MA243: Geometry

1. Lecture 1: Introduction, motivation, overview. Definition of a metric.

1.1. Practical matters.

- Lecturer: Helena Verrill, office: B1.35, email: H.A.Verrill@Warwick.ac.uk
- Lectures:
  - Monday 11-12: week 1 and 3, will be WLT, weeks 2 and 4-10: L3
  - Tuesday 11-12: L4
  - Friday 11-12: WOODS-SCAWEN
- Lecture Notes: A pdf of the lecture notes will be posted at the beginning of each week. The lecture capture will be available after the lectures.
- References: This course is based on the book:

  Reid and Szendrői (chapters 1-6) Geometry and Topology.

  For more details for the parts on hyperbolic geometry we also use:

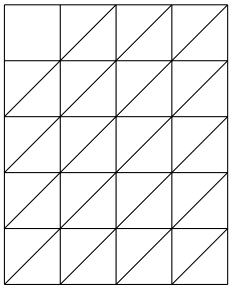
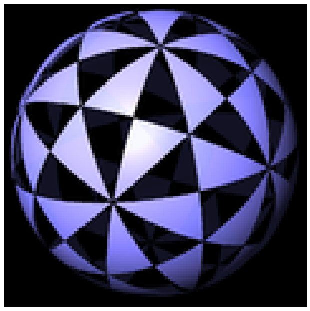
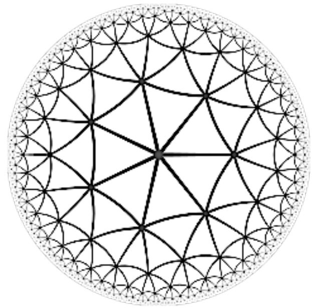
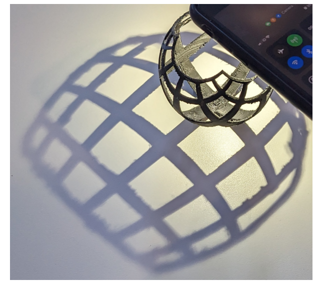
  Ratcliffe (parts of chapters 1-3) Hyperbolic Geometry.
- TAs: Rob Phillips and James Rawson
- Support Classes: Tuesday 4pm (Rob) and Thursday 2pm (James). Starting week 2, both in MS.04.
- Assignmnents: There are 4 assignments, contributing 15% to your final mark. The best 3 assignments count.
- Quizzes: There will be optional quizzes posted on moodle for you to check understanding. These will not contribute to your grade.
- Forum: Please post your questions to the moodle forum and join in dicusssions online.
- Goals: This course aims to introduce you to
  - **Euclidean space**,
  - **spherical geometry**,
  - **hyperbolic geometry**,
    - For these geometries, defining the
      - respective metrics (ie. how distance is measured),
      - proving they satisfy the triangle inequality
      - describing the groups of isometries
      - constrasting parallel lines and angles of triangles for these geometries.
  - **projective geometry**.
- Prerequisites:
  - Linear algebra: **vector space, basis, linear map, matrix, rank-nullity theorem**
  - Group theory: **group, identity, associtive, inverses**
  - Relations: **transitive, reflective, equivalence**
  - Hyperbolic function: **sinh, cosh**

  We will review these topics, but it will be useful if you can refresh your memories in advance.
- Exam: The exam counts for 85% of the mark. This will probably have a similar content to the exams for the last three years, as well as to the homework assignments.
- Vevox: We will have quizzes on vevox, the vevox ID will always be: 151-597-945

Let me know if you find any typos in these note, or have any questions!

1.2. Course contents overview:

Cartesian geometry –⋄– Spherical geometry –⋄– Hyperbolic geometry –⋄– Projective geometry

Comparison of 2-dimensional metric geometries:

	Euclidean	Spherical	Hyperbolic
Parallel lines through a point parallel to a line	unique	don't exist	not unique
Triangle angle sum	π	$> \pi$	$< \pi$

Klein's Erlangen program: describes geometry in terms of groups of transformations.

Geometry "=" Group theory

Geometry	Group
Euclidean $\mathbb{R}^n$	$O(n) \ltimes \mathbb{R}^n$
Spherical S^n	$O(n+1)$
Hyperbolic $\mathcal{H}^n$	$O^+(1,n)$
Projective $\mathbb{P}^n$	$PGL(n+1)$

1.3. Metric Spaces.

Definition 1. *A metric d on a set X is a function $d : X \times X \to [0, \infty)$ such that $\forall x, y \in X$, it is:*

(1) *Non-degenerate:*
$$d(x,y) = 0 \iff x = y$$
(2) *Symmetric:*
$$d(x,y) = d(y,x)$$
(3) *Satisfies the triangle inequality:*
$$d(x,y) \leq d(x,z) + d(z,y)$$

The pair (X,d) is called a metric space. The function d may also be called a distance function.

1.4. The Euclidean metric.
Definitions and notation. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, we write:

- $\mathbf{x} = (x_1, x_2, \ldots, x_m)$,
- $\mathbf{y} = (y_1, y_2, \ldots, y_m)$,
- *Inner product*: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^m x_i y_i = x_1y_1 + x_2y_2 + \cdots + x_my_m$
  This is the same as the dot product, $\mathbf{x} \cdot \mathbf{y}$.
- *Euclidean Norm*: $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ (Note, $|\lambda|$ is the abolute value of a real number.)
- *Euclidean metric*: $d(\mathbf{x}, \mathbf{y}) := ||\mathbf{x} - \mathbf{y}||$

Note, the Pythagorean theorem is true by definition!
To prove the Euclidean metric is actually a metric we need some results about the inner product.

Lemma 1 (Bilinearity of the inner product). *The inner product on $\mathbb{R}^m$ is bilinear and symmetric, that is for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$, we have*
$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle \text{ and } \langle \lambda \mathbf{a}, \mathbf{b} \rangle = \lambda \langle \mathbf{a}, \mathbf{b} \rangle \; \textit{and} \; \langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle$$
in particular $\langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle - 2 \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle$.

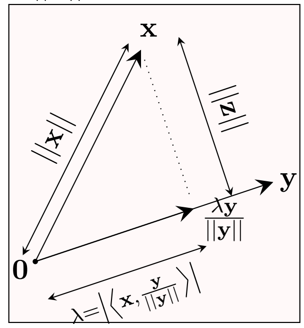
Proof. Follows from the rules of vector and matrix multiplication. Details left as an exercise. □

2. Lecture 2: The Euclidean metric. The Cauchy-Schwartz inequality.

LEMMA 2 (Cauchy-Schwartz inequality). *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle^2 \leq ||\mathbf{x}||^2 \cdot ||\mathbf{y}||^2$ *Or, equivalently,*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$$

When $\mathbf{y} \neq \mathbf{0}$*, there is equality if and only if* $\mathbf{x} = \lambda \mathbf{y}$ *for some* $\lambda \in \mathbb{R}$.



PROOF. If $\mathbf{y} = \mathbf{0}$, we get $0 \leq 0$, which is true. Now assume $\mathbf{y} \neq \mathbf{0}$. Recall from A-level maths, if the angle between $\mathbf{x}$ and $\mathbf{y}$ is θ, then:

$$\langle \mathbf{x}, \mathbf{y} \rangle = ||\mathbf{x}|| \cdot ||\mathbf{y}|| \cos(\theta).$$

The result follows since $\cos\theta \leq 1$.
For a proof from first principles: Let $\mathbf{z} := \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\mathbf{y}$. Using bilinearity, we have

$$0 \leq ||\mathbf{z}||^2 = \left\langle \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\mathbf{y}, \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\mathbf{y} \right\rangle = \langle \mathbf{x}, \mathbf{x} \rangle - 2\left\langle \mathbf{x}, \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\mathbf{y} \right\rangle + \left\langle \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\mathbf{y}, \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\mathbf{y} \right\rangle$$

$$= ||\mathbf{x}||^2 - 2\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\langle \mathbf{x}, \mathbf{y} \rangle + \left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}\right)^2 \langle \mathbf{y}, \mathbf{y} \rangle = ||\mathbf{x}||^2 - 2\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||}\right)^2 + \left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||}\right)^2.$$

Rearranged, this is $||\mathbf{x}||^2 \geq \left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||}\right)^2$, which gives the required result.
Equality is acheived if and only if $||\mathbf{z}|| = 0$. Because the norm is non-degenerate $||\mathbf{z}|| = 0$ if and only if $\mathbf{z} = \mathbf{0}$, which is the case if and only if

$$\mathbf{x} = \lambda \mathbf{y} \text{ where } \lambda = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}$$

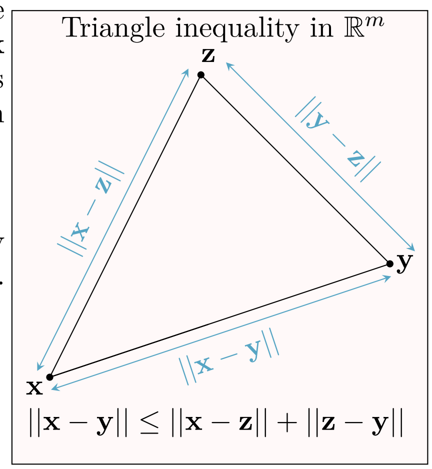
$\square$

To save on notation, we prove that the Euclidean metric is translation invariant.

LEMMA 3 (Translation invariance of Euclidean metric). *For* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m$ *we have* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}-\mathbf{z}, \mathbf{y}-\mathbf{z})$.

PROOF. $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = ||(\mathbf{x} - \mathbf{z}) - (\mathbf{y} - \mathbf{z})|| = d(\mathbf{x} - \mathbf{z}, \mathbf{y} - \mathbf{z})$. $\square$

LEMMA 4 (Triangle inequality for the Euclidean metric). *For all* $\mathbf{x}, \mathbf{y}, \mathbf{x} \in \mathbb{R}^n$ *we have* $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$*, with equality if and only if either* $\mathbf{z} = \mathbf{x}$*, or* $\mathbf{y} - \mathbf{z}$ *is a positive scalar multiple of* $\mathbf{z} - \mathbf{x}$.



We want to show that $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$. By translation invariance (Lemma 3), we can replace $\mathbf{x}, \mathbf{y}, \mathbf{z}$ by $\mathbf{0}, \mathbf{y}' = \mathbf{y} - \mathbf{x}$ and $\mathbf{z}' = \mathbf{z} - \mathbf{x}$ respectively. (because the distances between these points are the same as the distances between $\mathbf{x}, \mathbf{y}, \mathbf{z}$. The only reason to change is to cut down on notation.) So by definition of the metric, it is sufficient to prove

$$||\mathbf{y}'|| \leq ||\mathbf{z}'|| + ||\mathbf{y}' - \mathbf{z}'||.$$

This is equivalent to $||\mathbf{y}'||^2 \leq (||\mathbf{z}'|| + ||\mathbf{y}' - \mathbf{z}'||)^2$. Using the Cauchy Schwartz inequality, we have $||\mathbf{z}'|| \cdot ||\mathbf{y}' - \mathbf{z}'|| \geq |\langle \mathbf{z}', \mathbf{y}' - \mathbf{z}' \rangle| \geq \langle \mathbf{z}', \mathbf{y}' - \mathbf{z}' \rangle$. So

$$(||\mathbf{z}'|| + ||\mathbf{y}' - \mathbf{z}'||)^2 = ||\mathbf{z}'||^2 + 2||\mathbf{z}'|| \cdot ||\mathbf{y}' - \mathbf{z}'|| + ||\mathbf{y}' - \mathbf{z}'||^2$$

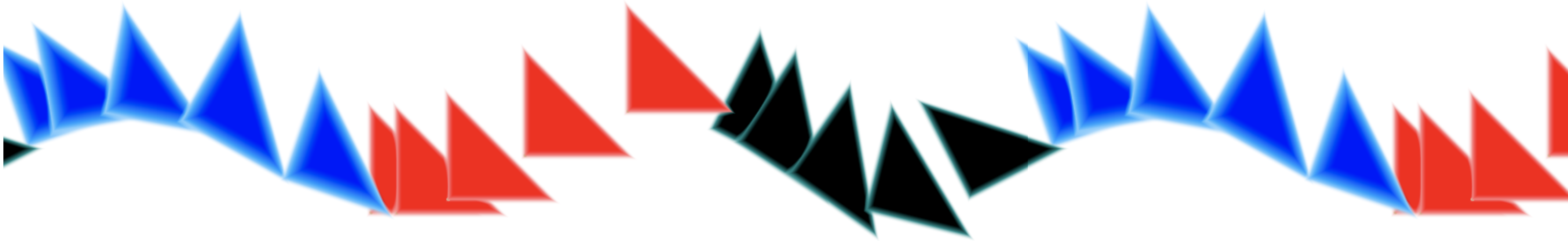
$$\geq ||\mathbf{z}'||^2 + 2|\langle \mathbf{z}', \mathbf{y}' - \mathbf{z}' \rangle| + ||\mathbf{y}' - \mathbf{z}'||^2 \tag{1}$$

$$\geq ||\mathbf{z}'||^2 + 2\langle \mathbf{z}', \mathbf{y}' - \mathbf{z}' \rangle + ||\mathbf{y}' - \mathbf{z}'||^2 \tag{2}$$

$$= \langle \mathbf{z}', \mathbf{z}' \rangle + 2\langle \mathbf{z}', \mathbf{y}' - \mathbf{z}' \rangle + \langle \mathbf{y}' - \mathbf{z}', \mathbf{y}' - \mathbf{z}' \rangle = \langle \mathbf{z}' - (\mathbf{y}' - \mathbf{z}'), \mathbf{z}' - (\mathbf{y}' - \mathbf{z}') \rangle = \langle \mathbf{y}', \mathbf{y}' \rangle = ||\mathbf{y}'||^2,$$

as required. Looking at the lines (1) and (2), we see equality is obtained if and only if both: $\mathbf{z}' = \mathbf{0}$ or $y' - \mathbf{z}'$ is a multiple of $\mathbf{z}'$, **<u>and</u>** $\langle \mathbf{z}', \mathbf{y}' - \mathbf{z}' \rangle$ is positive. In terms of the original $\mathbf{x}, \mathbf{y}, \mathbf{z}$, this is if and only if $\mathbf{z} - \mathbf{x} = \mathbf{0}$ or $\mathbf{y} - \mathbf{z}$ is a positive multiple of $\mathbf{z} - \mathbf{x}$. $\square$

Question: (Thinking about transformations) How are these triangles related?



(See `www.mathamaze.co.uk/MA243/MA243week1.html`, or video capture of discussion)

3. Lecture 3: Colinearity. Euclidean space. Isometries. Euclidean Isometries.

Theorem 1 (The Euclidean metric is a metric). *$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ on $\mathbb{R}^m$ is a metric on $\mathbb{R}^m$.*

Proof. We need to show this satisfies the three properties of metrics.
Non-degenerate: Since x_i and y_i are real, $(x_i - y_i)^2 = 0 \iff x_i = y_i$.
Thus $d(\mathbf{x}, \mathbf{y}) = 0 \iff x_i = y_i$ for $i = 1, \ldots m \iff \mathbf{x} = \mathbf{y}$.
Symmetric: Since $(x_i - y_i)^2 = (y_i - x_i)^2$, we have $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
Triangle inequality: was proved in Lemma 4. □

3.1. Isometries and Euclidean space.

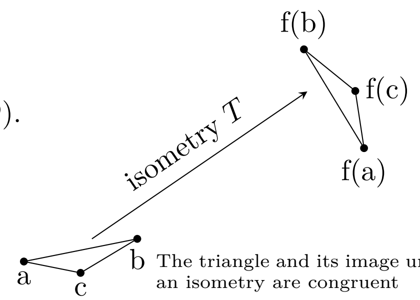
Definition 2. *If (X_1, d_1) and (X_2, d_2) are metric spaces, then a distance preserving map between (X_1, d_1) and (X_2, d_2) is a map*

$$f : X_1 \to X_2$$

such that for any points P, Q in X_1, we have

$$d_2(f(P), f(Q)) = d_1(P, Q).$$

An isometry is a bijective distance preserving map.
If f is an isometry, then (X_1, d_1) and (X_2, d_2) are said to be isometric.



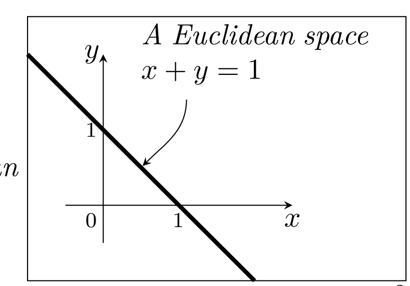
The triangle and its image under an isometry are congruent

Quiz: Does a distance preserving map have to be injective?

Definition 3. *A Euclidean space is a metric space which is isometric to $\mathbb{R}^n$, with the Euclidean metric, for some integer n. We use the notation $\mathbb{E}^n$ to denote the metric space $\mathbb{R}^n$ together with the Euclidean metric. If not specified otherwise, $\mathbb{R}^n$ is used to mean $\mathbb{E}^n$.*

WARNING: We are often confusing or conflating the concepts of vectors and points. We can add vectors in $\mathbb{R}^n$, but we can't add points in $\mathbb{E}^n$, unless we idenifity $\mathbb{E}^n$ with $\mathbb{R}^n$, but there are infinitely many ways to do this. Identifying a Eucidean space with $\mathbb{R}^n$ means choosing an origin and a basis.

Example 1.



The line

$$L = \{(x, y) \in \mathbb{R}^2 : x + y = 1\},$$

with metric given by the usual metric on $\mathbb{R}^2$ is a Euclidean space. We can choose our identification with $\mathbb{R}$ by chosing 0 and a direction.

Note that adding points on L does not make sense: If the points are added together using addition in $\mathbb{R}^2$, then the result is not on L. We can make addition on L make sense, but we would have to choose a zero.

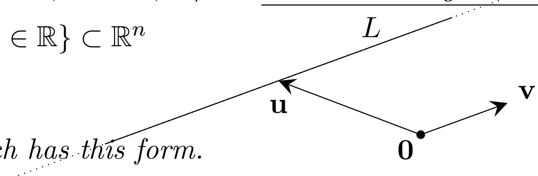
Definition 4. *A motion, also called a Euclidean isometry, is an isometry of a Euclidean space.*

Note: A motion does not move.

3.2. Lines and Collinearity.

Definition 5 (Definition of a line). *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq \mathbf{0}$. The line through $\mathbf{u}$ in direction $\mathbf{v}$ is*

$$L := \{\mathbf{u} + \lambda \mathbf{v} | \lambda \in \mathbb{R}\} \subset \mathbb{R}^n$$



A line in $\mathbb{R}^m$ is any subset of $\mathbb{R}^m$ which has this form.

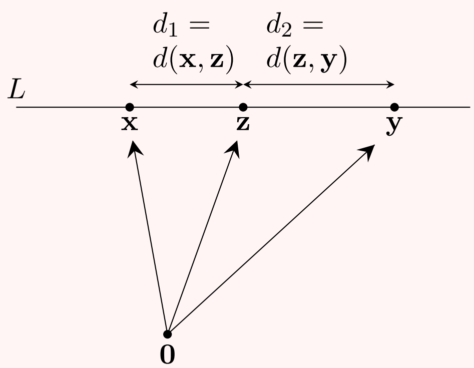
Definition 6 (Collinearity). *We say that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear if there is a line L with $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L$.*

Lemma 5 (Formula for collinearity). *If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are distinct points in $\mathbb{R}^n$, then they are colinear if and only if for some $\lambda \in \mathbb{R}$ we have*

$$\begin{aligned} \mathbf{z} &= \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) \\ &= (1-\lambda)\mathbf{x} + \lambda \mathbf{y} \\ &= \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y} \end{aligned}$$

where $\lambda_1 + \lambda_2 = 1$ and $\lambda_2 = \lambda$. Further, in the case of collinearity $|\lambda_1| : |\lambda_2| = d_2 : d_1$, where $d_1 = d(\mathbf{x}, \mathbf{z})$ and $d_2 = d(\mathbf{z}, \mathbf{y})$.

Quiz: In this picture, $\lambda = \lambda_2 \in (0, 1)$, and $\mathbf{z}$ is between $\mathbf{x}$ and $\mathbf{y}$.
Where is $\mathbf{z}$ on L if $\lambda > 1$ or < 0?



$\mathbf{z} = \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}$
$\lambda_1 + \lambda_2 = 1$
$|\lambda_1| : |\lambda_2| = d_2 : d_1$

Proof. (probably won't use green proof.)

$\Rightarrow$: Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear, on a line L. By definition of L, for some vectors $\mathbf{u}$ and $\mathbf{v}$, all points on L have the form $\mathbf{u} + \lambda \mathbf{v}$. So for some $\lambda_x, \lambda_y, \lambda_z \in \mathbb{R}$, we have

$$\mathbf{x} = \mathbf{u} + \lambda_x \mathbf{v},\ \mathbf{y} = \mathbf{u} + \lambda_y \mathbf{v},\ \mathbf{z} = \mathbf{u} + \lambda_z \mathbf{v}$$

Solving for $\mathbf{u}$ and $\mathbf{v}$, and setting $\lambda = (1 - \lambda_x)/(\lambda_y - \lambda_x)$ gives the result:

$$\mathbf{z} = \frac{\lambda_y \mathbf{x} - \lambda_x \mathbf{y}}{\lambda_y - \lambda_x} + \frac{\mathbf{x} - \mathbf{y}}{\lambda_x - \lambda_y} = \left[\frac{\lambda_y - 1}{\lambda_y - \lambda_x}\right] \mathbf{x} + \left[\frac{1 - \lambda_x}{\lambda_y - \lambda_x}\right] \mathbf{y}$$

Now we have $d(\mathbf{x}, \mathbf{z}) = ||\mathbf{x} - \mathbf{z}|| = ||\mathbf{x} - \lambda_1 \mathbf{x} - \lambda_2 \mathbf{y}|| = |\lambda_2| ||\mathbf{x} - \mathbf{y}||$, and similarly $d(\mathbf{x}, \mathbf{z}) = |\lambda_1| ||\mathbf{x} - \mathbf{y}||$, which gives $|\lambda_1| : |\lambda_2| = d_2 : d_1$.

$\Leftarrow$: If $\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$, then $\mathbf{z}$ lies on the line through $\mathbf{x}$ in the direction of $\mathbf{y} - \mathbf{x}$. This line also contains $\mathbf{y}$, by taking $\lambda = 1$.

$\square$

Alternative proof of $\Rightarrow$ (same idea, different notation):
Proof 2. (of Lemma 5)
$\Rightarrow$ $\mathbf{x}, \mathbf{y}, \mathbf{z}$ being collinear means they lie on a line $\mathbf{u} + \mathbb{R}\mathbf{v}$, with $\mathbf{v} \neq \mathbf{0}$. So, for some $\lambda_x, \lambda_y, \lambda_z \in \mathbb{R}$,

$$\mathbf{x} = \begin{bmatrix} \vert & \vert \\ \mathbf{u} & \mathbf{v} \\ \vert & \vert \end{bmatrix} \begin{pmatrix} 1 \\ \lambda_x \end{pmatrix},\ \mathbf{y} = \begin{bmatrix} \vert & \vert \\ \mathbf{u} & \mathbf{v} \\ \vert & \vert \end{bmatrix} \begin{pmatrix} 1 \\ \lambda_y \end{pmatrix},\ \mathbf{z} = \begin{bmatrix} \vert & \vert \\ \mathbf{u} & \mathbf{v} \\ \vert & \vert \end{bmatrix} \begin{pmatrix} 1 \\ \lambda_z \end{pmatrix}$$

So,

$$\begin{bmatrix} \vert & \vert & \vert \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \vert & \vert & \vert \end{bmatrix} = \begin{bmatrix} \vert & \vert \\ \mathbf{u} & \mathbf{v} \\ \vert & \vert \end{bmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \lambda_x & \lambda_y & \lambda_z \end{pmatrix} \tag{3}$$

we're assuming $\lambda_x \neq \lambda_y$ (since $\mathbf{x} \neq \mathbf{y}$), so the matrix

$$M := \begin{pmatrix} 1 & 1 & 1 \\ \lambda_x & \lambda_y & \lambda_z \end{pmatrix}$$

has rank 2, so has kernel of rank 1, (rank nullity theorem) spanned by some vector $(u_1, u_2, u_3)^T \in \mathbb{R}^3$. We can't have $u_3 = 0$, since this would imply that the first two columns are linearly dependent, implying $\lambda_x = \lambda_y$, implying $\mathbf{x} = \mathbf{y}$, which we assume is not the case.

So we can scale by $1/u_3$, and so replace (u_1, u_2, u_3) by $(-\lambda_1, -\lambda_2, 1)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.
Now multiplying both sides of Equation 3 by $(-\lambda_1, -\lambda_2, 1)^T$, we get:

$$\begin{bmatrix} | & | & | \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ | & | & | \end{bmatrix} \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \\ 1 \end{pmatrix} = \mathbf{0} \tag{4}$$

Rearranged, $\mathbf{z} = \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}$. Since $(-\lambda_1, -\lambda_2, 1)$ is in the kernel of M, we must have $\lambda_1 + \lambda_2 = 1$. This implies that we also have

$$\begin{bmatrix} | & | & | \\ \mathbf{z} & \mathbf{z} & \mathbf{z} \\ | & | & | \end{bmatrix} \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \\ 1 \end{pmatrix} = \mathbf{0} \tag{5}$$

Subtracting Equation 5 from Equation 4 gives

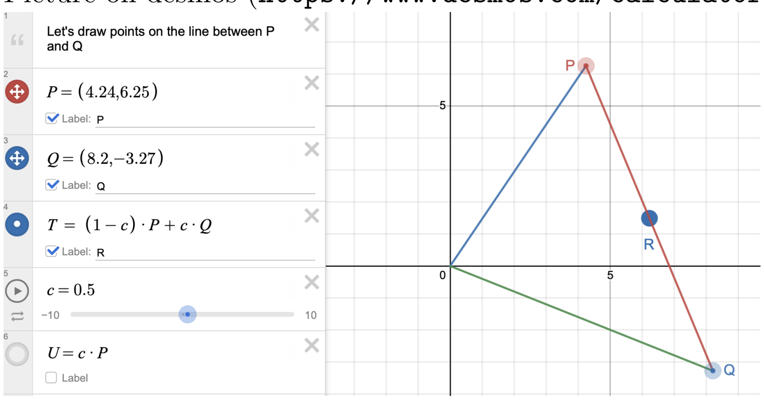
$$\begin{bmatrix} | & | & | \\ \mathbf{x} - \mathbf{z} & \mathbf{y} - \mathbf{z} & \mathbf{0} \\ | & | & | \end{bmatrix} \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \\ 1 \end{pmatrix} = \mathbf{0}$$

I.e., $\lambda_1(\mathbf{x} - \mathbf{z}) = -\lambda_2(\mathbf{y} - \mathbf{z})$ So

$$|\lambda_1|||\mathbf{x} - \mathbf{z}|| = |\lambda_2|||\mathbf{y} - \mathbf{z}||.$$

□

Picture on desmos (`https://www.desmos.com/calculator/lcgtswvxgm`):



4. Lecture 4: Collinearity in terms of the metric

For this class, before the new material:

(1) Recap on triangle inequality:

For all $\mathbf{x}, \mathbf{y}, \mathbf{x} \in \mathbb{R}^n$ we have $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$, with equality if and only if either $\mathbf{z} = \mathbf{x}$, or $\mathbf{y} - \mathbf{z}$ is a positive scalar multiple of $\mathbf{z} - \mathbf{x}$.

(2) Finish proof of Lemma 5:

If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are distinct points in $\mathbb{R}^n$, then they are colinear if and only if for some $\lambda \in \mathbb{R}$ we have

$$\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) = (1 - \lambda)\mathbf{x} + \lambda\mathbf{y} = \lambda_1\mathbf{x} + \lambda_2\mathbf{y}$$

where $\lambda_1 + \lambda_2 = 1$ and $\lambda_2 = \lambda$. In the case of collinearity $|\lambda_1| : |\lambda_2| = d(\mathbf{x}, \mathbf{z}) : d(\mathbf{z}, \mathbf{y})$.

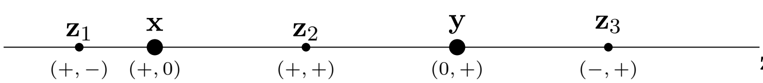
We need to finish the part about $|\lambda_1| : |\lambda_2| = d_2 : d_1$, and $\Leftarrow$. See notes for Lecture 3.

4.1. Note on collinearity and λ_1, λ_2.

In Lemma 5, we have collinearity of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and, providing $\mathbf{x}, \mathbf{y}$ are distinct, we can write

$$\mathbf{z} = \lambda_1\mathbf{x} + \lambda_2\mathbf{y}$$

with $\lambda_1 + \lambda_2 = 1$. The following picture shows possible signs of λ_1, λ_2:



$$\mathbf{z} = \lambda_1\mathbf{x} + \lambda_2\mathbf{y} \iff (\mathbf{x}, \mathbf{y}, \mathbf{z})\begin{pmatrix}\lambda_1\\ \lambda_2\\ -1\end{pmatrix} = \mathbf{0}$$

We can always scale the vector $(\lambda_1, \lambda_2, -1)$ so that one term is -1 and the other two are non-negative; the term which is -1 the corresponds to which of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ is "between" the other two. E.g., if $\mathbf{z} = 3\mathbf{x} - 2\mathbf{y}$, then we can scale $(3, -2, -1)$ to $(-1, 2/3, 1/3)$, and write as $\mathbf{x} = 2/3\mathbf{y} + 1/3\mathbf{z}$.

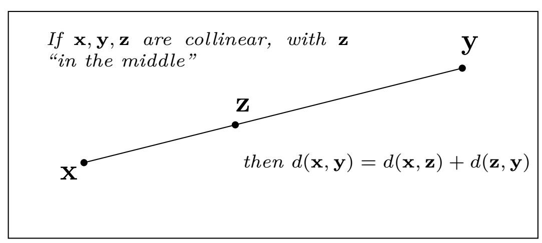
From now on, we identify $\mathbb{E}^m$ with $\mathbb{R}^m$, i.e., we choose and fix a zero. This is always possible, so the results apply to any Euclidean space. We now give two lemmas which we use to express collinearity in terms of the metric.

Now we can characterize collinearity in terms of the Euclidean metric:

Proposition 1. $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *are collinear if and only if we have equality in the triangle inequality*

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \tag{6}$$

up to permuting $\mathbf{x}, \mathbf{y}, \mathbf{z}$.



Proof. Easy case: If $\mathbf{y} = \mathbf{z}$, the last term is zero and the others are equal, so the result is true. Since we are proving the result up to permutaion of the points, we now assume that all points are distinct.

$\Leftarrow$: In the proof of the triangle inequality, we saw that there is equality in (6) if and only if $\mathbf{z} = \mathbf{x}$ (which we are now assuming is not the case) or $\mathbf{y} - \mathbf{z}$ is a positive multiple of $\mathbf{z} - \mathbf{x}$. I.e., for some $\lambda \in \mathbb{R}_{\geq 0}$, $\mathbf{y} - \mathbf{z} = \lambda(\mathbf{z} - \mathbf{x})$. Rearranged,

$$\mathbf{y} = \mathbf{z} - \lambda(\mathbf{x} - \mathbf{z}) \tag{7}$$

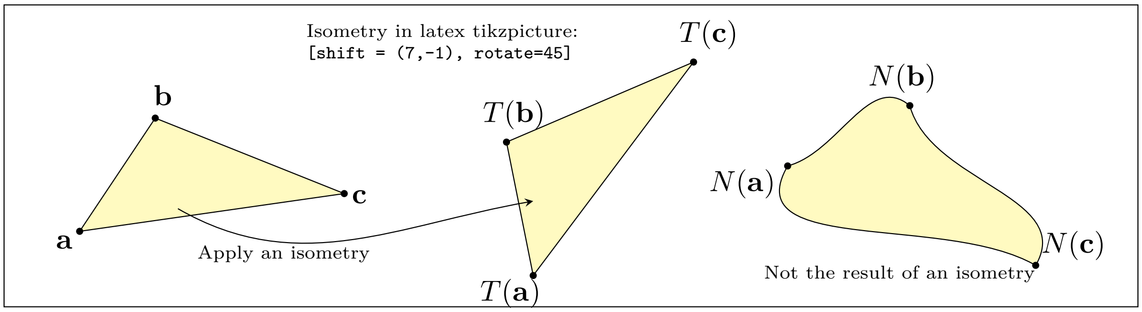
which by Lemma 5 implies $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear. (λ happens to be positive, so we don't yet have $\Rightarrow$.)

$\Rightarrow$: If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear, and distinct, then by Lemma 5, and following discussion, we can permute (rename) $\mathbf{x}, \mathbf{y}, \mathbf{z}$ so that $\mathbf{z} = \lambda_1\mathbf{x} + \lambda_2\mathbf{y}$ with $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$. Then

$$\mathbf{y} = \frac{1}{\lambda_2}\mathbf{z} - \frac{\lambda_1}{\lambda_2}\mathbf{x} = \mathbf{z} + \frac{\lambda_1}{\lambda_2}(\mathbf{z} - \mathbf{x})$$

So $\mathbf{y} - \mathbf{z}$ is a positive multiple of $\mathbf{z} - \mathbf{x}$, and so by Lemma 4 (triangle inequality) Equation (6) holds. $\square$

4.2. Isometries and Collinearity. We're going to show that isometries map lines to lines, and use this to write isometries in terms of matrices. Previously you have seen that vectors and matrices can be used to define "motions", but now we show that *all* isometries have this form.



