Successió de Fibonacci: terme general?

$u_0 = 0,\ u_1 = 1;\ u_k = u_{k-1} + u_{k-2}$, si $k \geq 2$:

n	0	1	2	3	4	5	6	7	.....
u_n	0	1	1	2	3	5	8	13	.....

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} u_{n-1} \\ u_{n-2} \end{pmatrix}, \quad \text{si } n \geq 2$$

$$\boxed{\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}} = A \begin{pmatrix} u_{n-1} \\ u_{n-2} \end{pmatrix} \overset{\text{si } n-1 \geq 2}{=} A \cdot A \begin{pmatrix} u_{n-2} \\ u_{n-3} \end{pmatrix} =$$

$$= A^2 \begin{pmatrix} u_{n-2} \\ u_{n-3} \end{pmatrix} \overset{\text{si } n-2 \geq 2}{=} A^2 \cdot A \begin{pmatrix} u_{n-3} \\ u_{n-4} \end{pmatrix} =$$

$$= A^3 \begin{pmatrix} u_{n-3} \\ u_{n-4} \end{pmatrix} =$$

....

$$= A^{n-1} \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \boxed{A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Calculem A^K:

diagonalitzem $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$p_A(x) = \det \begin{pmatrix} 1-x & 1 \\ 1 & -x \end{pmatrix} = (-x)(1-x) - 1 = x^2 - x - 1$$

arrels: $x = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2} \rightsquigarrow$

vaps: $\dfrac{1+\sqrt{5}}{2}$ $(=\alpha)$, $\dfrac{1-\sqrt{5}}{2}$ $(=\beta)$

veps de vap. α:

$$E_\alpha : \begin{pmatrix} 1-\alpha & 1 \\ 1 & -\alpha \end{pmatrix} \sim \begin{pmatrix} 1 & -\alpha \\ 1-\alpha & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\alpha \\ 0 & \underbrace{1-(1-\alpha)(-\alpha)}_{-\alpha^2+\alpha+1=0} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -\alpha \\ 0 & 0 \end{pmatrix}$$

Solució: $x = \alpha y,\ y \in \mathbb{R}$ $\left\{ \begin{pmatrix} \alpha y \\ y \end{pmatrix} = y \begin{pmatrix} \alpha \\ 1 \end{pmatrix},\ y \in \mathbb{R} \right.$

$$E_\alpha = \left\langle \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right\rangle$$

OBS: $A \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha+1 \\ \alpha \end{pmatrix} \overset{?}{=} \alpha \cdot \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^2 \\ \alpha \end{pmatrix}$

és cert ja que:

$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^2 = \alpha + 1$

veps de vap. β :

$$E : \begin{pmatrix} 1-\beta & 1 \\ 1 & -\beta \end{pmatrix} \sim \begin{pmatrix} 1 & -\beta \\ 1-\beta & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\beta \\ 0 & \underbrace{1-(1-\beta)(-\beta)}_{-\beta^2+\beta+1=0} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -\beta \\ 0 & 0 \end{pmatrix}$$

Solució : $x=\beta y,\ y \in \mathbb{R} \quad \left\{ \begin{pmatrix} \beta y \\ y \end{pmatrix} = y \begin{pmatrix} \beta \\ 1 \end{pmatrix},\ y \in \mathbb{R} \right.$

$$E_\beta = \left\langle \begin{pmatrix} \beta \\ 1 \end{pmatrix} \right\rangle$$

Per tant, si $P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}$, aleshores:

$$P^{-1} A P = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = D \quad , \quad P^{-1} = \frac{1}{\alpha-\beta} \cdot \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix}$$

$$\Rightarrow \quad A = P D P^{-1}$$

$$\Rightarrow$$

$$A^k = P \cdot D^k P^{-1} = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha^k & 0 \\ 0 & \beta^k \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix} \cdot \frac{1}{\alpha-\beta} =$$

$$= \frac{1}{\alpha-\beta} \cdot \begin{pmatrix} \alpha^{k+1} & \beta^{k+1} \\ \alpha^k & \beta^k \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix} =$$

$$= \boxed{\frac{1}{\alpha-\beta} \begin{pmatrix} \alpha^{k+1} - \beta^{k+1} & -\beta \cdot \alpha^{k+1} + \alpha \beta^{k+1} \\ \alpha^k - \beta^k & -\beta \cdot \alpha^k + \alpha \beta^k \end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{\alpha-\beta} \begin{pmatrix} \alpha^n - \beta^n & -\beta \cdot \alpha^n + \alpha \beta^n \\ \alpha^{n-1} - \beta^{n-1} & -\beta \alpha^{n-1} + \alpha \beta^{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\alpha-\beta} (\alpha^n - \beta^n) \\ \frac{1}{\alpha-\beta} (\alpha^{n-1} - \beta^{n-1}) \end{pmatrix}$$

$$\Rightarrow \quad u_n = \frac{1}{\alpha-\beta}\left(\alpha^n - \beta^n\right) \underset{\uparrow}{=} \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$\alpha-\beta = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

$$\boxed{u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

Observem que:

$$u_0 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right) = \frac{1}{\sqrt{5}}(1-1) = 0$$

$$u_1 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}-(1-\sqrt{5})}{2}\right) = 1$$

$$u_2 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right) = \frac{1}{\sqrt{5}}\left(\frac{1+5+2\sqrt{5}}{4} - \frac{1+5-2\sqrt{5}}{4}\right) = 1$$

etc.

SUCCESIONS RECURRENTS, EN GENERAL:

$(a_n)_{n\geq 0}$ tq. $a_0, \dots, a_{k-1}$ donats i

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \dots + \alpha_k a_{n-k}, \text{ si } n \geq k$$

$$\begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_{n-k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{k-1} & \alpha_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \\ \vdots \\ a_{n-k} \end{pmatrix}, \text{ si } n \geq k$$

$$\Rightarrow \begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_{n-k+1} \end{pmatrix} = A^{n-k+1} \begin{pmatrix} a_{k-1} \\ a_{k-2} \\ \vdots \\ a_0 \end{pmatrix}, \text{ si } n \geq k$$

$\Rightarrow$ obtenim a_n en funció de $a_0, \dots, a_{k-1}$ si coneixem la 1ª fila de A^{n-k+1}

Si A diagonalitza: $\exists P$ invertible, $A = PDP^{-1}$ i per tant, $A^r = PD^rP^{-1}$, $\forall r \geq 1$