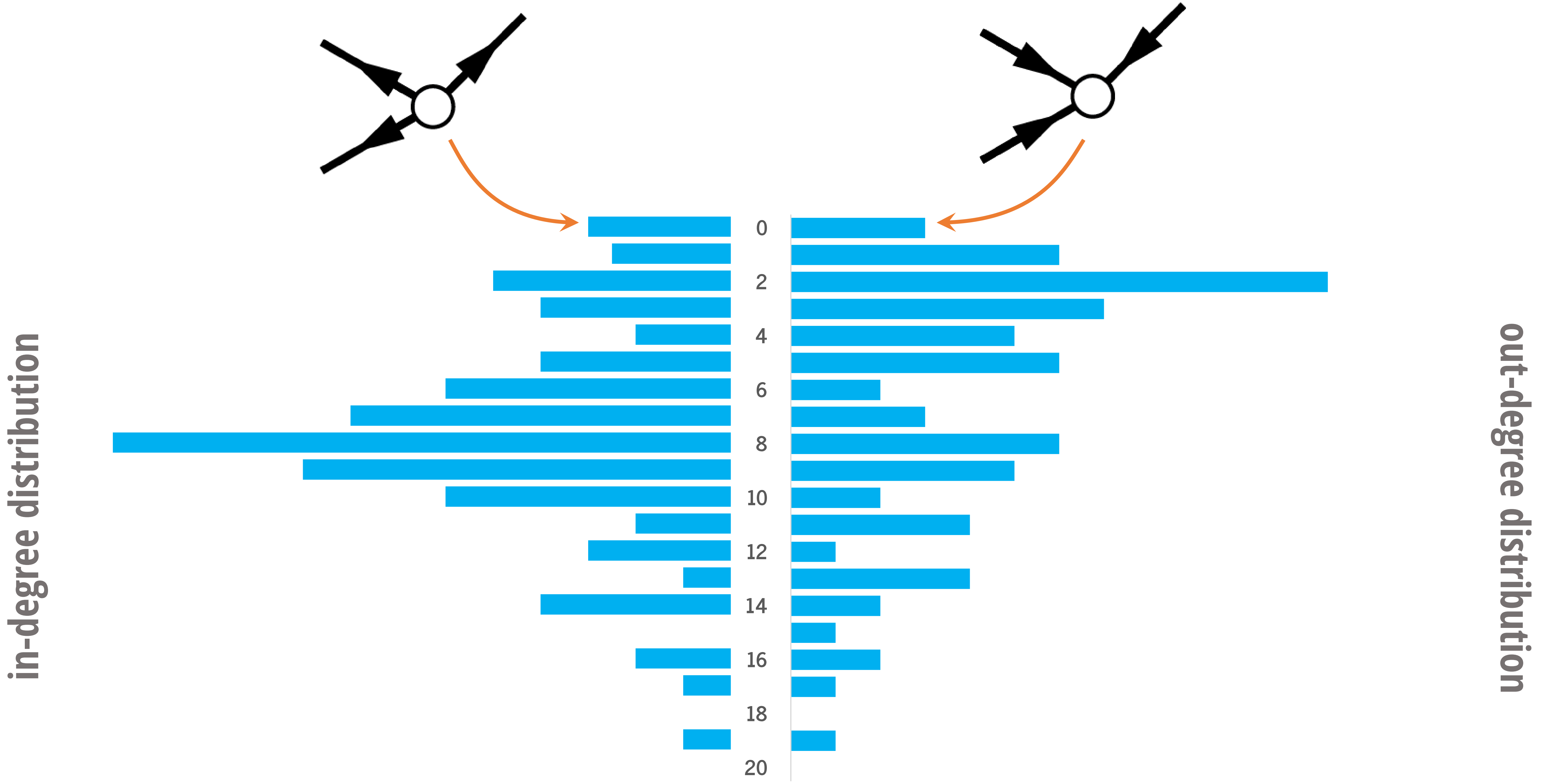
$$N_c = N_s + N_e + N_i$$



full degree distribution reduces prediction error by only:

	$n_c^{deg} - n_c^{src/snk}$
ER	2.7%
BA	0.6%
LA	0.7%
DD	7.3%
real	5.1%

Ruths, Ruths (2014)



in-degree distribution
out-degree distribution
0
2
4
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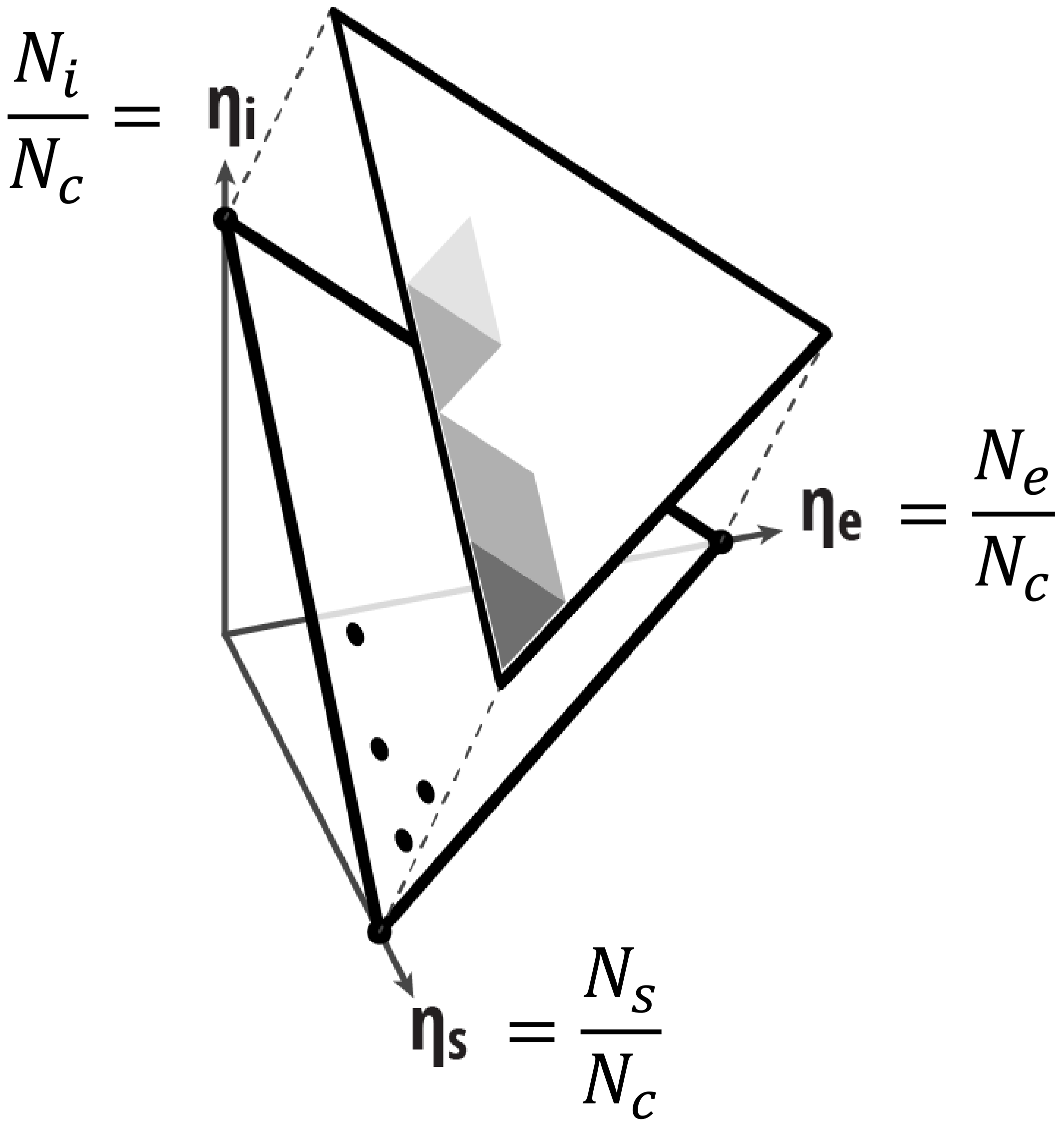
20