PROPOSITION 2. *If T is an isometry of $\mathbb{E}^m$, then T maps lines to lines, i.e., $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear then $T(\mathbf{x}), T(\mathbf{y}), T(\mathbf{z})$ are also collinear. Moreover, if $\mathbf{z} = (1-\lambda)\mathbf{x} + \lambda\mathbf{y}$, then $T(\mathbf{z}) = (1-\lambda)T(\mathbf{x}) + \lambda T(\mathbf{y})$.*

PROOF. By Propsition 1, and since T is distance preserving,

$$\begin{aligned}
\mathbf{x}, \mathbf{y}, \mathbf{z} \text{ are collinear } &\Rightarrow d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \text{ after possible permutation of } \mathbf{x}, \mathbf{y}, \mathbf{z} \\
&\Rightarrow d(T(\mathbf{x}), T(\mathbf{y})) = d(T(\mathbf{x}), T(\mathbf{z})) + d(T(\mathbf{z}), T(\mathbf{y})) \\
&\Rightarrow T(\mathbf{x}), T(\mathbf{y}), T(\mathbf{z}) \text{ are collinear by Proposition 1}
\end{aligned}$$

If $\lambda \in [0,1]$, the final statement follows from Lemma 5, which implies $(1-\lambda) : \lambda = d(\mathbf{z}, \mathbf{y}) : d(\mathbf{x}, \mathbf{z}) = d(T(\mathbf{z}), T(\mathbf{y})) : d(T(\mathbf{x}), T(\mathbf{z}))$. Cases $\lambda \notin [0,1]$ are obtained by reordering $\mathbf{x}, \mathbf{y}, \mathbf{z}$ appropriately. □

DEFINITION 7. *A* **linear map** *is a function $L : V \to W$ between two vector spaces over k, such that*

$$L(\lambda\mathbf{v} + \mu\mathbf{w}) = \lambda L(\mathbf{v}) + \mu L(\mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$, and for all $\lambda, \mu \in k$. For us, the field k is almost always $\mathbb{R}$; we might extend to $\mathbb{C}$ sometimes.

DEFINITION 8 (**Affine maps**). *A map $T : \mathbb{R}^n \to \mathbb{R}^k$ is affine if it is of the form $T(\mathbf{x}) = L(\mathbf{x}) + \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$ for some linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ linear and $\mathbf{b} \in \mathbb{R}^k$. Note that $\mathbf{b} = T(\mathbf{0})$.*

From linear algebra, you know that a linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ is described by a k by n matrix A.
Notation: For a $k \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^k$, write $T_{(A,\mathbf{b})}$ for the map

$$T_{(A,\mathbf{b})}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

for $\mathbf{x} \in \mathbb{R}^n$.

THEOREM 2. *The set of all invertible affine maps from $\mathbb{R}^n$ to $\mathbb{R}^n$,*

$$\mathit{Aff}(n) := \left\{ T_{(A,\mathbf{b})} : A \text{ is an } n \times n \text{ invertible matrix}, \mathbf{b} \in \mathbb{R}^n \right\}$$

is a group. The composition law is

$$T_{(A_1,\mathbf{b}_1)} \circ T_{(A_2,\mathbf{b}_2)} = T_{(A_1A_2, A_1\mathbf{b}_2 + \mathbf{b}_1)}$$

PROOF. Exercise. □

This group law is denoted

$$\mathrm{Aff}(n) \cong GL(n) \ltimes \mathbb{R}^n.$$

The notation $\ltimes$ is just a way of saying that this is a special kind of product, called the "semi direct product", not the direct product of two groups.
Exercise: What is the inverse of $T_{A,\mathbf{b}}$?
Note that the $GL(n)$ is a subgroup of $\mathrm{Aff}(n)$ – this is the subgroup of affine maps that fix the origin.
$\mathbb{R}^n$ is the subgroup of $\mathrm{Aff}(n)$ consisting of the translations.
Exercise: $GL(n)$ and $\mathit{Stab}(\mathbf{v})$ (maps fixing $\mathbf{v}$) are normal subgroups of $\mathrm{Aff}(n)$. Show by example that the subgroup of translations is not normal.
We are going to show that Euclidean isometries are affine. But not just any old affine map.

5. Lecture 5: Isometries of $\mathbb{R}^m$ in terms of linear maps

First a more general result about affine maps between real vector spaces which may have different dimension. This result shows that affine maps send lines to lines.

PROPOSITION 3. *Given a map $T : \mathbb{R}^n \to \mathbb{R}^k$, the following are equivalent*

(1) *T is affine.*
(2) *for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$T(\lambda\mathbf{x} + \mu\mathbf{y}) - T(\mathbf{0}) = \lambda(T(\mathbf{x}) - T(\mathbf{0})) + \mu(T(\mathbf{y}) - T(\mathbf{0})) \tag{†}$$

I.e., $\mathbf{x} \to T(\mathbf{x}) - \mathbf{b}$ is a linear map, and T is linear if $T(\mathbf{0}) = \mathbf{0}$.

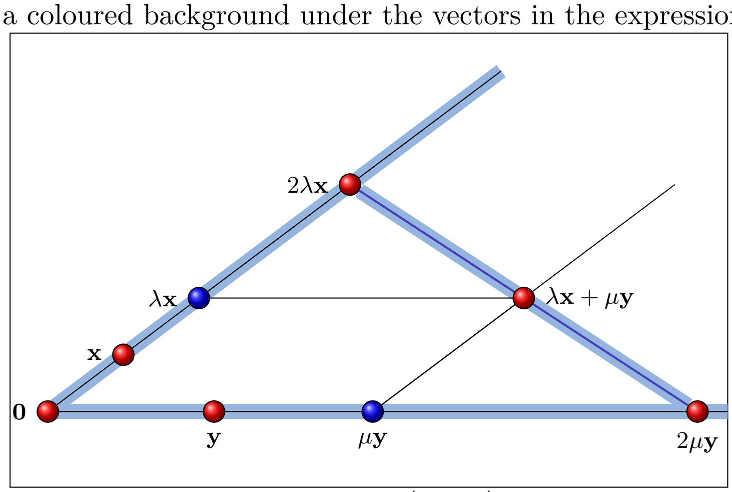
(3) *For all $\lambda \in \mathbb{R}$,*

$$T(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda T(\mathbf{x}) + (1-\lambda)T(\mathbf{y}) \tag{$*$}$$

PROOF. (1) $\iff$ (2): Define a map L by $L(\mathbf{x}) := T(\mathbf{x}) - T(\mathbf{0})$.

$$\begin{aligned} T \text{ is affine} &\iff L \text{ is linear, by definition of affine} \\ &\iff L(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda L(\mathbf{x}) + \mu L(\mathbf{y}), \text{ by defintion of linear} \\ &\iff T(\lambda\mathbf{x} + \mu\mathbf{y}) - T(\mathbf{0}) = \lambda(T(\mathbf{x}) - T(\mathbf{0})) + \mu(T(\mathbf{y}) - T(\mathbf{0})) \text{ since } L(\mathbf{x}) = T(\mathbf{x}) - T(\mathbf{0}). \end{aligned}$$

(2)$\Rightarrow$ (3): This is obtained by setting $\mu = 1 - \lambda$ in (Equation †); all terms involving $T(\mathbf{0})$ cancel.
(3)$\Rightarrow$ (2): By Lemma 5, (3) is telling us that T maps lines to lines, since $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is on the line through $\mathbf{x}$ and $\mathbf{y}$, and $T(\mathbf{z}) = \lambda T(\mathbf{x}) + (1-\lambda)T(\mathbf{y})$ means that $T(\mathbf{z})$ is on the line through $T(\mathbf{x})$ and $T(\mathbf{y})$. This is stronger than just colinearity: it says that the relation between the points on a line is preserved. Now we need to relate linearity to conservation of collinearity. We will use the relationship between the red points on the blue lines in the figure.
Since it may be unclear whether $a\mathbf{v}$ means the scalar a times the vector $\mathbf{v}$, or the vector $a\mathbf{v}$, I have put a coloured background under the vectors in the expressions of collinearity.



$2\lambda\mathbf{x}, \lambda\mathbf{x} + \mu\mathbf{y}$ and $2\mu\mathbf{y}$ are collinear, since

$$\boxed{\lambda\mathbf{x} + \mu\mathbf{y}} = \left(1 - \tfrac{1}{2}\right) \boxed{2\lambda\mathbf{x}} + \tfrac{1}{2} \boxed{2\mu\mathbf{y}}$$

So by Equation $*$, with $\lambda = \frac{1}{2}$,
(★)

$$T(\boxed{\lambda\mathbf{x} + \mu\mathbf{y}}) = \frac{1}{2}T(\boxed{2\lambda\mathbf{x}}) + \frac{1}{2}T(\boxed{2\mu\mathbf{y}}).$$

$\mathbf{0}, \mathbf{x}$ and $2\lambda\mathbf{x}$ are collinear, with

$$\boxed{2\lambda\mathbf{x}} = 2\lambda \boxed{\mathbf{x}} + (1 - 2\lambda) \boxed{\mathbf{0}}$$

So $T(\boxed{2\lambda\mathbf{x}}) = 2\lambda T(\boxed{\mathbf{x}}) + (1 - 2\lambda)T(\boxed{\mathbf{0}})$, so

$$\tfrac{1}{2}T(2\lambda\mathbf{x}) = \lambda T(\mathbf{x}) + \left(\tfrac{1}{2} - \lambda\right) T(\mathbf{0})$$

Similarly, $\frac{1}{2}T(2\mu\mathbf{y}) = \mu T(\mathbf{y}) + \left(\frac{1}{2} - \mu\right) T(\mathbf{0})$. Substituting in Equation ★ and rearranging gives the required expression (†) of linearity of $T(\mathbf{x}) - T(\mathbf{0})$. □

COROLLARY 1. *Every Euclidean isometry $T : \mathbb{R}^m \to \mathbb{R}^n$ is affine, and $n = m$ i.e.,T has the form $T(\mathbf{x}) = L(\mathbf{x}) + \mathbf{b}$ for some linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ and vector $\mathbf{b} \in \mathbb{R}^n$.*

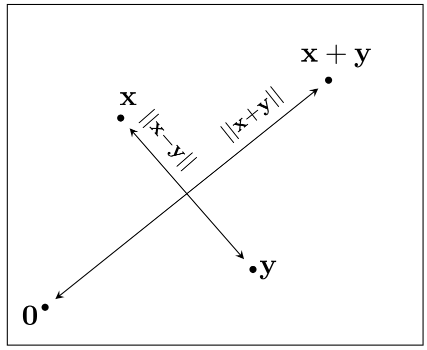
PROOF. By Proposition 2 T preserves collinearity, so (3) of Proposition 3 is satisfied, so since (1) and (3) are equivalent, T is affine. Since T is a bijection, L must also be a bijection. From linear algebra, e.g., use the rank-nullity theorem, this implies $n = m$. □

Next we work out what kinds of linear map L can be. It turns out that being an isometry implies extra conditions on L, so we want to understand what kind of L is possible.

5.1. Linear isometries have orthogonal matrices. We now find a characterisation of linear isometries L in terms of the matrix A corresponding to L. We need a result, which expresses the inner product in terms of the norm:

Lemma 6 (polarization identity for real inner products). *For $\mathbf{x}, \mathbf{y}$ in a vector space over $\mathbb{R}$ (or over a field with characteristic not 2),*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}||\mathbf{x}+\mathbf{y}||^2 - \frac{1}{4}||\mathbf{x}-\mathbf{y}||^2$$



Proof. Exercise in manipulating the inner product, and recalling the definition of the norm. $\square$

Proposition 4. *let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix A with respect to the standard basis. Then the following are equivalent:*

1. *L is an isometry*
2. *$||L(\mathbf{x})|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$, i.e., L is norm preserving*
3. *$\langle L(\mathbf{x}), L(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle \, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ i.e., L preserves the inner product*
4. *The matrix A is orthogonal, i.e., $A^T A = I$*

Proof. (1)⇒(2): Using definition of the metric, defining property of isometries, and linearity of L: $||\mathbf{x}|| = d(\mathbf{x}, \mathbf{0}) = d(L(\mathbf{x}), L(\mathbf{0})) = d(L(\mathbf{x}), \mathbf{0}) = ||L(\mathbf{x})||$.
(2)⇒(3): This follows from the polarisation inequality in Lemma 6.
(3)⇒(4): Now compute the elements of $A^T A$. By the formula for matrix multiplication

$$(A^T A)_{ij} = \sum_{k=1}^{n} (A^T)_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj} = \sum_{k=1}^{n} L(\mathbf{e}_i)_k L(e_j)_k = \langle L(\mathbf{e}_i), L(e_j)\rangle$$

So the entries of $A^T A$ are the inner products of the columns of A. Since the $\mathbf{e}_i$ are the standard basis, and by hypothesis (3), L preserves the inner product, we have

$$(A^T A)_{ij} = \langle L(\mathbf{e}_i), L(e_j)\rangle = \langle \mathbf{e}_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

These are the elements of the identity matrix, so A is orthogonal.
(4)⇒ (1): Assuming (4), A is invertible with inverse A^T, so L is a bijection. So we just need to check L is distance preserving. First check that L is norm preserving:

$$\begin{aligned} ||L(\mathbf{x})||^2 &= \langle L(\mathbf{x}), L(\mathbf{x})\rangle \\ &= (L(\mathbf{x}))^T L(\mathbf{x}) \;\text{ using matrix notation for dot product} \\ &= (A\mathbf{x})^T (A\mathbf{x}) \;\text{ writing } L(\mathbf{x}) = A\mathbf{x} \\ &= \mathbf{x}^T A^T A\mathbf{x} \;\text{ property of transpose} \\ &= \mathbf{x}^T \mathbf{x} \;\text{ since } A \text{ is orthogonal} \\ &= \langle \mathbf{x}, \mathbf{x}\rangle, \;\text{ by definition of inner product} \\ &= ||\mathbf{x}||^2, \;\text{ by definition of norm.} \end{aligned}$$

Now distance preservation follows from the linearity of L:

$$d(L(\mathbf{x}), L(\mathbf{y})) = ||L(\mathbf{x}) - L(\mathbf{y})|| = ||L(\mathbf{x}-\mathbf{y})|| = ||\mathbf{x}-\mathbf{y}|| = d(\mathbf{x}, \mathbf{y}).$$

Since A has inverse A^T, L is invertible, so L is an isometry. $\square$

6. Lecture 6: Isometry group of Euclidean space

Now we can characterize isometries in terms of their matrices. Proposition 4 deals with isometries which are linear maps. Corollary 1 showed how isometries are related to linear maps. So we have:

Corollary 2. *A Euclidean isometry (motion) $T : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$ for some unique orthogonal $n \times n$ matrix A and some vector $\mathbf{b} \in \mathbb{R}^n$.*
Conversely, if A is an orthogonal matrix, $A\mathbf{x} + \mathbf{b}$ is an isometry for all $\mathbf{b} \in \mathbb{R}^m$.

Proof.

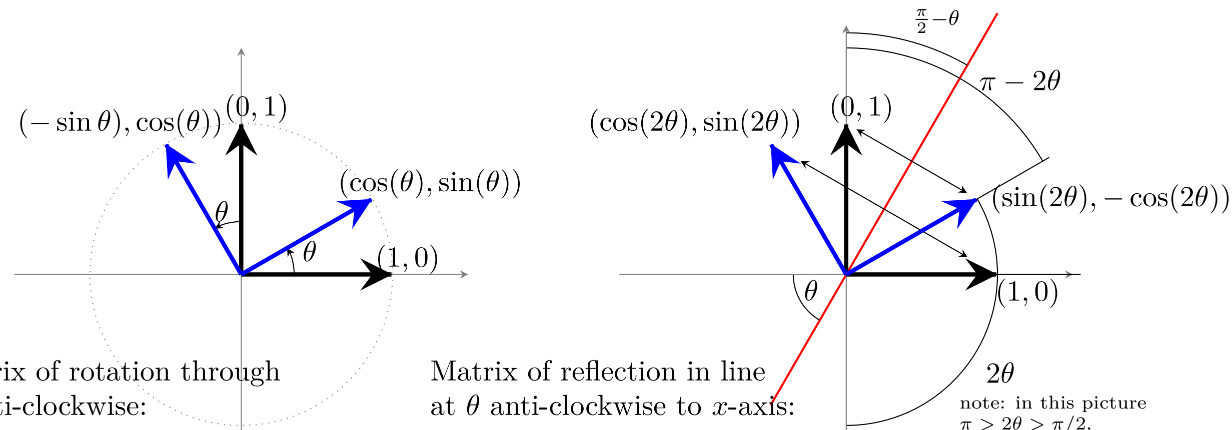
$$\begin{aligned} T \text{ is a motion } &\iff L(\mathbf{x}) := T(\mathbf{x}) - T(\mathbf{0}) \text{ is a linear isometry} \\ &\qquad\quad \text{(distance is translation invariant by Lemma 3)} \\ &\iff L(\mathbf{x}) = A\mathbf{x} \text{ for some orthogonal } A \text{ by Proposition 4} \\ &\iff T(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \text{ for some orthogonal } A \text{ and } \mathbf{b} = T(\mathbf{0}) \end{aligned}$$

The matrix corresponding to a linear map with respect to a given basis is unique (result from linear algebra). $\square$

Example 2. *The following matrices are orthogonal:* $I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

6.1. Isometries of $\mathbb{R}^2$.

Recall the matrices for rotations and reflections are defined as follows:



Matrix of rotation through θ anti-clockwise:

$$B_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Matrix of reflection in line at θ anti-clockwise to x-axis:

$$\rho_\theta := \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Note that $\rho_\alpha \circ \rho_\beta = B_{2(\beta-\alpha)}$

Theorem 3. *Let $T : \mathbb{E}^2 \to \mathbb{E}^2$ be an isometry. Then $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where, with respect to the standard basis, $A = B_\theta$ or ρ_θ and $\mathbf{b} \in \mathbb{R}^2$, so, T is a rotation or reflection, followed by a translation.*

Proof. By Corollary 2, $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$, with A orthogonal. Write A as a column matrix, $A = [\mathbf{v}, \mathbf{w}]$. Then $A^T A = I$ implies $||\mathbf{v}|| = ||\mathbf{w}|| = 1$ and $\langle \mathbf{v}, \mathbf{w}\rangle = 0$. Let $\mathbf{v} = (a, b)$. Since $a^2 + b^2 = 1$, and $a, b \in \mathbb{R}$, for some θ, $(a, b) = (\cos\theta, \sin\theta)$. Since $\langle \mathbf{v}, \mathbf{w}\rangle = 0$, and $||\mathbf{w}|| = 1$, $\mathbf{w} = \pm(-b, a)$. These give the two cases. $\square$

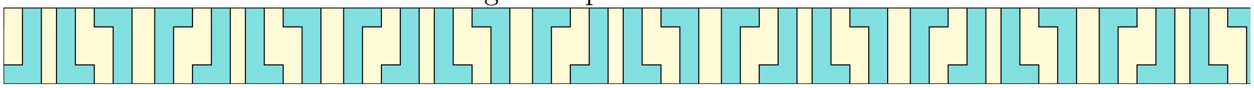
We can classify isometries in $\mathbb{E}^2$ further.

Theorem 4. *Every isometry of $\mathbb{E}^2$ has one of the following forms:*

- *Translation*
- *Rotation*
- *Reflection*
- *Glide reflection: reflection, followed by translation parallel to the line fixed by the reflection.*

Proof. Exercise. See [RS] Section 1.14 for more details. $\square$

6.2. Frieze groups: examples of isometries of $\mathbb{R}^2$. The motives in the following frieze patterns are related by translations, rotations, reflections or glide reflections - can you identify which of these isometries are involved in the following frieze pattern?



6.3. Isometry Groups.

Definition 9. *if (X, d) is a metric space, then Isom(X, d) is the group of all isometries from X to X, with group law being composition of functions.*

Exercise: Verify that $\mathrm{Isom}(X, d)$ is indeed a group.

6.4. The group of Euclidean isometries. We know from Corollary 2 that isometries of $\mathbb{E}^n$ have the form

$$\mathbf{x} \to A\mathbf{x} + \mathbf{b}$$

where A is an orthogonal matrix and $\mathbf{b}$ is a vector.

Definition 10 (The orthogonal group). *The group of $n \times n$ real orthogonal matrices is defined by*

$$O(n, \mathbb{R}) = \{A \text{ is an } n \times n \text{ real matrix } : A^T A = I_n\}$$

We will usually just write $O(n)$, since usually we will only consider matrices with real coefficients.

Lemma 7. *$O(n, \mathbb{R})$ is a group.*

Proof. Exercise. □

As sets,

$$\mathrm{Isom}(\mathbb{R}^n, d) \cong O(n, \mathbb{R}) \times \mathbb{R}^n$$
$$T \leftrightarrow (A, \mathbf{b}) = (T - T(\mathbf{0}), T(\mathbf{0}))$$

As group,

$$\mathrm{Isom}(\mathbb{R}^n, d) \cong O(n, \mathbb{R}) \ltimes \mathbb{R}^n$$

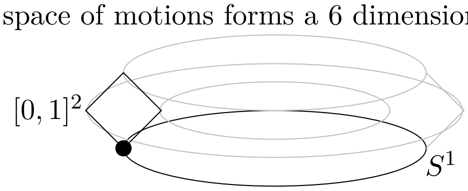
If $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ and $T'(\mathbf{x}) = A'\mathbf{x} + \mathbf{b}'$, with A and $A' \in O(n)$, and $\mathbf{b}, \mathbf{b}' \in \mathbb{R}^n$, we have

$$(T \circ T')\mathbf{x} = A(T'(\mathbf{x})) + \mathbf{b} = AA'\mathbf{x} + A\mathbf{b}' + \mathbf{b}$$

6.5. Topology of $\mathrm{Isom}(\mathbb{E}^2)$ [non-examinable; from [RS chapter 8]]. Now we have seen that all elements of $\mathrm{Isom}(\mathbb{E}^2)$ can be described by four parameters, $\theta \in [0, 2\pi), a \in \mathbb{R}, b \in \mathbb{R}, c \in \{1, -1\}$:

$$\mathbf{x} \mapsto \begin{pmatrix} \cos(\theta) & -c\sin(\theta) \\ \sin(\theta) & c\cos(\theta) \end{pmatrix} \mathbf{x} + \begin{pmatrix} a \\ b \end{pmatrix}$$

Since a rotation of 2π is the same as a rotation through 0, we may think of the rotations as being parameterised by points on the circle $S^1 = \{\mathbf{x} \in \mathbb{R}^2 || |\mathbf{x}|| = 1\}$. So, the elements of $\mathrm{Isom}(\mathbb{E}^2)$ correspond to points in $S^1 \times \mathbb{R}^2 \times \{\pm 1\}$. You can think of this as two disjoint copies of a kind of an infinite solid torus. Now a continuous "motion" in $\mathbb{R}^2$ will correspond to a continuous path in the component corresponding to rotations. (called $SO(2)$.) The motion of an object in $\mathbb{R}^2$ from time 0 to 1 could be be parameterised by a map from $[0, 1]$ into $S^1 \ltimes \mathbb{R}^2$. Our motions form a three dimensional topological space. For $\mathbb{R}^3$, the space of motions forms a 6 dimensional space. See [RS chapter 8] for more details.



sketch of $S^1 \times [0, 1]^2$, representing motions mapping $\mathbf{0}$ to a point in $[0, 1]^2$

7. Lecture 7: Complex Linear algbera

The goal of this section is to show that all isometries of $\mathbb{R}^n$ can be expressed as a composition of rotations and reflections on 1 and 2 dimensional mutually orthogonal subspaces of $\mathbb{R}^n$.

7.1. Extending a linear map from $\mathbb{R}^n$ to $\mathbb{C}^n$.

In order to prove results about orthogonal matrices, it's useful to work over $\mathbb{C}$.
If $L : \mathbb{R}^n \to R^m$ is a linear map, we can extend it to a map from $L : \mathbb{R}^n \to R^m$, by mapping $\mathbf{u} + i\mathbf{v}$ to $L(\mathbf{u}) + iL(\mathbf{v})$, for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The matrix describing the transformation is the same.

7.2. sesquilinear inner product.

For $\mathbf{z}$,$\mathbf{w} \in \mathbb{C}^n$ we define the sesquilinear inner product by

$$\langle \mathbf{z}, \mathbf{w} \rangle := \sum_{i=1}^{m} z_i \cdot \overline{w_i} = \mathbf{z}^T \cdot \overline{\mathbf{w}}$$

Where $\overline{a}$ is the complex conjugate of a. Note that $||\mathbf{v}||$ is a real number, and Cauchy-Schwartz still holds.

Lemma 8. *If L is a linear isometry of $\mathbb{R}^n$, with matrix A then the extension of L to $\mathbb{C}^n$ preserves the sesquilinear inner product.*

Proof. Take $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$. Then by definition,

$$\langle L(\mathbf{z}), L(\mathbf{w}) \rangle = \langle A\mathbf{z}, A\mathbf{w} \rangle = (A\mathbf{z})^T \cdot \overline{(A\mathbf{w})} = (\mathbf{z}^T A^T)(\overline{A}\overline{\mathbf{w}}) = \mathbf{z}^T A^T A\overline{\mathbf{w}} = \mathbf{z} I_n \overline{w} = \mathbf{z}^T \overline{\mathbf{w}} = \langle \mathbf{z}, \mathbf{w} \rangle$$

□

7.3. Direct sum.

If W is a vector space containing subspaces U and V, then we say W is the direct sum of U and and V, written $W = U \oplus V$ if $U \cap V = \{\mathbf{0}\}$ and every element of W can be written uniquely as a sum of an element of U and an element of V. E.g., $\mathbb{R}^2 = (\mathbb{R}\mathbf{e}_1) \oplus (\mathbb{R}\mathbf{e}_2)$, where $\mathbf{e}_1, \mathbf{e}_2$ is a basis for $\mathbb{R}^2$. We can extend this concept, writing $W = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ where U_i are subspaces of a vector space V, $U_i \cap U_j = \{\mathbf{0}\}$ for $i \neq j$, and any element of W can be written uniquely as $\mathbf{w} = \mathbf{u}_1 + \cdots + \mathbf{u}_n$ with $\mathbf{u}_i \in U_i$.

7.4. Orthogonal complement.

If V is a vector space, with a subspace W, then the orthogonal complement of W in V is the vector subspace:

$$W^\perp := \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \ \forall \mathbf{w} \in W\}$$

If V is a complex vector space, the inner product here means the sesquilinear inner product. In MA251 you will see a proof that $V = W \oplus W^\perp$. This involves finding an orthonormal basis for W.

Lemma 9. *If L is an isometry of $\mathbb{R}^n$, and W is a subspace of $\mathbb{R}^n$ such that $L(W) = W$, then also $L(W^\perp) = W^\perp$. This result also holds for L extended to $\mathbb{C}^n$ and W a subspace of $\mathbb{C}^n$.*

Proof. Note that since L is an isometry, it is a bijection, so has an inverse L^{-1}.

$$\begin{aligned}
\mathbf{v} \in L(W^\perp) &\iff L^{-1}(\mathbf{v}) \in W^\perp \\
&\iff \left\langle L^{-1}(\mathbf{v}), \mathbf{w} \right\rangle = 0 \, \forall \mathbf{w} \in W \\
&\iff \left\langle L(L^{-1}(\mathbf{v})), L(\mathbf{w}) \right\rangle = 0 \, \forall \mathbf{w} \in W, \text{ by Lemma 8} \\
&\iff \langle \mathbf{v}, L(\mathbf{w}) \rangle = 0 \, \forall \mathbf{w} \in W \\
&\iff \langle \mathbf{v}, \mathbf{w} \rangle = 0 \, \forall \mathbf{w} \in W, \text{ since } L(W) = W \\
&\iff \mathbf{v} \in W^\perp
\end{aligned}$$

□

Corollary 3. *Suppose L is an isometry of $\mathbb{R}^n$, and W is a subspace with $L(W) = W$. Now let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p$ be a basis for W and $\mathbf{w}_{p+1}, \mathbf{w}_{p+2}, \ldots, \mathbf{w}_n$ be a basis for $W^\perp$. Then with respect to this basis, the matrix for L has the form:*

$$\begin{pmatrix} \textit{matrix of } L|_W & 0 \\ 0 & \textit{matrix of } L|_{W^\perp} \end{pmatrix}$$

Proof. Let L have matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to this basis, so if $\mathbf{w}_1 \in W$, $\mathbf{w}_2 \in W^\perp$, then

$$L(\mathbf{w}_1 + \mathbf{w}_2) = A\mathbf{w}_1 + B\mathbf{w}_2 + C\mathbf{w}_1 + D\mathbf{w}_2,$$

with $A\mathbf{w}_1, B\mathbf{w}_2 \in W$, $C\mathbf{w}_1, D\mathbf{w}_2 \in W^\perp$. Then $L(W) = W \Rightarrow C = 0$ and $L(W^\perp) = W^\perp \Rightarrow B = 0$.

$\square$

8. Lecture 8: Normal form Theorem

LEMMA 10. *Let $\lambda \in \mathbb{C}$ be an eigen value of an isometry L of $\mathbb{R}^n$, with eigen vector $\mathbf{z}$ in $\mathbb{C}^n$. Then*

(1) $|\lambda| = 1$
(2) *The complex conjugate $\overline{\lambda}$ is also an eigenvalue, with eigenvector $\overline{\mathbf{z}}$.*
(3) *If $\lambda \neq \mu$ are eigenvalues with eigen vectors $\mathbf{z}$ and $\mathbf{w}$ respectively, then $\langle \mathbf{z}, \mathbf{w} \rangle = 0$.*

PROOF. Since the norm is non degenerate, and eigen vectors are non-zero, (1) follows from:

$$\langle \mathbf{z}, \mathbf{z} \rangle = \langle A\mathbf{z}, A\mathbf{z} \rangle = \langle \lambda\mathbf{z}, \lambda\mathbf{z} \rangle = \lambda\overline{\lambda}\,\langle \mathbf{z}, \mathbf{z} \rangle$$

Since A is real, (2) follows from:

$$A\overline{\mathbf{z}} = \overline{A}\overline{\mathbf{z}} = \overline{A\mathbf{z}} = \overline{\lambda}\overline{\mathbf{z}}$$

To prove (3), first compute:

$$\langle \mathbf{z}, \mathbf{w} \rangle = \langle A\mathbf{z}, A\mathbf{w} \rangle = \langle \lambda\mathbf{z}, \mu\mathbf{w} \rangle = \lambda\overline{\mu}\,\langle \mathbf{z}, \mathbf{w} \rangle$$

So, either $\langle \mathbf{z}, \mathbf{w} \rangle = 0$ or $1 = \lambda\overline{\mu}$. If $1 = \lambda\overline{\mu}$, then $\mu = \lambda\mu\overline{\mu} = \lambda|\mu|^2 = \lambda$, since by (1) $|\mu| = 1$. But this contradicts $\lambda \neq \mu$, and so $\langle \mathbf{z}, \mathbf{w} \rangle = 0$. ☐

THEOREM 5 (Normal form Theorem). *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry, with matrix $A \in O(n, \mathbb{R})$. Then there is an orthonormal basis of $\mathbb{R}^n$ with respect to which L has a matrix of the form*

$$\begin{bmatrix} I_k & & & & \\ & -I_m & & & \\ & & B_1 & & \\ & & & \ddots & \\ & & & & B_\ell \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix, and B_i is the 2×2 matrix representing a rotation of θ_i degrees counter clockwise about origin

$$B_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

PROOF. We use induction on n. By Theorem 3, we already know the result for $n = 2$, since either $A = B_\theta$ already has the required form, or $A = \rho_\theta$ has eigen values 1 and -1, and orthogonal eigen vectors $(\cos\theta, \sin\theta)$ and $(\sin\theta, -\cos\theta)$. In this case, with respect to this basis L has matrix $\mathrm{diag}(1, -1)$.
Now suppose $n > 2$. The characteristic polynomial of A has at least one root, and so there is at least one eigen vector $\mathbf{v}$ of L, with eigen value λ. By Lemma 10, either $\lambda = \pm 1$ or λ is not real.
Case 1: If $\lambda = \pm 1$, (fixed sign choice if both are possible) then $\mathrm{rank}(A \mp I) < n$, so $W = \ker(A \mp I) \subset \mathbb{R}^n$ has dimension $\dim W = d \geq 1$, (rank-nullity theorem) then $L|_W = \pm I_d$. By Corollary 3, since $\dim W^\perp < n$, the result follows by induction.
Case 2: if λ is not real, then $\mathbf{v} = \mathbf{x} + i\mathbf{y}$, with $\mathbf{x}, \mathbf{y}$ both non-zero real vectors, since otherwise $\lambda \neq \overline{\lambda}$ would both be eigen values of the same vector, by Lemma 10, which is impossible.
Let $W = \mathbf{x}\mathbb{R} \oplus \mathbf{y}\mathbb{R}$. By Lemma 10, we have $\langle \mathbf{v}, \overline{\mathbf{v}} \rangle = 0$, so $0\langle \mathbf{x} + i\mathbf{y}, \mathbf{x} - i\mathbf{y} \rangle = (\mathbf{x} + i\mathbf{y})^T(\mathbf{x} + i\mathbf{y}) = ||\mathbf{x}||^2 - ||\mathbf{y}||^2 + 2i\langle \mathbf{x}, \overline{\mathbf{y}} \rangle$, and by comparing imaginary parts, we see that $\mathbf{x}, \mathbf{y}$ are orthogonal; by comparing real parts, they have the same length, and so by scaling, which will preserve the eigen value of $\mathbf{v}$, we can assmume $\mathbf{x}, \mathbf{y}$ are orthonormal (length one and orthogonal). Since $|\lambda| = 1$, we have $\lambda = cos\theta + i\sin\theta$ for some $\theta \in [0, 2\pi)$. By considering real and imaginary parts of $L(\mathbf{x}) + iL(\mathbf{y}) = L(\mathbf{x} + i\mathbf{y}) = L(\mathbf{v}) = \lambda\mathbf{v} = (\cos\theta + i\sin\theta)(\mathbf{x} + i\mathbf{y})$, we obtain that L is represented by B_θ with respect to the orthonormal basis $\mathbf{x}, \mathbf{y}$.
By Corollary 3, the result follows by induction. ☐

9. Lecture 9: Reflections

Definition 11. *A hyperplane of $\mathbb{R}^n$ is an affine subspace of dimension $n-1$, which has the form*

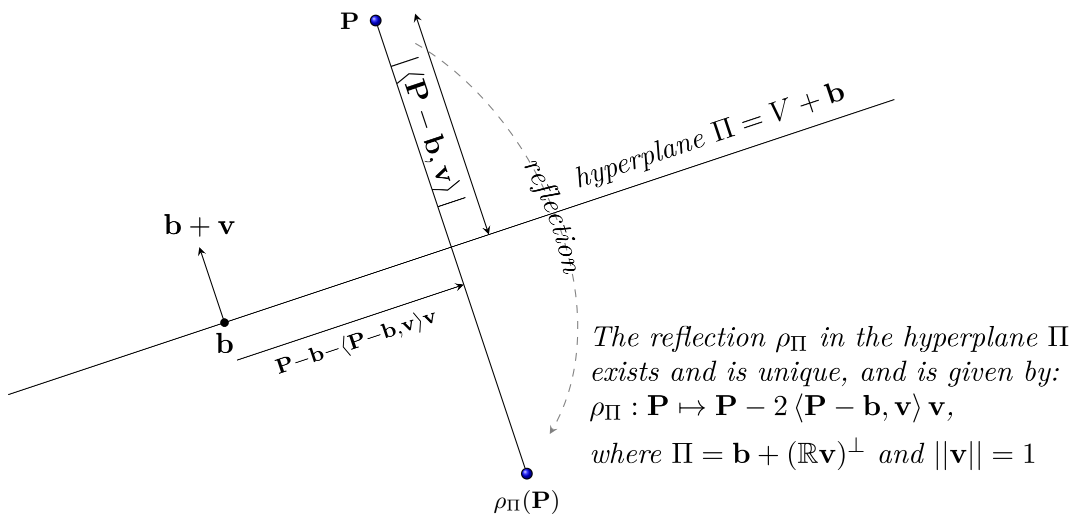
$$\Pi = V + \mathbf{b} = \{\mathbf{b}+\mathbf{v} : \mathbf{v}\in V\} = \{\mathbf{b}+\lambda_1\mathbf{v}_1+\cdots+\lambda_{m-1}\mathbf{v}_{n-1}|\lambda_i\in\mathbb{R}\} = (\mathbb{R}\mathbf{v}_n)^\perp + \mathbf{b}.$$
$$= \{\mathbf{v}\in\mathbb{R}^n : \langle \mathbf{v},\mathbf{v}_n\rangle = \beta\}$$

where $\mathbf{b}\in\mathbb{R}^n$ and $V\subseteq\mathbb{R}^n$ is a vector subspace of dimension $n-1$, with basis $\mathbf{v}_1,\ldots,\mathbf{v}_{n-1}$, and $\mathbf{v}_n$ is orthogonal to V, and $\beta = \langle \mathbf{v}_n,\mathbf{b}\rangle$.
For any $\mathbf{c}\in\mathbb{R}^n$, $\Pi+\mathbf{c} = V+(\mathbf{b}+\mathbf{c})$. ($\Pi$ is a coset of V as an additive subgroup of $\mathbb{R}^n$.)

