Tutorial 3

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1 Vector Spaces

A vector space V is a set equipped with 2 operations:

- Addition: given $v, w \in V,\ v + w \in V$
- Scalar Multiplication: given $v \in V$ and $c \in \mathbb{R}$, $c \cdot v \in V$

These 2 operations are required to satisfy the following axioms $\forall\ u, v, w \in V$ and all scalars $c, d \in \mathbb{R}$:

- Commutativity of Addition: $v + w = w + v$
- Associativity of Addition: $(u + v) + w = u + (v + w)$
- Additive Identity: $\exists$ a zero element $\mathbf{0} \in V \ni v + \mathbf{0} = v = \mathbf{0} + v$
- Additive Inverse: $\forall\ v \in V,\ \exists\ (-v) \in V \ni v + (-v) = \mathbf{0} = (-v) + v$
- Distribuvity: $(c + d) \cdot v = cv + dv$ and $c \cdot (v + w) = cv + cw$
- Associativity of Scalar Multiplication: $c \cdot (dv) = (cd) \cdot v$
- Unit for Scalar Multiplication: the scalar $1 \in \mathbb{R}$ satisfies $1v = v$

1.1 Subspace

A subspace W of a vector space V is a subset $W \subset V$ and is a vector space on its own.

2 Inner Products

The most basic form of inner products is the inner product:

$$\langle v; w \rangle = v \cdot w = v_1 w_1 + v_2 w_2 + ... = \sum_{i=1}^{n} v_i w_i$$

The Euclidean norm or length of a vector v is given by:

$$||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$

Definition: An inner product on the vector space V is a pairing that takes two vectors $v,\ w \in V$ and produces $\langle v; w \rangle \in \mathbb{R}$ which is required to satisfy the following axioms, given $u, v, w \in V$ and $c, d, k \in \mathbb{R}$:

- Additivity:

$$\langle u + v; w \rangle = \langle u + v \rangle + \langle u + w \rangle$$

- Homoneneity:

$$\langle ku + v; w \rangle = k \langle u + v; w \rangle$$

- Bilinearity:

$$\langle c \cdot u + d \cdot v; w \rangle = c \langle u; v \rangle + d \langle v; w \rangle$$

- Symmetry:

$$\langle v; w \rangle = \langle w; v \rangle$$

- Positivity:

$$\langle v; v \rangle > 0 \text{ whenever } v \neq 0 \text{ while } \langle 0; 0 \rangle = 0$$

3 The Cauchy-Schwarz Inequality

Every inner product satisfies the Cauchy-Schwarz inequality:

$$v \cdot w = ||v|| \, ||w|| \cos\theta$$

$$|\cos\theta| \leq 1$$

$$v \cdot w \leq ||v|| \, ||w||$$

4 Orthogonal Vectors

Definition: Two elements $v, w \in V$ of an inner product space V are called orthogonal if $\langle v; w \rangle = 0$. Note that the property of orthogonality depends on which type of inner product is used.

4.1 Orthogonal Set

Definition: A non-empty set in $\mathbb{R}^n$ is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal.

5 The Triangle Inequality

Given $u, v \in V$ and any scalar k $\in \mathbb{R}$, then $||u + v|| \leq ||u|| + ||v||$.

6 Parallelogram Equation for Vectors

Given $u, v \in R^n$, then $||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2)$.

7 Norms

Definition: A norm on the vector space V assigns a real number $||v||$ to each vector $v \in V$ subject to the axioms of Positivity, Homogeneity and Triangle Inequality.

The 1st norm of a vector $v = (v_1, v_2, ..., v_n)^T$ is given by the sum of the absolute values of its entries:

$$||v||_1 = |v_1| + |v_2| + ... + |v_n|$$

The max or ∞ norm is equal to its maximal entry in absolute values:

$$||v||_\infty = Sup(|v_1|, |v_2|, ..., |v_n|)$$

In general, the p-norm is given by:

$$||v||_p = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p}$$

The triangle inequality for norms:

$$\sqrt[p]{\sum_{i=1}^{n} |v_i + w_i|^p} = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p} + \sqrt[p]{\sum_{i=1}^{n} |w_i|^p}$$

8 Unit Vectors

If V is a fixed normed vector space, the elements $u \in V$ with unit norm $||u|| = 1$ are known as unit vectors. If v is any non-zero vector, to obtain a unit vector u parallel to v:

$$u = \frac{v}{||v||}$$

9 Orthogonal Bases

Definition: A basis $u_1, u_2, ..., u_n$ of V is called orthogonal if $\langle u_i; u_j \rangle = 0 \; \forall \; i \neq j$. The basis is called orthogonormal if each vector has unit length that is $||u_i|| = 1 \; \forall \; i = 1, 2, ..., n$.

If $v_1, v_2, ..., v_n$ is an orthogonal basis of V, then the normalised vectors $u_i = v_i / ||v_i|| \; \forall \; i = 1, 2, ..., n$ form an orthogonormal basis of V.

If $v_1, v_2, ..., v_n \in V$ are non-zero and mutually orthogonal ($\langle v_i; v_j \rangle = 0 \; \forall \; i \neq j$), then they are linearly independent.

10 Vector Norms

Definition: Let $v : \mathbb{C}^n \to \mathbb{R}$. Then v is a vector norm if $\forall\, x, y \in \mathbb{C}$:

- Positive definite: $x \neq 0 \implies v(x) > 0$
- Homogeneity: $v(\alpha x) = |\alpha| v(x)$
- Obeys triangle inequality: $v(x + y) \leq v(x) + v(y)$