control profile



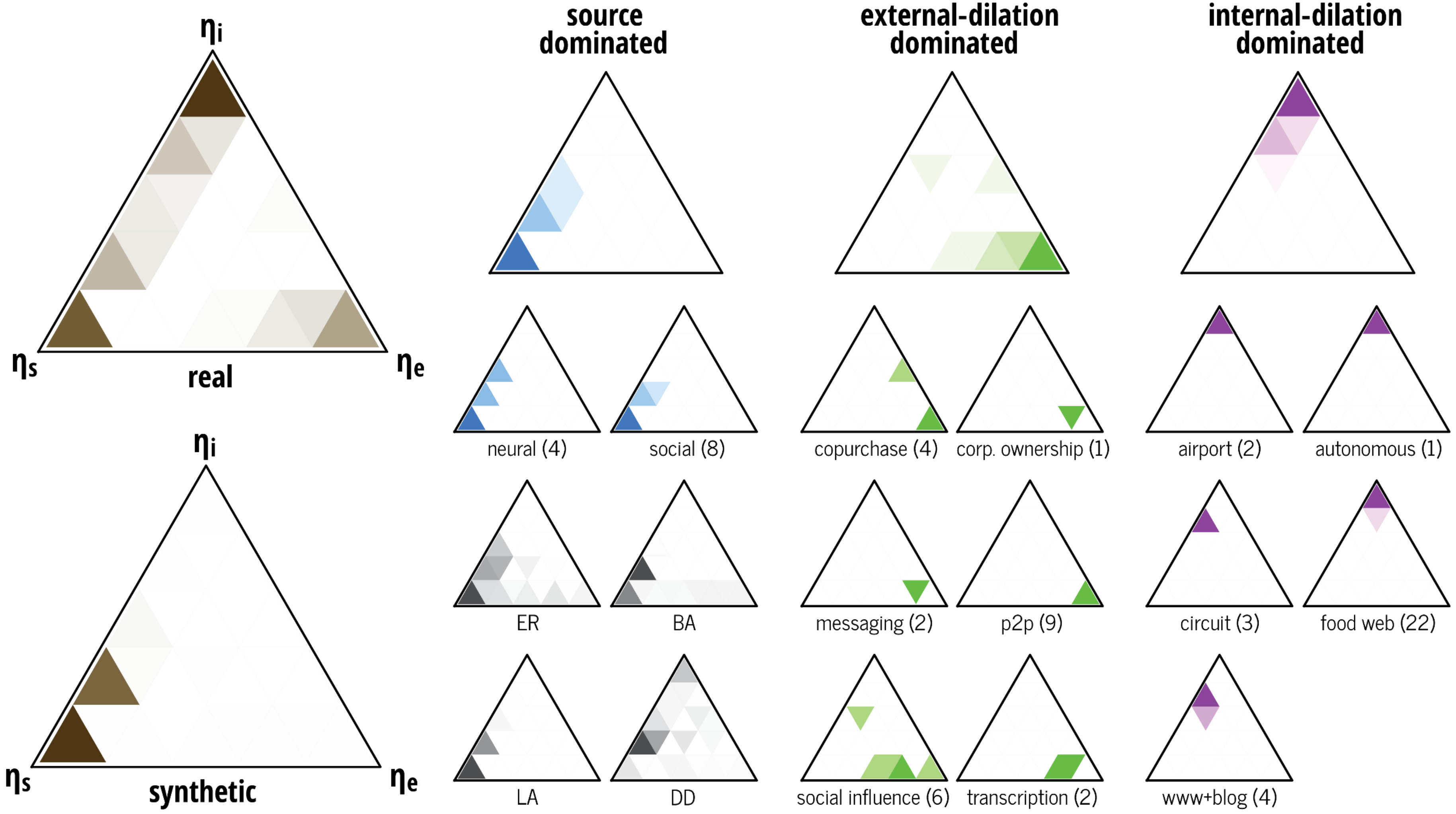
(η_s, η_e, η_i)