Definition 12. *For a map $T:\mathbb{R}^n\to\mathbb{R}^n$, the set of fixed points of T is*

$$\mathrm{Fix}(T) := \{\mathbf{x}\in\mathbb{R}^n : T(\mathbf{x}) = \mathbf{x}\}$$

Definition 13. *Let Π be a hyperplane. A* **reflection** *in Π is a Euclidean isometry $\rho_\Pi:\mathbb{R}^n\to\mathbb{R}^n$ such that $\mathrm{Fix}(\rho_\Pi) = \Pi$.*



Theorem 6. *The reflection ρ_Π in the hyperplane Π exists and is unique.*

Proof. Let $\Pi = V+\mathbf{b}$. Pick $\mathbf{v}$ with $V^\perp = \mathbb{R}\mathbf{v}$. Take a basis of $\mathbb{R}^n$ consisting of $\mathbf{v}$ together with a basis for V. With respect to this basis, we define a linear map ρ with matrix

$$A = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_{n-1}\end{pmatrix}.$$

This fixes V, and is not the identity, so is a reflection. Exercise: the conjugate $\rho_\Pi := T^{-1}\rho T$ of ρ by $T(\mathbf{x}) = \mathbf{x}-\mathbf{b}$, is a reflection in Π. Thus ρ_Π exists. If $R\neq \mathrm{id}$ fixes Π, then $S := T\circ R\circ T^{-1}$ fixes V. By Lemma 9, S fixes $V^\perp = \mathbb{R}\mathbf{v}$, so $S(\mathbf{v}) = \alpha\mathbf{v}$ for some $\alpha\in\mathbb{R}$. By Proposition 4, $|\alpha| = 1$, so since $S\neq\mathrm{id}$, $\alpha = -1$, and S has matrix A, so $S = \rho$, so $R = \rho_\Pi$, so the reflection in Π is unique. $\square$

Stuff that I would have covered in week 3 if there was more time

LEMMA 11. *$\rho_\Pi(\mathbf{P}) = \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v}$, where $\Pi = \mathbf{b} + (\mathbb{R}\mathbf{v})^\perp$ and $||\mathbf{v}|| = 1$.*

PROOF. Exercise. $\square$

LEMMA 12. *Given distinct points $\mathbf{P}$ and $\mathbf{Q} \in \mathbb{R}^n$, there exists a reflection ρ with $\rho(\mathbf{P}) = \mathbf{Q}$.*

PROOF. Let $\mathbf{v} = (\mathbf{P} - \mathbf{Q})/||\mathbf{P} - \mathbf{Q}||$. Let $\mathbf{b} = \frac{1}{2}\mathbf{P} + \frac{1}{2}\mathbf{Q}$. By Lemma 5, $\mathbf{b}$ is a point on the line between $\mathbf{P}$ and $\mathbf{Q}$. Let $\Pi = \mathbb{R}\mathbf{v}^\perp + \mathbf{b}$. Then $\rho_\Pi(\mathbf{P}) = \mathbf{Q}$, because using the formula in Lemma 11, $\rho(\mathbf{P}) = \mathbf{P} - 2\left\langle \mathbf{P} - \frac{1}{2}\mathbf{P} - \frac{1}{2}\mathbf{Q}, \frac{(\mathbf{P}-\mathbf{Q})}{||\mathbf{P}-\mathbf{Q}||}\right\rangle \frac{(\mathbf{P}-\mathbf{Q})}{||\mathbf{P}-\mathbf{Q}||} = \mathbf{P} - \langle \mathbf{P} - \mathbf{Q}, \mathbf{P} - \mathbf{Q}\rangle \frac{(\mathbf{P}-\mathbf{Q})}{||\mathbf{P}-\mathbf{Q}||^2} = \mathbf{Q}$. $\square$

LEMMA 13. *If $W = U \oplus V$ for some vector spaces U, V and W, and $R : U \to U$ is a linear reflection, then $R \oplus I : W \to W$ is also a reflection.*

PROOF. The map $R \oplus I$ is defined by $R \oplus I(\mathbf{u} + \mathbf{v}) = R(\mathbf{u}) + \mathbf{v}$ for $\mathbf{u} \in U, \mathbf{v} \in V$. R has order 2 and fixes a codimension 1 subspace U_0 of U. Hence $R \oplus I$ also has order 2, and fixes $U_0 \oplus V$ which is a hyperplane in W. $\square$

Reflections Generate $\mathrm{Isom}(\mathbb{E}^n)$

PROPOSITION 5. *Any linear isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ is the product of at most n reflections.*

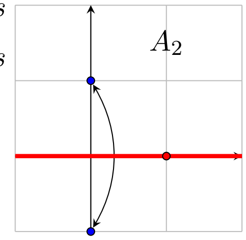
PROOF. This result hold when $n = 1$, since by Lemma 10 in this case $\lambda = \pm 1$, corresponding to 0 or 1 reflections. Now suppose $n > 1$. If $T \neq I$, we're done. If $T \neq I$, there is some $\mathbf{P} \neq \mathbf{0}$ with $T(\mathbf{P}) = \mathbf{Q} \neq \mathbf{P}$. By Lemma 12, there is a reflection R_1 with $R_1(\mathbf{Q}) = \mathbf{P}$, so $R_1 \circ T$ fixes $\mathbf{P}$.
Now note that R_1 is linear. Not all reflections are linear (they are all affine). In general, a reflection might be in a plane which is not through the origin. However, in this case, the hyperplane perpendicular to the line from P to Q and midway between them does pass through the origin. From previous discussion, the plane that you have to reflect in to get from P to Q is perpendicular to the vector $P - Q$. I.e., it is a subspace V, plus some vector $\mathbf{b}$. And the mid point between P and Q is $\mathbf{b} = (P + Q)/2$. So the mirror plane is $V + \mathbf{b}$. If we can show that $P + Q$ is perpendicular to $P - Q$, then $P + Q$ lies in the reflection hyperplane, i.e., $\mathbf{b}$ is in V, and so then $V + \mathbf{b} = V$. And reflecting in a hyperplane through the origin is linear, because the hyperplane is fixed, and so $\mathbf{0}$ is fixed. We know reflections are affine (linear plus constant), so if $\mathbf{0}$ maps to $\mathbf{0}$, then the reflection is linear. So we just have to check that $\langle P + Q, P - Q\rangle = 0$. But $\langle P + Q, P - Q\rangle = \langle P, P\rangle - \langle Q, Q\rangle = ||P||^2 - ||Q||^2$. But isometries preserve length, so since $Q = T(P)$, we have $||Q|| = ||T(P)|| = ||P||$, so $P + Q$ is perpendicular to $P - Q$, as required, and R_1 is linear.
Since R_1 and T are both linear, $R_1 \circ T$ is linear. And the space $W = \mathbb{R}\mathbf{P}$ is also fixed by $R_1 \circ T$. By Lemma 9 $W^\perp$ is fixed by $R_1 \circ T$. By induction $(R_1 \circ T)|_{W^\perp}$ is a product of at most $n - 1$ reflections, $(R_1 \circ T)|_{W^\perp} = R_2 \circ \cdots \circ R_k$ with $k \leq n$. By Lemma 13, these reflections can be considered as reflections of $\mathbb{R}^n$, which all fix W.
Now we have $(R_k \circ \cdots \circ R_2 \circ \mathbb{R}_1 \circ T)|_{W^\perp} = I|_{W^\perp}$, but since $R_1 \circ T|_W = I|_W$, and all R_i are trivial on W, then also $(R_k \circ \cdots \circ R_2 \circ \mathbb{R}_1 \circ T)|_W = I|_W$, So since this map is trivial on both W and $W^\perp$, it is the identity, and so $R_k \circ \cdots \circ R_2 \circ \mathbb{R}_1 \circ T = I$ from which we obtain the result

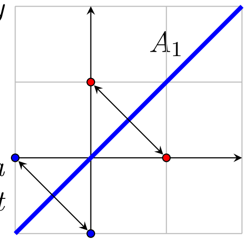
$$T = R_1 \circ R_2 \circ \cdots \circ R_k.$$

$\square$

THEOREM 7. *Any isometry of $\mathbb{E}^n$ is the product of at most $n + 1$ reflections.*

PROOF. Suppose $T(\mathbf{0}) \neq \mathbf{0}$. Then by Lemma 12, there is some reflection R_1 with $R_1(T(\mathbf{0})) = \mathbf{0}$, so $R_1 \circ T$ fixes $\mathbf{0}$, and so is a linear isometry. By Proposition 7, $R_1 \circ T$ is the product of at most n reflections, and hence T is the product of at most n reflections. $\square$





Example 3. *Let T be a rotation through $90°$ about the origin of $\mathbb{R}^2$. Then T has matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Take any point which is not fixed, e.g., $\mathbf{P} = (1,0)$. There is a reflection mapping $(1,0)$ to $T(1,0) = (1,0)$, which has matrix*

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now $A_2 := A_1 \circ A$ must fix the line $(1,0)\mathbb{R}$ pointwise. The orthogonal complement, $(0,1)\mathbb{R}$ must also be fixed by $A_1 \circ A$, so either is the identity, which would imply $A_1 = A$, which is not the case, or $A_1 \circ A$ is a reflection; we can check

$$A_1 \circ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So A_2 is a reflection in the x-axis, and $A = A_1 \circ A_2$. So T can be acheieved by a reflection in $y = 0$ followed by a reflection in $x = y$. If we choose a different point $\mathbf{P}$, we will get a different decomposition.

9.1. Classification of isometries of $\mathbb{R}^3$. Using the Normal form theorem, all isometries of $\mathbb{R}^3$ are one of the following:

(1) Translation, by a vector $\mathbf{v} \in \mathbb{R}^3$.
(2) Rotation about some axis
(3) Twist: A rotation about some axis, followed by a translation in the direction of this axis
(4) Reflection in some plane
(5) Glide: Reflection in a plane, followed by a translation by a vector parallel to the plane.
(6) Rotary relection: Rotation followed by a reflection in a plane perpendicular to the rotation axis.

Proof is omitted. See course text book, [RS].

9.2. Euclid's parallel postulate and angles in triangles.

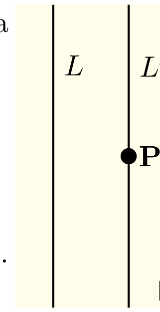
9.3. Euclid's fifth postulate.

PROPOSITION 6. *Given a line L and a point $\mathbf{P} \in \mathbb{R}^2$, with $\mathbf{P} \notin L$, there is a unique line L' with $\mathbf{P} \in L$ and $L \cap L' = \varnothing$.*

PROOF. Given $L = \mathbb{R}\mathbf{v} + \mathbf{b}$, let $L' = \mathbb{R}\mathbf{v} + \mathbf{P}$. This contains $\mathbf{P}$ and is a line by construction. Suppose $\mathbf{w} \in L \cap L'$. Then $\mathbf{w} = \alpha\mathbf{v} + \mathbf{b} = \beta\mathbf{v} + \mathbf{P}$, which implies $\mathbf{P} = (\alpha - \beta)\mathbf{v} + \mathbf{b} \in L$, a contradiction. Hence $L \cap L' = \varnothing$, and the required L' exists.
Suppose there was another line $L'' = \mathbb{R}\mathbf{v}' + \mathbf{P}$ also through $\mathbf{P}$ with $L \cap L' = \varnothing$.
To find $\mathbf{w} \in L \cap L''$, we want to solve $\mathbf{w} = \mathbf{b} + \alpha\mathbf{v} = \mathbf{P} + \beta\mathbf{v}'$, which means solving

$$\mathbf{b} - \mathbf{P} = (\mathbf{v} \quad \mathbf{v}') \begin{pmatrix} -\alpha \\ \beta \end{pmatrix},$$

which is possible if the matrix $[\mathbf{v}\mathbf{v}']$ is invertible, which is the case if $\mathbf{v}$ and $\mathbf{v}'$ are independent.
So, if $L \cap L'' = \varnothing$, then $\mathbf{v}'$ is a multiple of $\mathbf{v}$, and so $L' = L''$. Hence L' is unique. □



9.4. Angles and Triangles. We now define angles and recall that the sum of angles in a Euclidean triangle is π. This is equivalent to the parallel postulate, given Euclid's other axioms and postulates.

DEFINITION 14. *If $\mathbf{A}, \mathbf{B}$ are points in $\mathbb{R}^n$, the* <u>**line segment**</u> *between $\mathbf{A}$ and $\mathbf{B}$ is defined by*

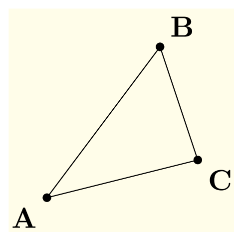
$$[\mathbf{A}, \mathbf{B}] := \{(1-\lambda)\mathbf{A} + \lambda\mathbf{B} : \lambda \in [0,1]\}$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are points in $\mathbb{R}^n$, then their <u>**triangle**</u> *is*

$$\Delta(\mathbf{ABC}) := [\mathbf{A}, \mathbf{B}] \cup [\mathbf{B}, \mathbf{C}] \cup [\mathbf{C}, \mathbf{A}].$$

We define the (measure of the) <u>**angle**</u> *$\angle(BAC)$ to be $\theta \in [0, \pi)$ such that*

$$\cos(\theta) := \frac{\langle \mathbf{B} - \mathbf{A}, \mathbf{C} - \mathbf{A} \rangle}{||\mathbf{B} - \mathbf{A}||||\mathbf{C} - \mathbf{A}||} \in [-1, 1]$$



This is well defined by the Cauchy-Schwartz inequality, Lemma 2.
There are other possible ways to define angle, for example, as defined by Euclid.

LEMMA 14. *The angles in a triangle are preserved by an isometry.*

PROOF. This follows from Proposition 4, that an isometry is norm and inner product preserving. □

PROPOSITION 7. *The angles of a triangle $\Delta\mathbf{ABC}$ sum to π.*

PROOF. See [RS] page 19. □

Week 4: Preamble to lecture proper

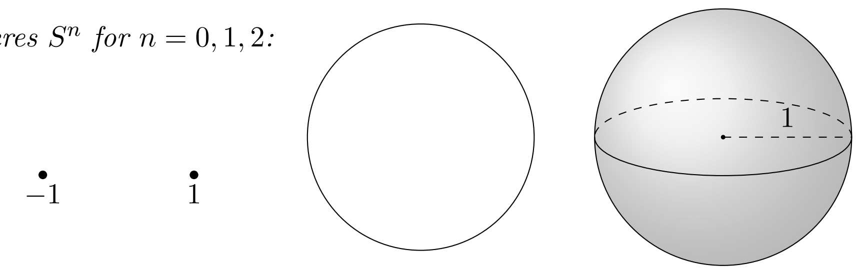
Note, I am currently skipping the rest of the results about reflections, because some questions about spherical geometry are on the homework, so I want to make sure this is covered. We will try and fit in some more about reflections if there is time at the end of this or some other lecture. The result about reflections generating all isometries will not be used later in the course.
Also in the lecture, following the results of the beginning of course evaluation, I will give a very short recap on the academic goals of the course.
Also, given that we are now discussing another metric, I will talk a little bit about metrics in general, especially about when metric spaces can be isometric, because this is relevant to the homework this week.

10. Lecture 10: Spherical geometry: The sphere and the spherical metric

Definition 15. *The n-dimensional sphere of radius $r \geq 0$ is defined by*

$$\begin{aligned} S^n_r &:= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = r^2\} \\ &= \{\mathbf{x} \in \mathbb{R}^{n+1} : ||\mathbf{x}|| = r\} \\ S^n &:= S^n_1 \end{aligned}$$

Example 4. *Spheres S^n for $n = 0, 1, 2$:*

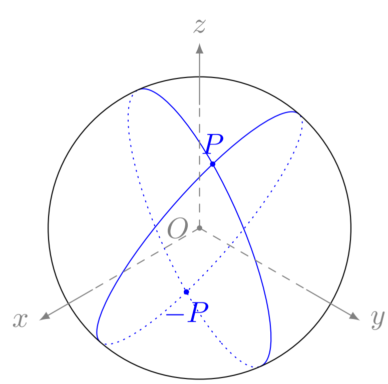


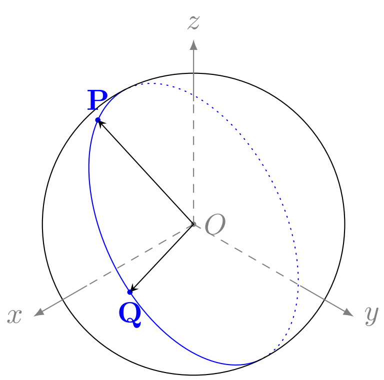
S^0 is two points S^1 is a circle S^2 is a sphere

Definition 16. *A <u>spherical line</u> or <u>great circle</u> is the intersection of S^n_r with a 2-dimensional vector subspace of $\mathbb{R}^{n+1}$.*

Definition 17. $\mathbf{P}, \mathbf{Q} \in S^n_r$ *are <u>antipodal</u> if* $\mathbf{Q} = -\mathbf{P}$.

In the figure (right), $\mathbf{P}$ and $-\mathbf{P}$ are shown on two great circles, with 4 possible paths along great circles. These lines all have the same length. Antipodal points have infinitely many "shortest paths" between them, and so are a special case.





Lemma 15. *If $\mathbf{P}$ and $\mathbf{Q} \in S^n_r$ are not antipodal, then there is a unique great circle containing both of them*

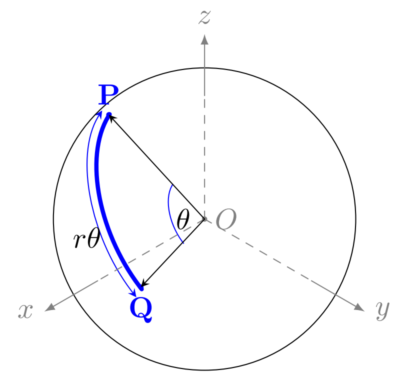
Proof. $\mathbf{P}$ and $\mathbf{Q}$ have length r and so are non-zero. If they are not antipodal, then they are not multiples of each other, since a line $\mathbb{R}\mathbf{v}$ only intersects S^n_r at the points $\lambda\mathbf{v}$ where $||\lambda\mathbf{v}|| = |\lambda|||\mathbf{v}|| = R$, which has only two solutions, $\lambda = \pm R/||\mathbf{v}||$. Thus the only two dimensional vector subspace containing $\mathbf{P}$ and $\mathbf{Q}$ is

$$\mathbb{R}\mathbf{P} \oplus \mathbb{R}\mathbf{Q} = \{\alpha\mathbf{P} + \beta\mathbf{Q} : \alpha, \beta \in \mathbb{R}\}$$ □

Definition 18. *The* <u>*spherical distance*</u> *between two points* $\mathbf{P}, \mathbf{Q} \in S^n_r$ *is the length of the shortest arc of a great circle joining them. We will show that this distance is a metric, called the* <u>*spherical metric*</u>*, which we can also define by*

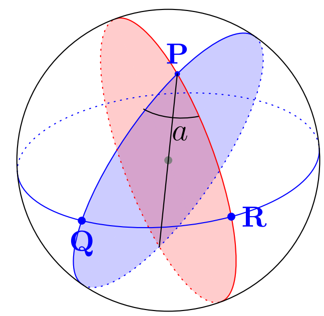
$$d_{S^n_r}(\mathbf{P}, \mathbf{Q}) := r \cos^{-1}\left(\frac{\langle \mathbf{P}, \mathbf{Q} \rangle}{r^2}\right)$$

where we take $\cos^{-1}$ *to be in* $[0, \pi]$.



Notes: Spheres of different radii are not isometric.
A sphere can be projected to the plane; there are lots of ways to represent a sphere in the plane.
IMPORTANT CONVENTION: For simplicity, from now on, we take the <u>radius $r = 1$.</u>
In order to prove that d_{S^n} is a metric, we need to prove the triangle inequality.

Definition 19. *A* <u>*spherical triangle*</u> a *in* S^n_r *consists of three distinct vertices,* $\mathbf{P}, \mathbf{Q}, \mathbf{R} \subset S^n$ *and three arcs of great circles joining them, which do not intersect except at the vertices, and a specified area enclosed by these arcs. Note that there are* 8 *triangles with vertices* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$.
The <u>*spherical angle*</u> *at* $\mathbf{P}$ *between the lines* $\mathbf{PQ}$ *and* $\mathbf{PR}$ *is the dihedral angle between the planes containing these lines. In the figure, this is the angle between the red and blue planes. This angle can also be computed as the angle between the tangent lines to the sphere that lie in these two planes and touch the sphere at* $\mathbf{P}$.



11. Lecture 11: Comparison of lines in $\mathbb{R}^2$ and S^2

Euclid's fifth postulate: Given a line L and a point $\mathbf{P} \in \mathbb{R}^2$, with $\mathbf{P} \notin L$, there is a unique line L' with $\mathbf{P} \in L$ and $L \cap L' = \varnothing$.
There are no parallel lines in S^2. Here, by lines we mean great circles, and by parallel, we mean lines with no intersection. Thus Euclid's parallel postulate does not hold for S^n. More precisely:

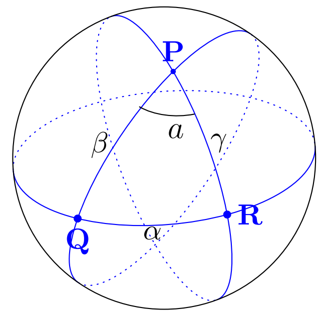
Proposition 8. *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are two distinct great circles on* S^2*, then the intersection* $\mathcal{C} \cap \mathcal{D}$ *is equal to a pair of antipodal points.*

Proof. By definition, there are 2-dimensional subspaces V and W in $\mathbb{R}^3$ such that $\mathcal{C} = S^2 \cap V$ and $\mathcal{D} = S^2 \cap W$. Since $\mathcal{C} \neq \mathcal{D}$, we must have $V \neq W$. This implies that $W \cap V$ is one dimensional, since if it was zero dimensional, the space $V \oplus W$ would be four dimesional, but this is a subpace of $\mathbb{R}^3$, which is three dimensional. Since $W \neq V$, $W \cap V$ has dimension strictly less than 2. Hence $\dim(V \cap W) = 1$ and for some $\mathbf{v}$, with $||\mathbf{v}|| = 1$, we have $W \cap V = \mathbb{R}\mathbf{v}$, and so $\mathcal{C} \cap \mathcal{D} = S^2 \cap V \cap W = S^2 \cap \mathbb{R}\mathbf{v} = \{\mathbf{v}, -\mathbf{v}\}$, which is a set of two antipodal points. $\square$

Note: Great circles in S^n for $n > 2$ do not always intersect.

12. Lecture 12: Main formula of spherical trigonometry

(This was covered partly in lecture 11 and partly in lecture 12)



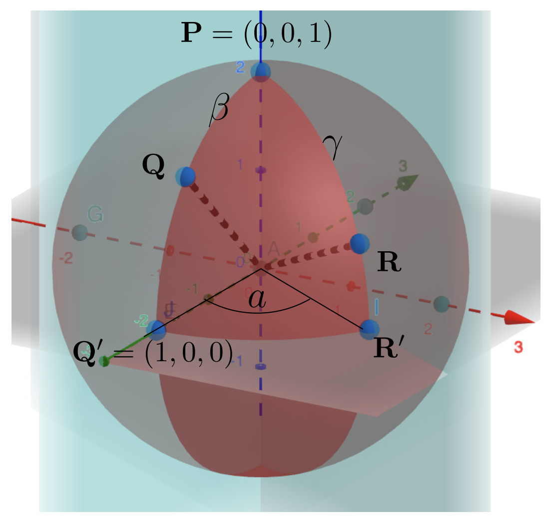
Proposition 9 (Main formula of spherical trigonometry). *Let α, β, γ be the side lengths of a spherical triangle with vertices $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in S^2$ on the unit sphere.*

$$\alpha := d(\mathbf{Q}, \mathbf{R}), \quad \beta := d(\mathbf{P}, \mathbf{Q}), \quad \gamma := d(\mathbf{P}, \mathbf{R}),$$

where $d = d_{S^2}$ is the spherical metric. Let a be the spherical angle between arcs $\mathbf{PQ}$ and $\mathbf{PR}$. Then

$$\cos\alpha = \cos\beta \cdot \cos\gamma + \sin\beta \cdot \sin\gamma \cdot \cos a$$

Proof. Note that the distance on the sphere, in Definition 18, is given in terms of the inner product. By Proposition 4, any linear isometry of $\mathbb{R}^3$ preserves the inner product, so, if we apply a linear isometry of $\mathbb{R}^3$ to the sphere and the points on it, the distances between the points will be unchanged. Therefore, we may apply an isometry to map the vertices of the triangle to more convenient positions.



There is an isometry L of $\mathbb{R}^3$ which maps $\mathbf{P}$ to $(0, 0, 1)$, since these points are equidistant from $\mathbf{0}$. This may be followed by a rotation about the z axis to map $\mathbf{Q}$ to a point in the xz plane (the plane where $y = 0$). Thus, without loss of generality, we may assume $\mathbf{P} = (0, 0, 1)$, and $\mathbf{Q}$ has y coordinate 0. We can use this to write the Euclidean coordinates of $\mathbf{Q}$ and $\mathbf{R}$ in terms of a, β and γ, as in the figure:

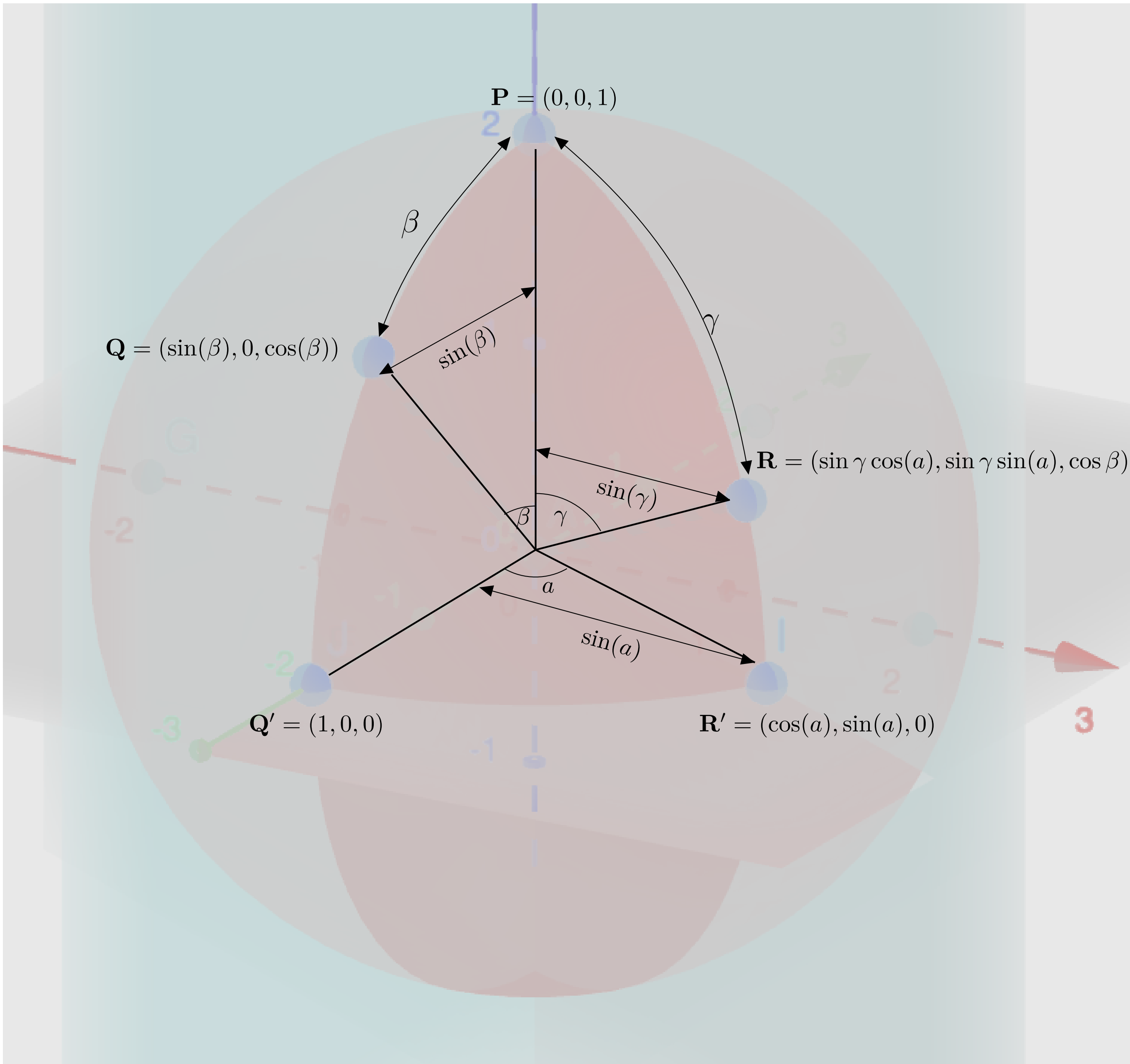
$$\begin{array}{ccccc} \mathbf{Q} & = & (\sin\beta, & 0, & \cos\beta) \\ \mathbf{R} & = & (\sin\gamma\cos a, & \sin\gamma\sin a, & \cos\gamma) \end{array}$$

By definition,

$$\cos\alpha = \langle \mathbf{Q}, \mathbf{R} \rangle = \sin\beta\sin\gamma\cos a + \cos\beta\cos\gamma$$

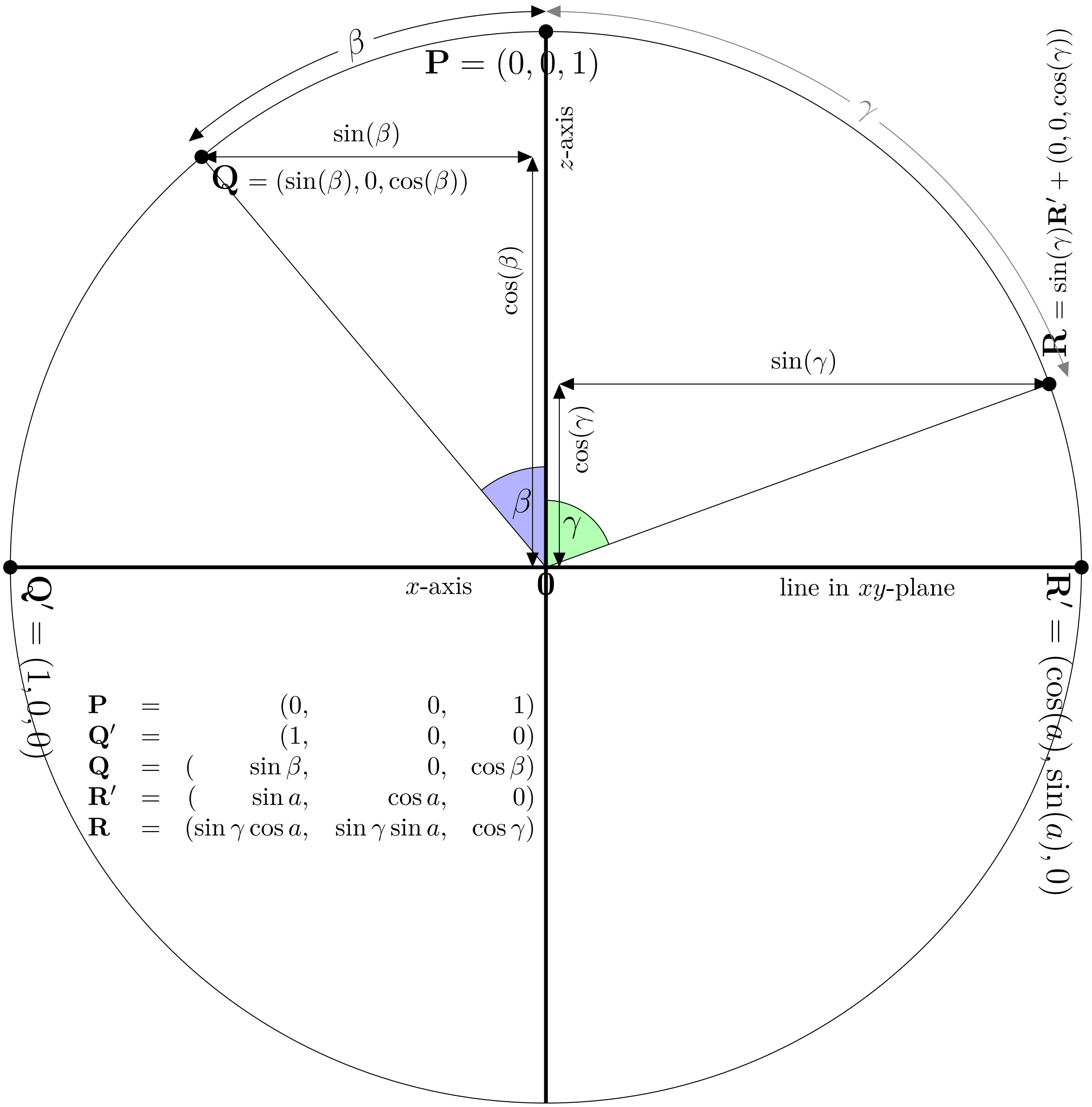
which gives the required result. $\square$

Note, since it may be hard to see why $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ can be put in this form, there are futher pictures on the next two pages.

