Tutorial 1

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1 Question 1

$$\text{Let } A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 6 & 6 \\ 3 & -3 & 2 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 6 & 6 \\ -3 & 2 \end{bmatrix} \; M_{12} = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} \; M_{13} = \begin{bmatrix} 4 & 6 \\ 3 & -3 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \; M_{22} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \; M_{23} = \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} 2 & -1 \\ 6 & 6 \end{bmatrix} \; M_{32} = \begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix} \; M_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

$$\text{Cofactor matrix: } \begin{bmatrix} (6\cdot 2 - 6\cdot -3) & -(4\cdot 2 - 6\cdot 3) & (4\cdot -3 - 6\cdot 3) \\ -(2\cdot 2 - -1\cdot -3) & (1\cdot 2 - -1\cdot 3) & -(1\cdot -3 - 2\cdot 3) \\ (2\cdot 6 - -1\cdot 6) & -(1\cdot 6 - -1\cdot 4) & (1\cdot 6 - 2\cdot 4) \end{bmatrix} = \begin{bmatrix} 30 & 10 & -30 \\ -1 & 5 & 9 \\ 18 & -10 & -2 \end{bmatrix}$$

$$det(A) = 1(30) + 2(10) - 1(-30) = 80$$

2 Question 2

$$\text{(a)} \begin{vmatrix} 2 & -3 & 4 \\ 5 & 1 & -6 \\ -7 & 8 & -9 \end{vmatrix} = 2\begin{vmatrix} 1 & -6 \\ 8 & -9 \end{vmatrix} + 3\begin{vmatrix} 5 & -6 \\ -7 & -9 \end{vmatrix} + 4\begin{vmatrix} 5 & 1 \\ -7 & 8 \end{vmatrix} = 2(39) + 3(-87) + 4(47) = 5$$

$$\text{(b)} \begin{vmatrix} 5 & 0 & 7 \\ 8 & 1 & -2 \\ 5 & 2 & -9 \end{vmatrix} = 5\begin{vmatrix} 1 & -2 \\ 2 & -9 \end{vmatrix} + 7\begin{vmatrix} 8 & 1 \\ 5 & 2 \end{vmatrix} = 5(-5) + 7(11) = 52$$

$$\text{(c)} \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} = a\begin{vmatrix} d & e \\ e & f \end{vmatrix} - b\begin{vmatrix} b & e \\ c & f \end{vmatrix} + c\begin{vmatrix} b & d \\ c & e \end{vmatrix} = a(df - e^2) - b(bf - ec) + c(be - dc)$$

$$= adf - ae^2 - b^2f + bec + bec - dc^2 = adf - ae^2 - b^2f + 2bec - dc^2$$

3 Question 3

$$\text{(a)} \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

$$= xyz(x-y)(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y & y+z & z^2 \end{vmatrix} = xyz(x-y)(y-z) \begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix}$$

$$= xyz(x-y)(y-z)(z-x)$$

$$\text{(b)} \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = \begin{vmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a+z \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & a & -a \\ x & x+y & a+z \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ x & x+y & a+z+x+y \end{vmatrix} = a^2(a+x+y+z)$$

$$\text{(c)} \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = a^2b^2c^2 \cdot -1 \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} = 4a^2b^2c^2$$

4 Question 4

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0 \implies \begin{vmatrix} a+x & a-x & 0 \\ a-x & a+x & -2x \\ a-x & a-x & 2x \end{vmatrix} = 0 \implies \begin{vmatrix} a+x & a-x & 0 \\ a-x & a+x & -2x \\ 2(a-x) & 2a & 0 \end{vmatrix} = 0$$

$$\implies 2x[2a \cdot (a+x) - 2(a-x)^2] = 0 \implies 2x[a \cdot (a+x) - (a-x)^2] = 0$$

$$\implies 2x[a^2 + ax - a^2 + 2ax - x^2] = 0 \implies x[3ax - x^2] = 0 \implies x^2(3a - x) = 0$$