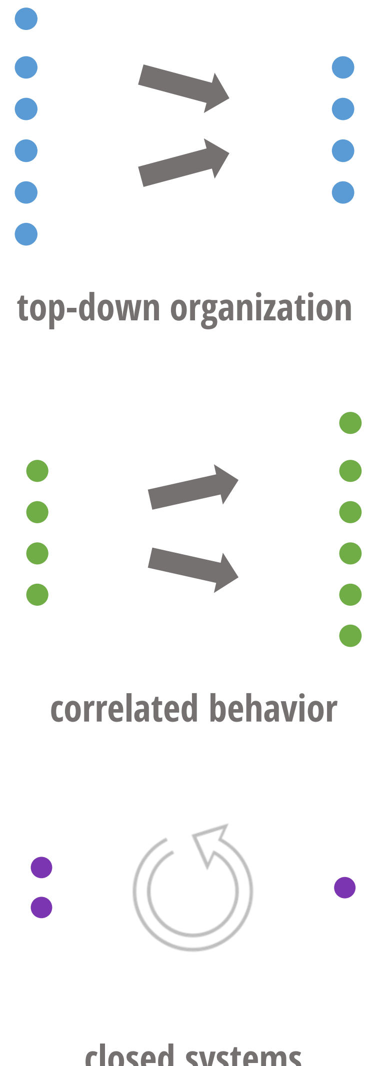
$$1 = \eta_s + \eta_e + \eta_i$$

Ruths & Ruths, Science (2015)

functional breakdown

$$1 = \eta_s + \eta_e + \eta_i$$





Ruths, Ruths (2014)