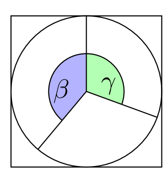

$\mathbf{P} = (0,0,1)$
β
γ
$\mathbf{Q} = (\sin(\beta), 0, \cos(\beta))$
$\sin(\beta)$
$\mathbf{R} = (\sin\gamma\cos(a), \sin\gamma\sin(a), \cos\beta)$
$\sin(\gamma)$
β
γ
a
$\sin(a)$
$\mathbf{Q}' = (1,0,0)$
$\mathbf{R}' = (\cos(a), \sin(a), 0)$

12.1. "Cut out" proof of Main theorem for spherical trigonometry.

$$\cos\alpha = \cos(a)\sin(\beta)\sin(\gamma) + \cos(\beta)\cos(\gamma)$$



Cut out the circle, and fold along the vertical axis. As you "open or close" the circle, the angles a, which is the dihedral angle between the two planes (the plane containing $\mathbf{0}, \mathbf{P}$ and $\mathbf{R}$ and the plane containing $\mathbf{0}, \mathbf{P}, \mathbf{Q}$), and the angle α, which is the angle between the two vectors $\mathbf{Q}$ and $\mathbf{R}$ will vary.
The minimum possible value of $\cos(a)$ is -1, which is obtained before the paper is folded, ie., lying flat, unfolded. In this case, either $\alpha = \beta + \gamma$, when $\beta + \gamma \leq \pi$, or, if $\beta + \gamma > \pi$, as in this little picture, then if $a = \pi$, we still have $\cos(\alpha) = \cos(\beta + \gamma)$, so $\beta + \gamma = 2\pi - \alpha$.

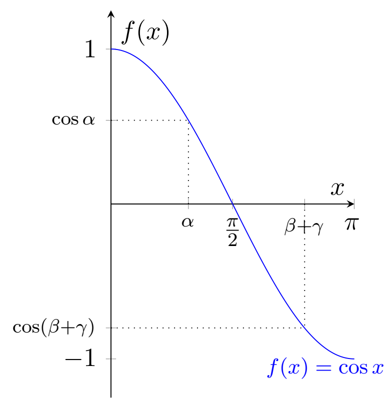


Lemma 16. *The spherical distance satisfies the triangle inequality. That is, with* $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \alpha, \beta, \gamma$ *and* a *as in Proposition 9, we have* $\alpha \leq \beta + \gamma$*, with equality, or with* $\alpha + \beta + \gamma = 2\pi \iff a = \pi$*. Furthermore,* $a = 0 \iff \alpha = |\beta - \gamma|$.

Proof. Since $\mathbf{P}$, $\mathbf{Q}, \mathbf{R}$ are contained in a vector space V of dimension 3, and so lie on the unit sphere $S^2 \cong V \cap S^n$, it is enough to prove the result for $n = 2$. We may assume $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are distinct, since otherwise the result is easy.

By definition, $\alpha, \beta, \gamma \in [0, \pi]$, so $\sin\alpha, \sin\beta \geq 0$. Since $\cos a \geq -1$, by Proposition 9

$$(\dagger\dagger) \qquad \begin{aligned} \cos\alpha &= \cos\beta \cdot \cos\gamma + \sin\beta \cdot \sin\gamma \cdot \cos a \\ &\geq \cos\beta \cdot \cos\gamma - \sin\beta \cdot \sin\gamma = \cos(\beta + \gamma) \end{aligned}$$



Case $\beta + \gamma \leq \pi$: By definition, $\alpha \in [0, \pi]$. On $[0, \pi]$, $\cos(x)$ is strictly decreasing, so $\cos\alpha \geq \cos(\beta + \gamma) \Rightarrow \alpha \leq \beta + \gamma$. There is equality in $(\dagger\dagger)$ if and only if $\cos(a) = -1$, which is if and only if $a = \pi$.

Case $\beta + \gamma > \pi$: Since $\alpha \in [0, \pi]$, the inequality is immediate. Equality in $(\dagger\dagger)$ occurs $\iff a = \pi$, but now $\beta + \gamma > \pi$ so $\cos(\beta + \gamma) = \cos(\alpha) \iff \alpha = 2\pi - (\beta + \gamma) \iff \alpha + \beta + \gamma = 2\pi$. The final statement follows similarly, since $a = 0 \iff \cos(\alpha) = \cos(\beta - \gamma)$. □

Proposition 10. *The spherical distance* d_{S^n} *is a metric on* S^n

Proof. Symmetry follows from symmetry of the inner product on $\mathbb{R}^n$. Non-degeneracy follows since $\cos^{-1} \in [0, \pi]$, so $d(\mathbf{P}, \mathbf{Q}) = 0 \iff \langle \mathbf{P}, \mathbf{Q} \rangle = 1$, and since $||\mathbf{P}|| = ||\mathbf{Q}|| = 1$, this implies $\mathbf{P} = \mathbf{Q}$. The triangle inequality is proved in Lemma 16. □

13. Lecture 13: Spherical isometries

In this section we write $d := d_{S^n}$. The goal is to show that $\text{Isom}(S^n, d) \cong O(n+1)$.
Easy direction first:

LEMMA 17. *If T is a linear Euclidean isometry of $\mathbb{R}^{n+1}$, then the restriction $T|_{S^n}$ to S^n is a spherical isometry.*

PROOF. Since T preserves the Euclidean distance, and fixes $\mathbf{0}$, it must map S^n to S^n, since this is the set of points distance 1 from $\mathbf{0}$. Since T has an inverse which is also a Euclidean isometry, and so also maps S^n to S^n, then T restricted to S^n is a bijection. By Proposition 4 T preserves the inner product, and thus preserves the spherical distance $\cos^{-1}(\langle \mathbf{x}, \mathbf{y}\rangle)$ on S^n. Hence T_{S^n} is a spherical isometry ☐

Isometries are given by linear maps

LEMMA 18. *A bijective map $f : \mathbb{R}^k \to \mathbb{R}^k$, which preserves the standard inner product on $\mathbb{R}^k$ is a linear isometry of $\mathbb{R}^k$.*

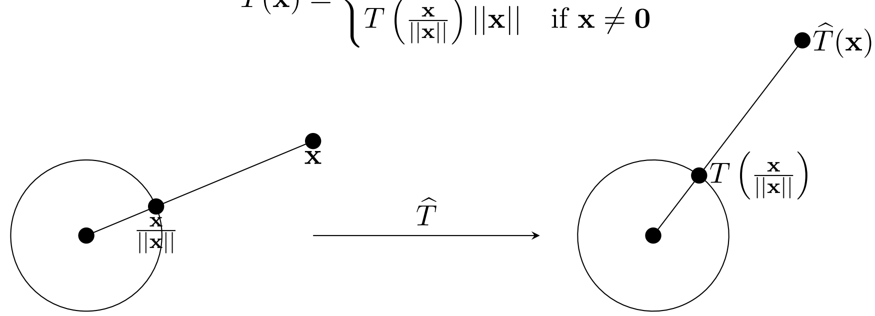
PROOF. If the inner product is preserved, then since $d(\mathbf{x}, \mathbf{y}) = \sqrt{\langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle}$, then f must preserve distance: $||f(\mathbf{x}) - f(\mathbf{y})||^2 = \langle f(\mathbf{x}) - f(\mathbf{y}), f(\mathbf{x}) - f(\mathbf{y})\rangle = \langle f(\mathbf{x}), f(\mathbf{x})\rangle - 2\langle f(\mathbf{x}), f(\mathbf{y})\rangle + \langle f(\mathbf{y}), f(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{x}\rangle - 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = ||\mathbf{x} - \mathbf{y}||$. Given that f is bijective, f must be a Euclidean isometry. Since the inner product is preserved, $f(\mathbf{0}) = \mathbf{0}$. By Corollary 1 isometries of $\mathbb{R}^k$ fixing $\mathbf{0}$ are given by linear maps. ☐

Remark: We can drop the assumption that f is a bijection in Lemma 18, as this follows since if $\sum a_i f(\mathbf{e}_i) = \mathbf{0}$, then by taking the inner product with $f(\mathbf{e}_j)$, we find $a_j = 0$ for $1 \le j < k$.

PROPOSITION 11. *If T is an isometry of S^n, then T extends to a Euclidean isometry $\widehat{T}$ of $\mathbb{R}^{n+1}$, that is, there is a Euclidean isometry $\widehat{T}$ of $\mathbb{R}^{n+1}$, and $\widehat{T}|_{S^n} = T$.*

PROOF. Given an isometry T of S^n, define $\widehat{T}$ by radial extension, i.e., by

$$\widehat{T}(\mathbf{x}) = \begin{cases} \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0} \\ T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}|| & \text{if } \mathbf{x} \neq \mathbf{0} \end{cases} \tag{8}$$



We will show that $\widehat{T}$ is an isometry of $\mathbb{R}^{n+1}$. First, $\widehat{T}$ is a bijection, because $\widehat{T^{-1}}$ is the inverse of $\widehat{T}$: First, note that since T and T^{-1} preserve the S^n, only $\mathbf{0}$ maps to $\mathbf{0}$, so we assume $\mathbf{x} \neq \mathbf{0}$. Then, using the fact that $||T(\mathbf{y})|| = 1$ for all $\mathbf{y} \in S^n$, we have

$$\begin{aligned}\widehat{T^{-1}}\left(\widehat{T}(\mathbf{x})\right) &= \widehat{T^{-1}}\left(T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}||\right) = T^{-1}\left(\frac{\mathbf{z}}{||\mathbf{z}||}\right)||\mathbf{z}||, \text{ where } \mathbf{z} = T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}|| \\ &= T^{-1}\left(T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)\right)||\mathbf{x}||, \text{ since } ||\mathbf{z}|| = ||\mathbf{x}|| \\ &= \left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}|| = \mathbf{x}.\end{aligned}$$

Now we show $\widehat{T}$ preserves the inner product on $\mathbb{R}^{n+1}$. Since T preserves the spherical metric, for $\mathbf{x}, \mathbf{y} \in S^n$, $\langle \mathbf{x}, \mathbf{y}\rangle = \cos(d_{S^n}(\mathbf{x}, \mathbf{y})) = \cos(d_{S^n}(T(\mathbf{x}), T(\mathbf{y})) = \langle T(\mathbf{x}), T(\mathbf{y})\rangle$, so T preserves the inner product restricted to S^n. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, since otherwise the result is easy.

$$\begin{aligned}
\left\langle \widehat{T}(\mathbf{x}), \widehat{T}(\mathbf{y}), \right\rangle &= \left\langle T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}||, T\left(\frac{\mathbf{y}}{||\mathbf{y}||}\right)||\mathbf{y}||\right\rangle \\
&= \left\langle T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right), T\left(\frac{\mathbf{y}}{||\mathbf{y}||}\right)\right\rangle ||\mathbf{x}||||\mathbf{y}||, \text{ since the inner product is bilinear} \\
&= \left\langle \frac{\mathbf{x}}{||\mathbf{x}||}, \frac{\mathbf{y}}{||\mathbf{y}||}\right\rangle ||\mathbf{x}||||\mathbf{y}||, \text{ since } T \text{ preserves the inner product on } S^n \\
&= \langle \mathbf{x}, \mathbf{y} \rangle \text{ by bilinearity}
\end{aligned}$$

Hence by Lemma 18 $\widehat{T}$ is linear Euclidean isometry. □

Corollary 4. *There is a group isomorphism*

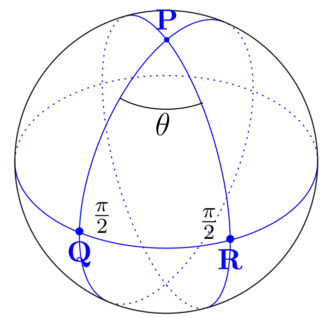
$$\mathrm{Isom}(S^n, d_{S^n}) \cong O(n+1)$$

Proof. This follows since Lemma 17 tells us we have a map $O(n+1) \to \mathrm{Isom}(S^n, d_{S^n})$. This map must be injective, since S^n contains a basis for $\mathbb{R}^{n+1}$, so for $A \in O(n+1)$, the mapping $T_A|_{S^n}$ determines the map T_A. Proposition 11 tell us that this map is surjective. □

14. Lecture 14: Spherical Triangles

In this section we restrict to S^2

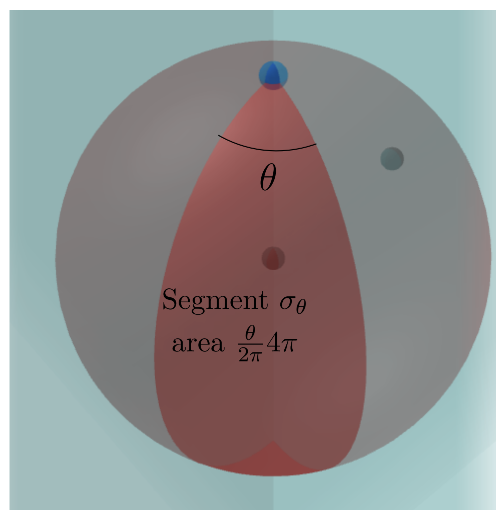
In the Euclidean plane, the angles of a triangle sum to π.
We now show that the angles of a (non-trivial) spherical triangle sum to a number strictly greater than π.
First, note that the angle sum for a spherical triangle is not constant. For example, the triangle in the figure, where $\mathbf{P} = (0,0,1)$ and $\mathbf{Q} = (1,0,0)$ and $\mathbf{R} = (\cos\theta, \sin\theta, 0)$ can have angle sum $\pi + \theta$ for any $\theta \in (0, 2\pi)$.
We will show that the angle sum is determined by the area of the triangle.



We will make some assumptions about area, in particular, assume that you will learn about a rigourous definition of area in terms of integration in analysis.

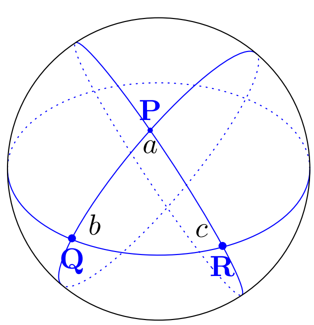
Assumptions:

(1) The total surface area of a unit sphere is 4π.
(2) The area of a region of S^2 is invariant under application of any isometry of S^2.
(3) If X and Y are two non intersecting regions on S^2, then $\text{area}(X \cup Y) = \text{area}(X) + \text{area}(Y)$
(4) The segment of a sphere, σ_θ that lies between two semi-circular segments of great circles, from $\mathbf{P}$ to $-\mathbf{P}$, meeting at angle θ has area 2θ.
Let Σ_θ be the union of σ_θ and $-\sigma_\theta$. So Σ_θ has area 4θ.



Theorem 8. *(Girard's theorem) Let* $\Delta\mathbf{PQR}$ *be a spherical triangle on* S^2*, with* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ *distinct points, and such that the interior of* $\Delta\mathbf{PQR}$ *does not intersect the great circles which contain the sides of* $\Delta\mathbf{PQR}$*, and with non-zero area. Then*

$$\textit{area}\,(\Delta\mathbf{PQR}) = a + b + c - \pi$$



Proof. Notice that $\text{area}(\Delta\mathbf{PQR}) = \text{area}(-\Delta\mathbf{PQR})$, since $\mathbf{x} \mapsto -\mathbf{x}$ is an isometry, and area is preserved by isometries. Also notice that since we assume that the triangle does not contain any of the extended edges in its interior, these triangles are disjoint. Notice that S^2 is the union of Σ_a, Σ_b and Σ_c, and that

$$\sigma_a \cap \sigma_b = \sigma_b \cap \sigma_c = \sigma_a \cap \sigma_b = \Delta\mathbf{PQR}$$

and similarly for the images of these areas under $\mathbf{x} \mapsto -\mathbf{x}$.
Let

$$S_a = \sigma_a \setminus \Delta\mathbf{PQR}$$
$$S_b = \sigma_b \setminus \Delta\mathbf{PQR}$$
$$S_c = \sigma_c \setminus \Delta\mathbf{PQR}$$

Then we have a disjoint union:

$$S^2 = S_a \cup S_b \cup S_c \cup (-S_a) \cup (-S_b) \cup (-S_c) \cup \Delta\mathbf{PQR} \cup (-\Delta\mathbf{PQR}).$$

Since we have a disjoint union $\sigma_a = S_a \cup \Delta\mathbf{PQR})$, we have $\text{area}(\sigma_\theta) = \text{area}(S_\theta) + \text{area}(\Delta\mathbf{PQR})$, for $\theta = a, b, c$, so

$$\text{area}(S^2) = \text{area}(\Sigma_a) + \text{area}(\Sigma_a) + \text{area}(\Sigma_a) - 4\,\text{area}(\Delta\mathbf{PQR})).$$

By assumption Σ_θ has area 2θ so

$$4\pi = 4(a+b+c) - 4\,\text{area}(\Delta\mathbf{PQR}),$$

and the result follows. □

Remark: This result also holds if we do not make the restriction that the triangle does not contain the extensions of its edges in its interior, but then $\Delta\mathbf{PQR}$ and $(-\Delta\mathbf{PQR})$ can not be guaranteed to be disjoint, and so we need to consider several different cases. To save time, I will not prove the general case, which is not needed for the following result.

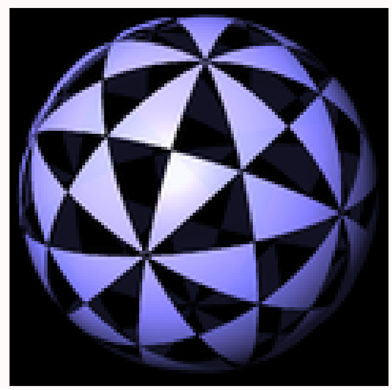
COROLLARY 5. *The sum of the angles of a spherical triangle, satisfying the hypothesis of Theorem 8, is strictly greater than π.*

PROOF. This follows from Theorem 8. □

Schwarz triangles: Suppose Δ is a spherical triangle, and G is the group generated by reflections in the sides of Δ. Suppose these reflections result in a tiling of the sphere – this means that $S^2 = \cup_{T\in G} T(\Delta)$ and $T_1(\Delta)^\circ \cap T_2(\Delta)^\circ \neq \varnothing \iff T_1 = T_2$, where X° means the interior of X. Then the angles must divide π. This means that if Δ tiles S^2 through reflections, then the angles are $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ for positive integers p, q, r such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

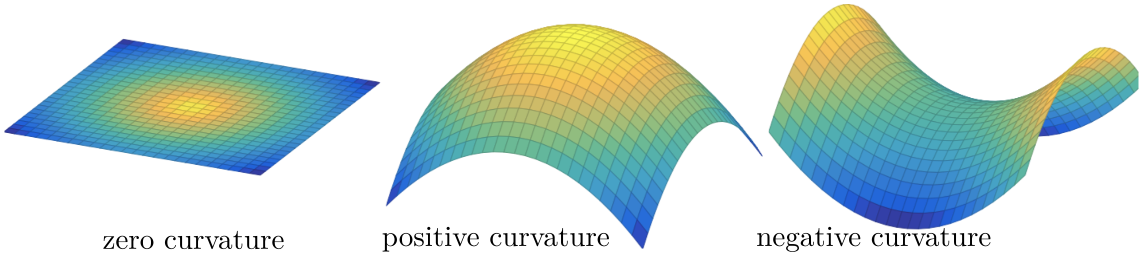
The only solutions are $(p, 2, 2), (3, 3, 2), (4, 3, 2), (5, 3, 2)$. The later is shown on the right.



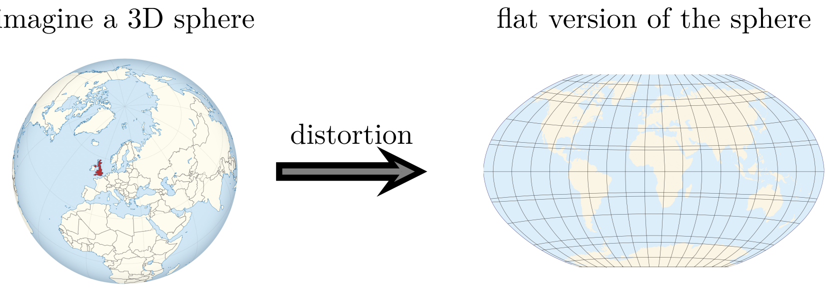
(You will probably see more of this if you take the third year course on reflection groups.)

15. Lecture 15: Hyperbolic Geometry

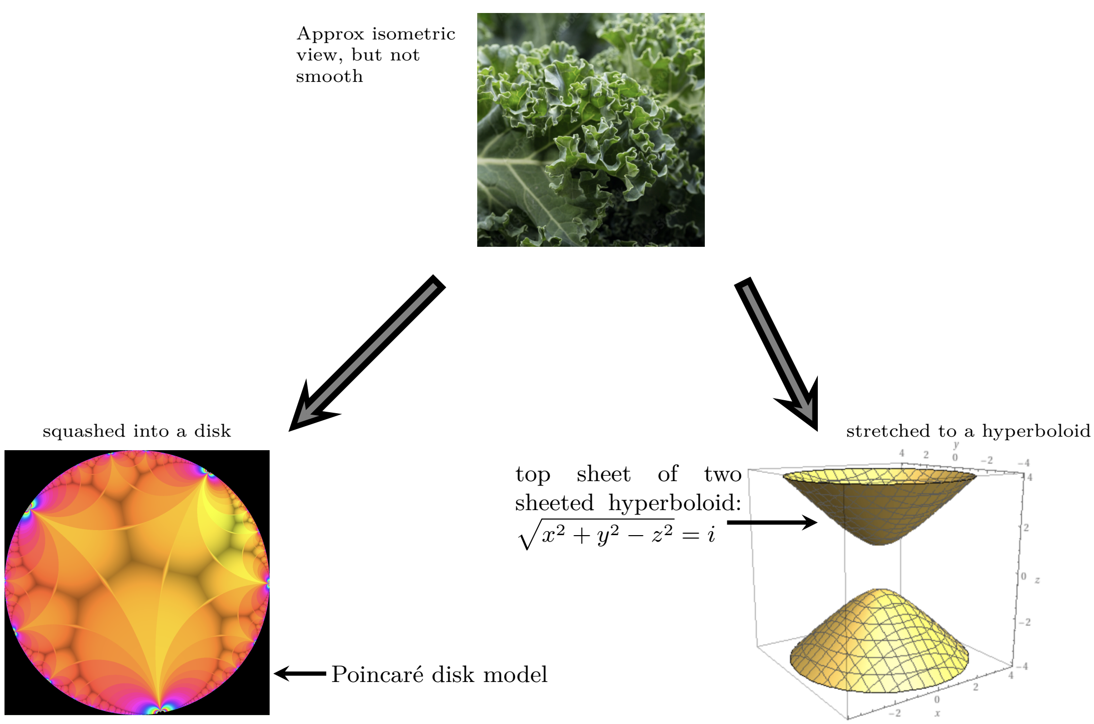
Motivation. So far we have looked at Euclidean space, which is flat, zero curvature, and the sphere, which is curved – positive curvature. Hyperbolic space has everywhere **constant** negative curvature. Investigating these concepts in more detail is beyond the scope of this course.



Distortion. There is no way to draw the sphere on the plane without some distortion. This gives rise to various different map projections. (Possibly topic for a second year essay).
Example, of a non isometric map from S^2 to $\mathbb{R}^2$:



Similarly, the hyperbolic plane can't be mapped isometrically to $\mathbb{R}^2$ or $\mathbb{R}^3$. Depicting it in $\mathbb{R}^2$ or $\mathbb{R}^3$ must give distortions. There are several standard ways to put the hyperbolic plane in Euclidean space, two of which are as follows, which either "squash" into a disk, or "stretch" onto a hyperboloid:
We will use the hyperboliod model, since it's closest to the sphere, which we have just studied. The disadvantage of this model is that it does not preserve angles (not conformal).



Hyperbolic space

We will use the model of hyperbolic space defined as follows. An advantage of this definition is that results are very similar to results in the case of the sphere, just with a "twist", which is to replace the usual Euclidean metric $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ with the the Lorentz inner product. We define

$$J_n := \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{pmatrix}$$

Note that J_n has n rows and columns and acts on $\mathbb{R}^n$. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x}^T J \mathbf{y}$ means $\mathbf{x}^T J_n \mathbf{y}$.

Definition 20. *The Lorentz inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined by:*

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle_L &:= -x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots x_n y_n \\ &= \mathbf{x}^T J \mathbf{y} \end{aligned}$$

Definition 21. *The Lorentz norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is given by*

$$||\mathbf{x}||_L := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_L}$$

We always take this root to be positive or positive imaginary, or zero. WARNING: $\langle \mathbf{x}, \mathbf{x} \rangle$ may be negative, so $||\mathbf{x}||_L$ may be imaginary, so is not a norm.

Definition 22. *The n-dimensional hyperbolic space is the metric space with underlying set*

$$\begin{aligned} \mathcal{H}^n &:= \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : -x_1^2 + x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = -1 \text{ and } x_1 > 0 \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : ||\mathbf{x}||_L = i \text{ and } x_1 > 0 \right\} \end{aligned}$$

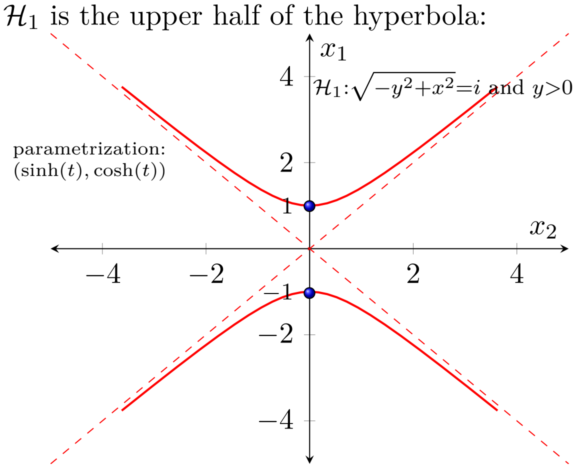
and with metric defined by

$$d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y}) := \cosh^{-1}\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle_L}{||\mathbf{x}||_L ||\mathbf{y}||_L} \right) = \cosh^{-1}\left(- \langle \mathbf{x}, \mathbf{y} \rangle_L \right)$$

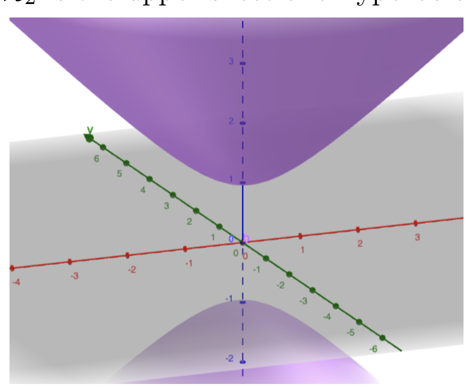
where we have used that on $\mathcal{H}^n$, $||\mathbf{x}||_L = i$. We will review the hyperbolic cosine cosh shortly.

Examples: The x_1-axis is usually drawn pointing up.

$\mathcal{H}_1$ is the upper half of the hyperbola:



$\mathcal{H}_2$ is the upper sheet of a hyperboloid:[a]



[a] https://www.geogebra.org/m/xqnc3sdm

The Goal of this chapter is to

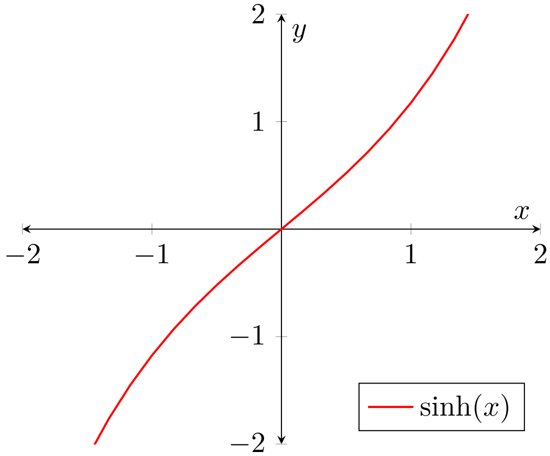
1. show that the hyperbolic metric is a metric
2. determine the group of hyperbolic isometries
3. investigate the failure of the parallel postulate
4. show that triangles have angles sum less than π

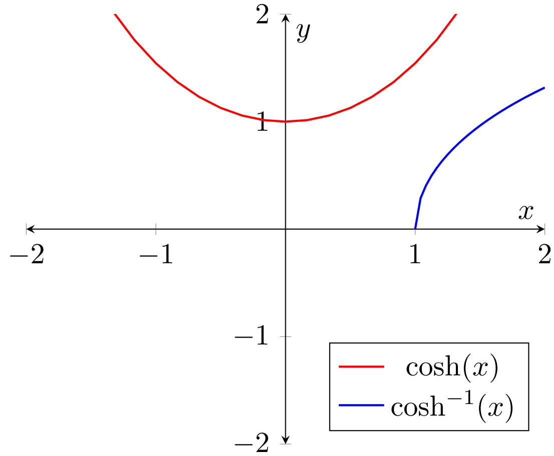
First we need to understand hyperbolic sine and cosine, and check d_{H^n} is well defined.

Hyperbolic trigonometry [please review for homework]

DEFINITION 23. *The hyperbolic sine and cosine,* sinh *and* cosh *are defined by*

$$\boxed{\sinh(\theta) = \frac{\exp(\theta)-\exp(-\theta)}{2} = -i\sin(i\theta)} \qquad \boxed{\cosh(\theta) = \frac{\exp(\theta)+\exp(-\theta)}{2} = \cos(i\theta)}$$

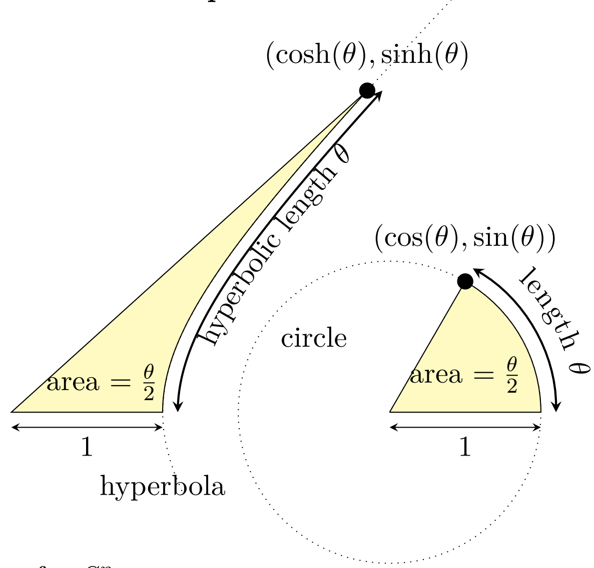




$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$
$$y \mapsto \ln(y + \sqrt{y^2+1})$$

$$\cosh^{-1} : [1,\infty) \to [0,\infty)$$
$$y \mapsto \ln(y + \sqrt{y^2-1})$$

Geometric interpretation:



Formulae: These can be derived from properties of the exponential, or the relationship with cos and sin, and formulae you already know

hyperbolic

$\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$
$\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)$
$\cosh'(a) = \sinh(a) \qquad \sinh'(a) = \cosh(a)$
$\cosh^2(a) - \sinh^2(a) = 1$

comparison with sine and cosine:

trigonometric

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
$\cos'(a) = -\sin(a) \qquad \sin'(a) = \cos(a)$
$\cos^2(a) + \sin^2(a) = 1$

The definition of the metric for $\mathcal{H}^n$ is comparable to that for S^n:

$$d_{S^n}(\mathbf{x},\mathbf{y}) = \cos^{-1}\left(\frac{\langle \mathbf{x},\mathbf{y}\rangle}{||\mathbf{x}||||\mathbf{y}||}\right) \qquad d_{\mathcal{H}^n}(\mathbf{x},\mathbf{y}) = \cosh^{-1}\left(\frac{\langle \mathbf{x},\mathbf{y}\rangle_L}{||\mathbf{x}||_L||\mathbf{y}||_L}\right)$$

So results about the hyperbolic metric are analagous to results for the spherical metric.
To show that $d_{\mathcal{H}^n}(\mathbf{x},\mathbf{y})$ is well defined, we need to prove that if $\mathbf{x},\mathbf{y} \in \mathcal{H}^n$, then $\frac{\langle \mathbf{x},\mathbf{y}\rangle_L}{||\mathbf{x}||_L||\mathbf{y}||_L} \in [1,\infty)$. We will follow [R3.7–3.10], and find an isometry which makes these results easier to prove. First we need to investigate the Lorentz inner product, and the maps of $\mathbb{R}^{n+1}$ which preserve it. Eventually we will prove that these are all linear. We follow [RS, Appendix B] for Lorentz linear algebra.

Lemma 19. *There is a parameterization*

$$f : \mathbb{R} \to \mathcal{H}^1$$
$$f : (t) \mapsto (\cosh(t), \sinh(t)) \quad t \in [0, \infty)$$

There is a parameterization

$$f : \mathbb{R}^2 \to \mathcal{H}^2$$
$$f : (t, \theta) \mapsto (\cosh(t), \cos(\theta)\sinh(t), \sin(\theta)\sinh(t)), \quad t \in [0, \infty), \theta \in [0, 2\pi),$$

where $\mathbb{R}^2$ is given with polar coordinates, and $\mathcal{H}^2$ with the Cartesian coordinates of $\mathbb{R}^3$.

