Detailed Analysis of δ (Temporal Discount Factor) and δ^* (Critical Threshold)

This document provides a structured explanation of two core concepts—δ **(Temporal Discount Factor)** and δ^* **(Critical Threshold)**—as introduced in the memo *"Complex Contagion and Temporal Discounting."* Both parameters serve as fundamental metrics for evaluating the **temporal value of diffusion** and determining the **optimal network topology** in influence propagation processes.

1. Temporal Discount Factor (δ) and Objective Function V

1.1 Definition and Role of δ

δ is the **Temporal Discount Factor**, representing the rate at which the future value of diffusion is discounted relative to the present. It quantifies the temporal preference of the evaluator and satisfies:

$$0 \leq \delta \leq 1$$

A smaller δ implies a strong preference for immediate diffusion, while a larger δ values future diffusion more equally.

1.2 Objective Function V (Total Discounted Cumulative Activation Size)

δ defines the objective function V, representing the **Total Discounted Cumulative Activation Size** (or **Discounted Cumulative Reward**). At each discrete time step t, let c_t denote the number of newly activated nodes (marginal influence gain). Then:

$$V = c_1\delta + c_2\delta^2 + c_3\delta^3 + \cdots = \sum_{t=1}^{T} c_t \delta^t$$

V integrates both **speed** (how fast diffusion occurs) and **scale** (total reach) into a single dynamic performance metric weighted by time.

1.3 Interpretation by δ Values

- **Long-Term Horizon ($\delta \approx 1$):** The future value diminishes negligibly. V approximates the total final reach, i.e., $\sum c_t$, emphasizing complete diffusion across the network.
- **Short-Term Horizon ($\delta \approx 0$):** Future activations are heavily discounted. V highlights the early-stage spread speed and the success of cascade initiation.

2. Critical Threshold (δ^*): Concept and Strategic Implications

2.1 Definition and Role of δ^*

δ^* is the **Critical Temporal Discount Factor**, a strategic tipping point where the relative advantage between network structures reverses. Formally, it satisfies:

$$V_{\text{Clustered}} = V_{\text{Random}} \quad \text{when } \delta = \delta^*$$

Below δ^*, clustered structures dominate due to reinforcement; above δ^*, random structures dominate due to reach. Thus, δ^* quantifies the temporal trade-off between **reinforcement** and **reach** in Complex Contagion dynamics.

2.2 Optimal Network Structure by δ^*

δ Range	Optimization Focus	Dominant Network	CC Dynamics Explanation
$\delta < \delta^*$	Reinforcement Maximization (Early Success)	Clustered Network	Redundant ties provide social reinforcement necessary to meet activation thresholds, enabling successful early-stage initiation and short-term propagation.
$\delta > \delta^*$	Reach/Speed Maximization (Long-Term Reach)	Random Network	Long ties facilitate exponential information propagation across the network, maximizing diffusion speed and total long-term influence.

2.3 Research Significance of δ^*

1. **Quantification of Contagion Dynamics:** δ^* distinguishes the temporal regimes—reinforcement-driven ($\delta < \delta^*$) versus reach-driven ($\delta > \delta^*$)—under temporal constraints.
2. **Guidance for Influence Maximization (IM):** For short-term interventions, use clustered structures; for long-term objectives, adopt random or long-tie networks.
3. **Diagnostic Criterion for Diffusion Phenomena:** δ^* helps classify whether observed diffusion patterns are dominated by rapid spread or intensive reinforcement.

3. Extended Interpretation of δ

Beyond being a modeling parameter, δ can also capture the **intrinsic temporal characteristics** of the behavior or information itself.

- **Intrinsic Temporal Value of Behavior:** Behaviors that quickly lose relevance or utility (short lifespan T) inherently correspond to lower δ. Behaviors with enduring validity correspond to higher δ.
- **Interpretation of V:** V measures how effectively a diffusion process aligns with both its temporal constraints and its structural mechanisms (reinforcement vs. reach), functioning as a **Total Success Score**.

4. Conclusion

δ governs the **temporal weighting of diffusion value**, while δ^* defines the **critical temporal boundary** where optimal contagion mechanisms shift. Analyzing δ^* thus provides not only a sensitivity analysis but a theoretical foundation for **temporal optimization of social diffusion mechanisms**.