controllable subspace

Kalman rank condition: $rank[B, AB, A^2B, \dots, A^{n-1}B]$

generic rank: $grank[\tilde{A}, \tilde{B}]$

integer linear program → control centrality

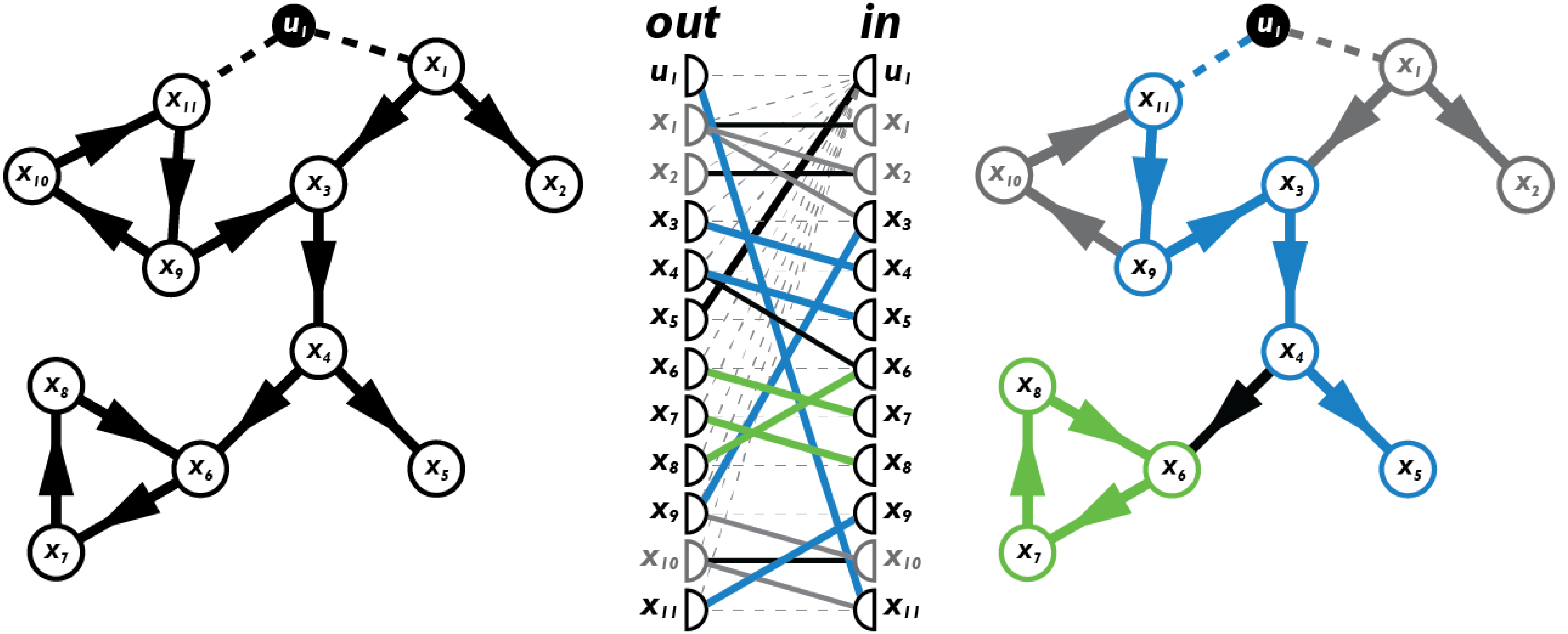
[Hosoe (1980), Poljak (1990)] [Liu, Slotine, Barabasi (2012)]

$\leftrightarrow$

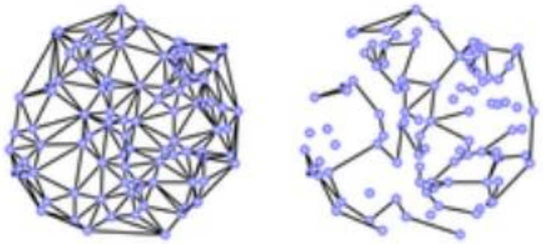
weighted maximum matching → robustness of controllablity

[??, Ruths (2014, 2016)] [??, Ruths (2014, 2016)]