Proof. For the first parameterisation, we use the fact that $\cosh(t)^2 - \sinh(t)^2 = 1$, and that $\cosh(t)$ is strictly increasing on $[0, \infty)$, and symmetric, and that $\sinh(-t) = -\sinh(t)$.
For the parameterisation of $\mathcal{H}^2$, this is the inverse of the projection from $\mathbb{R}^3$ given by $(x_1, x_2, x_3) \to (x_2, x_3)$. This is a bijection between $\mathcal{H}^2$ and $\mathbb{R}^2$, since for any (x_2, x_3) there is a unique solution to

$$(\star) \qquad -x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_1 > 0.$$

In polar coordinates,

$$(x_2, x_3) = (r\cos(\theta), r\sin(\theta)),$$

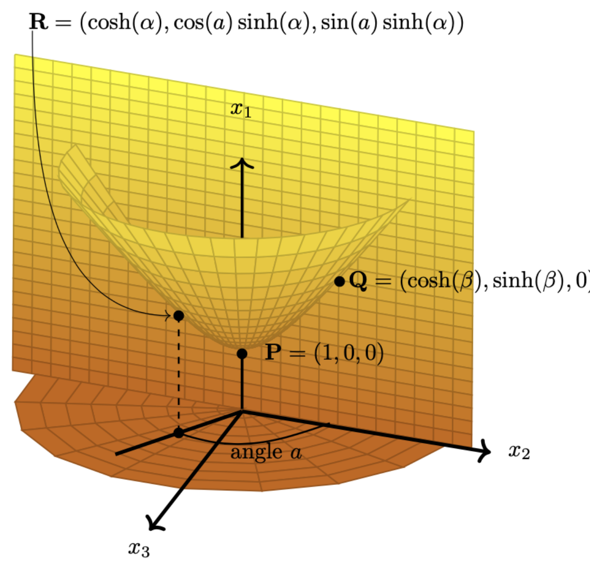
so $(\star)$ becomes

$$-x_1^2 + r^2 = -1, \quad x_1 > 0,$$

the equation for $\mathcal{H}^1$, parametrised by

$$(x_1, r) = (\cosh(t), \sinh(t)),$$

from which the result follows by substitution. □



15.1. The hyperbolic metric is well defined.

Lemma 20. *For $\mathbf{x}, \mathbf{y} \in \mathcal{H}^n$ we have $-\langle \mathbf{x}, \mathbf{y} \rangle_L \geq 1$, and $\langle \mathbf{x}, \mathbf{y} \rangle > 0$.*

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Let $\widehat{\mathbf{x}} = (1, x_2, \ldots, x_n)$ and $\widehat{\mathbf{y}} = (1, y_2, \ldots, y_n)$. Then since $\mathbf{x}, \mathbf{y} \in \mathcal{H}^n$, we have

$$(9) \qquad \begin{aligned} x_1^2 &= 1 + x_2^2 + x_3^2 + \cdots + x_n^2 = ||\widehat{\mathbf{x}}||^2 \\ y_1^2 &= 1 + y_2^2 + y_3^2 + \cdots + y_n^2 = ||\widehat{\mathbf{y}}||^2, \end{aligned}$$

where the norm on the right is the Euclidean norm. So we have by the Cauchy-Schwartz inequality for the Euclidean norm, Lemma 2,

$$\langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle^2 \leq ||\widehat{\mathbf{x}}||^2 ||\widehat{\mathbf{y}}||^2 = (x_1 y_1)^2$$

i.e.,

$$(10) \qquad |1 + x_2 y_2 + x_3 y_3 + \ldots x_n y_n| \leq |x_1 y_1|$$

so

$$1 + x_2 y_2 + x_3 y_3 + \ldots x_n y_n \leq |x_1 y_1| = x_1 y_1$$

where we have used that x_1, y_1 are positive, since $\mathbf{x}, \mathbf{y} \in \mathcal{H}^n$. Rearranged, we get

$$\langle \mathbf{x}, \mathbf{y} \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n \leq -1,$$

which multiplying through by -1 gives the first inequality.

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}|x_2y_2 + x_3y_3 + \cdots + x_ny_n|^2 &\le (x_2^2 + \cdots x_n^2)(y_2^2 + \cdots y_n^2) = (x_1^2 - 1)(y_1^2 - 1)\\ &= x_1^2y_1^2 - y_1^2 - x_1^2 + 1\\ &< x_1^2y_1^2,\end{aligned}$$

so we have

$$x_1y_1 > |x_2y_2 + x_3y_3 + \cdots + x_ny_n|$$

where we have used that $x_1 \ge 1$ and $y_1 \ge 1$, which follows from Equation 9. So

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y}\rangle &= x_1y_1 + (x_2y_2 + x_3y_3 + \ldots x_ny_n)\\ &\ge x_1y_1 - |x_2y_2 + x_3y_3 + \ldots x_ny_n| > 0,\end{aligned}$$

which gives the second inequality. □

Corollary 6. *$d_{\mathcal{H}^n}$ is well defined.*

Proof. This is immediate from Lemma 20 and the definition of $d_{\mathcal{H}^n}$. □

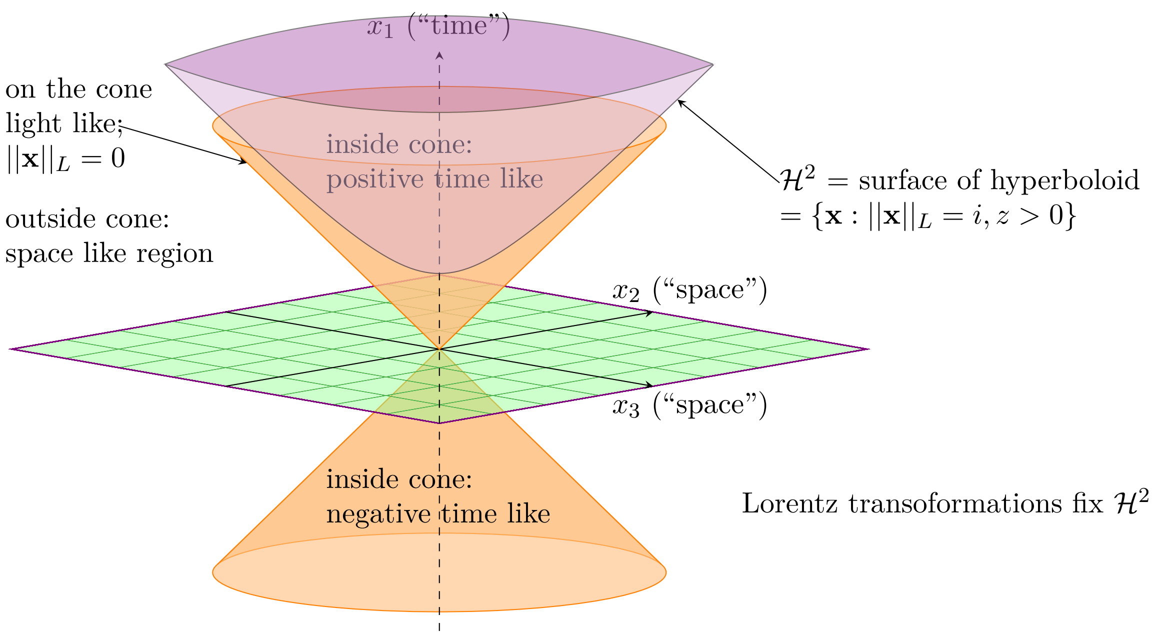
15.2. Terminology inspired by special relativity: Time, space, and light like vectors.

Definition 24.

$$\text{A point } \mathbf{x} \in \mathbb{R}^n \text{ is } \begin{cases} \underline{\text{space-like}} \text{ if} & ||\mathbf{x}||_L \in (0,\infty), \quad \text{i.e., if } \langle \mathbf{x}, \mathbf{x}\rangle_L > 0\\ \underline{\text{light-like}} \text{ if} & ||\mathbf{x}||_L = 0, \quad \text{i.e., if } \langle \mathbf{x}, \mathbf{x}\rangle_L = 0\\ \underline{\text{time-like}} \text{ if} & ||\mathbf{x}||_L \in i(0,\infty), \quad \text{i.e., if } \langle \mathbf{x}, \mathbf{x}\rangle_L < 0\\ \underline{\text{positive}} \text{ if} & x_1 > 0\\ \underline{\text{negative}} \text{ if} & x_1 < 0\end{cases}$$

A time-like or space-like vector $\mathbf{x} \in \mathbb{R}^n$ is positive if $x_1 > 0$ and negative if $x_1 < 0$.

Note that vectors in $\mathcal{H}^n$ are all time-like.



16. Lecture 16: The Lorentz group

Definition 25. *A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is a* <u>*Lorentz transformation*</u> *if it preserves the Lorentz inner product:*

$$\langle T(\mathbf{x}), T(\mathbf{y})\rangle_L = \langle \mathbf{x}, \mathbf{y}\rangle_L \, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

We say that T is <u>*positive*</u> *if $\mathbf{x}$ is positive time-like if and only if $T(\mathbf{x})$ is positive time-like.*

Lemma 21. *If $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a positive, bijective Lorentz transformation, then $T|_{\mathcal{H}^n} \in \mathrm{Isom}(\mathcal{H}^n, d_{\mathcal{H}^n})$.*

Proof. Since $\mathcal{H}^n = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{x}\rangle_L = -1 \text{ and } \mathbf{x} \text{ is positive}\}$, T preserves $\mathcal{H}^n$. Since T is bijective, it has a bijective inverse, which is also a positive Lorentz transformation, and so $T|_{\mathcal{H}^n}$ is a bijection. Distance on $\mathcal{H}^n$ is given in terms of the Lorentz inner product, so T is distance preserving. Hence $T|_{\mathcal{H}^n}$ is an isometry. □

Euclidean isometries correspond to orthogonal matrices. The Lorentzian analogy is given by:

Definition 26. *An $n \times n$ real matrix A is* <u>*Lorentz orthogonal*</u> *if*

$$A^T J A = J := \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{pmatrix}$$

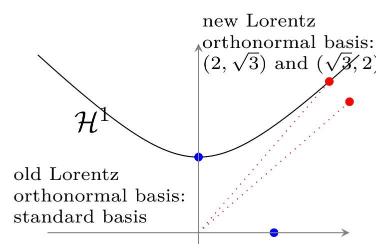
Definition 27. *The* <u>*Lorentz group*</u> *is the group of Lorentz orthogonal $n+1 \times n+1$ matrices, denoted*

$$O(1,n) = \{A \in M_{n+1 \times n+1} | A^T J A = J\}$$

The <u>*positive Lorentz group*</u> *is the subgroup of $O(1,n)$ which maps positive time like vectors bijectively to positive time like vectors. This is denoted $O^+(1,n)$.*

Lemma 22. *$O(1,n)$ and $O^+(1,n)$ are groups.*

Proof. Exercise. □



Example:

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} \in O(1,1)$$

We check this by showing that:

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} \begin{pmatrix} -2 & -\sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By Lemma 24 this means that the columns of this matrix form a Lorentz orthonormal basis for $\mathbb{R}^2$.

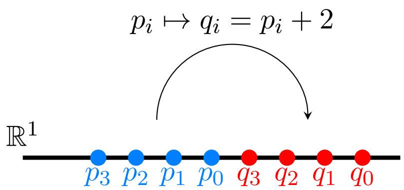
Lorentz translation

More generally for any $\gamma \in \mathbb{R}$, $A_\gamma = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}$ defines an element of $O^+(1,1)$, which we can think of as inducing a translation on $\mathcal{H}^1$, since we have

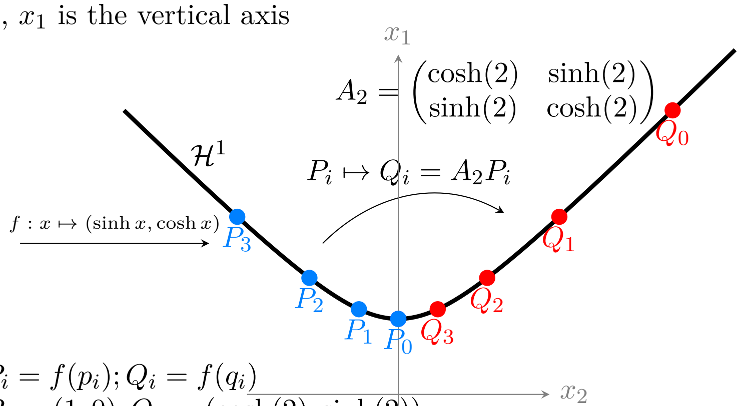
$$\begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} \cosh x \\ \sinh x \end{pmatrix} = \begin{pmatrix} \sinh\gamma\sinh x + \cosh\gamma\cosh x \\ \cosh\gamma\sinh x + \sinh\gamma\cosh x \end{pmatrix} = \begin{pmatrix} \cosh(x+\gamma) \\ \sinh(x+\gamma) \end{pmatrix}$$

The spaces $(\mathbb{E}^1, d_{\mathbb{E}^1})$ and $(\mathcal{H}^1, d_{\mathcal{H}^1})$ are isometric, via the paramterisation $x \mapsto (\sinh x, \cosh x)$.

WARNING: first component, x_1 is the vertical axis



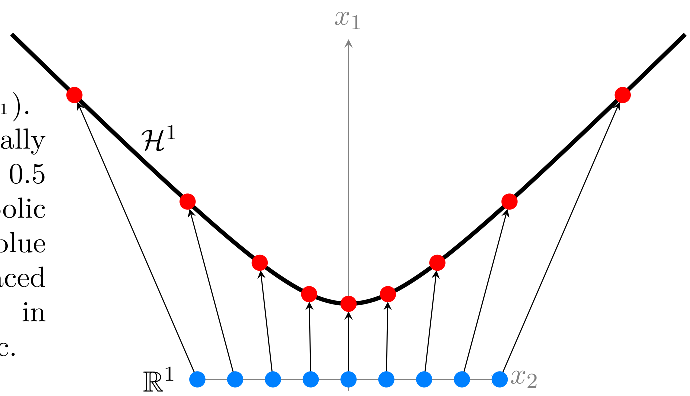
p_0	p_1	p_2	p_3
0	-0.5	-1	-1.5



$P_i = f(p_i); Q_i = f(q_i)$
$P_0 = (1, 0); Q_0 = (\cosh(2), \sinh(2))$

Since $\cosh(x) \in [1, \infty)$, the image of A_γ applied to any element of $\mathcal{H}^1$ is in $\mathcal{H}^1$, and by linearity, A_γ preserves all positive elements of $\mathbb{R}^2$, and so $A_\gamma \in O^+(1, n)$.
Quiz: What do you think a "reflection" matrix for $\mathcal{H}^1$ looks like?
Note that the parameterisation is not a projection to the x_2 axis:

illustration of the isometry from $(\mathbb{R}^1, d_{R^1})$ to $(\mathcal{H}^1, d_{\mathcal{H}^1})$. The red dots are equally spaced, distance 0.5 apart in the hyperbolic metric, and the blue dots are equally spaced distance 0.5 apart in the Euclidean metric.



To Do:

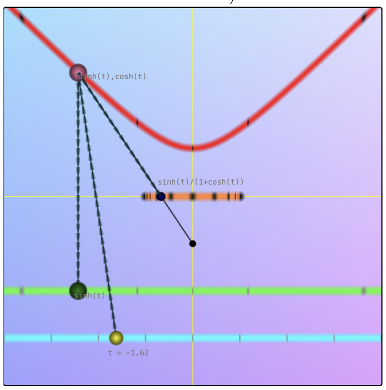
Important results we hope to have time to prove later:

- The hyperbolic metric is a metric
- All isometries of $\mathcal{H}^n$ are given by elements of $O(1, n)$.
- For a hyperbolic triangle with angles a, b, c and area A, we have $A = \pi - a - b - c$.

Things we will actually do:

- Think about lines in the hyperbolic plane, by projection to the plane $x_1 = 0$.
- Think about what are all the isometries of the hyperbolic plane, e.g., translation, rotation, etc

Some different views of the hyperbolic line. The hyperbolic line is isometric to $\mathbb{R}$ with the Euclidean metric, under the map $t \mapsto (\cosh(t), \sinh(t))$. So the simplest way to think about $\mathcal{H}^1$ would be to imagine the usual Euclidean line.
However, $\mathcal{H}^2$ can not be embedded smoothly and isometrically in $\mathbb{R}^3$, so is harder to imagine. There are several common models to work with. Since these models are easier to understand in the analogous one dimensional cases, the following picture shows 4 different ways to view $\mathcal{H}^1$.



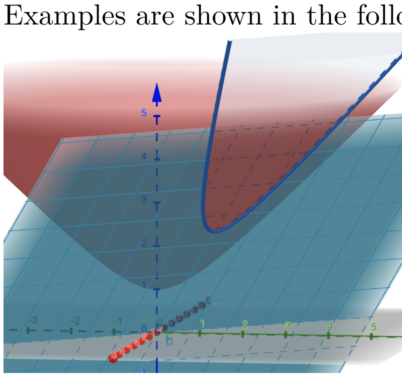
Lower, cyan line	$t \in \mathbb{R}$
	parameter, mapping to red upper hyperbola
Red hyperbola	$(\cosh(t), \sinh(t))$
	vertical component first.
	(point, mapping to either orange or green lines)
Green line	$\sinh(t)$
	vertical projection from hyperbola
Orange segment $(-1, 1)$	$\sinh(t)/(1 + \cosh(t))$, Poincare line.
	Projection of the hyperbola from the point $(0, -1)$.

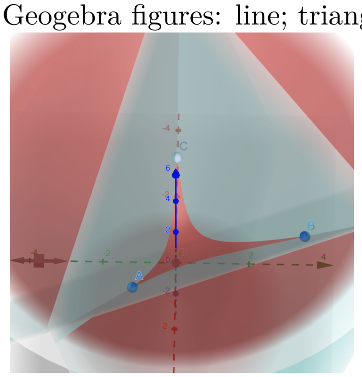
17. Lecture 17: Hyperbolic lines

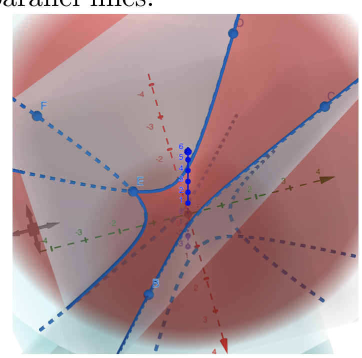
We need to prove the triangle inequality, and discuss lines and triangles. It will be useful to introduce some terminology about hyperbolic lines.

Definition 28. *A 2-dimensional sub-vector space V of $\mathbb{R}^{n+1}$ is called a Lorentz plane if it contains a timelike vector. A hyperbolic line is the intersection of $\mathcal{H}^n$ with any Lorentz plane.*

Hyperbolic line segments are the shortest distances between two points in the hyperbolic metric. (We do not have time to prove this.)
Examples are shown in the following Geogebra figures: line; triangle; parallel lines. [1]







Lemma 23. *For $\mathbf{P} \neq \mathbf{Q} \in \mathcal{H}^n$, there is a unique hyperbolic line L containing $\mathbf{P}$ and $\mathbf{Q}$.*

Proof. Let V be the subvector space of $\mathbb{R}^{n+1}$ spanned by $\mathbf{P}$ and $\mathbf{Q}$, and set $L = \mathcal{H}^n \cap V$. □

We will show that Euclid's parallel postulate does not hold in $\mathcal{H}^2$.

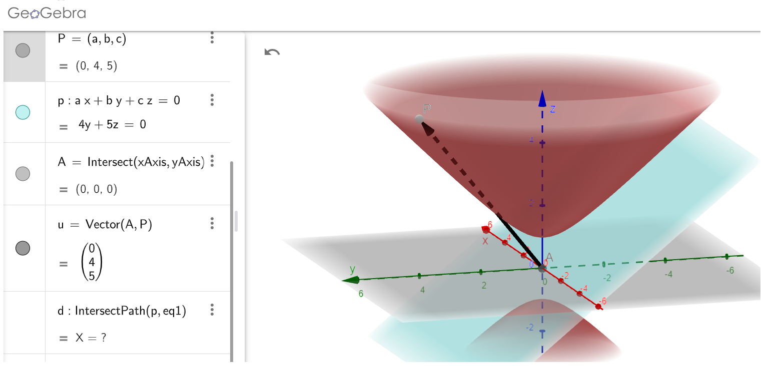
Definition 29. *A one dimensional subvector space $V = \mathbb{R}\mathbf{v}$ of $\mathbb{R}^n$ is called time-like, space-like or light-like depending on whether $\mathbf{v}$ is time-like, space-like or light-like.*

Definition 30. *Let $L_1 = \mathcal{H}^2 \cap \Pi_1$ and $L_2 = \mathcal{H}^2 \cap \Pi_2$ be two distinct lines in $\mathcal{H}^2$, for two dimensional subvector spaces Π_1 and Π_2. Let $\mathbf{v} \in \Pi_1 \cap \Pi_2 \setminus \{\mathbf{0}\}$, and $V = \mathbb{R}\mathbf{v} = \Pi_1 \cap \Pi_2$.*

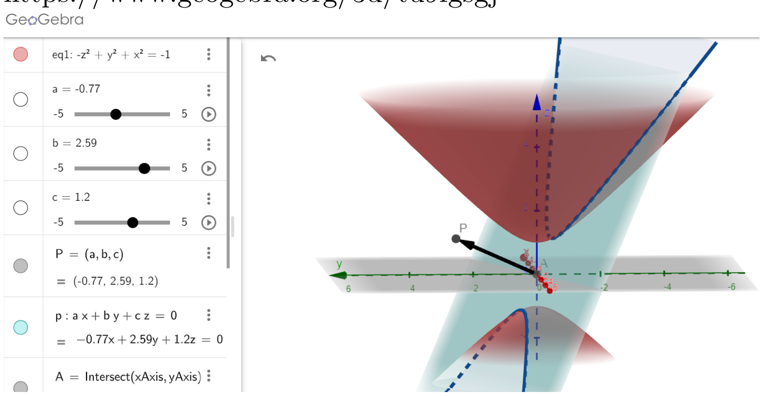
1. *V is time-like: L_1 and L_2 intersect at a point $\mathbf{x}$, with $\{\mathbf{x}\} = V \cap \mathcal{H}^2$.*
2. *V is space-like: L_1 and L_2 are parallel and diverge.*
3. *V is light-like: L_1 and L_2 are ultraparallel.*

[1] `www.geogebra.org/3d/xk4cauzn` ; `www.geogebra.org/3d/ktnpt5ae` ; `www.geogebra.org/3d/bdtqpp8v`

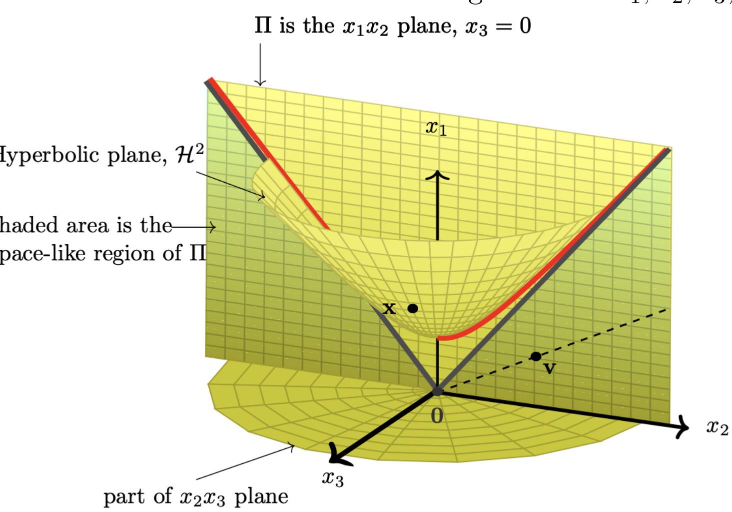
If a plane does not contain a time like vector, it can't intersect $\mathcal{H}^2$, as for example in this case:



Any plane has the form $(\mathbb{R}\mathbf{v})^\perp$ for some vector $\mathbf{v}$. As long as $\mathbf{v}$ is space line, we're guaranteed that $\mathcal{H}^2 \cap (\mathbb{R}\mathbf{v})^\perp$ is non empty.
You can move the point P in the following geogebra graph to see this for yourself:
https://www.geogebra.org/3d/tu9fgsgj



WARNING: We are using variables x_1, x_2, x_3, whereas in geogebra these are called z, y, x.

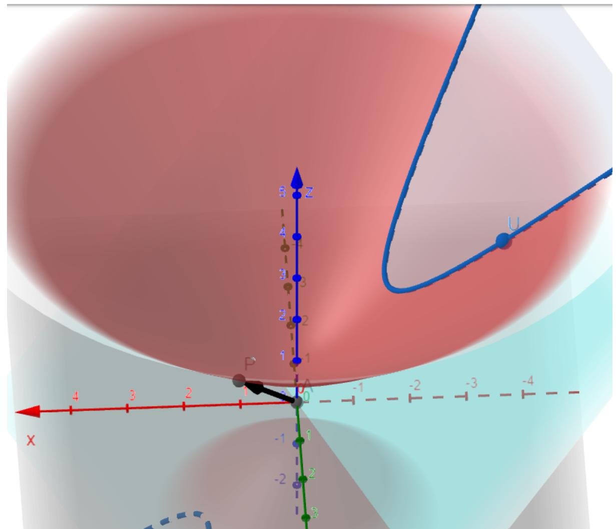


THEOREM 9. *Let $\mathbf{x} \in \mathcal{H}^2$, and let L be a line in $\mathcal{H}^2$ which does not contain $\mathbf{x}$. Then there are infinitely many lines in $\mathcal{H}^2$ which pass through $\mathbf{x}$ and do not intersect L.*

PROOF. Let Π be a 2 dimensional vector subspace such that $L = \Pi \cap \mathcal{H}$.
We will find infinitely many lines L_t, containing $\mathbf{x}$ and disjoint from Π, where t may take infinitely many different values, and $L_t \neq L_t'$ for $t \neq t'$.
Each L_t is of the form $\Pi_t \cap \mathcal{H}^2$ for some subspace Π_t.
We are going to find a way to choose Π_t so that $\Pi_t \cap \Pi$ is space like. This means that $\Pi_t \cap \Pi$ can not contain any time like vector, and so in particular can't contain any point in $\mathcal{H}^2$, and so $\text{Ł}_t \cap L = \varnothing$. We will give the sequence of steps of the argument together with illustrations from geogebra.

step 1: Pick a point $\mathbf{u}$ in L

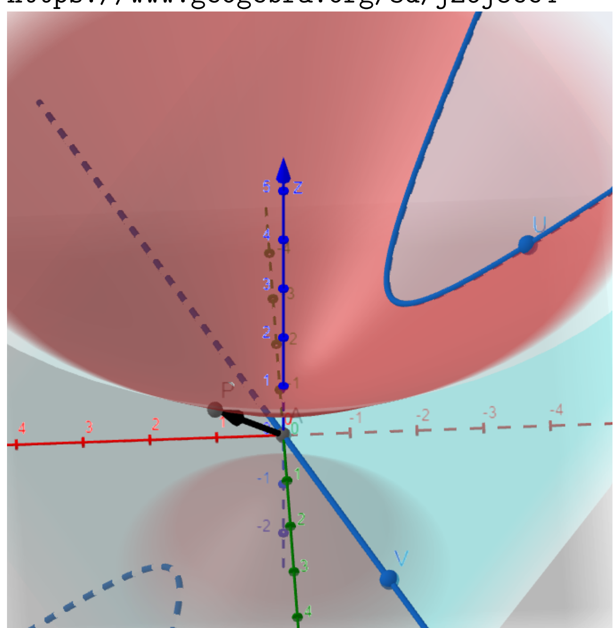
https://www.geogebra.org/3d/jz6j8cc4



step 2:

- Pick a point $\mathbf{v}$ in Π that is space like.
- This can be any point, but to be specific, let's take a non zero point on the line where Π intersects the plane $\{x_1 = 0\}$.
- Since the two planes Π and $\{x_1 = 0\}$ are two different two dimensional subspaces of $\mathbb{R}^3$, they must intersect in a line, which is spanned by some vector $\mathbf{v}$.
- Since we've chosen $\mathbf{v} = (0, v_2, v_3) \neq (0, 0, 0)$ we must have $\langle \mathbf{v}, \mathbf{v} \rangle_L = v_2^2 + v_3^2 > 0$, so $\mathbf{v}$ is a space like point on Π

https://www.geogebra.org/3d/jz6j8cc4

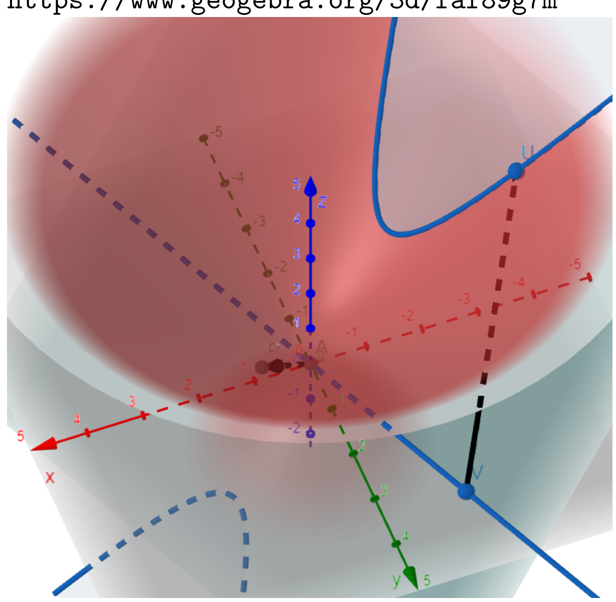


step 3: Consider the line segment S from $\mathbf{v}$ to $\mathbf{u}$. From our description of lines in Euclidean space, we know this is the set of points of the form

$$P_t = (1 - t)\mathbf{v} + t\mathbf{u}$$

My goal is to find infinitely many space like points on this line segment S. (This is a Euclidean line, not a hyperbolic line). I could find other space like points, but this is one possible method.

https://www.geogebra.org/3d/faf89g7m



step 4

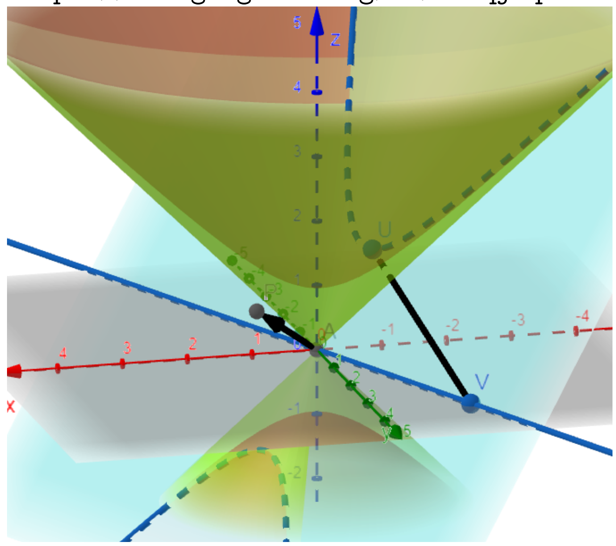
Note that

- $\mathbf{v}$ is space like, and
- $\mathbf{u}$ is time like.
- And there is a continuous function $g(t)$ from $[0,1]$ to $\mathbb{R}$ mapping t to $g(t) := \langle P_t, P_t \rangle_L$
- Since $g(0) = \langle \mathbf{v}, \mathbf{v} \rangle_L = \alpha > 0$ and $g(0) = \langle \mathbf{u}, \mathbf{u} \rangle_L = -1$, for any value of $a \in (0, \alpha)$, by the intermediate value theorem, there is some point $P_{t_a} \in [0,1)$ with $g(t_a) = a$, i.e., P_{t_a} is a space like point on Π, and since these points all have different values of their Lorentz norm, they are different points. So we have infinitely many space like points in Π on this line segment.

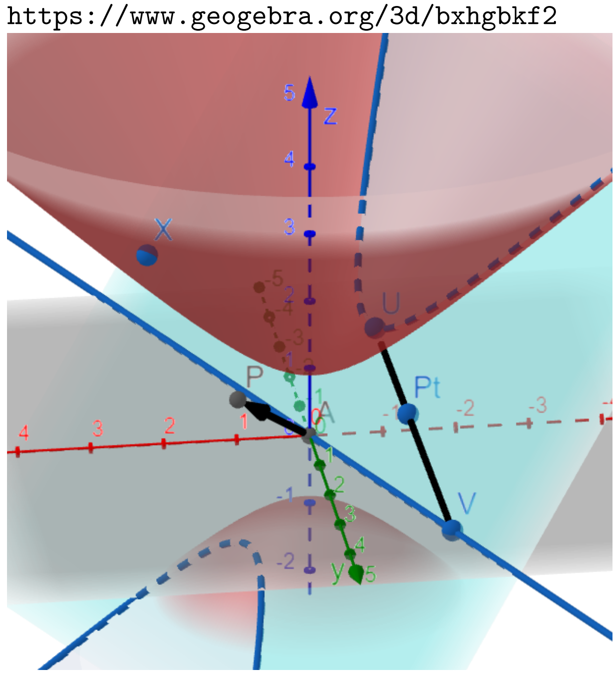
We can add the light cone to emphasise that points on S which are not in the light cone are space like points on Π.

It's not that easy to see, but the green area is the light cone.

https://www.geogebra.org/3d/xfsqjeq5



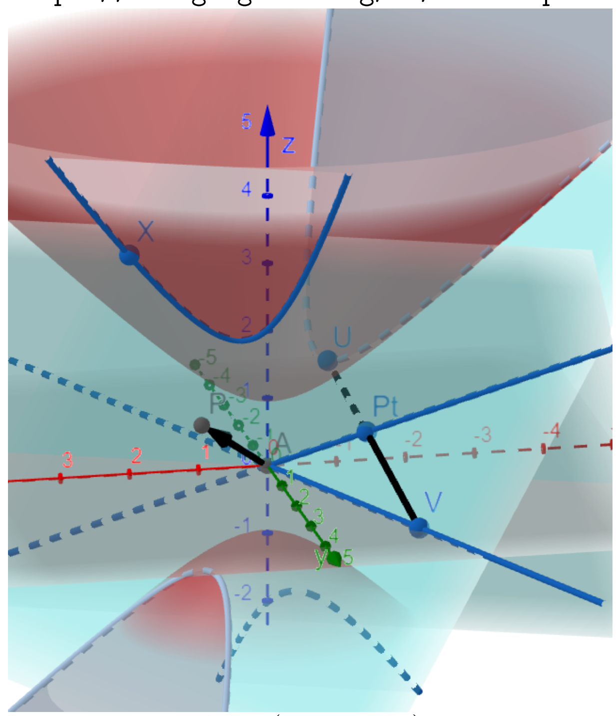
https://www.geogebra.org/3d/bxhgbkf2



step 5

Now consider one of the space like points P_t, and the point $\mathbf{x}$ on $\mathcal{H}$.

https://www.geogebra.org/3d/uakw5h7p



step 6

- Now consider one of the space like points P_t, and the point $\mathbf{x}$ on $\mathcal{H}$.
- We let the plane Π_t be the span of the vectors $\mathbf{x}$ and P_t.
- This plane intersects Π in the subspace $\mathbb{R}P_t$ spanned by the vector P_t.
- This is because Π contains P_t, since P_t was chosen to be on Π and Π_t contains P_t since Π_t was defined to be the span of P_t and $\mathbf{x}$.
- Since $\mathbb{R}P_t \subset \Pi \cap \Pi_t$, we must have $\mathbb{R}P_t = \Pi \cap \Pi_t$, since otherwise they would be equal, but $\mathbf{x}$ is in Π_t by construction, but is not in L, and thus not in Π by assumption.

