1 Series

Given an infinite series:

$$\sum_{k=1}^{\infty} a_k$$

- the numbers a_k are the terms of that series
- the sequence of partial sums is the sequence:

$$(S_n) = \left(\sum_{k=1}^{n} a_k\right)_{n=1}^{\infty} = a_1 + a_2 + a_3 + ... + a_n$$

The series converges to $L \in \mathbb{R}$, written $\sum_{k=1}^{\infty} a_k = L$, if $(S_n) \to L$, and diverges (to $\infty, -\infty$ or does not exist) if (S_n) does. Likewise, we say the series is bounded or monotone if S_n is.

2 Geometric Series

A geometric series can be written in the form:

$$\sum_{n=1}^{\infty} a \cdot r^n = a + ar + ar^2 + ...$$

Its partial sum, S_n can be written as:

$$S_n = \frac{a(1-r^n)}{1-r} \implies \lim_{n\to\infty} S_n = \frac{a}{1-r}$$

A geometric series will always converge if $|r| < 1$.

3 Harmonic Series

A harmonic series can be written in the form:

$$\sum_{k=1}^{\infty} \frac{1}{K} = 1 + \frac{1}{2} + \frac{1}{3} + ...$$

This series diverges.

4 P-Series

If p is a real constant, a p-series can be written in the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + ...$$

A p-series converges if $p > 1$ and diverges if $p \leq 1$.

5 The N-th Term Test

$$\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges. As such, if } \sum_{n=1}^{\infty} a_n \text{ converges, } \lim_{n\to\infty} a_n = 0$$

This test only guarantees that a series diverges if the terms do not go to 0 in the limit. It cannot prove convergence.

6 Comparison Test For Convergence

$$\text{Given } \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n.\text{ Then if } a_n,\ b_n \geq 0,\ a_n \leq b_n\ \forall\ n \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges, then}$$
$$\sum_{n=1}^{\infty} a_n \text{ converges. Similarly, if } \sum_{n=1}^{\infty} b_n \text{ diverges, then } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

7 Limit Comparison Test

$$\text{Given } \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n,\text{ then, if } a_n,\ b_n > 0\ \forall\ n > 0 \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} \text{ is positive and finite, then}$$
$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ are either both convergent or divergent.}$$

8 The Ratio Test

$$\text{Given } \sum_{n=1}^{\infty} a_n.\text{ If } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L,\text{ then the series :}$$

$$i.\text{ converges if } L < 1$$

$$ii.\text{ diverges if } L > 1$$

$$iii.\text{ may or may not converge if } L = 1$$