Statistics Assignment

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1 Negative Binomial Distribution (NBD)

A Bernoulli trial is an experiment that can result in either a 'success' or a 'failure', but not both.

Consider a sequence of Bernoulli trials with probability of success p and probability of failure q such that $0 \leq p \leq 1$ and $p + q = 1$. If X is the number of failures before the r^{th} success, X is said to follow a NBD with parameters r and p, denoted by:

$$X \sim \text{NBin}(r, p)$$

Note:

- Mean is always greater than variance for a NBD. This is known as over-dispersion
- A random variable D which follows a NBD can also be defined as the number of trials until the r^{th} success. In such a case, $D = (X + r)$
- The terms 'success' and 'failure' in a NBD are arbitrary. As such, a NBD can also be described as modeling the number of successes before a desired number of failures. In this case, the roles of p and q are reversed

1.1 Probability Mass Function Of NBD

$$P(X = n) = \binom{n + r - 1}{r - 1} p^r q^n, \text{ for } n \in \mathbb{N}, \text{ where } q = 1 - p$$

1.2 Expected Value and Variance of NBD

$$E(X) = \frac{r(1-p)}{p}$$

$$Var(X) = \frac{r(1-p)}{p^2}$$

1.3 Assumptions of NBD

- Experiment must have 2 mutually exclusive outcomes denoted as 'success' or 'failure'
- Probability of success must be constant for each trial
- Each trial must be independent
- The experiment must have a finite number of success(es)

1.4 Relationship between Binomial Distribution (BD) and NBD

Consider n independent Bernoulli trials with the same probability of success p. If Y is the number of successes, it is said to follow a binomial distribution with parameters n and p denoted by:

$$Y \sim Bin(n, p)$$

Upon comparison, both the BD and NBD are based upon independent Bernoulli trials. However, they differ in what they are counting. The BD counts the number of successes in a fixed number of trials n while the NBD counts the number of failures until a fixed number of successes r.

1.5 Relationship between Geometric Distribution (BD) and NBD

Consider a sequence of Bernoulli trials, each with the same probability of success. If Z is the number of failures before the first successful trial,then Z is said to follow a geometric distribution(GD) with parameter p representing probability of success. This is denoted by:

$$Z \sim Geo(p)$$

When comparing a GD with a NBD,it becomes apparent that the NBD generalizes a GD where instead of waiting for just the first success,we can wait for any predetermined number r of successes.In other words,A GD is a special case of NBD where r is equal to 1.

1.6 Illustration of NBD

To illustrate, we will use data from Statistics Mauritius pertaining to grades of student in Economics A Level during the 2023 seating. Below is a summary of the data collected, along with mean and variance.

Grade	**Point Range**	f_i
A*	129-180	75
A	113-129	261
B	95-112	435
C	83-95	419
D	71-83	513
E	60-71	490
F	0-60	495

From the above data, we can calculate the probability that a random student has obtained a credit:

$$P(credit) = \frac{1703}{2688} = 0.6338$$

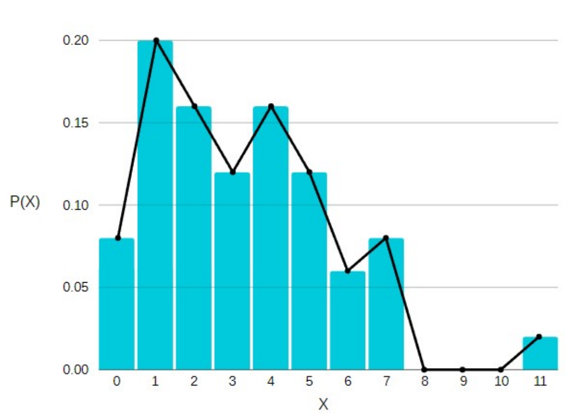
Since $E(X) < Var(X)$, the NBD can be applied.

Scenario: Consider an event where A-Level economics students are gathered. The event organiser wants a group of 5 students who obtained a credit and starts approaching the attendees about their grades one by one . Let the random variable X represent the number of students approached who have not got a credit until the organiser is able to achieve his goal.

Note: probability that a student obtains a credit is 0.6336.

Using the rnbinom function in R we have generated the following data for our distribution:

x	0	1	2	3	4	5	6	7	8	9	10	11
f	4	10	8	6	8	6	3	4	0	0	0	1



1.6.1 Issues with this application

Since the data set originally pertains to students sitting for the exams in 2023, it may not be suitable for any other application except for scenarios concerning that batch of students specifically.