Let $L_t = \mathcal{H} \cap \Pi_t$. Since this contains the time like point $\mathbf{x}$, it is a line. (not empty).
We're now going to double check that the L_t lines are the infinite collection of lines we're looking for. A projected to $\mathbb{R}^2$ picture looks like:

- The lines L_t and L are disjoint, since the intersection is $L_t \cap L = \Pi_t \cap \mathcal{H} \cap \Pi \cap \mathcal{H} = \Pi_t \cap \Pi \cap \mathcal{H} = \mathbb{R}P_t \cap \mathcal{H} = \varnothing$, where the final step is because all points on $\mathbb{R}P_t$ are space like (or light like at the origin), but all points on $\mathcal{H}$ are time like. So the intersection is empty.
- Now we just have to check that the lines L_t are all different. I.e., if $t_1 \neq t_1$ then $L_{t_1} \neq L_{t_2}$. This is because the planes Π_{t_1} and Π_{t_2} are different, which is because the vectors P_{t_1} and P_{t_2} are not colinear. If they were colinear, then $\mathbf{0}$ would be a point on the line through P_{t_1} and P_{t_2} which is the same as the line through $\mathbf{u}$ and $\mathbf{v}$ by construction. But the origin can't be on the line between these points, or else we would be able to write $\mathbf{0} = (1-t)\mathbf{u} + t\mathbf{v}$ for some real number t, but then $\mathbf{u}$ and $\mathbf{v}$ would be linearly dependent, but this contradicts one being time like and the other being space like. since all non-zero multiples of a space like vector are space like, and similarly for time like vectors.

  So, Π_{t_1} and Π_{t_2} are different planes. This implies their intersections with $\mathcal{H}^2$ are different, since if not, they would have two points in common, but any two points will span a plane, unless they are colinear. Two points on $\mathcal{H}^2$ can not be colinear, since for any timelike $\mathbf{y}$, there is a unique α with $\alpha\mathbf{y}$ on $\mathcal{H}$.

  So, L_t are different for different t.

Hence there are infinitely many lines L_t which contain $\mathbf{x}$ but do not intersect L, as required.

$\square$

18. Lecture 18: Sketching hyperbolic lines

(We first will spend some time going over the proof covered at the end of last lecture)

We can't easily draw on a hyperboloid when we only have flat paper.
To represent hyperbolic lines in $\mathcal{H}^2$, we can project from $\mathcal{H}^2$ to the $x_1 = 0$ plane.
The projection map is given by

$$\pi : \mathcal{H}^2 \to \mathbb{R}^2$$
$$(x_1, x_2, x_3) \mapsto (x_2, x_3)$$

In this section, we describe the images of lines in $\mathcal{H}^2$ under this projection map.
Suppose Π is a plane given by $ax_1 + bx_2 + cx_3 = 0$.
For there to be a time like point on Π, so that L is non-empty, we must have $-a^2 + b^2 + c^2 > 0$.
Then on $L = \Pi \cap \mathcal{H}^2$, we have $ax_1 = -bx_2 - cx_3$, and also $x_1^2 = 1 + x_2^2 + x_3^2$, and $x_1 > 0$, so we obtain

$$(bx_2 + cx_3)^2 = a^2(1 + x_2^2 + x_3^2)$$

which is a quadratic equation, which defines a conic section.

$$(b^2 - a^2)x_2^2 + 2bcx_2x_3 + (c^2 - a^2)x_3^2 = a^2$$

We can factor the left hand side to get

$$(b^2 - a^2)(x_2 - \alpha_1 x_3)(x_2 - \alpha_2 x_3) = a^2$$

where

$$\alpha_1, \alpha_2 = \frac{1}{2(b^2 - a^2)}\left(-2bc \pm \sqrt{4b^2c^2 - 4(b^2 - a^2)(c^2 - a^2)}\right) = \frac{-2bc \pm 2a\sqrt{b^2 + c^2 - a^2}}{2(b^2 - a^2)},$$

And since $b^2 + c^2 - a^2 > 0$, there are two distinct asymptotes, given by $x_2 = \alpha_1 x_3$ and $x_2 = \alpha_2 x_3$, so this conic is a hyperbola.
Example lines:
case 1: If $a = 0$, then the line is $0 = bx_2 + cx_3$, which is a straight line through the origin.
E.g., taking $c = 0$ we have a hyperbolic line which projects to $x_2 = 0$.
case 2: Now consider the case $a \neq 0$. Since we have $x_1 > 0$, then $-(bx_2 + cx_3)/a = x_1 > 0$, so we can use this to determine which half of the hyperbola is obtained.
case 3: Now consider the point at $P = (\sqrt{2}, 1, 0)$ on $\mathcal{H}^2$. This point projects to $(1, 0)$. For the plane $ax_1 + bx_2 + cx_3 = 0$ to pass through P, we need $a\sqrt{2} + b = 0$, so we take $a = 1$, and $b = -\sqrt{2}$. c can be anything. The projection of the line on $\mathbb{R}^2$ is

$$(-\sqrt{2}x_2 + cx_3)^2 = (1 + x_2^2 + x_3^2)$$

So

$$2x_2^2 + 2\sqrt{2}cx_2x_3 + c^2x_3^2 = 1 + x_2^2 + x_3^2$$

So

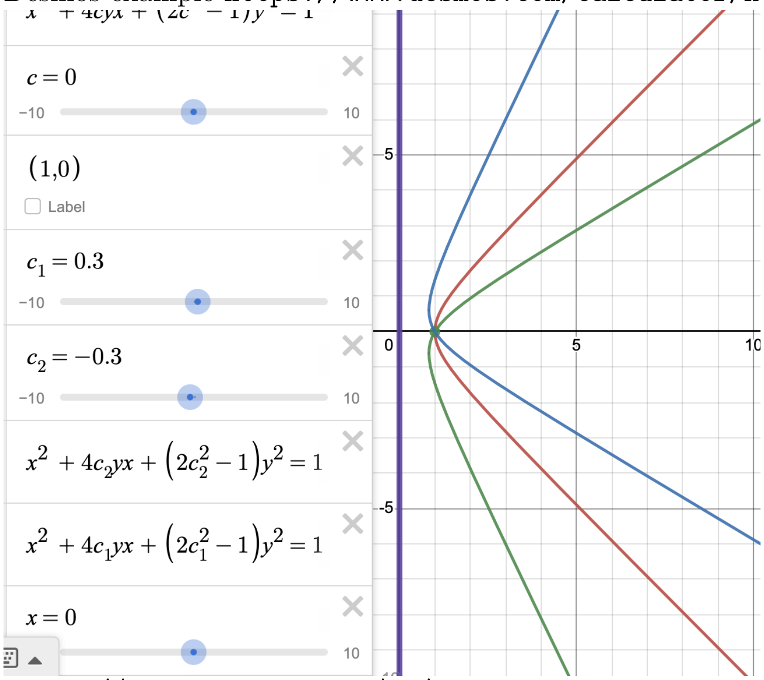
$$x_2^2 + 2\sqrt{2}cx_2x_3 + (c^2 - 1)x_3^2 = 1$$

If we replace c by $\sqrt{2}c$, we have

$$x_2^2 + 4cx_2x_3 + (2c^2 - 1)x_3^2 = 1$$

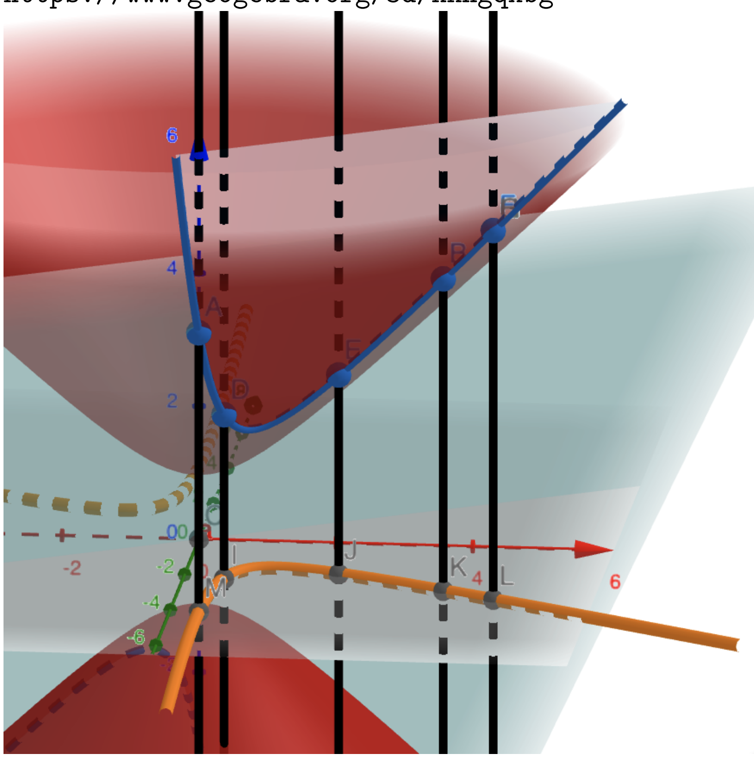
We will sketch some of these lines in class.

Desmos example `https://www.desmos.com/calculator/kts5zeg1cb`



(Desmos coordintes are x, y, but we use x_2, x_3 for these) The line L is given by $x = 0$, the point $P = (1, 0)$. The hyperbolic lines are given by $x_2^2 + 4cx_2x_3 + (2c^2 - 1)x_3^2 = 1$

`https://www.geogebra.org/3d/kkhgqxbg`



In Geogebra, define a hyperboliod, and a Lorentz plane. These intersect in a hyperbolic line L

If you pick 5 points on L, and project to the plane $x_1 = 0$, then you can use the geogebra tool to draw a conic through these points, which is the orange line in this picture.

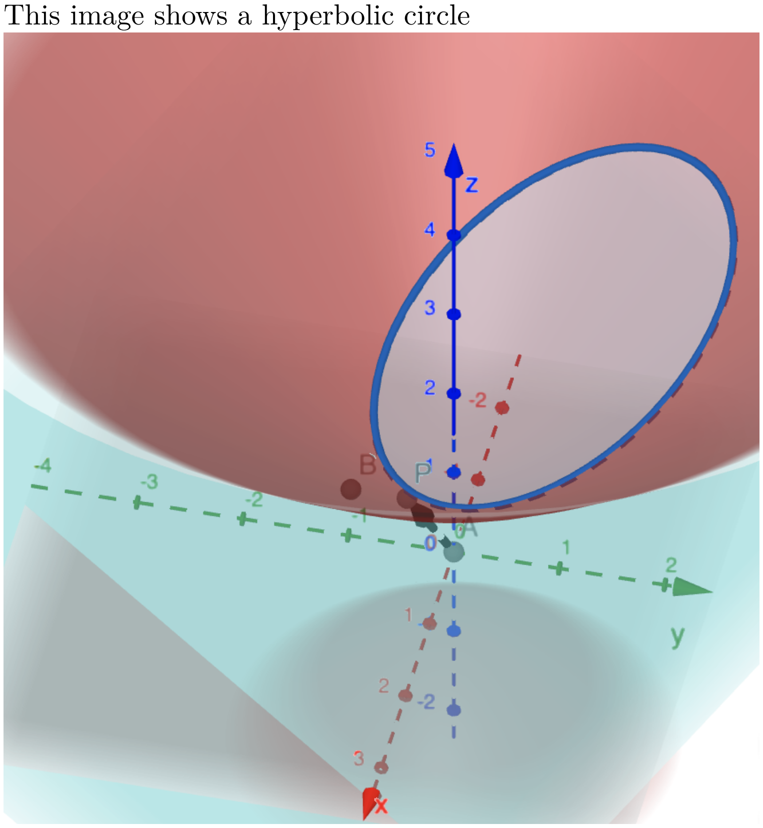
19. Lecture 19: Different Kinds of Intersections of planes with $\mathcal{H}^2$

A plane through the origin in $\mathbb{R}^3$ which contains a time like vector intersects $\mathcal{H}^2$ in a hyperbolic line. There are some other kinds of planes, with different curves obtained in $\mathcal{H}^2$.

plane in $\mathbb{R}^3$	curve on $\mathcal{H}^2$
planes through the origin	
Lorentz plane	hyperbolic line
Plane with no time like vector	empty set
planes not through the origin containing time like vector	
normal is light like	circles
normal is space like	hypercycles
normal is time like	horocycles

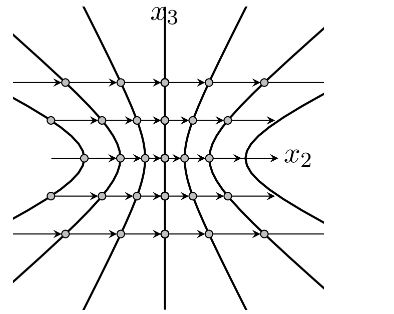
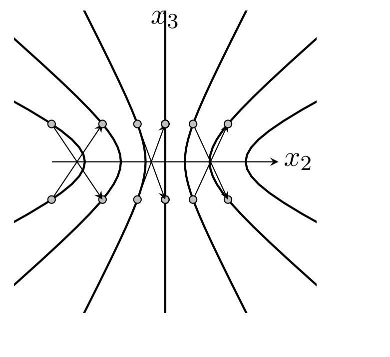
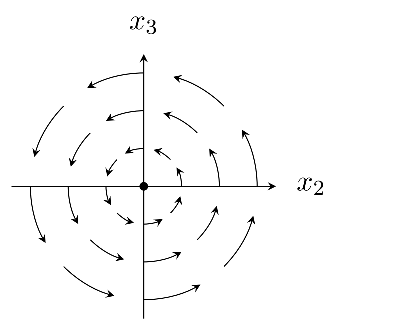
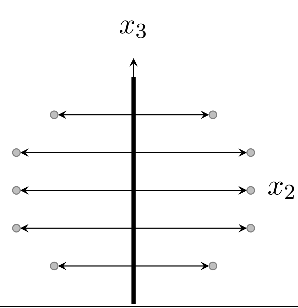
Hypercycles and horocycles both come in sets which are obtained from sets of parallel planes.
Each hypercycles is equidistant to some hyperbolic line, which you might consider them to be parallel to, in some sense, and each hypercycle generally has different amounts of curvature.
Horocycles are all congruent to each other, but not to any hyperbolic line. They are also somewhat similar to sets of parallel lines, but they are not straight lines, and not equidistant to any hyperbolic line.

A geogebra example: `https://www.geogebra.org/3d/bmcussgf`
This image shows a hyperbolic circle



Hyperbolic isometries of $\mathcal{H}^2$

Recall that if A is in $O^+(1,n)$, then A defines an Isometry of $\mathcal{H}^n$. This means that $A^TJA = J$ and T_A maps $\mathcal{H}^2$ to $\mathcal{H}^2$.
We have not yet proved that all isometies of $\mathcal{H}^n$ have this form, but this does turn out to be true.
Almost every Lorentz transformation of $\mathcal{H}^2$ is given by one of the following matrices, up to a Lorentz orthonormal change of basis, i.e., from the standard basis to another Lorentz orthormal basis.

matrix	description	fixed points	eigen values	picture – projection to x_2, x_3 plane
$\begin{pmatrix} \cosh(\beta) & \sinh(\beta) & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	Lorentz translation (by β on the x_2 axis); orientation preserving (direct); hyperbolic	none (in $\mathcal{H}^2$)	$1, \exp(\pm\beta)$	
$\begin{pmatrix} \cosh(\beta) & \sinh(\beta) & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 \\ 0 & 0 & -1 \end{pmatrix}$	Lorentz glide (in x_2 direction); orientation reversing (indirect / opposite)	none (in $\mathcal{H}^2$)	$-1, \exp(\pm\beta)$	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$	rotation (about the point $(1,0,0)$); orientation preserving (direct); elliptic	one point	$1, \exp(\pm i\theta)$	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	reflection (in the line $x_2 = 0$); orientation reversing (indirect)	a line	$1, 1, -1$	

Note, each of the above matrices has an eigen value ± 1 with an eigen vector which is space-like or time-like. There are elements of $O^+(1,2)$ with light-like eigen vectors with eigen value 1, such as

$$\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

These correspond to "parabolic" transformations of $\mathcal{H}^2$, which preserve the orientation, and don't fix any points of $\mathcal{H}^2$, but whereas a Lorentz translation fixes a space-like vector when extended to $\mathbb{R}^3$, parabolic transformations fix a light-like vector.

Proof: See [RS, 3.11(page 47) and 3.16(page 58) and B.3 page 189], and the RS erratum

You are expected to be able to

- Check all the above are elements of $O^+(1,2)$.
- given an element of $O^+(1,2)$, determine which kind of transformation it corresponds to, e.g., by considering fixed points and eigen values
- sketch images of simple shapes e.g., lines, points, triangle, under the action of transformations
- compute the composition of hyperbolic isometries
- find the matrix for a hyperbolic isometry given a description

Lorentz orthogonal matrices

Definition 31. *A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ is <u>Lorentz orthogonal</u> if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = 0$ for $i \neq j$. It is <u>Lorentz orthonormal</u> if*

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } 2 \leq i = j \leq n \\ -1 & \text{if } i = j = 1 \end{cases}$$

Lemma 24.

(1) *The canonical basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is Lorentz orthonormal.*
(2) *If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are Lorentz orthonormal, then they form a basis of $\mathbb{R}^n$.*
(3) *$\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a Lorentz orthonormal basis if and only if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = J_{i,j}$, the element in the i^{th} row, j^{th} column of J.*

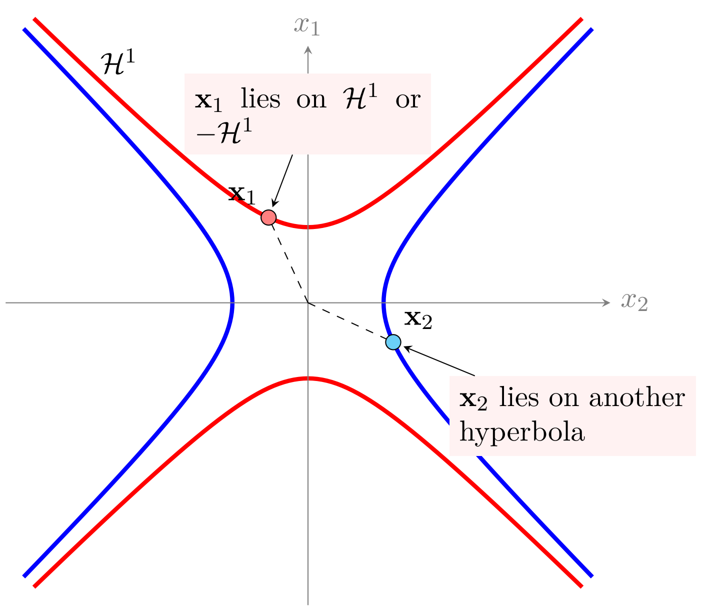
Proof. Exercise. ☐

For the Euclidean norm, the orthonormal vectors were on a sphere. For the Lorentz metric, we have the first element of a Lorentz orthonormal basis is in $\mathcal{H}^n$ or in $-\mathcal{H}^n$. The other elements of a Lorentz orthonormal basis satisfy

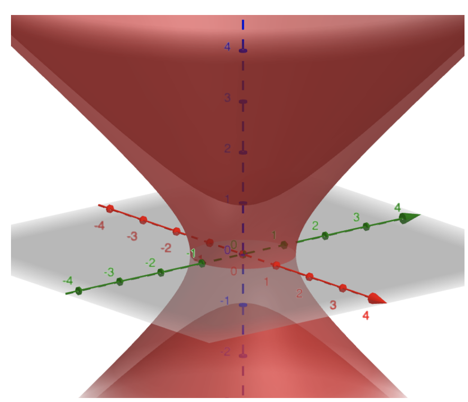
$$\langle \mathbf{x}, \mathbf{x} \rangle_L = 1 = -x_1^2 + x_2^2 + \cdots + x_n^2,$$

which means they lie on another hyperoloid, as in the figure:

Suppose $\mathbf{x}_1, \mathbf{x}_2$ is a Lorentz orthnormal basis for $\mathbb{R}^2$



surfaces where elements of a Lorentz orthonomal basis of $\mathbb{R}^3$ live:



20. Lecture 20: Sketch proof that $\text{Isom}(\mathcal{H}^2, d_{\mathcal{H}^2}) \cong O^+(1,n)$

We won't have time to prove this result. We already know that $O^+(1,n) \subset \text{Isom}(\mathcal{H}^2, d_{\mathcal{H}^2})$, so we just need to prove the reverse. An outline is as follows:

(1) All linear Lorentz transformations correspond to elements of $O(1,n)$.
(2) All Lorentz transformations are linear
(3) Any Isometry of $\mathcal{H}^n$ extends to a Lorentz transformation of $\mathbb{R}^{n+1}$.

(1) is almost by definition.
(3) can be acheived by using the fact that $\mathcal{H}^n$ does contain a basis for $\mathbb{R}^{n+1}$, so we can extend linearly, and just have to check it actually works.
We don't have time to prove all these steps in detail, but we can do (2). See course text book [R] or last year's notes for (1) and (3).

Proposition 12. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation then*

(1) *the image of the canonical basis $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ is Lorentz orthonormal.*
(2) *T is linear.*

Proof. $\Rightarrow$ Suppose T is a Lorentz transformation.

$$\begin{aligned} \langle T(\mathbf{e}_i), T(\mathbf{e}_j)\rangle_L &= \langle \mathbf{e}_i, \mathbf{e}_j\rangle_L \text{ since } T \text{ is a Lorentz transformation} \\ &= J_{i,j} \text{ by Lemma 24, which implies } T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n) \text{ is Lorentz orthonormal.} \end{aligned}$$

Suppose $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i \in \mathbb{R}^n$, and $T(\mathbf{x}) = \sum_{i=1}^n b_i T(\mathbf{e}_i)$. By orthnormality of the $\mathbf{e}_i$ and the $T(\mathbf{e}_i)$,

$$\begin{aligned} -b_1 = \langle T(\mathbf{x}), T(\mathbf{e}_1)\rangle_L &= \langle \mathbf{x}, \mathbf{e}_1\rangle_L = -x_1 \\ b_i = \langle T(\mathbf{x}), T(\mathbf{e}_i)\rangle_L &= \langle \mathbf{x}, \mathbf{e}_i\rangle_L = x_i \text{ for } 2 \le i \le n \end{aligned}$$

Thus $b_i = x_i$ for $i = 1, \ldots, n$ so T is linear.

□

Corollary 7. *A Lorentz transformation is bijective.*

Proof. This is because a Lorentz transformation is linear and maps a basis to a basis. □

Assuming that we now know that Lorentz transformations correspond to elements of $O(1,n)$, to finish off our proof that $\text{Isom}(\mathcal{H}^2, d_{\mathcal{H}^2}) \cong O^+(1,n)$ we need to know which elements of $O(1,n)$ preserve the sign of x_1.

Lemma 25. *If $A \in O(1,n)$ then $A^T \in O(1,n)$.*

Proof. $A \in O(1,n) \Rightarrow A^T JA = J \Rightarrow JA^T JA = I \Rightarrow (JA^T J)^{-1} = A \Rightarrow A(JA^T J) = I \Rightarrow AJA^T J = I \Rightarrow AJA^T = J \Rightarrow A^T \in O(1,n)$. □

Proposition 13. *For $A = (a_{i,j}) \in O(1,n)$, we have $A \in O^+(1,n) \iff a_{1,1} > 0$.*

Proof. Let A have columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.
$\Rightarrow$: If $A \in O^+(1,n)$, then $T(\mathbf{e}_1) = \mathbf{a}_1 = (a_{1,1}, a_{2,1}, \ldots, a_{n,1}) \in \mathcal{H}^n$, so $a_{1,1} > 0$.
$\Leftarrow$: Denote the first column of A^T by $\mathbf{a}_1'$. By Lemma 25, $A^T \in O(1,n)$, and so by definition of $O(1,n)$, $\mathbf{a}_1'$ is in $\mathcal{H}^n$ or in $-\mathcal{H}^n$. Since $a_{1,1} > 0$, we have $\mathbf{a}_1' \in \mathcal{H}^n$. The first element of the vector $A\mathbf{x}$ is $\langle \mathbf{a}_1', \mathbf{x}\rangle = \beta \langle \mathbf{a}_1', \mathbf{x}/\beta\rangle$, where this is the usual inner product, and $\beta = -i||\mathbf{x}||_L > 0$. Since $\mathbf{x}/\beta \in \mathcal{H}^n$, by Lemma 20 $\langle \mathbf{a}_1', \mathbf{x}/\beta\rangle$ is positive, so $A\mathbf{x}$ is also positive, so we are done. □

Loose end: In our description of hyperbolic lines, we used that Lorentz planes are orthogonal to space like vectors. We will now prove this.

LEMMA 26. *If* $\mathbf{w}$ *is Lorentz orthogonal to a time-like vector, then* $\mathbf{w}$ *is space like. In other words: If* $\mathbf{x} \in \mathbb{R}^n$ *with* $\langle \mathbf{x}, \mathbf{x} \rangle_L < 0$*, and* $\mathbf{w} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *with* $\langle \mathbf{x}, \mathbf{w} \rangle_L = 0$*, then* $\langle \mathbf{w}, \mathbf{w} \rangle_L > 0$*.*

PROOF. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Let $\mathbf{x}' = \mathbf{x} - x_1\mathbf{e}_1$ and $\mathbf{w}' = \mathbf{w} - w_1\mathbf{e}_1$. Since $\langle \mathbf{x}', \mathbf{x}' \rangle_L \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle_L = -x_1^2 + \langle \mathbf{x}', \mathbf{x}' \rangle_L$, we must have $x_1 \neq 0$. If $w_1 = 0$, then since $\mathbf{w} \neq 0$, we have $\langle \mathbf{w}, \mathbf{w} \rangle_L = \langle \mathbf{w}', \mathbf{w}' \rangle_L > 0$, and we are done. So we may assume $w_1 \neq 0$. The vector $x_1\mathbf{w} - w_1\mathbf{x}$ has zero $\mathbf{e}_1$ component. This vector can not be zero, or $x_1\mathbf{w} = w_1\mathbf{x}$, but then $0 > \langle \mathbf{x}, \mathbf{x} \rangle_L = \left\langle \mathbf{x}, \frac{x_1}{w_1}\mathbf{w} \right\rangle_L = \frac{x_1}{w_1} \langle \mathbf{x}, \mathbf{w} \rangle_L = 0$, a contradiction. Now

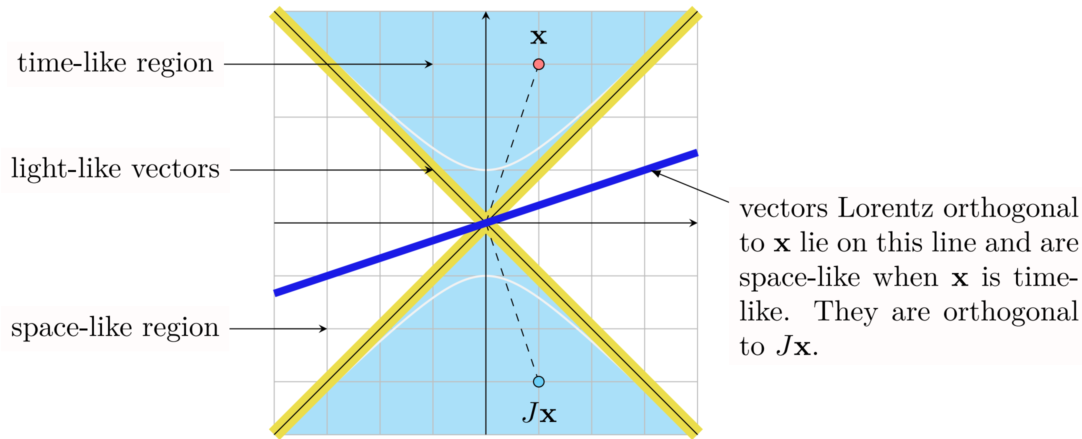
$$0 \leq ||x_1\mathbf{w} - w_1\mathbf{x}||_L^2 = \langle x_1\mathbf{w} - w_1\mathbf{x}, x_1\mathbf{w} - w_1\mathbf{x} \rangle_L = x_1^2 \langle \mathbf{w}, \mathbf{w} \rangle_L + w_1^2 \langle \mathbf{x}, \mathbf{x} \rangle_L$$

so

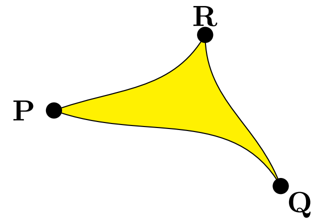
$$\langle \mathbf{w}, \mathbf{w} \rangle_L \geq -\frac{w_1^2}{x_1^2} \langle \mathbf{x}, \mathbf{x} \rangle_L > 0.$$

□

Examples. In $\mathcal{H}^1$, if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{w} = (w_1, w_2)$, then $\langle \mathbf{x}, \mathbf{w}, \rangle_L = 0$ means $x_1w_2 = x_2w_2$. For fixed $\mathbf{x}$ this gives an equation for a line through the origin on which $\mathbf{w}$ must lie.



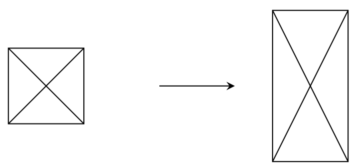
21. Lecture 21: Triangle Inequality (for $\mathcal{H}^2$)



21.1. Hyperbolic triangles.

DEFINITION 32. *A hyperbolic triangle, denoted $\Delta\mathbf{PQR}$, consists of three distinct, non-collinear points, $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ on $\mathcal{H}^n$, and the finite hyperbolic line segments joining each pair of points, and the finite area enclosed by these lines.*
Note, collinear is used to mean points are contained in the same hyperbolic line.

21.2. Hyperbolic angles: Motivation from spherical geometry. In order to prove the triangle inequality, we will use angles in triangles. We are not going to give a proper definition of angle in this course. (See course text book [R] (Ratcliffe) for comprehensive definition.)
Problem: Our model of hyprbolic space does not define a conformal embedding of $\mathcal{H}^2$ in $\mathbb{R}^3$. Conformal means angle preserving.
We haven't even defined what angles are, but you will have to trust that whatever they are, they are not preserved by stretching. Example:



The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ does not preserve angles

So we can't just measure hyperbolic angles by taking the dihedral angle between corresponding planes. What do we do?
Recall

$$\text{For a spherical triangle:} \quad \cos(a) = \frac{\langle \mathbf{P}\times\mathbf{R}, \mathbf{P}\times\mathbf{Q}\rangle}{||\mathbf{P}\times\mathbf{R}||||\mathbf{P}\times\mathbf{Q}||}.$$

By analogy,

$$\text{For a hyperbolic triangle:} \quad \cos(a) = \frac{\langle \mathbf{P}\times_L\mathbf{R}, \mathbf{P}\times_L\mathbf{Q}\rangle_L}{||\mathbf{P}\times_L\mathbf{R}||_L||\mathbf{P}\times_L\mathbf{Q}||_L}. \tag{11}$$

It turns out the is invariant under Lorentz transformation. In this expression,

DEFINITION 33. *Let $J = J_3$ For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, the <u>Lorentz cross-product</u> of $\mathbf{x}, \mathbf{y}$ is*

$$\mathbf{x}\times_L\mathbf{y} := J\cdot(\mathbf{x}\times\mathbf{y}) = \begin{pmatrix} -x_2y_3 + x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} = (J\mathbf{y})\times(J\mathbf{x}) = \left|\begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}\right|$$

Note about invariance:

LEMMA 27. *(Binet–Cauchy identity.) For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$,*

$$\langle \mathbf{x}\times\mathbf{y}, \mathbf{z}\times\mathbf{w}\rangle = \langle \mathbf{x}, \mathbf{z}\rangle\cdot\langle \mathbf{y}, \mathbf{w}\rangle - \langle \mathbf{x}, \mathbf{w}\rangle\cdot\langle \mathbf{y}, \mathbf{z}\rangle$$

The following is a version of this identity for the Lorentz inner product, where the difference with the usual formula is highlighted in red.

LEMMA 28. *(Binet–Cauchy identity for Lorentz inner product.) For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, we have*

$$\langle \mathbf{x}\times_L\mathbf{y}, \mathbf{z}\times_L\mathbf{w}\rangle_L = -\langle \mathbf{x}, \mathbf{z}\rangle_L\cdot\langle \mathbf{y}, \mathbf{w}\rangle_L + \langle \mathbf{x}, \mathbf{w}\rangle_L\cdot\langle \mathbf{y}, \mathbf{z}\rangle_L$$

This result tells us that the expression on the right hand side of Equation 11 is invariant under Lorentz transformations, since the Lorentz inner product is.
Suppose $\mathbf{x} = (1, 0, 0)$, then we have

$$\mathbf{x}\times_L\mathbf{y} = \left|\begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}\right| = \left|\begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 0 \\ y_1 & y_2 & y_3 \end{pmatrix}\right| = \left|\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 0 \\ y_1 & y_2 & y_3 \end{pmatrix}\right| = \mathbf{x}\times\mathbf{y}$$

So the angles are the same as in Euclidean space when $\mathbf{x} = \mathbf{0}$.
So, we're going to translate $\mathbf{P}$ to $(1, 0, 0)$ by a Lorentz transformation so we can measure the triangle angle at $\mathbf{P}$.

