Differential Equations: First Order ODE

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1 ODEs

An ordinary differential equation is an equations involving an unknown function and its derivatives. The order of the differential equation is the order of the highest derivative. A linear ODE of order n is a differential equation written in the form:

$$a_n(x) \cdot \frac{d^n y}{dx^n} + a_{n-1}(x) \cdot \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1(x) \cdot \frac{dy}{dx} + a_o(x) \cdot y = f(x) \tag{1}$$

The equation is said to be homogeneous if $f(x) = 0$. Otherwise, it is non-homogeneous. The equation is linear when:

- y and its derivatives are multiplied by a function x or a constant
- the right hand side is a function of x

1.1 Examples of Differential Equations

First order, linear and non-homogeneous:

$$\frac{dy}{dx} = x + y$$
$$\frac{dy}{dx} - y = x$$

Second order, non-linear and homogeneous:

$$\frac{d^2 y}{dx^2} + xy \cdot \frac{dy}{dx} + 3y = 0$$

First order, non-linear and non-homogeneous:

$$\frac{d\theta}{dt} + 2t = 0$$

2 General Solution of an ODE

The general solution of an ODE is the relationship between the dependent and the independent variables obtained by integration. A first order ODE will involve one constant whilst a second order has two constants.
A particular solution is the solution to an initial value problem (IVP) where the constants of integration is found from the initial conditions given.

2.1 Method of Seperation of Variables

A differential equation is said to be seperable if:

$$\frac{dy}{dx} = f(x, y) = a(x) \cdot b(y)$$

1. Rewrite the equation as:

$$\frac{dy}{b(y)} = a(x) \cdot dx$$

2. Integrate on both sides:

$$\int \frac{dy}{b(y)} = \int a(x) \cdot dx$$

$$\text{General solution: } B(y) = A(x) + C$$

3. If you are given IVP, use the initial conditions to find the particular solution.

2.2 Example 1

Find the particular solution of the following ODE given $y(1) = 3$.

$$\frac{dy}{dx} = \frac{y^2 - 1}{x}$$

Solution:

$$\frac{dy}{y^2 - 1} = \frac{dx}{x}$$

$$\int \frac{dy}{y^2 - 1} = \int \frac{dx}{x}$$

Using partial fraction to rewrite:

$$\frac{1}{y^2 - 1} = \frac{1}{2(y-1)} - \frac{1}{2(y+1)}$$

$$\int (\frac{1}{2(y-1)} - \frac{1}{2(y+1)}) = \int \frac{dx}{x}$$

$$\frac{1}{2} \cdot ln(\frac{y-1}{y+1}) = ln(x) + C$$

$$\text{General Solution: } \frac{y-1}{y+1} = Bx^2, \text{ where } B = lnC$$

$$\text{When } x = 1,\ y = 3,\ \implies B = \frac{1}{2}$$

$$\therefore y = \frac{2 + x^2}{2 - x^2} \leftarrow \text{Particular Solution}$$

2.3 Example 2

$$x(y-1)\ dx + y(x-1)\ dy = 0$$

$$x(y-1) = -dx + y(x-1)\ dy$$

$$\frac{x}{x-1}\ dx = -\frac{y}{y-1}$$

$$\text{Integrating on both sides: } \int \frac{x}{x-1}\ dx = \int -\frac{y}{y-1}$$

$$\int 1 + \frac{1}{x-1}\ dx = -\int 1 + \frac{1}{y-1}\ dy$$

$$x + \ln{(x-1)} = -[y + \ln{(y-1)}] + C$$

$$x + y = -\ln{[A(x-1)(y-1)]}$$

$$A(x-1)(y-1) = e^{x+y} \leftarrow \text{General Solution}$$

2.4 Example 3

$$\frac{dy}{dx} = x^3(y^2+1)$$

$$\frac{dy}{y^2+1} = x^3\ dx$$

$$\int \frac{dy}{y^2+1} = \int x^3\ dx$$

$$\tan^{-1} y = \frac{1}{4}x^4 + C$$

$$y = \tan{(\frac{x^4}{4} + C)}$$