The Very Friendly Sequence

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1 Introduction

Sequences and their convergence form some of the most fundamental building blocks in real analysis. It is not surprising that mathematicians enjoy observing sequences with odd behaviors. While many sequences calmly converge to a number, others bounce around from value to value and many sputter off to infinity - never to been seen again! Still there are other, more chaotic examples, who's behavior is difficult to even define correctly. This paper is an observation of one such sequence to which I even bequeathed a name (more on that later). To begin, lets lay some ground work so that we can better understand this very odd sequence of numbers.

2 Sequence Convergence

We define sequence convergence in the following way: We say that a sequence a_n converges to a point a in $\mathbb{R}$ if for all $\varepsilon > 0$ there exists some N such that $\mid a_n - a \mid < \varepsilon$ for all $n > N$. Essentially, what we are saying is that if you consider a sequence of numbers $a_n = (a_1, a_2, a_3, ...)$, if all of the sequence elements after the a_n entry are within ε of a, then our sequence converges. Check out the following example:

Show that $a_n = (\frac{1}{n} | n \in \mathbb{N})$ converges to 0.

Allow $\varepsilon > 0$ and consider when $N = \frac{1}{\varepsilon}$. Using our definition, whenever $N < n$:

$$\begin{aligned} \mid \frac{1}{n} - 0 \mid &= \mid \frac{1}{n} \mid \\ &= \frac{\mid 1 \mid}{\mid n \mid} < \frac{1}{N} = \varepsilon. \end{aligned}$$

□

Short and sweet but there is a lot going on here! Specifically where the "<" sign occurs this is because as n becomes larger and larger than N the fraction with n as the denominator becomes smaller and smaller. Think about the fraction $\varepsilon = \frac{1}{2}$ and then consider $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$ In this case $N = 2$ and $n = 3, 4, 5, ...$ The real power in this proof is that we aren't picking an N and we are instead picking an ε. How close is our sequence to 0? It is ε close. And just exactly how close is that? When we pick an ε, by the density of $\mathbb{R}$, there is always another number we could have picked that is closer to where the sequence converges. We could go closer and closer and... I think that you get the idea! Mathematicians have mostly drawn the line to say that convergence is based on being "epsilon close".

3 Subsequence Convergence

Lets think about our sequence a_n again:

$$a_n = \{\frac{1}{n} | n \in \mathbb{N}\} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...).$$

Notice that n starts at 1 and goes odd towards positive infinity.

We define a **subsequence** (S_n) of a sequence (B_n) in the following way: Allow $n_1 < n_2 < n_3 < ...$ be an increasing sequence of integers. Then $S_n = \{b_{n_1}, b_{n_2}, b_{n_3}, ...\}$ is called a subsequence of B_n.

Just like their counterparts, subsequences can converge as well. Consider the sequence

$$T_n = (1, \pi, \frac{1}{100}, 1, \pi, \frac{1}{100}, 1, \pi, \frac{1}{100}, ...).$$

If we composed a subsequence of all of all the "1" entries, and two other subsequences filled with the π and $\frac{1}{100}$ entries respectively, then we would have three different subsequences converging to three very different places. These subsequences would look like this:

$$S_1 = (1, 1, 1, ...)$$
$$S_\pi = (\pi, \pi, \pi, ...)$$
$$S_{\frac{1}{100}} = \left(\frac{1}{100}, \frac{1}{100}, \frac{1}{100}, ...\right).$$

An important point is that a sequence converges to a point a if and only if <u>all</u> of its subsequences also converge to the same point a. "How does this work?", "This is crazy talk!?", you might say. Well, Let me show you.

For the sake of contradiction assume that a sequence $A_n \to c$ and a subsequence of A_n, $S_n \to d$ for $c, d \in \mathbb{R}$. By definition, for $\varepsilon > 0$ for all $n > N$, $| a_n - a | < \varepsilon$. Allow $M = | c - d |$ and allow $\varepsilon = \frac{M}{2}$. Thus every entry past a point N is between $c \pm \varepsilon$ or $c \pm \frac{M}{2}$. This is a contradiction because as $n \to \infty$, $S_n \to d$ and $A_n \to c$. Most importantly, since $M = | c - d |$ and $\varepsilon = \frac{M}{2}$, there are entries in the sequence S_n such that $| s_n - c | > \varepsilon$, such as s_{n+1}.