Lemma 29. *For points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathcal{H}^2$, there is some Lorentz transformation T such that for some θ, β and γ we have*

$$\begin{aligned} T\mathbf{x} &= (\quad 1, \quad 0, \quad 0) \\ T\mathbf{y} &= (\quad \cosh(\beta), \quad \sinh(\beta), \quad 0) \\ T\mathbf{z} &= (\quad \cosh(\gamma), \quad \cos(\theta)\sinh(\gamma), \quad \sin(\theta)\sinh(\gamma)) \end{aligned}$$

Proof. Recall from our parameterisation of $\mathcal{H}^2$, we know that we can write

$$\mathbf{x} = (\cosh(\delta), \cos(c)\sinh(\delta), \sin(c)\sinh(\delta))$$

Let

$$A_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{pmatrix} \quad \text{and} \quad B_w = \begin{pmatrix} \cosh(w) & \sinh(w) & 0 \\ \sinh(w) & \cosh(w) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

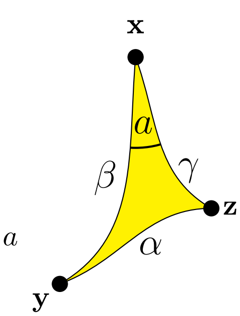
These are hyprbolic rotation and translation matrices.
We have $A_{-c}\mathbf{x} = (\cosh(\delta), \sinh(\delta), 0)$.
And $B_{-\delta}A_{-c}\mathbf{x} = (\cosh(0), \sinh(0), 0) = (1, 0, 0)$.
Now we have that $B_{-\delta}A_{-c}\mathbf{y} = (\cosh(\beta), \cos(b)\sinh(\beta), \sin(b)\sinh(\beta))$ for some β and b.
Now apply A_{-b}, which fixes $(1, 0, 0)$, and all the images of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ under $T = A_{-b}B_{-\beta}A_{-c}$ have the required form.

$\square$

Proof of the triangle inequality

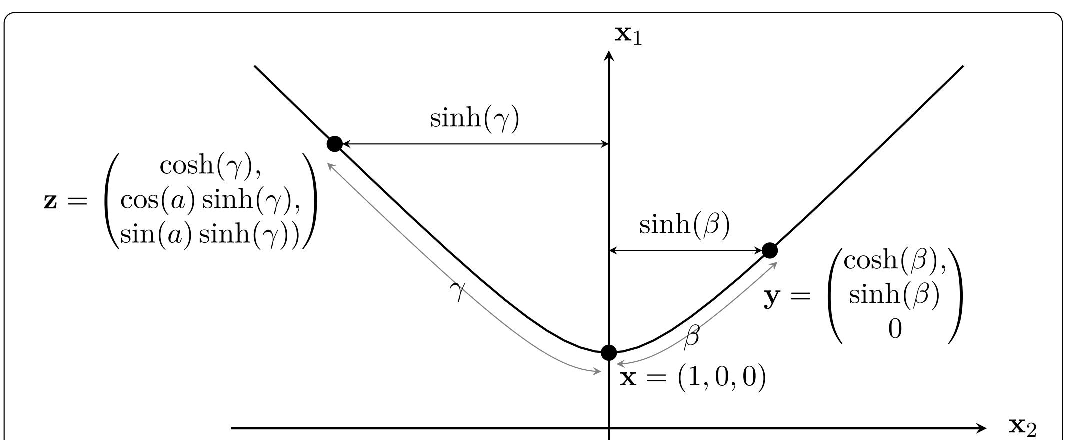
Theorem 10. *(Main theorem of hyperbolic trigonometry) For distinct points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathcal{H}^n$, let*

$$\begin{aligned} \alpha &= d_{\mathcal{H}^n}(\mathbf{z}, \mathbf{y}) \\ \beta &= d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{z}) \\ \gamma &= d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y}) \\ a &= \text{ the hyperbolic angle of } \Delta\mathbf{xyz} \text{ at } \mathbf{x}. \end{aligned}$$



Then

$$\cosh\alpha = \cosh\beta \cdot \cosh\gamma - \sinh\beta \cdot \sinh\gamma \cdot \cos a$$



cut out and fold along x_1 axis; a is the angle between the two sides of the paper.
See how the distance between $\mathbf{z}$ and $\mathbf{y}$ varies as this angle varies.

22. Lecture 22: $d_{\mathcal{H}^n}$ is a metric, and intro to projective space

Triangle inequality in hyperbolic space

Proof of Main theorem of hyperbolic trigonometry. Since angles and lengths are invariant under Lorentz transformations we may assume $\mathbf{x}, \mathbf{y}, \mathbf{z}$ have the form as in Lemma 29.

$$\begin{array}{rlrrr} \mathbf{x} & = (& 1, & 0, & 0) \\ \mathbf{y} & = (& \cosh(\beta), & \sinh(\beta), & 0) \\ \mathbf{z} & = (& \cosh(\gamma), & \cos(\theta)\sinh(\gamma), & \sin(\theta)\sinh(\gamma)) \end{array}$$

Since we are in the special case where $\mathbf{x} = (1, 0, 0)$, the angle between the sides $\mathbf{xz}$ and $\mathbf{xy}$ is the angle between the corresponding planes, which is $a = \theta$.
Now we can compute α, β, γ in terms of the coordinates of $\mathbf{x}, \mathbf{y}, \mathbf{z}$:

$$\begin{aligned} -\langle \mathbf{x}, \mathbf{y} \rangle_L &= \cosh(\beta) \\ -\langle \mathbf{x}, \mathbf{z} \rangle_L &= \cosh(\gamma) \\ -\langle \mathbf{z}, \mathbf{y} \rangle_L &= \cosh(\beta)\cosh(\gamma) - \cos(a)\sinh(\beta)\sinh(\gamma) \end{aligned}$$

Combining these gives the required result:

$$\cosh(\alpha) = \cosh(\gamma)\cosh(\beta) - \sinh\beta \cdot \sinh\gamma \cdot \cos a$$

□

Corollary 8. *(The triangle inequality for the hyperbolic metric.) For $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathcal{H}^n$, we have*

$$d_{\mathcal{H}^n}(\mathbf{y}, \mathbf{z}) \le d_{\mathcal{H}^n}(\mathbf{y}, \mathbf{x}) + d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{z})$$

Proof. Use notation of Theorem 10. Since $\cos(a) \ge -1$

$$\cosh(\alpha) \le \cosh(\gamma)\cosh(\beta) + \sinh\beta \cdot \sinh\gamma = \cosh(\beta + \gamma).$$

Since cosh is increasing on $[0, \infty)$, this implies that $\alpha \le \beta + \gamma$, the required result. □

Corollary 9. *The hyperbolic metric is a metric.*

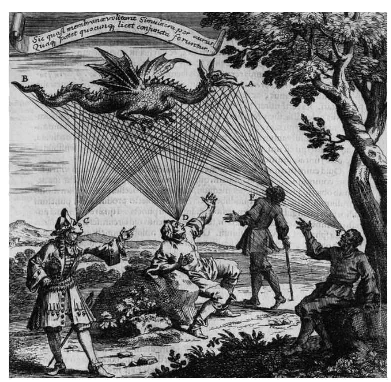
Proof. Recall that

$$d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y}) = \cosh^{-1}(-\langle \mathbf{x}, \mathbf{y} \rangle_L),$$

which we showed was well defined in Lemma 20.
Non degeneracy: If $d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y}) = 0$, then $-\langle \mathbf{x}, \mathbf{y} \rangle_L = 1$. In Lemma 20, we showed that $-\langle \mathbf{x}, \mathbf{y} \rangle_L \ge 1$. Since Lorentz transformations preserve the Lorentz inner product, and we saw there is a Lorentz transformation mapping $\mathbf{x}$ and $\mathbf{y}$ to the form in Lemma 29, where $\mathbf{x} = (1, 0, 0)$ and $-\langle \mathbf{x}, \mathbf{y} \rangle_L = \cosh(\beta)$ which is only 1 if $\beta = 0$ and $\mathbf{x} = \mathbf{y}$.
Symmetry: This is immediate from the fact that the Lorentz inner product is symmetric on $\mathbb{R}^{n+1}$.
Triangle inequality: This is Corollary 8.
Hence $d_{\mathcal{H}^n}$ satisfies the requirments to be a metric. □

Next: projective geometry!

(this is a wikipeadia picture from a page about emission theory, so not really about projective geometry, but it reminded me of projective geometry).



Projective geometry

We take 3 views of this subject: Algebraic; Classical perspectivities; Combinatorial/Axiomatic.

22.1. Algebraic Definition of Projective Space.

DEFINITION 34. *Let V be an $n+1$ dimensional vector space over a field k, so $V \cong k^{n+1}$. Let $V^* = V \setminus \{\mathbf{0}\}$. (This definition also works with $V = A^{n+1}$ for any division ring A. A division ring is a ring where $ax = b$ has a solution $x \in A$ for any non-zero $a, b \in A$. Most of the time we take $k = \mathbb{R}$.)*
Define an equivalence relation on V^*, by defining for $\mathbf{x}, \mathbf{y} \in V$,

$$\mathbf{x} \sim \mathbf{y} \iff k\mathbf{x} = k\mathbf{y}.$$

Exercise: check that this is an equivalence relation, namely reflexive, transitive and symmetric.
The projective space of V is the set of equivalence classes in V^* under this relation:

$$\mathbb{P}^n(k) := \mathbb{P}(V) := V^*/\sim$$

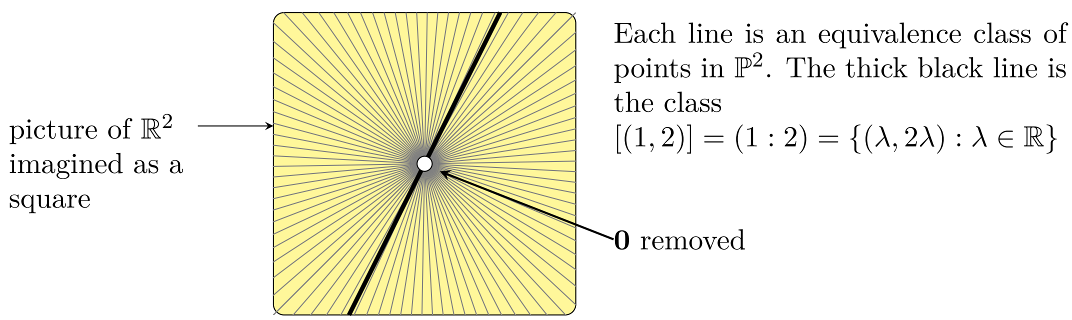
For $\mathbf{x} = (x_1, x_2, \ldots, x_{n+1}) \in V^*$, we denote the equivalence class of $\mathbf{x}$ by

$$[\mathbf{x}] = (x_1 : x_2 : \cdots : x_{n+1}) = \{\mathbf{v} \in V^* : \mathbf{v} \sim \mathbf{x}\}$$

When $k = \mathbb{R}$, we write $\mathbb{P}^n = \mathbb{P}(\mathbb{R}^{n+1})$, and call this n-dimensional real projective space.

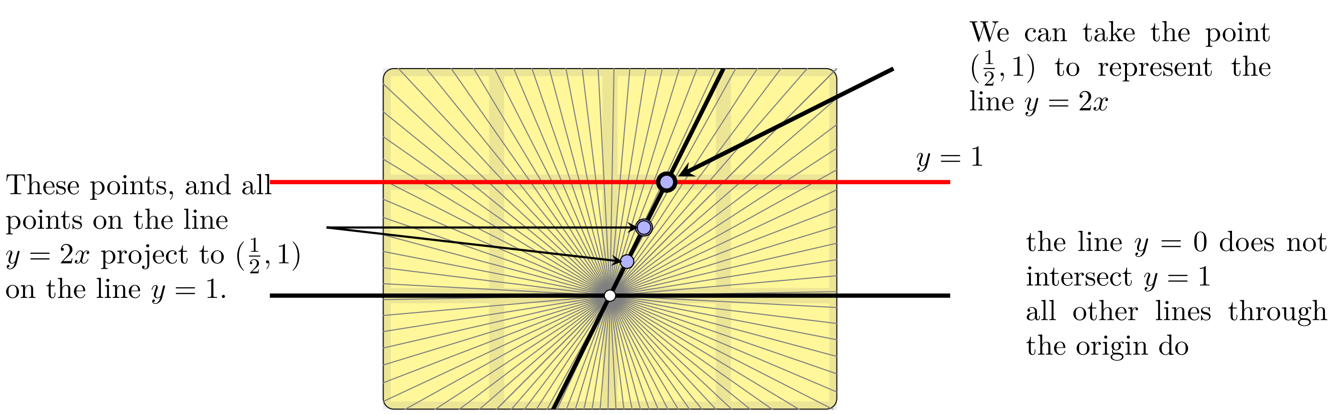
LEMMA 30. *There is a bijection between $\mathbb{P}(V)$ and the set of lines through the origin in V.*
For $\mathbf{x}, \mathbf{y} \in V^*$, $\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda\mathbf{y}$ for some $\lambda \in k \setminus \{0\}$.

PROOF. Exercise. $\square$

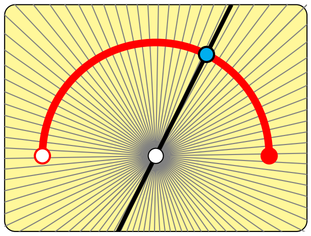


The projective line $\mathbb{P}^1(\mathbb{R})$

22.2. Equivalence class representatives for the projective line. Consider the red line in this figure, given by $y = 1$. Then (nearly) every equivalence class of lines in $\mathbb{P}^2$ intersects this line exactly once. E.g., the line $[(1,2)]$ intersects at $(\frac{1}{2}, 1)$. All points except $(1:0)$ can be written in the form $[(x,1)]$ for some $x \in \mathbb{R}$. So $\mathbb{P}^1 = \{[(x:1)] : x \in \mathbb{R}\} \cup \{(1:0)\}$, and we can think of $(1:0)$ as a "point at infinity".

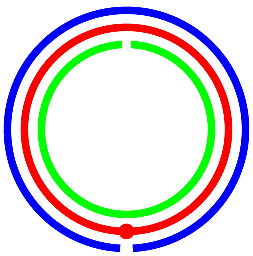


22.3. Semi-circle identified with $\mathbb{P}^1$.



Points on the semi-circle correspond to lines through $\mathbf{0}$. E.g., the point at $(\cos(\theta), \sin(\theta))$ corresponds to the line $y = \tan(\theta)x$

In the above picture, the red and white dots at the end of the semi circle correspond to the same line, $y = 0$, so these points are identified. Topologically (in topology, you can bend and stretch but not tear a geometric object), $\mathbb{P}^1$ is a circle, which you can see after identifying these points, i.e., stretching round and gluing together as in the next picture. We can also think of the $\{x = 1\}$ and $\{y = 1\}$ projections of $\mathbb{P}^1$ as real lines that together cover the whole of of $\mathbb{P}^1$, as in the following figure:



The red line represents all of $\mathbb{P}^1$.
The blue line represents the part of $\mathbb{P}^1$ that projects to $\{y = 1\}$
The green line represents the part of $\mathbb{P}^1$ that projects to $\{x = 1\}$
This is a schematic diagram, only topologically equivalent to $\mathbb{P}^1$.

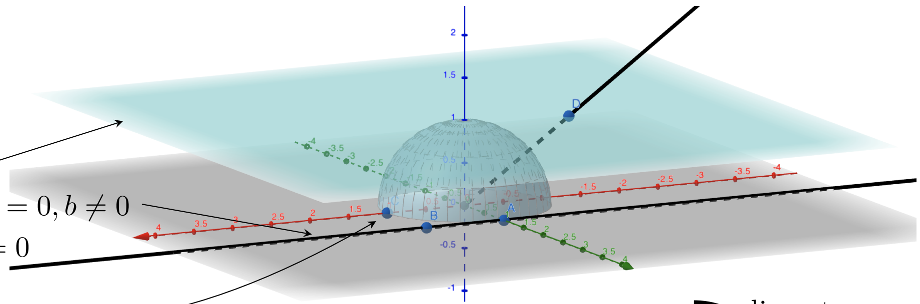
23. Lecture 23: The projective plane; real and Fano.

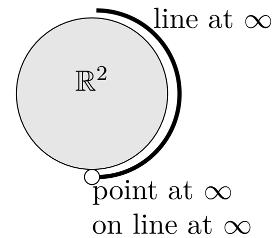
The projective plane $\mathbb{P}^2(\mathbb{R})$

Every point in $\mathbb{P}^2$ can be represented by a point on S^2, but this includes each point twice. So we only need the upper hemisphere $Z \geq 0$. And we only need one half of the $S^1 = S^2 \cap \{z = 0\}$.
Here is a Geogebra picture `https://www.geogebra.org/3d/katc2xrq`

Every point $(a : b : c)$ in $\mathbb{P}^2$ is represented by unique point on either

$$\begin{cases} \text{the plane} & Z = 1, \text{ if } c \neq 0 \\ \text{the line} & Z = 0, Y = 1 \text{ if } c = 0, b \neq 0 \\ \text{the point} & (1,0,0) \text{ if } b = c = 0 \end{cases}$$



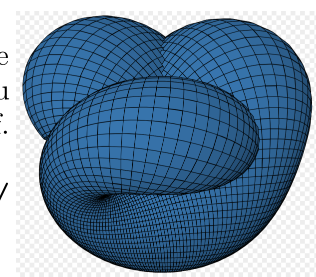


$$\mathbb{P}^2 \cong \overbrace{\{Z = 1\}}^{\mathbb{R}^2} \cup \overbrace{\{Z = 0, Y = 1\}}^{\mathbb{R}} \cup \overbrace{\{(1,0,0)\}}^{\text{point}}$$

You can think of $\mathbb{P}^n$ as $\mathbb{R}^n$ with a point added at ininfity for every radial direction in $\mathbb{R}^n$.

Topology of the projective plane $\mathbb{P}^2(\mathbb{R})$

If you take the hemisphere, and distort it to "glue together" points on opposite side of the circular boundary on $Z = 0$, which lie on the same line in $\mathbb{R}^3$, you get a closed surface. This surface only can be viewed in $\mathbb{R}^3$ if it crosses itself. One model is called "Boy's surface".
(Picture from Wikipedia, Ggleizer `https://commons.wikimedia.org/wiki/File:BoysSurfaceKusnerBryant.svg`)



Lines and linear subspaces in projective geometry

For a vector space V, over a field k, there is a projection map from $V^* = V \setminus \{\mathbf{0}\}$ to $\mathbb{P}(V)$ given by

$$\pi : V^* \to \mathbb{P}(V)$$

$$\mathbf{v} \mapsto k\mathbf{v}$$

Definition 35. *If V is a vector space V, with a $k+1$ dimensional vector subspace W, then the <u>k-dimensional projective linear subspace</u> $\mathbb{P}(W)$ of $\mathbb{P}(V)$ is the image under π of W in $\mathbb{P}(V)$, that is*

$$\mathbb{P}(W) := \{\mathbf{x} \in \mathbb{P}(V) : \mathbf{x} \subset W\} = \pi(W)$$

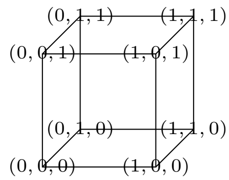
Warning: the dimension of $\mathbb{P}(W)$ is one less than the dimension of W.

Definition 36. *Let W be a subvector space of V.*

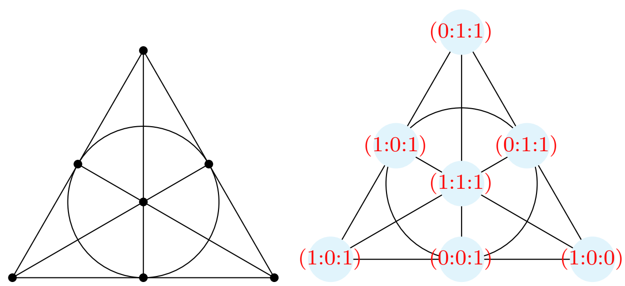
$$\mathbb{P}(W) \text{ is a } \begin{cases} \textit{point} & \textit{if } \dim(W) = 1 \\ \textit{line} & \textit{if } \dim(W) = 2 \\ \textit{plane} & \textit{if } \dim(W) = 3 \end{cases}$$

Lecture 23/24: The Fano plane and Projectivities

23.1. Fano Plane. In this section, we discuss the smallest possible projective plane.
Let V be the vector space $(\mathbb{F}_2)^3$ over the finite field of two elements, $\mathbb{F}_2 = \{0, 1\}$. What are the points and lines in $\mathbb{P}(V)$?
The elements of V are $(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)$, which can be pictured as points at the vertices of a cube:



The points of $\mathbb{P}(V)$ are given by the one dimensional subspaces of V, which have the form $\mathbb{F}_2\mathbf{v} = \{0, \mathbf{v}\}$ for non zero $\mathbf{v}$. So there are 7 of these.
Lines are given by the two dimensional subspaces. Each of these has the form $\{0, \mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}\}$ for some $\mathbf{v}, \mathbf{w} \in V$. For example, the plane $P = \{(0,0,0), (0,1,0), (1,0,1), (1,1,1)\}$. So, there are $\binom{7}{2}$ ways to choose two points $\mathbf{w}$ and $\mathbf{v}$, but any three non-zero points in P span P, so this would count the planes 3 times, so the number of planes is $\frac{1}{3}\binom{7}{2} = 7$.
It's not a coincidence that this is the same as the number of points, since each plane can also be defined by the vector it is orthogonal to, e.g., P above is orthogonal to $(1,0,1)$.
We can draw a diagram showing the points and lines as follows. Right shows labels for the points, with a vector in V which generates each point in $\mathbb{P}(V)$. This is called the **Fano plane**.



23.2. Working in other finite fields. If we work in the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, which has p elements, each line in $\mathbb{F}_p^n$ has elements $\{\mathbf{0}, \mathbf{v}, 2\mathbf{v}, 3\mathbf{v}, \ldots, (p-1)\mathbf{v}\}$, i.e., There are $p-1$ non-zero points on any line. Since $\mathbb{F}_p^n \setminus \{\mathbf{0}\}$ has $p^n - 1$ points, this means it contains $(p^n-1)/(p-1)$ distinct lines, which are points of $\mathbf{P}^{n-1}(\mathbb{F}_p)$.
To determine the number of lines in $\mathbf{P}^{n-1}(\mathbb{F}_p)$, we have to count the number of planes in $\mathbb{F}_p^n$. Each plane is spanned by 2 a basis of non-zero vectors, $\mathbf{u}, \mathbf{v}$, which would be $(p^n-1)(p^n-p)$ sets of ordered basis, because once the first, $\mathbf{u}$ is choosen, that rules out p choices for $\mathbf{v}$, since $\mathbf{v}$ can not be a multiple of $\mathbf{u}$. but each plane would be counted multiple times. The number of times a plane is counted is equal to the number of different choices of (ordered) basis. Once one non-zero basis vector is chosen, there are are $p^2 - p$ choices for the remainder, so that's a total of $(p^2-1)(p^2-p)$ possible basis. So this means there are $\frac{(p^n-1)(p^n-p)}{(p^2-1)(p^2-p)}$ lines.

Definition 37. *For a vector space V and a subset $\Sigma \subset \mathbb{P}(V)$, the projective cone of Σ is given by*

$$\widetilde{\Sigma} := \bigcup_{L \in \Sigma} L = \{\mathbf{v} \in V : \mathbf{v} \in L \subset V \text{ for some line } L \in \Sigma\},$$

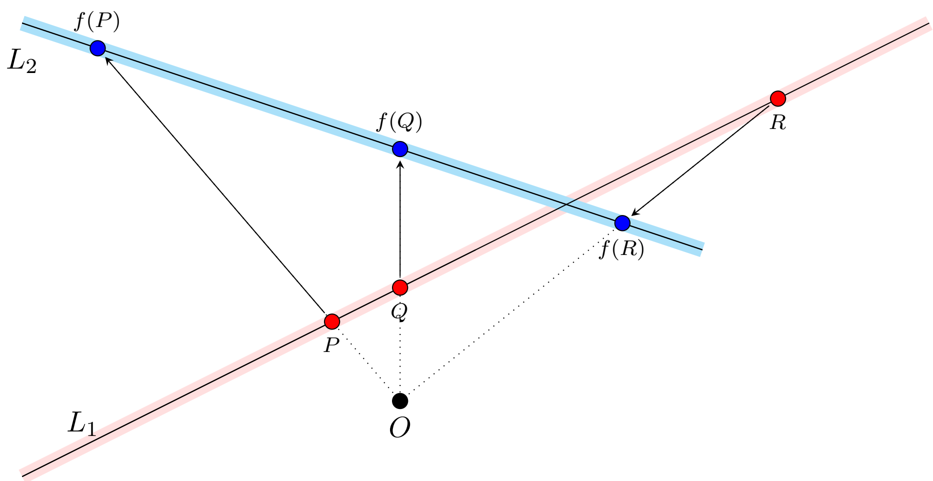
i.e., elements of Σ may be considered to be lines, and we take their union.
The linear span or span of Σ is is the smallest projective linear subspace of $\mathbb{P}(V)$ containing Σ,

$$\langle \Sigma \rangle := \mathbb{P}(span(\widetilde{\Sigma})),$$

24. Lecture 24: Projective transformations and projective frames of reference

Perspectivities

DEFINITION 38. *A perspectivity f is a map between two distinct hyperplanes (linear subspaces of dimension $n-1$), Π_1 and Π_2 in $\mathbb{P}^n$, given by projection from a point $O \notin \Pi_1 \cup \Pi_2$. That is, if $P \in \Pi_1$, then $f(P) \in \Pi_2$ is the point such that $0, P, f(P)$ all lie on the same line.*



(Next week we will prove this is well defined, i.e., there is a unique intersection point.)

Projective linear maps

DEFINITION 39. *The real projective general linear group $PGL(n) = PGL(\mathbb{R}^n)$ is the group of invertible real matrices up to scalar muliplication, that is the equivalence classes of the set of matrices in $GL(n)$ with the relation*

$$A \sim B \iff A = \lambda B \text{ for some } \lambda \in \mathbb{R}^*$$

Example, in $PGL(2)$, we have

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

DEFINITION 40. *For $A \in PGL(n+1)$ the map*

$$T_A : \mathbb{P}^n \to \mathbb{P}^n$$
$$[\mathbf{v}] \mapsto [A\mathbf{v}]$$

is called a projective transformation, or a projectivity or a projective linear map.

T_A is well defined, since for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$, if $\mathbf{x} \sim \mathbf{y}$, that is, if $[\mathbf{x}] = [\mathbf{y}]$, then $\mathbf{x} = \lambda \mathbf{y}$ for some $\lambda \in \mathbb{R}$, so $A\mathbf{x} = A\lambda\mathbf{y} = \lambda A\mathbf{y}$, so $[A\mathbf{x}] = [A\mathbf{y}]$.
Also, if $A \sim B$, then $T_A = T_B$, since if $A = \lambda B$, then $[A\mathbf{x}] = [\lambda A\mathbf{x}] = [B\mathbf{x}]$. So, projective linear maps of $\mathbb{P}^n$ correspond to elements of $PGL(n+1)$.

Big result

Perspectivities are the same thing as projective linear maps.

Projective ~~basis~~ frame of reference

$\mathbb{P}^n$ is not a vector space, so it doesn't have a basis, but it's very close to a vector space. Instead of a basis, we have a projective frame of reference.

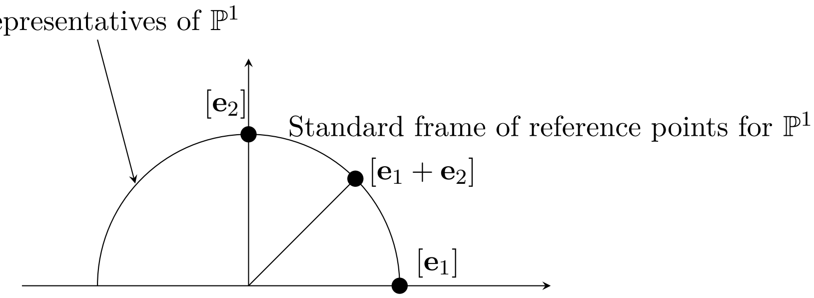
DEFINITION 41. *For a point $P \in \mathbb{P}(V)$, the "lift" of P, is a non-unique vector $\widetilde{P}$ is given by:*

$$\widetilde{P} = \textit{any choice of non zero } \mathbf{v} \in P$$

For $s+1$ points $P_0, P_1, \ldots, P_s$ in $\mathbb{P}(V)$, the dimension of $\langle\{P_0, P_1, \ldots, P_s\}\rangle = \pi(Span(\widetilde{P_0}, \ldots, \widetilde{P_0})$ is at most s, (by definition of projective dimension).
We say $P_0, P_1, \ldots, P_s$ are linearly independent if $\dim \langle\{P_0, P_1, \ldots, P_s\}\rangle = s$.

DEFINITION 42. *A projective frame of reference for $\mathbb{P}^n$ is an ordered set of $n+2$ points, $P_0, P_1, \ldots, P_{n+1} \in \mathbb{P}^n$, any $n+1$ of which are linearly independent (and so span $\mathbb{P}^n$)*

DEFINITION 43. *The standard frame of reference for $\mathbb{P}^n$ is given by $[\mathbf{e}_1], \ldots, [\mathbf{e}_{n+1}]$ together with $[\mathbf{e}_1 + \cdots + \mathbf{e}_{n+1}]$, where $\mathbf{e}_i$ are the standard basis for $\mathbb{R}^{n+1}$.*



Example: Any three distinct points P_1, P_2, P_3 in $\mathbb{P}^1$ is a frame of reference for $\mathbb{P}^1$, since being distinct means that $\widetilde{P_1}$, $\widetilde{P_2}$, $\widetilde{P_3}$ are pairwise linearly independent.
Motivation: Why do we need $n+2$ points? This is because $n+2$ points is enough to determine a projecive linear map of $\mathbb{P}^n$, but $n+1$ is not. This might be suprising, since the images of n points is enough to determine a linear map on $\mathbb{R}^n$.
Example: There are infinitely many projective linear maps $\mathbb{P}^1 \to \mathbb{P}^1$ with

$$(1:0) \mapsto (1:0)$$
$$(0:1) \mapsto (0:1)$$

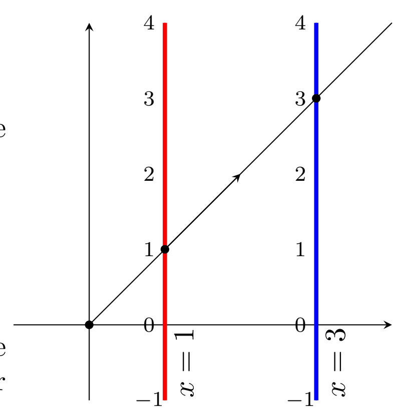
namely, any diagonal map, of the form

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

However, these maps are not all the same map, since we have

$$A : (1:0) \mapsto (1:0)$$
$$A : (0:1) \mapsto (0:1)$$
$$A : (1:1) \mapsto (\alpha:\beta)$$

If we fix the image of $(1:1)$, then we uniquely determine the projective linear map.



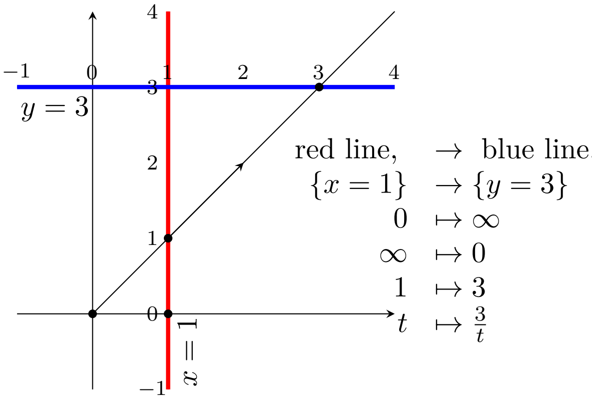
Example: The radial projection, with centre $\mathbf{0}$, from the line $x = 1$ to the line $x = 3$ takes a point with $y = 1$ to a point with $y = 3$. We can paramaterise points on these lines by their y-coordinates. If we complete these lines to form copies of $\mathbb{P}^1$, with point with parameter y maping to $(y:1)$, then the map is given by $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{P}^1 \to \mathbb{P}^1$

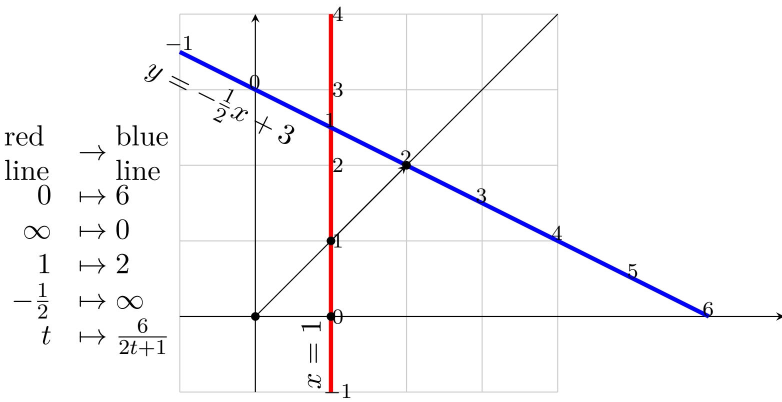
Example: The map $t \mapsto \frac{1}{t}$ from $\mathbb{R}$ to $\mathbb{R}$ can be extended to a map on $\mathbb{P}^1$, given by

$$\left[\begin{pmatrix} t \\ 1 \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ t \end{pmatrix}\right],$$

extended to a map on $\mathbb{P}^1$ by allowing the map to apply to $(1:0)$. This map has the effect of switching 0 and ∞, thought of as the points $(0:1)$ and $(1:0)$.

