Tutorial 1

January 22, 2024

1 Question 1

a. There exists at least one subsequence of a_n that does not converge.
b. $\exists\, x_n \geq 5 \ni x_n \notin S$.
c. $x \notin A \cup B$.
c. $x \notin A \cap B$.
d. $\exists\, x \in (a, b) \ni f(x) < 0$.
e. For all $K \in \mathbb{R}$, $\exists\, x \in [-a, a] \ni |f(x) \leq k|$

2 Question 2

$$\text{Prove that } \lim_{n\to\infty} \frac{n}{n+1} = 1. \text{ Given } \epsilon > 0, \text{by the Archimedean property of } \mathbb{R},\ \exists\, K \in \mathbb{N} \ni$$
$$\frac{1}{K+1} < \epsilon \implies \frac{1}{n+1} \leq \frac{1}{K+1} < \epsilon\ \forall\, n \geq K \implies \left|\frac{n}{n+1} - 1\right| = \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right|$$
$$= \left|\frac{1}{n+1}\right| = \frac{1}{n+1} < \epsilon\ \forall\, n \geq K.$$

$$\text{Prove that } \lim_{n\to\infty} \frac{n}{n+2\sqrt{2}} = 1. \text{ Given } \epsilon > 0, \text{by the Archimedean property of } \mathbb{R},\ \exists\, K \in \mathbb{N} \ni$$
$$\frac{2\sqrt{2}}{K+2\sqrt{2}} < \epsilon \implies \frac{2\sqrt{2}}{n+2\sqrt{2}} \leq \frac{2\sqrt{2}}{K+2\sqrt{2}} < \epsilon\ \forall\, n \geq K \implies \left|\frac{n}{n+2\sqrt{2}} - 1\right| = \left|\frac{2\sqrt{2}}{n+2\sqrt{2}}\right| =$$
$$\frac{2\sqrt{2}}{n+2\sqrt{2}} < \epsilon\ \forall\, n \geq K.$$

3 Question 3

4 Question 4

5 Question 5

Since (a_n) is a null sequence, $\lim_{n\to\infty} a_n = 0 \implies$ given $\epsilon > 0,\ \exists\ K \in \mathbb{N} \ni |a_n - 0| = |a_n| < \epsilon\ \forall\ n \geq K \implies K|a_n| < K \cdot \epsilon \implies |b_n - l| < K|a_n| < K \cdot \epsilon$. Yep death.

6 Question 6

7 Question 7 (Sus)

7.1 Part 1

$$\text{Given } \epsilon > 0,\ \text{choose } N \in \mathbb{N} \ni N \geq \frac{\epsilon^2}{8} - \frac{3}{2}.\ \text{Then, } \forall\ m, n > N,$$
$$|\sqrt{3+2m} - \sqrt{3+2n}| = |-\sqrt{3+2n} + \sqrt{3+2m}| \leq \sqrt{3+2n} + \sqrt{3+2m} <$$
$$\sqrt{3 + 2 \cdot (\frac{\epsilon^2}{8} - \frac{3}{2})} + \sqrt{3 + 2 \cdot (\frac{\epsilon^2}{8} - \frac{3}{2})} = \sqrt{3 + \frac{\epsilon^2}{4} - 3} + \sqrt{3 + \frac{\epsilon^2}{4} - 3} = \epsilon$$
$$\implies a_n \text{ is a Cauchy sequence } \implies a_n \text{ is bounded.}$$

7.2 Part 2

$$\text{Let } f(x) = \sqrt{3+2x}.\ \ f'(x) = \frac{1}{\sqrt{3+2x}}. \text{ Since } a_0 = 0,\ \ \sqrt{3+2x} \text{ will always be positive.}$$

$$\implies f'(x) \text{ will always be positive } \implies f(x) \text{ is an increasing function.}$$

$$\implies a_n \text{ is a monotone increasing sequence by comparison.}$$

Since a_n is monotone and bounded, it is convergent according to the monotone convergence theorem.

8 Question 8 (Sus)

8.1 Part 1

$$\text{Given } \epsilon > 0,\ \text{choose } N \in \mathbb{N} \ni N \geq \frac{\epsilon^2}{12} - \frac{4}{3}. \text{ Then, } \forall\, m, n > N,$$

$$|\sqrt{4+3m} - \sqrt{4+3n}| = |-\sqrt{4+3n} + \sqrt{4+3m}| \leq \sqrt{4+3n} + \sqrt{4+3m} <$$

$$\sqrt{4+3\cdot(\frac{\epsilon^2}{12} - \frac{4}{3})} + \sqrt{4+3\cdot(\frac{\epsilon^2}{12} - \frac{4}{3})} = \sqrt{4+\frac{\epsilon^2}{4} - 4} + \sqrt{4+\frac{\epsilon^2}{4} - 4} = \epsilon$$

$$\implies a_n \text{ is a Cauchy sequence } \implies a_n \text{ is bounded.}$$

8.2 Part 2

$$\text{Let } f(x) = \sqrt{4+3x}.\ \ f'(x) = \frac{1}{\sqrt{4+3x}}. \text{ Since } a_0 = 6,\ \ \sqrt{4+3x} \text{ will always be positive.}$$

$$\implies f'(x) \text{ will always be positive } \implies f(x) \text{ is an increasing function.}$$

$$\implies a_n \text{ is a monotone increasing sequence by comparison.}$$