controllable subspace



robustness of network controllability



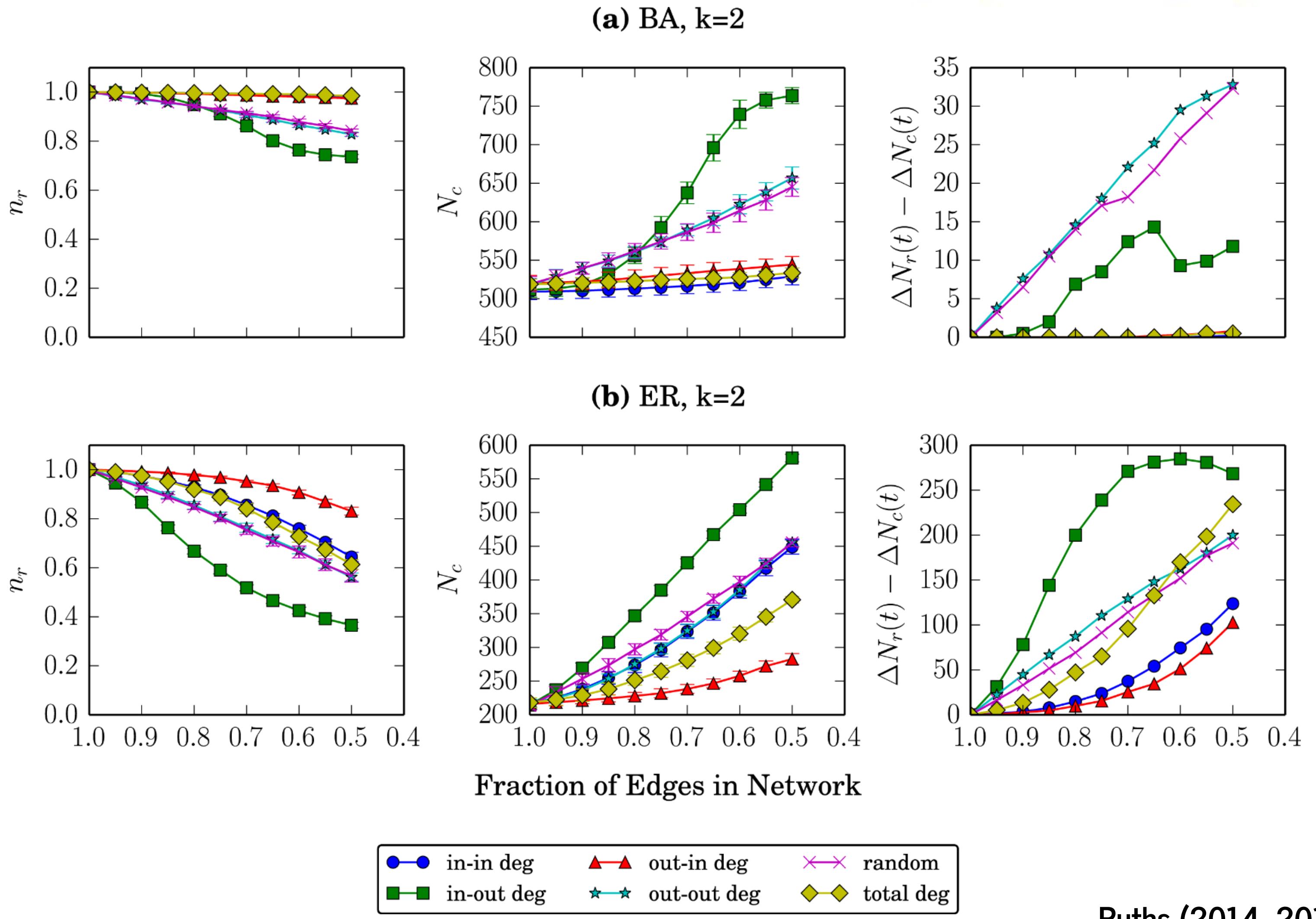
control-based robustness

increase of controls required with failure or attack

reachability-based robustness

decrease in control due to failure or attack

across network classes
attack types
control reconfigurations



Ruths (2014, 2016)

a bilinear model

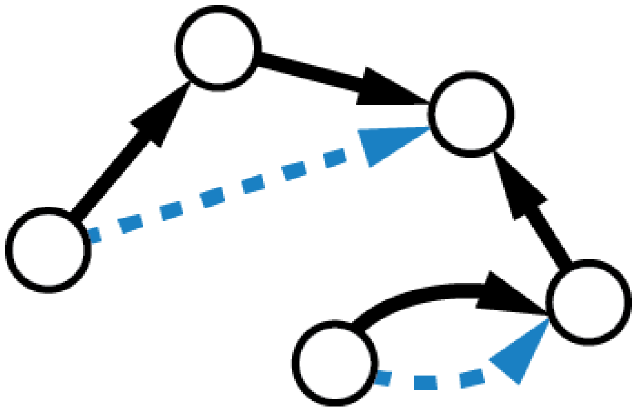
$$\frac{d}{dt}x = Ax(t) + u(t)Bx(t) = (A + uB)x$$