1.7 Goodess of fit test for NBD

Based on the sampled data, the mean and variance can be calculated:

$$\mu = \frac{\sum x \cdot f}{\sum f} = \frac{163}{50} = 3.26$$

$$\sigma^2 = \frac{(x_i - \mu)^2}{n-1} = 5.46$$

Since mean < variance, the data does satisfy the conditions for a NBD.

x	O_i	E_i (4 d.p)	$(O_i - E_i)^2/E_i$ (4 d.p)
0	4	5.106	0.2394
1	10	9.3534	0.04476
2	8	10.2813	0.5062
3	6	8.7898	0.8855
4	8	6.4412	0.3772
5	6	4.2481	0.4124
6	3	2.5942	0.0635
7	4	1.4936	0.2060
8	0	0.8209	0.8209
9	0	0.4345	0.4345
10	0	0.2229	0.2290
11	1	0.1113	0.0689
	50	49.8968	15.2882

$$x^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 15.2882$$

$$x^2_{11}(0.05) = 19.675$$

Since $x^2 < x^2_{11}(0.05)$, assuming a 5% significance level, there is statistical evidence that the random variable X does in fact follow a negative binomial distribution.

2 Hypergeometric Distribution (HD)

Consider a population of N objects which are divided into 2 types: type A and type B. There are n objects of type A and N - n objects of type B. Suppose a random sample of size r is taken (without replacement) from the entire population of N objects. If X is the number of objects of type A in the sample,then X follows a HD with parameters n, $N - n$ and r denoted by:

$$X \sim \text{HGeom}(n, N-n, r)$$

2.1 Probability Mass Function of HD

$$p(k) = \frac{{}^{n}C_{k} \cdot {}^{(N-n)}C_{(r-k)}}{{}^{N}C_{r}}, \quad \text{for } max\{0, r-(N-n)\} \leq k \leq min\{r, n\}$$

2.2 Expected Value and Variance of HD

$$E(X) = \frac{nr}{N}$$

$$Var(X) = \frac{nr}{N} \cdot \frac{N-r}{N} \cdot \frac{N-n}{N-1}$$

2.3 Assumptions of a HD

- Finite population
- Population can be separated into 2 types
- Sampling is done without replacement (dependent trials).

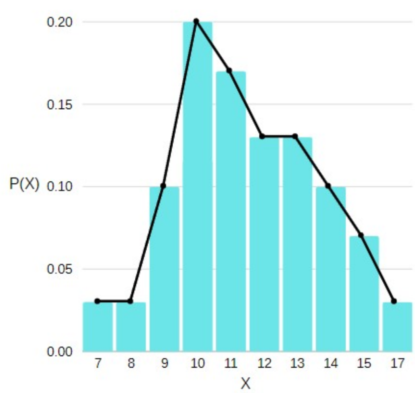
2.4 Illustration of a HD

Here, we will use data from MES concerning the number of students who sat for the A-Level exams in 2023 in Rodrigues. The following table shows amount of students classified by gender:

Male	**Female**	**Total**
94	150	244

After this, the following data was generated using the rhyper function in R, called with:

x	7	8	9	10	11	12	13	14	15	17
f	1	1	3	6	5	4	4	3	2	1



2.5 Relationship between HD and BD

A BD is not suitable for the random variable X as the Bernoulli trials (where a "success" is represented by Type A) are dependent (since sampling is done without replacement).However if the sample size is small enough (< 5 % of population size),the BD can be used to provide a reasonable approximation for a HD.

2.6 Goodness of Fit Test for HD

x	O_i	E_i (4 d.p)	$(O_i - E_i)^2/E_i$
7	1	0.9165	0.0076
8	1	1.7910	0.3493
9	3	2.9187	0.0023
10	6	4.0077	0.9904
11	5	4.6722	0.0230
12	4	4.6515	0.0913
13	4	3.9711	0.0002
14	3	2.9148	0.0025
15	2	1.8423	0.0135
17	1	0.4704	0.5963
	30	28.1562	2.0764

$$x^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 2.0764$$

$$x_9^2(0.05) = 16.919$$

Since $x^2 < x_9^2(0.05)$, assuming a 5% significance level, there is statistical evidence that the random variable X does follow a hypergeometric distribution.