El Recap De Comer

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1 Real Numbers

1.1 Completeness

Completeness is a property of $\mathbb{R}$ that implies that there are no ”gaps” in the real number line. It states that every non-empty set of $\mathbb{R}$ that is bounded has a supremum and an infimum in $\mathbb{R}$.

1.2 Archimedean Property of $\mathbb{R}$

It states that $\forall\ \epsilon > 0,\ \exists\ n \in \mathbb{N} \ni \frac{1}{n} < \epsilon$. This means that there are no infinitesimally small elements in the real line, no matter how small ϵ gets, we will always be able to find an even smaller positive real number in the form $\frac{1}{n}$.

1.3 Boundedness

A set of real numbors is said to be **bounded** if it it bounded above and below. A subset S of $\mathbb{R}$ is said to be bounded $\iff |s| < \alpha\ \forall\ s \in S \implies -\alpha < s < \alpha\ \forall\ s \in S$.

1.4 Infimum and Supremum

Let S be a subset of $\mathbb{R}$. S is said to be **bounded above** if there is a real number $\alpha \ni \alpha \geq s\ \forall\ s \in S$. Such a number α is called an **upperbound** of S. S is **bounded below** if $\exists\ \beta \in \mathbb{R} \ni \beta \leq s\ \forall\ s \in S$. β is called a **lowerbound** of set S.

α is the **least upperbound / supremum** of $S \iff$:

- $\alpha \geq s\ \forall\ s \in S$
- given $\epsilon > 0\ \exists\ s \in S \ni \alpha - \epsilon < s$

β is the **greatest upperbound / infinum** of $S \iff$:

- $\beta \leq s\ \forall\ s \in S$
- given $\epsilon > 0\ \exists\ s \in S \ni \beta + \epsilon > s$

2 Sequences

2.1 Limits And Convergence

Consider an infinite sequence $(a_n) = a_1, a_2, a_3, \ldots$ of real numbers. The sequence is said to converge if $\exists\ L \in \mathbb{R} \ni$ for any given $\epsilon > 0\ \exists\ K \in \mathbb{R} \ni |x_n - L| < \epsilon\ \forall\ n \geq K$. In this case, we say that (x_n) has limit L and we write:

$$\lim_{n\to\infty} x_n = L$$

2.2 Example

Using the definition of the limit of a sequence, prove that $\lim_{n\to\infty}(1/n) = 0$.

Proof: Given $e > 0$, by the Archimedean property of $\mathbb{R}$, $\exists\ K \in \mathbb{N} \ni 1/K < \epsilon$. Then, if $n \geq K$, then $1/n \leq 1/K < \epsilon \implies |x_n - 0| = |1/n - 0| = 1/n \leq 1/K < \epsilon$. This proves, by definition, that $\lim_{n\to\infty} \frac{1}{n} = 0$.

2.3 Propositions

- Every convergent sequence is bounded. Thus, if (a_n) is convergent then $\exists\ \alpha \in \mathbb{R} \ni |a_n| < \alpha\ \forall\ n$.

- Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then:

$$\lim_{n\to\infty}(a_n + b_n) = a + b$$
$$\forall\ k \in \mathbb{R},\ \lim_{n\to\infty}(k \cdot a_n) = k \cdot a$$
$$\lim_{n\to\infty}(a_n \cdot b_n) = ab$$
$$\text{if } b \neq 0,\ \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$$

- Let $(a_n), (b_n)$ and (c_n) be sequences satisfying $(a_n) \leq (b_n) \leq (c_n)\ \forall\ n > k$, where k is some fixed positive integer. If (a_n) and (c_n) both converge to the same limit l, then (b_n) also converges to l.

- A sequence (a_n) is said to **tend to infinity** $[(a_n) \to \infty]$ if given $K \in \mathbb{R}$, $\exists\ N \in \mathbb{N} \ni n \geq N \implies a_n > K$.