$$\therefore x = 0, 0, 3a$$

5 Question 5

$$\text{(a) Let } A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\text{Cofactor Matrix of } A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & -1 \\ 0 & -4 & -3 \end{bmatrix}$$

$$det(A) = 1(3) - 2(-3) - 2(4) = 1$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & -3 & -3 \end{bmatrix}$$

$$\text{(b) Let } B = \begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$$

$$\text{Cofactor Matrix of } B = \begin{bmatrix} -30 & 0 & 10 \\ 0 & -50 & -10 \\ -10 & 0 & 10 \end{bmatrix}$$

$$det(B) = -30(5) + 10(5) = -100$$

$$\therefore A^{-1} = -\frac{1}{100}\begin{bmatrix} -30 & 0 & -10 \\ 0 & -50 & 0 \\ 10 & -10 & 10 \end{bmatrix} = \begin{bmatrix} 3/10 & 0 & 1/10 \\ 0 & 1/2 & 0 \\ -1/10 & 1/10 & -1/10 \end{bmatrix} =$$

$$\text{(a) Let } A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\text{Cofactor Matrix of } A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & -1 \\ 0 & -4 & -3 \end{bmatrix}$$

$$det(A) = 1(3) - 2(-3) - 2(4) = 1$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & -3 & -3 \end{bmatrix}$$

$$\text{(c) Let } C = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 3 & 4 \\ 3 & 2 & 2 \end{bmatrix}$$

$$\text{Cofactor Matrix of } C = \begin{bmatrix} -2 & 4 & -1 \\ -2 & 3 & 0 \\ 7 & -12 & 2 \end{bmatrix}$$

$$det(C) = -2(6) + 4(4) - 1(3) = 1$$

$$\therefore C^{-1} = \begin{bmatrix} -2 & -2 & 7 \\ 4 & 3 & -12 \\ -1 & 0 & 2 \end{bmatrix}$$

6 Question 6

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \quad A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$\text{Cofactors Matrix of } A = \frac{1}{9}\begin{bmatrix} 72 & -9 & -36 \\ -36 & -36 & -63 \\ -9 & 72 & -36 \end{bmatrix} = \begin{bmatrix} 8 & -1 & -4 \\ -4 & -4 & -7 \\ -1 & 8 & -4 \end{bmatrix}$$

$$det(A) = -8(8) + 1(-1) + 4(-1) \div 9 = -\frac{1}{9}$$

$$A^{-1} = -\frac{1}{9}\begin{bmatrix} 8 & 4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix} = A^{-1} = \frac{1}{9}\begin{bmatrix} -8 & -4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} = A^T$$

7 Question 7

$$\text{Let } A = \begin{bmatrix} 1 & 3 & a \\ 3 & k & 6 \\ -1 & 5 & 1 \end{bmatrix} \quad \text{For } A^{-1} \text{ to not exist, } det(A) = 0.$$

$$det(A) = 1\begin{bmatrix} k & 6 \\ 5 & 1 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 \\ -1 & 1 \end{bmatrix} + 4\begin{bmatrix} 3 & k \\ -1 & 5 \end{bmatrix} = 0 \implies$$

$$k - 30 - 27 + 4(15 + k) = 0 \implies k - 57 + 60 + 4k = 0 \implies k = -\frac{3}{5}$$

$$\therefore \text{A has an inverse when } k \neq 0.$$

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$$\text{Let } B = \begin{bmatrix} 1 & 3 & a \\ 3 & 1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$$

$$det(B) = 1 - 30 - 27 + 4(15 + 1) = 8$$

$$\text{Cofactor Matrix of B } = \begin{bmatrix} -29 & -9 & 16 \\ 17 & 5 & -8 \\ 14 & 6 & -8 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{8}\begin{bmatrix} -29 & 17 & 14 \\ -9 & 5 & 6 \\ 16 & -8 & -8 \end{bmatrix}$$