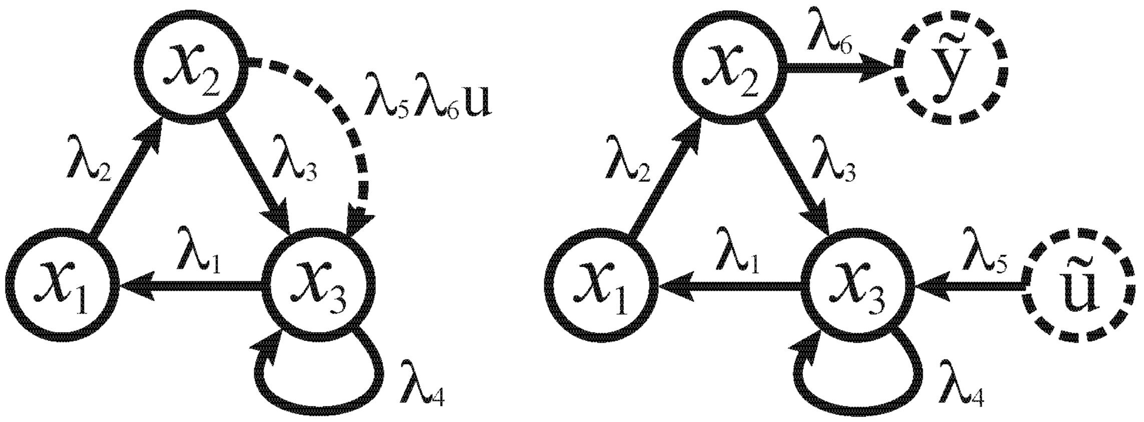
linear model corresponds to control applied exogenously

control directly modulates the state of the nodes/agents

bilinear model corresponds to control applied endogenously

control modulates the interaction between nodes/agents

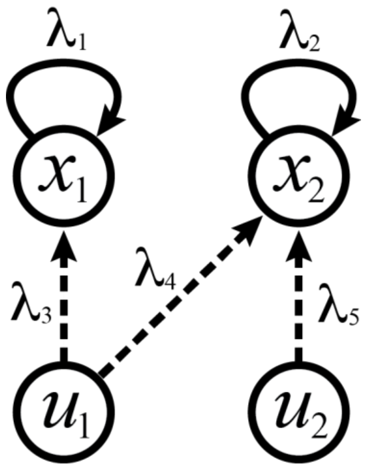




Ghosh, Ruths (2016)

closing the gap between structural controllability and controllability

what are the parameter sets for which a system is structurally controllable but not classically controllable?



$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad B = \begin{bmatrix} \lambda_3 & 0 \\ \lambda_4 & \lambda_5 \end{bmatrix}$$

$$C = \begin{bmatrix} \lambda_3 & 0 & \lambda_1\lambda_3 & 0 \\ \lambda_4 & \lambda_5 & \lambda_2\lambda_4 & \lambda_2\lambda_5 \end{bmatrix}$$

$$\left(\lambda_i \ldots \lambda_{i+\ell_1}\right)^{1/\ell_1} = \left(\lambda_j \ldots \lambda_{j+\ell_2}\right)^{1/\ell_2}$$

$$\psi(\boldsymbol{\lambda}) = (\lambda_3\lambda_5)^2 + [\lambda_3\lambda_4(\lambda_2 - \lambda_1)]^2 + (\lambda_2\lambda_3\lambda_5)^2 + (\lambda_1\lambda_3\lambda_5)^2 + (\lambda_1\lambda_2\lambda_3\lambda_5)^2$$

$= 0$ if $\lambda_3 = 0$ or $\lambda_4 = \lambda_5 = 0$ or $\lambda_5 = 0, \lambda_1 = \lambda_2$

Ruths (2015)

feasibility of control (input energy)

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \longrightarrow x(T) = e^{AT}x_0 + \int_0^T e^{A(T-t)}Bu(t)\,dt$$

bounds on energy based on eigenvalues
of the reachability Grammian matrix

$$W_T = \int_0^T e^{A(T-t)}BB'e^{A'(T-t)}dt$$

Yan, Ren, Lai, Lai, Li (2012)

resources

zen python library http://zen.networkdynamics.org/

online videos and datasets at http://justinruths.com