1 Negative Binomial Distribution (NBD)

A Bernoulli trial is an experiment that can result is either a 'success' or a 'failure', but not both.

Consider a sequence of Bernoulli trials with probability of success p and probability of failure q such that $0 \leq p \leq 1$ and $p + q = 1$. If X is the number of failures before the r^{th} success, X is said to follow a NBD with parameters r and p, denoted by:

$$X \sim \text{NBin}(r, p)$$

1.1 Probability Mass Function Of NBD

$$P(X = n) = \binom{n + r - 1}{r - 1} p^r q^n, \text{ for } n \in \mathbb{N}, \text{ where } q = 1 - p$$

1.2 Expected Value and Variance of NBD

$$E(X) = \frac{r(1-p)}{p}$$

$$Var(X) = \frac{r(1-p)}{p^2}$$

1.3 Conditons for NBD

- Experiment must 2 mutually exclusive outcomes denoted as 'success' or 'failure'
- Probability of success must be constant for each trial
- Each trial must be independent
- The experiment must have a finite number of success(es)

1.4 Relationship between Binomial Distribution and NBD

Consider n independent Bernoulli trials with the same probability of success p. If Y is the number of successes, it is said to follow a binomial distribution with parameters n and p.

$$Y \sim Bin(n, p)$$

Upon comparison, both the BD and NBD are based upon independent Bernoulli trials. However, they differ in what they are counting. The BD counts the number of succeses in a fixed number of trials n while the NBD counts the number of failures until a fixed number of succeses r.

1.5 Illustration of NBD

To illustrate, we will use data from Statistics Mauritius pertaining to grades of student in Economics A Level during the 2023 seating. Below is a summary of the data collected, along with mean and variance.

Grade	**Point Range**	f_i	x_i	$f_i \cdot x_i$	$(x_i - \mu)^2$	$f_i \cdot (x_i - \mu)^2$
A*	129-180	75	154.5	11587.5	5731.441922	429858.1441
A	113-129	261	120.5	31450.5	1739.414392	453987.1564
B	95-112	435	103.5	45022.5	610.4006273	265524.2729
C	83-95	419	89	37291	104.1682985	43646.51705
D	71-83	513	77	39501	3.2174056	1650.529073
E	60-71	490	65.5	32095	176.7227999	86594.17197
F	0-60	495	30	14850	2380.826409	1178509.072

$$\mu = \frac{\sum f_i \cdot x_i}{\sum f_i} = 78.79$$

$$\sigma = \frac{1}{\sum f_i} \cdot \sum f_i \cdot (x_i - \mu)^2 = 915.09$$

Scenario: Consider an event where A-Level economics students are gathered, the event organiser wants an r^{th} number of students who obtained A* and starts approaching attendees about their grades. Let the random variable X represent the probability that the organiser gathers those students after n number of attempts. From this, the following NBD can be constructed:

$$X \sim \text{NBD}(3, 0.0279)$$

Taking a sample of 50, the rnbinom function in R outputs the following:

x	0	1	2	3	4	5	6	7	8	9	10	11
f	4	10	8	6	8	6	3	4	0	0	0	1

1.5.1 Bar Chart

1.5.2 Issues with this application

2 Hypergeometric Distribution (HD)

Consider a population of N objects which are divided into 2 types: type A and type B. There are n objects of type A and N - n objects of type B. Suppose a random sample of size r is taken (without replacement) from the entire population of N objects. If X is the number of objects of type A in the sample,then X follows a HD with parameters n, $N-n$ and r denoted by:

$$X \sim \text{HGeom}(n, N-n, r)$$

2.1 Probability Mass Function of HD

$$p(k) = \frac{{}^{n}C_k \cdot {}^{(N-n)}C_{(r-k)}}{{}^{N}C_r}, \;\; \text{for } max\{0, r-(N-n)\} \leq k \leq min\{r, n\}$$

2.2 Expected Value and Variance of HD

$$E(X) = \frac{nr}{N}$$

$$Var(X) = \frac{nr}{N} \cdot \frac{N-r}{N} \cdot \frac{N-n}{N-1}$$

2.3 Conditions for HD

- Finite population
- Population can be seperated into 2 types
- Sampling is done without replacement (dependent trials).

3 Goodness of Fit

3.1 NBD

Based on the sampled data, the mean and variance can be calculated:

$$\mu = \frac{\sum x \cdot f}{\sum f} = \frac{163}{50} = 3.26$$

$$\sigma^2 = \frac{(x_i - \mu)^2}{n-1} = 5.46$$

$$\therefore \text{mean} < \text{variance}$$

x	O_i	E_i (4 d.p)	$(O_i - E_i)^2/E_i$ (4 d.p)
0	4	5.106	0.2394
1	10	9.3534	0.04476
2	8	10.2813	0.5062
3	6	8.7898	0.8855
4	8	6.4412	0.3772
5	6	4.2481	0.4124
6	3	2.5942	0.0635
7	4	1.4936	0.2060
8	0	0.8209	0.8209
9	0	0.4345	0.4345
10	0	0.2229	0.2290
11	1	0.1113	0.0689
	50	49.8968	15.2882

$$x^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 15.2882$$

$$x^2_{ll}(0.05) = 19.675$$

Since $x^2 < x^2_{ll}(0.05)$, assuming a 5% significane level, the random variable X does in fact follow a negative binomial distribution.