Statistics Assignment

Vishanraj Daby
2314620

Ihsaan Ramjanee
2315007

Zakariyya Kurmally
2315839

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Contents

1 Negative Binomial Distribution (NBD)

A Bernoulli trial is an experiment that can result in either a 'success' or a 'failure', but not both.

Consider a sequence of Bernoulli trials with probability of success p and probability of failure q such that $0 \leq p \leq 1$ and $p + q = 1$. If X is the number of failures before the r^{th} success, X is said to follow a NBD with parameters r and p, denoted by:

$$X \sim \text{NBin}(r, p)$$

Note:

- Variance is always greater than mean for a NBD. This is known as over-dispersion
- A random variable D which follows a NBD can also be defined as the number of trials until the r^{th} success. In such a case, $D = (X + r)$
- The terms 'success' and 'failure' in a NBD are arbitrary. As such, a NBD can also be described as modeling the number of successes before a desired number of failures. In this case, the roles of p and q are reversed

1.1 Probability Mass Function Of NBD

$$P(X = n) = \binom{n+r-1}{r-1} p^r q^n, \text{ for } n \in \mathbb{N}, \text{ where } q = 1 - p$$

1.2 Expected Value and Variance of NBD

$$E(X) = \frac{r(1-p)}{p}$$

$$Var(X) = \frac{r(1-p)}{p^2}$$

1.3 Assumptions of NBD

- Experiment must have 2 mutually exclusive outcomes denoted as 'success' or 'failure'
- Probability of success must be constant for each trial
- Each trial must be independent
- The experiment must have a finite number of success(es)

1.4 Relationship between NBD and other distributions

1.4.1 Binary Distribution (BD)

Consider n independent Bernoulli trials with the same probability of success p. If Y is the number of successes, it is said to follow a binomial distribution with parameters n and p denoted by:

$$Y \sim Bin(n, p)$$

Upon comparison, both the BD and NBD are based on independent Bernoulli trials. However, they differ in what they are counting. The BD counts the number of successes in a fixed number of trials n while the NBD counts the number of failures until a fixed number of successes r.

1.4.2 Geometric Distribution (GD)

Consider a sequence of Bernoulli trials, each with the same probability of success. If Z is the number of failures before the first successful trial,then Z is said to follow a geometric distribution(GD) with parameter p representing probability of success. This is denoted by:

$$Z \sim Geo(p)$$

When comparing a GD with a NBD,it becomes apparent that the NBD generalizes a GD where instead of waiting for just the first success,we can wait for any predetermined number r of successes.In other words,A GD is a special case of NBD where r is equal to 1.

1.5 Illustration of NBD

To illustrate, we will use data from MES pertaining to grades of student in Economics A Level during the 2023 seating. Below is a summary of the data collected:

Grade	**Point Range**	f
A*	129-180	75
A	113-129	261
B	95-112	435
C	83-95	419
D	71-83	513
E	60-71	490
F	0-60	495

From the above data, we can calculate the probability that a random student has obtained a credit:

$$P(credit) = \frac{1703}{2688} = 0.6336$$

Note: a credit is awarded to students who have gotten at least a D.

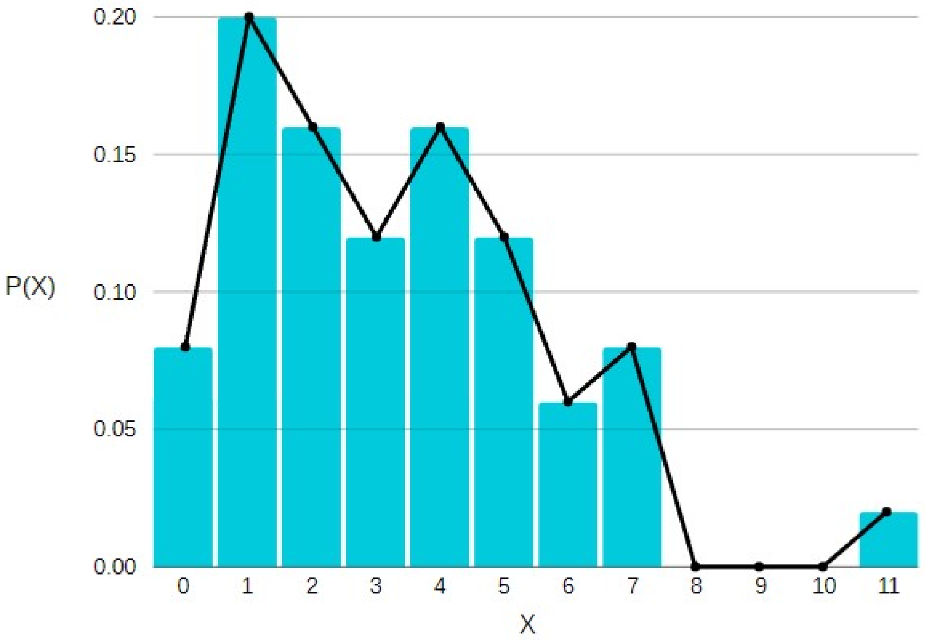
Scenario: Let 'success' be a student who was awarded a credit in A-Level Economics, and 'failure' be a student who was not able to get a credit.

Our goal is to calculate the probability of X which is a random variable representing the number of failures until we obtain 5 successes.

Note: probability that a student obtains a credit is 0.6336.

Using the rnbinom(50, 5, 0.6336) function in R we have generated the following data for our distribution:

x	0	1	2	3	4	5	6	7	8	9	10	11
f	4	10	8	6	8	6	3	4	0	0	0	1
P(X=x)	0.08	0.20	0.16	0.12	0.16	0.12	0.06	0.08	0	0	0	0.02



1.6 Issues with this application

Since we have used the rhyper function in R to generate our distribution data, it may not reflect actual real life data.