4 Our Very Friendly Sequence

Now has come the time for me to introduce to you my new friend. In fact this sequence is so friendly I have named it "**The Very Friendly Sequence**" or TVFS. From here on out we will refer to TVFS as F_n - because it is just so friendly! Meet your new friend:

$$F_n = \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6} ...\right).$$

Okay, so what is the big deal? It just looks like a bunch of fractions strung together! What if I told you that this sequence's subsequences converge to every real number in the interval $[0, 1]$. So every rational and irrational number in $[0, 1]$ is the convergence point of some subsequence contained in F_n. It is also pretty neat that while the sequence has subsequences converging to every real number in $[0, 1]$, the sequence itself does not converge at all!

For ease of digestion, lets break this proof down into parts. Allow (S_n) to be an arbitrary subsequence of (F_n). For our main course we will first show that for $a, b \in [0, 1]$, that $S_n \to a$ for $a \in \mathbb{Q}$ and for dessert that $S_n \to b$ for $b \notin \mathbb{Q}$. Lastly as a digestif, we will show that there exist subsequences of F_n converging to

0 as well. *Bon Appetit!*

The table is now set and we are ready to begin the first part of our proof. Allow $a \in \mathbb{Q}$ such that $0 < a < 1$ where $a = \frac{m}{n}$ such that m and n are relatively prime and $m < n$. Looking at our sequence of numbers it includes exactly these types of quotients. Thus there exists an $a \in A_n$. Now we must look for further sequence entries. Consider $\frac{k}{k} \cdot \frac{m}{n}$ for $k \in \mathbb{Z}$. Since $\frac{k}{k} = 1$, we are keeping the ratio of m and n less than one while finding new elements in our sequence. Thus creating a subsequence of the following form:

$$S_n = \left\{ \frac{km}{kn} \middle| m < n, k \in \mathbb{N} \right\} = \left(\frac{m}{n}, \frac{2m}{2n}, \frac{3m}{3n}, ... \right) = \left(\frac{m}{n}, \frac{m}{n}, \frac{m}{n}, ... \right).$$

The proof is of convergence is simple. We just need to show that a sequence composed solely of $\frac{m}{n}'s$ converges to $\frac{m}{n}$.

Allow $\varepsilon > 0$

$$\left| \frac{m}{n} - \frac{m}{n} \right| = 0 < \varepsilon$$

$\square$

Boom. Done.

Now lets consider something a bit more challenging. Allow $b \notin \mathbb{Q}$. We want to show that there exists a subsequence S_n of F_n such that $S_n \to b$. Consider that every irrational number can be expressed in its decimal form such that $\beta = 0.b_1b_2b_3b_4....$ Where $0 \leq b_n < 10$ and $b_n \in \mathbb{N}$. The b_n's being put together does not mean multiplication – it is just a listing. For instance, $\frac{\pi}{12} = 0.26179938779...$ does not mean that you are multiplying the numbers together after the decimal.

$\star$ Note: By the Archimedian Principle, for all $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Since for all $n \in \mathbb{N}$, $n < 10^n$; $\frac{1}{10^n} < \frac{1}{n} < \varepsilon$.

And so our proof beings.

Allow $\varepsilon > 0$ and $\beta \notin \mathbb{Q}$. We also consider some $\frac{1}{10^n}$ such that $\frac{1}{10^n} < \varepsilon$. Next consider $f_{n+1} = \frac{b_1b_2b_3...b_n+1}{10^{n+1}}$ and notice that

$$\begin{aligned} |\ \beta - f_{n+1}| &= |\ (0.b_1b_2b_3...b_nb_{n+1}...) - 0.b_1b_2b_3...b_n| \\ &= 0.000...0b_{n+1}b_{n+2}.... = \frac{b_{n+1}}{10^{n+1}} + \frac{b_{n+2}}{10^{n+2}} + \frac{b_{n+3}}{10^{n+3}} + ... < \frac{1}{10^n} < \varepsilon. \end{aligned}$$

$\square$

So there we have it! A full course meal of subsequence convergence. *Delicious!*