- The sequence (a_n) is said to **tend to minus infinity** $[(a_n) \to -\infty]$ if given $K \in \mathbb{R}$, $\exists\ N \in \mathbb{N} \ni n \geq N \implies a_n < K$.

2.4 Monotonic Sequences

Let (x_n) be a sequence of real number. We say that:

- (x_n) is monotonic increasing if $x_{n+1} \geq x_n\ \forall\ n \in \mathbb{N}$.

- (x_n) is monotonic strictly increasing if $x_{n+1} > x_n\ \forall\ n \in \mathbb{N}$.

- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n\ \forall\ n \in \mathbb{N}$.

- (x_n) is monotonic strictly decreasing if $x_{n+1} < x_n\ \forall\ n \in \mathbb{N}$.

2.5 Monotone convergence theorem

If (x_n) is bounded and monotone then:

- if it is bounded above and increasing:

$$\lim_{n \to \infty} x_n = Sup(x_n)$$

- if it is bounded below and decreasing:

$$\lim_{n \to \infty} x_n = Inf(x_n)$$

Proof: Suppose that (x_n) is bounded above and increasing. By the completeness property of $\mathbb{R}$, $Sup(x_n)$ exists. Let $\alpha = Sup(x_n)$ and $\epsilon > 0$ be arbitrary. Then, by the properties of the supremum, $\exists\ x_K \ni \alpha - \epsilon < x_K \leq \alpha$. Since (x_n) is increasing and α is an upperbound for the range of the sequence, it follows that $x_K \leq x_n \leq \alpha\ \forall\ n \geq K \implies \alpha - \epsilon < x_n < \alpha + \epsilon\ \forall\ n \geq K \implies -\epsilon < x_n - \alpha < \epsilon \implies |x_n - \alpha| < \epsilon\ \forall\ n \geq K$.

If (x_n) is not bounded and monotone then:

- if it is increasing, then $(x_n) \to \infty$.
- if it is decreasing, then $(x_n) \to -\infty$.

2.6 Subsequences

Let (x_n) be a sequence. A subsequence of (x_n) is a sequence of the form $(x_{n_1}, x_{n_2}, x_{n_3}, ...)$ where $n_1 < n_2 < n_3 < ...$ is a sequence of strictly increasing natural numbers. A subsequence of (x_n) will be denoted by (x_{n_k}).

2.7 Theorems

Let (x_n) be a sequence.

- if $(x_n) \to L$ then $(x_{n_k}) \to L$.
- if (x_n) has two subsequences converging to distinct limits, (x_n) is divergent.
- if (x_n) has a subsequence that diverges, then (x_n) is divergent.
- if (x_n) is bounded, then (x_{n_k}) is also bounded.
- if (x_n) is monotonic increasing, then (x_{n_k}) is also monotonic increasing.

2.8 Bolzano - Weierstrass theorem

It states that every bounded sequence has a convergent subsequence. **Proof:** Let (a_n) be a bounded sequence. Say (a_{n_k}) is a monotone subsequence. Since all the terms of (a_n) are bounded, a_{n_k} is also bounded. By the Monotone Convergence Theorem, the subsequence a_{n_k}

2.9 Cauchy Sequences

A sequence a_n is cauchy if $\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \ni |a_m - a_n| < \epsilon\ \forall\ m,\ n > N$. A sequence converges $\iff$ it is Cauchy.

2.10 Example

Prove $a_n = \frac{1}{n}$ is Cauchy. **Proof:**

$$\text{Given } \epsilon > 0, \text{by the Archimedean property of } \mathbb{R},\ \exists\, N \in \mathbb{N} \ni N > \frac{2}{\epsilon} \implies \frac{1}{N} < \frac{\epsilon}{2}. \text{ Then,}$$
$$\forall\, m, n > N, |a_m - a_n| = |\frac{1}{m} - \frac{1}{n}| = |\frac{1}{m} - +\frac{1}{n}| \leq |\frac{1}{m}| + |\frac{-1}{n}| = \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$