1.7 Goodness of fit test for NBD

Based on our distribution data, the mean and variance can be calculated:

$$E(x) = \frac{\sum x \cdot f}{\sum f} = \frac{163}{50} = 3.26$$

$$Var(X) = \frac{(x_i - \mu)^2}{n-1} = 5.46$$

Since mean < variance, the data does satisfy the over-dispersion assumption of a NBD.

The Chi-Square test will be used to determine if our distribution follows a NBD allowing, a significance level of 5%.

x	O_i	E_i (4 d.p)	$(O_i - E_i)^2/E_i$ (4 d.p)
0	4	5.106	0.2394
1	10	9.3534	0.04476
2	8	10.2813	0.5062
3	6	8.7898	0.8855
4	8	6.4412	0.3772
5	6	4.2481	0.4124
6	3	2.5942	0.0635
7	4	1.4936	0.2060
8	0	0.8209	0.8209
9	0	0.4345	0.4345
10	0	0.2229	0.2290
11	1	0.1113	0.0689
	50	49.8968	15.2882

$$x^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 15.2882$$

$$x^2_{11}(0.05) = 19.675$$

Since $x^2 < x^2_{11}(0.05)$, assuming a 5% significance level, there is statistical evidence that the random variable X does in fact follow a negative binomial distribution.

2 Hypergeometric Distribution (HD)

Consider a population of N objects which are divided into 2 types: type A and type B. There are n objects of type A and N - n objects of type B. Suppose a random sample of size r is taken (without replacement) from the entire population of N objects. If X is the number of objects of type A in the sample,then X follows a HD with parameters n, $N - n$ and r denoted by:

$$X \sim \text{HGeom}(n, N - n, r)$$

2.1 Probability Mass Function of HD

$$P(X = k) = \frac{{}^{n}C_{k} \cdot {}^{(N-n)}C_{(r-k)}}{{}^{N}C_{r}}, \text{ for } max\{0, r - (N - n)\} \leq k \leq min\{r, n\}$$

2.2 Expected Value and Variance of HD

$$E(X) = \frac{nr}{N}$$

$$Var(X) = \frac{nr}{N} \cdot \frac{N - r}{N} \cdot \frac{N - n}{N - 1}$$

2.3 Assumptions of a HD

- Finite population
- Population can be separated into 2 types
- Sampling is done without replacement (dependent trials).

2.4 Relationship between HD and BD

A BD is not suitable for the random variable X as the Bernoulli trials (where a “success” is represented by Type A) are dependent (since sampling is done without replacement).However if the sample size is small enough ($<$ 5 % of population size),the BD can be used to provide a reasonable approximation for a HD.

2.5 Illustration of a HD

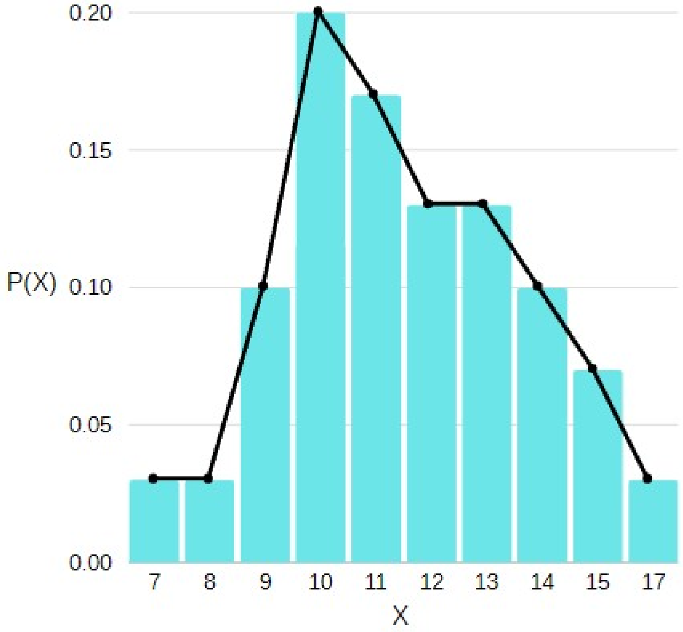
Here, we will use data from MES concerning the number of students who sat for the A-Level exams in 2023 in Rodrigues. The following table shows amount of students classified by gender:

Male	**Female**	**Total**
94	150	244

Scenario: Let a sample of 30 students be taken from the population without replacement. Our goal is to calculate the probability of X, which represents the number of males in the sample

The data for our distribution was generated using the rhyper(30, 94, 150, 30) function in R:

x	7	8	9	10	11	12	13	14	15	17
f	1	1	3	6	5	4	4	3	2	1
P(X=x)	0.03	0.03	0.10	0.20	0.17	0.13	0.13	0.10	0.07	0.03



2.6 Issues with this application

This particular distribution as well may not be suitable to model other applications since it only concerns itself with Rodrigues year 2023.

2.7 Goodness of Fit Test for HD

The Chi-Square test will be used to determine if a HD is suitable for our distribution, allowing a 5% level of significance.

x	O_i	E_i (4 d.p)	$(O_i - E_i)^2/E_i$
7	1	0.9165	0.0076
8	1	1.7910	0.3493
9	3	2.9187	0.0023
10	6	4.0077	0.9904
11	5	4.6722	0.0230
12	4	4.6515	0.0913
13	4	3.9711	0.0002
14	3	2.9148	0.0025
15	2	1.8423	0.0135
17	1	0.4704	0.5963
	30	28.1562	2.0764

$$x^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 2.0764$$

$$x_9^2(0.05) = 16.919$$

Since $x^2 < x_9^2(0.05)$, assuming a 5% significance level, there is statistical evidence that the random variable X does follow a hypergeometric distribution.