Any map $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : \mathbb{P}^1 \to \mathbb{P}^1$, for $\alpha, \beta \neq 0$ also switches $(0:1)$ and $(1:0)$, and corresponds to $t \mapsto \frac{\alpha}{\beta t}$.
By specifying the image of $(1:1)$, the map is uniquely determined.
The diagram on the right shows a projection from $\{x = 1\}$ to $\{y = 3\}$, which corresponds to the map $\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} : \mathbb{P}^1 \to \mathbb{P}^1$ sending $(1:1) \mapsto (3:1)$



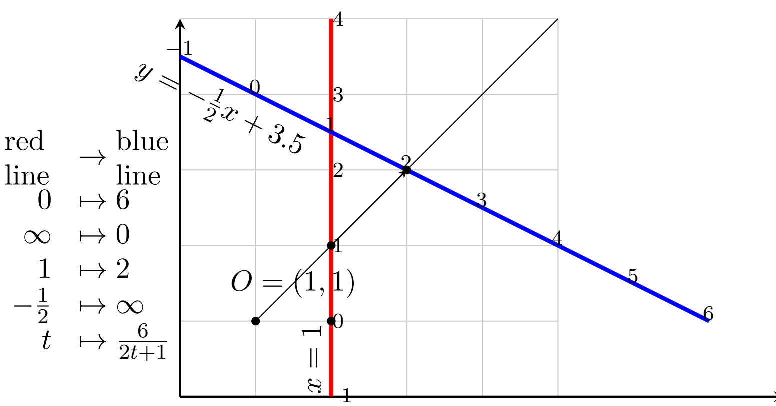


Example: In the figure on the right, we project from the line $\{x = 1\}$ to the line $\{y = -\frac{1}{2}x + 3\}$.
We parameterise $\{x = 1\}$ by the y coordinate, and $\{y = -\frac{1}{2}x + 3\}$ by the x coordinate, though we could choose different parameterisations.
With respect to this parameterisation, as a map from $\mathbb{P}^1$ to $\mathbb{P}^1$, this map is given by the matrix $\begin{pmatrix} 0 & 6 \\ 2 & 1 \end{pmatrix}$, and corresponds to $t \mapsto \frac{6}{2t+1}$ as a map on $\mathbb{R}$

The maps in these examples have been written as maps from $\mathbb{P}^1$ to $\mathbb{P}^1$, but drawn as maps from $\mathbb{R} \to \mathbb{R}$. To extend from $\mathbb{R}$ to $\mathbb{P}^1$, we should embed each copy of $\mathbb{R}$ in a copy of $\mathbb{P}^1$, which can be acheived by replacing $\mathbb{R}^2$ in these figures by $\mathbb{P}^2$.

Example

In all the above examples, the coordinates could be translated, and we would still have a projection. That is, you don't have to project from zero, e.g.:



Example: In the figure on the right, we project from the point $(1, 1)$ from the line $\{x = 2\}$ to the line $\{y = -\frac{1}{2}x + 3.5\}$.
We parameterise $\{x = 1\}$ by the y coordinate-1, and $\{y = -\frac{1}{2}x + 3.5\}$ by the x coordinate-1, though we could choose different parameterisations.
With respect to this parameterisation, as a map from $\mathbb{P}^1$ to $\mathbb{P}^1$, this map is given by the matrix $\begin{pmatrix} 0 & 6 \\ 2 & 1 \end{pmatrix}$, and corresponds to $t \mapsto \frac{6}{2t+1}$ as a map on $\mathbb{R}$

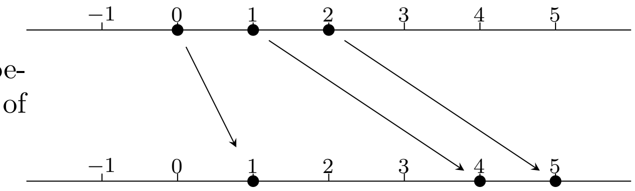
Note, if we were just told that in the above example, 1 (on the red line) maps to 2 (on the blue line) and 0 (on the red line) maps to 6 (on the blue line), by considering the intersection of the lines between these points, we see the projection must be from $(1, 1)$.
If we consider $\mathbb{R}^2$ extended to $\mathbb{P}^2$, by assuming this plane is the $Z = 1$ plane in $\mathbb{R}^3$, and embedding as $(x, y) \mapsto (x : y : 1)$, then we have $O = (1 : 1 : 1)$.
Quiz: Given this embedding, the blue line in the above extends to the line L_2 given by $Y = -X/2 + 3.5Z$ in $\mathbb{P}$. which point is in L_2, but not in the set of points where $Z = 0$?

An examples of a perspectivity

We will show that any three distinct points P_1, P_2, P_3 in $\mathbb{P}^1$ can be projected onto any other three distinct points Q_1, Q_2, Q_3. This is called **three transitivity** of the action of $PGL(2)$ on $\mathbb{P}^1$. This result will follow from Theorem 13 which shows that there is some A maping the standard frame of reference to P_1, P_2, P_3, and B maps the standard frame of reference to Q_1, Q_2, Q_3, then BA^{-1} maps P_1, P_2, P_3 to Q_1, Q_2, Q_3.

Example:

How do I project between these set of points???



Algebraic description of problem: Let's just write the algebra first. I.e., suppose we just want a projecive linear transformation given by a matrix $A \in PGL(2)$, and with

$$T_A(0:1) = (1:1), \quad T_A(1:1) = (4:1), \quad T_A(2:1) = (5:1)$$

This means we want A so that

$$A\begin{pmatrix}0\\1\end{pmatrix} \sim \begin{pmatrix}1\\1\end{pmatrix}, \quad A\begin{pmatrix}1\\1\end{pmatrix} \sim \begin{pmatrix}4\\1\end{pmatrix}, \quad A\begin{pmatrix}2\\1\end{pmatrix} \sim \begin{pmatrix}5\\1\end{pmatrix}$$

where $\sim$ means equal up to non-zero scalar multiplication. i.e., we want some A and some constants $\lambda_1, \lambda_2, \lambda_3$ with

$$A\begin{pmatrix}0\\1\end{pmatrix} = \lambda_1\begin{pmatrix}1\\1\end{pmatrix}, \quad A\begin{pmatrix}1\\1\end{pmatrix} = \lambda_2\begin{pmatrix}4\\1\end{pmatrix}, \quad A\begin{pmatrix}2\\1\end{pmatrix} = \lambda_3\begin{pmatrix}5\\1\end{pmatrix}$$

You can either solve this directly, or by working via the standard frame of reference. It's probably easier to work via the standard frame of reference.

This means finding A as a composition of two maps, say B^{-1} and C, with

$$\{0,1,2\} \xrightarrow{B^{-1}} \{\infty, 0, 1\} \xrightarrow{C} \{1,4,5\}$$

Or, written as projective points, with the inclusion $\mathbb{R} \to \mathbb{P}^1$ given by $t \mapsto (t:1)$.

$$\{(0:1),(1:1),(2:1)\} \xrightarrow{B^{-1}} \{(1:0),(0:1),(1:1)\} \xrightarrow{C} \{(1:1),(4:1),(5:1)\}$$

(Note, we could also choose to include $\mathbb{R}^1$ in $\mathbb{P}^1$ as $(1:t)$, or many other ways; eventually this will lead to the same formula for the map on $\mathbb{R}^1$.)

Step 1: Find the matrix B:

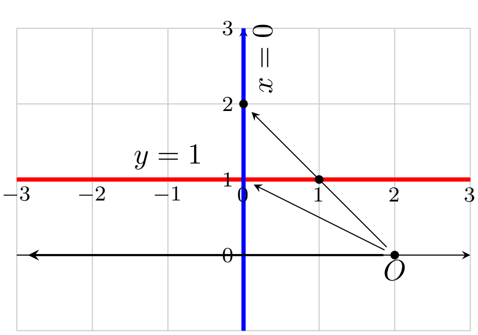
Since B maps $(1:0)$ to $(0:1)$ and $(0:1)$ to $(1:1)$, it has the form:

$$B = \begin{pmatrix}0 & \beta\\ \alpha & \beta\end{pmatrix}$$

Since B maps $(1:1)$ to $(2:1)$, we require $\beta : \alpha+\beta = 2:1$, i.e., $2(\alpha+\beta) = \beta$, so $\beta = -2\alpha$, and the matrix is

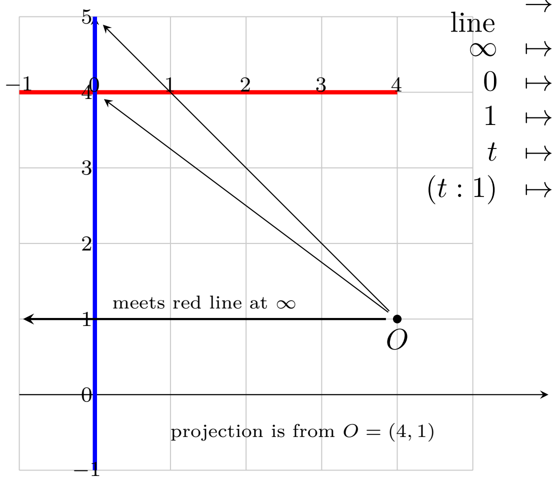
$$B = \begin{pmatrix}0 & -2\\ 1 & -2\end{pmatrix}$$

You don't have to represent this geometrically, but one interpretation is as on the right, which is a projection from the point $O = (2,0)$ between the lines $\{y=1\}$ and $\{x=0\}$.



red line	$\to$	blue line
∞	$\mapsto$	0
0	$\mapsto$	1
1	$\mapsto$	2
t	$\mapsto$	$\frac{-2}{t-2}$
$(t:1)$	$\mapsto$	$(-2 : t-2)$
$(t:s)$	$\mapsto$	$(-2s : t-2s)$

Step 2: Find the matrix C:



$$\begin{array}{rl} \text{red line} & \to \text{blue line} \\ \infty & \mapsto 1 \\ 0 & \mapsto 4 \\ 1 & \mapsto 5 \\ t & \mapsto \frac{t+4}{t+1} \\ (t:1) & \mapsto (t+4 : t+1) \end{array}$$

To project $[\mathbf{e}_1], [\mathbf{e}_2], [\mathbf{e}_1 + \mathbf{e}_2]$ to $(1:1), (4:1), (5:1)$, we need a matrix of the form $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, since then $T_A([\mathbf{e}_1]) = (1:1)$, and $T_A([\mathbf{e}_2]) = (4:1)$. To obtain $T_A((1:1)) = (5:1)$, we need $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, i.e., $\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, i.e.,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

so the required element of $PGL(2)$ is $\begin{pmatrix} 1 & -16 \\ 1 & -4 \end{pmatrix}$ Note that we have ignored the determinant factor -3, when computing the inverse of $\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$, since elements of $PGL(2)$ are only defined up to multiplcation by non-zero scalar.

Step 3: Compute $A = CB^{-1}$
Now to map $(0:1), (1:1), (2:1)$ to $(1:1), (4:1), (5:1)$, we take

$$A = CB^{-1} = \begin{pmatrix} 1 & -16 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -16 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 1 & 1 \end{pmatrix}$$

So, the final solution to the problem of finding the matrix is:

$$A = \begin{pmatrix} 7 & 1 \\ 1 & 1 \end{pmatrix}$$

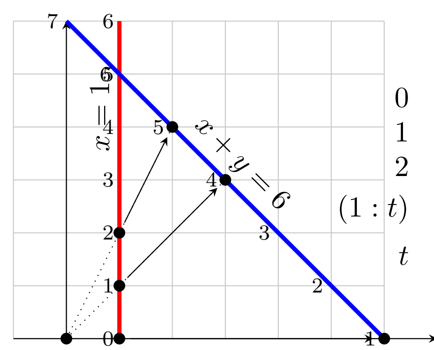
The final solution to finding a perspectivity mapping $0, 1, 2 \to 1, 4, 5$ is the map

$$t \mapsto \frac{1+7t}{1+t}$$

Note, the matrix computation is in $PGL(2)$, so any scale factors of the whole matrix are ignored.
We can check the result, e.g.,

$$T_A((2:1)) = \left[\begin{pmatrix} 7 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] = \left[\begin{pmatrix} 15 \\ 3 \end{pmatrix} \right] = (5:1)$$

Note, although the element of $PGL(2)$ is unique, the picture is not unique, because we could choose different parameterisations, or project from different points, to different lines. One example projection with this algberaic description is to the right.

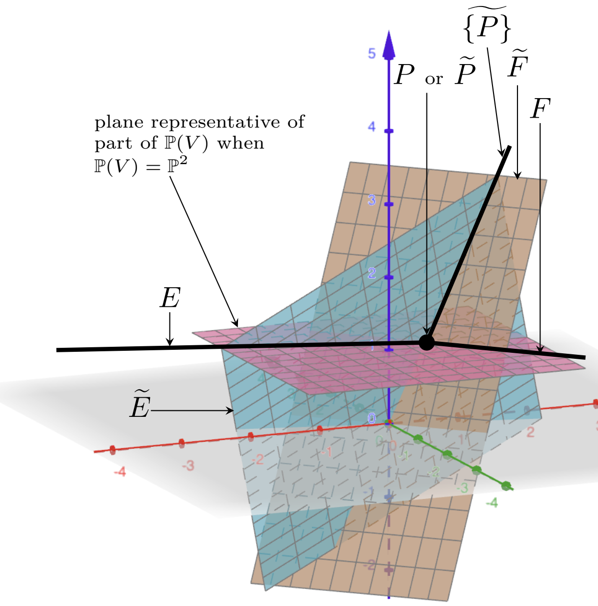


$$\begin{array}{rl} 0 & \mapsto 1 \\ 1 & \mapsto 4 \\ 2 & \mapsto 5 \\ (1:t) & \mapsto (1+t : 1+7t) \\ t & \mapsto \frac{1+7t}{1+t} \end{array}$$

In this examples, all we need to know is that a perspectivity is given by a projective linear map. Then this map can be determined by knowing the images of three points.
You don't have to draw the pictures. However, if you already know the lines and the point from which you are projecting, then a diagram may help you find the images of ∞, 0 and 1 (projectively $(1:0), (0:1), (1:1)$), and so then you only need to compute one matrix, and get A without having to compute two matrices and multiply.

25. Lecture 25: Projective dimension formula

Recall: For a projective linear subspace $E \subset \mathbb{P}^n$, with cone $\widetilde{E} \subset \mathbb{R}^{n+1}$, we have $E = \pi(\widetilde{E})$, and $\dim E = \dim \widetilde{E} - 1$.



Theorem 11. *For projective linear subspaces E and F of $\mathbb{P}(V)$,*

$$\dim(E \cap F) = \dim E + \dim F - \dim \langle E, F\rangle$$

with the convention $\dim \varnothing = -1$, and $\langle E, F\rangle$ is the linear subspace of $\mathbb{P}(V)$ spanned by E and F.

Proof. Let $n = \dim E$, $m = \dim F$ and $k = \dim E \cap F$. Then $\dim \widetilde{E} = n+1$ and $\dim \widetilde{F} = m+1$. We also have $\widetilde{E \cap F} = \widetilde{E} \cap \widetilde{F}$ (proof: exercise). Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}$ be a basis for $\widetilde{E} \cap \widetilde{F}$. Then we can extend this to a basis $\mathcal{B}_E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \ldots, \mathbf{v}_{n+1}\}$ of $\widetilde{E}$, and a basis $\mathcal{B}_F = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}, \mathbf{w}_{k+2}, \ldots, \mathbf{w}_{m+1}\}$ of $\widetilde{F}$. The vectors in $\mathcal{B} := \{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}, \mathbf{w}_{k+2}, \ldots, \mathbf{w}_{m+1}\}$ clearly span $\langle \widetilde{E}, \widetilde{F}\rangle$. Suppose $\sum_{i=1}^{n+1} a_i\mathbf{v}_i + \sum_{i=k+1}^{m+1} b_i\mathbf{w}_i = \mathbf{0}$, then $\sum_{i=1}^{n+1} a_i\mathbf{v}_i = -\sum_{i=k+1}^{m+1} b_i\mathbf{w}_i$, so since $\sum_{i=1}^{n+1} a_i\mathbf{v}_i \in \widetilde{E}$ and $\sum_{i=k+1}^{m+1} b_i\mathbf{w}_i \in \widetilde{F}$, we have $\sum_{i=k+1}^{n+1} b_i\mathbf{w}_i \in \widetilde{E} \cap \widetilde{F}$, and so can be written in terms of the basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$, but since this gives a relationship between elemtents of the basis $\mathcal{B}_F$, this must be the trivial relationship, since $\mathcal{B}_F$ is a basis, so all the coefficients must be zero. So $\mathcal{B}$ must be a basis for $\mathrm{span}(\widetilde{E}, \widetilde{F})$, and so

$$\dim(\mathrm{span}(\widetilde{E}, \widetilde{F})) = \overbrace{(n+1)}^{\mathbf{v}_1,\ldots,\mathbf{v}_{n+1}} + \overbrace{(m+1)-(k+1)}^{\mathbf{w}_{k+2},\ldots,\mathbf{w}_{m+1}}$$

$$\Rightarrow \dim\langle E, F\rangle + 1 = (\dim E + 1) + (\dim F + 1) - (\dim E \cap F + 1)$$

and so the result follows. $\square$

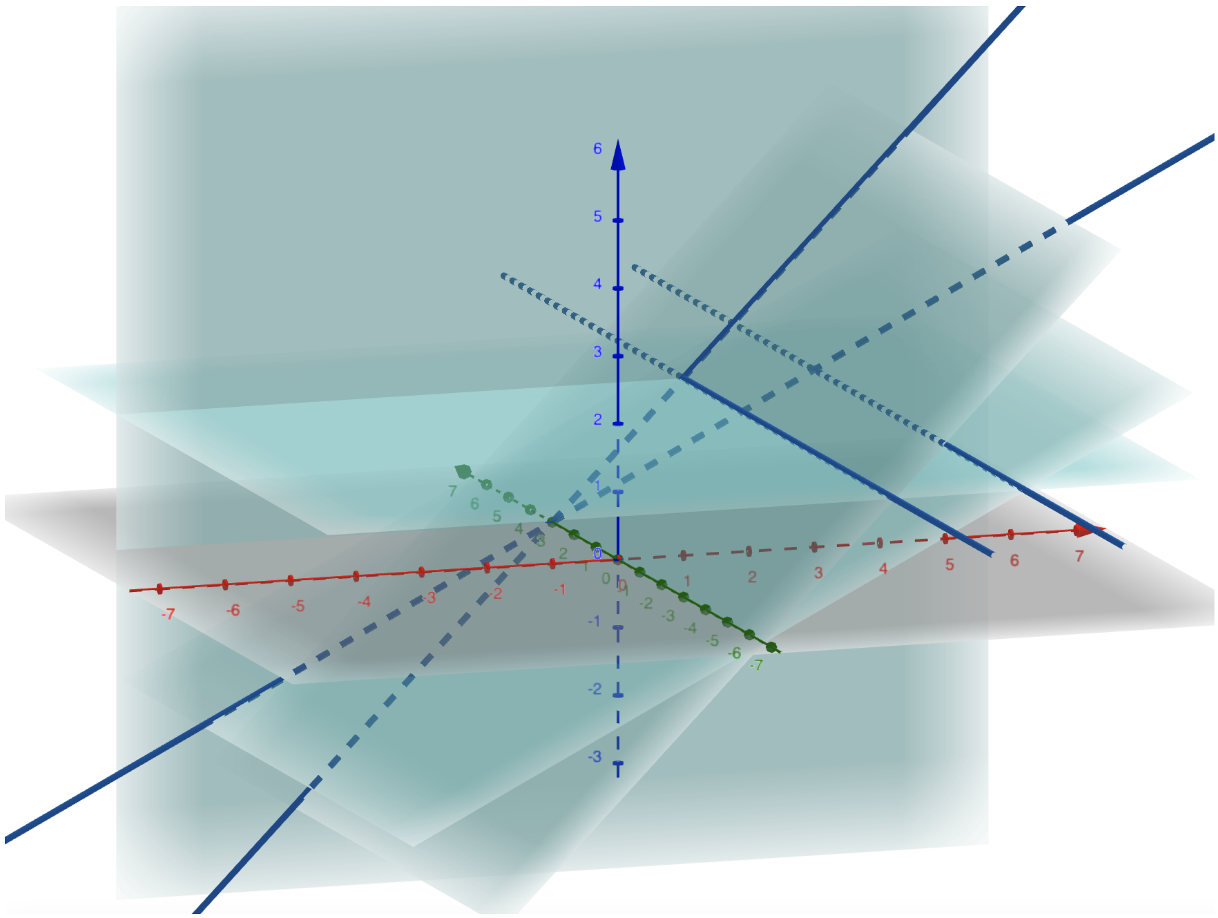
Remarks: (probably for after lecture reading)
Quiz: What is the corresponding result if E and F are replaced by vector subspaces U and V of $\mathbb{R}^n$?
Note: We have not defined the concept of dimension for subspaces which are not projective linear subspaces, and this concept will not be covered in the course. So, while there is a comparable statement about linear subspaces of $\mathbb{R}^n$, with an inequality instead of equality, we are not going to talk about that.
Example: If $E = L_1$ and $F = L_2$ are lines in $\mathbb{P}^2$, then they have projective dimension 1. In general, their intersection is a point $P = L_1 \cap L_2$, which has projective dimension 0. The union spans all of $\mathbb{R}^3$, which has dimension 2, so, provided E and F are distinct lines, we get the expected result, as shown in the above figure.

$$\overbrace{\dim(L_1 \cap L_2)}^{0} = \overbrace{\dim L_1}^{1} + \overbrace{\dim L_2}^{1} - \overbrace{\dim\langle L_1, L_2\rangle}^{2}$$

In the figure,(`https://www.geogebra.org/3d/vry2xuh5`) E is a line in $\mathbb{P}^2$, so it's drawn as a line in a plane representing $\mathbb{P}^2$, for psychological reasons. This is similar to how when we work with numbers modulo 5, we write things like 3, when really we mean $\{5\lambda + 3 : \lambda \in \mathbb{Z}\}$
Warning: The union of two vector subspaces is not a subspace. The span $\langle E, F\rangle$ is not the same as the union $E \cup F$. Because of this, we can't use an inclusion exculsion formula for the dimension of $D \cap E \cap F$ if D, E, F are projective linear subspaces. There is a formula, but you can't just automatically assume it's the same as the form you may have seen in probability or combinatorics. We can instead write, e.g., $\dim(E \cap F \cap D) = \dim E + \dim F + \dim D - \dim\langle F, D\rangle - \dim\langle E, F \cap D\rangle$, and more symmetric, but more complicated formulas are also possible.

Theorem 12. *Any two distinct lines L_1 and L_2 in $\mathbb{P}^2$ intersect in a point.*

Proof. The lines L_1 and L_2 are projections of planes through the origin, $\widetilde{L_1}$ and $\widetilde{L_2}$ in $\mathbb{R}^3$, to $\mathbb{P}^2$. Since $\widetilde{L_1}$ and $\widetilde{L_2}$ have dimension 2, and are distinct, their intersection must have dimension 1, by Theorem 11, since the dimension of the span of $\widetilde{L_1}$ and $\widetilde{L_2}$ is 3, since it can't be greater, since it's in $\mathbb{R}^3$, and can't be less, since $L_1 \neq L_2$. So the dimension of $L_1 \cap L_2$ is zero, and so $L_1 \cap L_2$ is a point. $\square$



Remarks: (Mostly for after lecture reading, but hopefully there is time in class to write the equations for the above pictured example, which shows the lines $X = Z$ and $X = 2Z$, which meet at the point $(0 : 1 : 0)$.)

Theorem 12 shows that lines always meet in $\mathbb{P}^2$. This can also be seen as a conseqence of the theorem we proved that spherical lines always intersect, since $\mathbb{P}^2$ can be considered to be the sphere S^2 with antipodal points identified. We can also consider parallel lines in $\mathbb{R}^2$ considered as a subset of $\mathbb{P}^2$ to meet in the "line at infinity", which can be seen by projecting $\mathbb{P}^2$ to a different plane: (`https://www.geogebra.org/3d/gnkzhywe`)

In this figure, the lines $x = 1$ and $x = 2$ are parallel, drawn on a copy of $\mathbb{R}^2$ which is the plane $Z = 1$ in $\mathbb{R}^3$ in this figure. x, y are the coordinates on this plane. These lines are (radial) projections of the planes $X = Z$ and $X = 2Z$ in $\mathbb{R}^3$, where $\mathbb{R}^3$ has coordinates X, Y, Z. These planes project to $x = z$ and $x = 2z$ on the plane $Y = 1$, where they meet at $(x, y) = (0, 0)$.

Similarly, lines $y = mx + c$ for fixed m, and varying c, are all parallel in the plane $Z = 1$. These lines are projections of $Y = mX + cZ$ in $\mathbb{R}^3$. If we project to $X = 1$, we get lines $y = m + cz$, which all meet at $(y, z) = (m, 0)$. You may also wish to consider these lines as projections to the sphere.

Note: Lines in $\mathbb{P}^3$ can be skew; skew lines do not intersect. A corresponding result to Theorem 12 for $\mathbb{P}^3$ would be e.g. that a plane and a line, not contained in that plane, in $\mathbb{P}^3$ always intersect in a point, since their span must be 3 dimensional, so their intersecion has dimension $2 + 1 - 3 = 0$, which means we have a point.

26. Lecture 26: Projective frames of reference and projective linear maps

THEOREM 13. *There is a bijection between projective transformations of $\mathbb{P}^n$ and projective frames of reference of $\mathbb{P}^n$, defined as follows:*

$$\phi : PGL(n+1) \to \{\textit{projective frames of reference}\}$$
$$T \mapsto \{T([\mathbf{e}_1]), T([\mathbf{e}_2]), \ldots, T([\mathbf{e}_{n+1}]), T([\mathbf{e}_1 + \cdots + \mathbf{e}_{n+1}])\}$$

PROOF. We need to show that this map is well defined, injective and surjective.
Well defined: The image of the standard projective frame of reference under T must also be a projective frame of reference, because T is invertible, so maps any basis of $\mathbb{R}^{n+1}$ to another basis.
Surjectivity: Suppose we are given a projective frame of reference, $P_0, P_1, \ldots, P_{n+1}$. By definition, any choice of $n+1$ of these these are linearly independent in $\mathbb{P}^n$, which means that a choice of lifts, $\mathbf{a}_0 = \widetilde{P_0}, \mathbf{a}_1 = \widetilde{P_1}, \ldots \mathbf{a}_n = \widetilde{P_n}$ is linearly independent in $\mathbb{R}^{n+1}$, and so for any nonzero $\alpha_0, \ldots, \alpha_n$, the matrix A with columns $\alpha_i \mathbf{a}_i$, is invertible.

$$A = \begin{pmatrix} | & | & \ldots & | \\ \alpha_0 \mathbf{a}_0 & \alpha_1 \mathbf{a}_1 & \ldots & \alpha_n \mathbf{a}_n \\ | & | & \ldots & | \end{pmatrix}$$

We know that $A\mathbf{e}_i$ is the ith column of A, and so for $i = 0, \ldots, n$,

$$[T_A(\mathbf{e}_i)] = [A\mathbf{e}_i] = [\alpha_i \mathbf{a}_i] = [\mathbf{a}_i] = P_i$$

We want

$$\left[T_A\left(\sum_{i=1}^{n} \mathbf{e}_i\right)\right] = P_{n+1}.$$

Let $\mathbf{a}_{n+1} = \widetilde{P_{n+1}}$, so $P_{n+1} = [\mathbf{a}_{n+1}]$. We have $[T_A(\sum_{i=1}^n \mathbf{e}_i)] = [A(\sum_{i=1}^n \mathbf{e}_i)] = [\sum_{i=1}^n \alpha_i \mathbf{a}_i]$, so it's sufficient to solve $\sum_{i=1}^n \alpha_i \mathbf{a}_i = \mathbf{a}_{n+1}$, i.e., $A(\alpha_0, \alpha_1, \ldots, \alpha_n)^T = \mathbf{a}_{n+1}$. Since A is invertible, set

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = A^{-1} \mathbf{a}_{n+1}.$$

Since $\mathbf{a}_0, \ldots, \mathbf{a}_{n+1}$ is a projective frame of reference, all α_i in this solution must be non-zero, since otherwise there would be a set of n vectors from $\mathbf{a}_0, \ldots, \mathbf{a}_n$ which was not linearly independent. Hence $A \in PGL(n+1)$. So $\phi(T_A) = \{P_0, \ldots, P_{n+1}\}$. Thus ϕ is surjective.
Injectivity:
Suppose that $\phi(T_A) = \phi(T_B)$ for matrices $A, B \in PGL(n)$. Then for $i = 0, \ldots, n$, we have

$$T_A(\mathbf{e}_i) = T_B(\mathbf{e}_i),$$

so the columns of B are multiples of the columns of A, so for some diagonal matrix $\mathrm{Diag}(\beta_0, \ldots, \beta_n)$,

$$B = A \cdot \mathrm{Diag}(\beta_0, \ldots, \beta_n).$$

Since $\phi(T_A) = \phi(T_B)$, we have $[T_A(\sum \mathbf{e}_i)] = [T_B(\sum \mathbf{e}_i)]$ so

$$\left[A \sum \mathbf{e}_i\right] = \left[B \sum \mathbf{e}_i\right] = \left[A \sum \beta_i \mathbf{e}_i\right].$$

We know that A is invertible, so we can't have $A \sum \mathbf{e}_i = \mathbf{0}$, since then we would have $\sum \mathbf{e}_i = A^{-1}\mathbf{0} = \mathbf{0}$, but we know that $\sum \mathbf{e}_i \neq \mathbf{0}$. So for some scalar λ,

$$\lambda A \sum \mathbf{e}_i = A \sum \beta_i \mathbf{e}_i$$

rearranging, $A \sum (\lambda - \beta_i)\mathbf{e}_i = \mathbf{0}$ and since A is invertible, $\sum(\lambda - \beta_i)\mathbf{e}_i = \mathbf{0}$ which is only possible if $\lambda = \beta_i$ for each $i = 0, \ldots, n$, which implies $B = \lambda A$, and so T_A and T_B are equal as elements of $PGL(n)$. $\square$

27. Lecture 27: Cross ratio

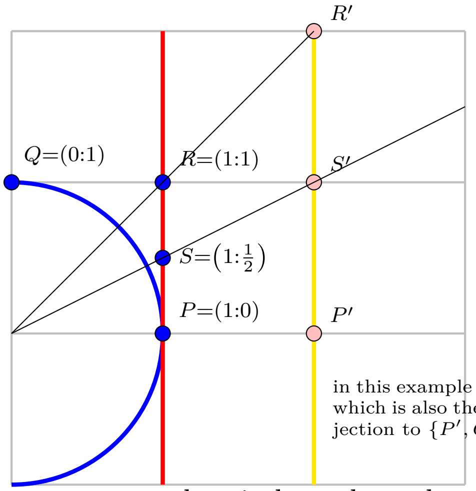
Any three points on $\mathbb{P}^1$ can be mapped by a projective linear transformation to any other three points, so the distance between points under projection is not preserved. But the maps are not totally random. What characterises a projection, in terms of what is preserved? The answer is the cross ratio.
We will show that the **cross ratio** of four points has to be the same for it to be possible to map a set of four points on a line to another set of four points on a line. So the cross ratio is an invariant of projective linear maps.

Definition 44. *Let P, Q, R, S be four distinct points on a line in $\mathbb{P}^n$. Choose an approrpiate basis such that $P = (1 : 0)$ and $Q = (0 : 1)$ (followed by $n-1$ zeros, which we can ignore). Since $\widetilde{P}$ and $\widetilde{Q}$ span the line that R and S lie on, and since $R, S \neq P$, there are $\lambda, \mu \in \mathbb{R}$ such that we can write with respect to this basis:*

$$R = (1 : \lambda), \quad S = (1 : \mu)$$

Then the **<u>cross ratio</u>** *is*

$$\{P, Q; R, S\} = \frac{\lambda}{\mu}$$

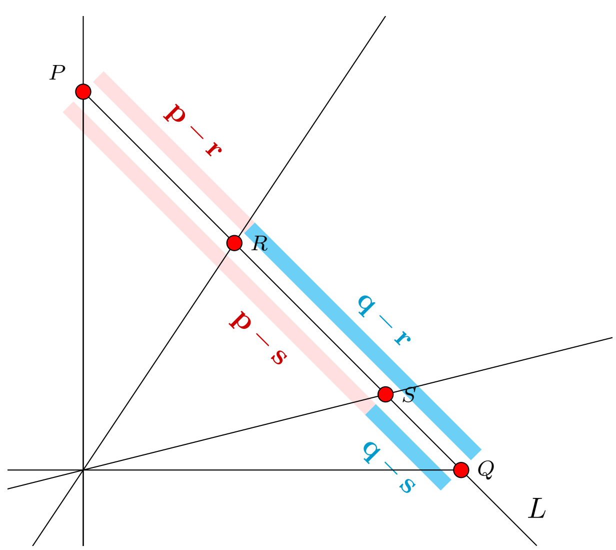


Warning: The cross ratio will change when the points are permuted, so it depends on the order P, Q, R, S are given in.
Note: We could also map P, Q, R to the standard projective frame of reference for $\mathbb{P}^1$, then the cross ratio is the image of S.
It is convenient not to have to make a change of basis in order to compute the cross ratio, so the following result is useful:

Proposition 14. *The cross ratio is well defined. Moreover, if P, Q, R, S are points in $\mathbb{P}^1$, then they can be considered as lines in $\mathbb{R}^2$. Let L be any line in $\mathbb{R}^2$ not through the origin. Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ be the vector coordinates of the intersections of P, Q, R, S with L. Then*

$$\{P, Q; R, S\} = \left(\frac{\mathbf{p}-\mathbf{r}}{\mathbf{p}-\mathbf{s}}\right)\left(\frac{\mathbf{q}-\mathbf{s}}{\mathbf{q}-\mathbf{r}}\right),$$

where the ratios of vectors make sense, because they all lie in the direction of L, and we define $\frac{\lambda \mathbf{v}}{\mathbf{v}} = \lambda$.



Proof. In order to compute the cross ratio, we apply a change of basis transformation T_A, so that

$$T_A(P) = (1:0), \quad T_A(Q) = (0:1), \quad T_A(R) = (1:\lambda).$$

This means that

$$T_A(\mathbf{p}) = \alpha(1,0), \quad T_A(\mathbf{q}) = \beta(0,1), \quad T_A(\mathbf{r}) = \gamma(1,\lambda), \quad T_A(\mathbf{s}) = \delta(1,\mu),$$

and then by definition,

$$\{P,Q;R,S\} = \frac{\lambda}{\mu}.$$

Note that any other change of basis matrix B mapping P and Q to $(1:0)$ and $(0:1)$ must differ by $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} A$ for some a, b. So $T_B(\mathbf{r}) = \gamma(a, b\lambda) = \gamma a(1, b/a\lambda)$ and $T_B(\mathbf{s}) = \gamma(a, b\mu) = \gamma a(1, b/a\mu)$, so the ratio of the second coordinates of $\mathbf{r}$ and $\mathbf{s}$ is still λ/μ, and so the cross ratio is well defined.
Since $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ are all chosen to be on a common line L, we must have that for some constants λ_1, λ_2, by collinearity property of points in $\mathbb{R}^2$, we have

$$\mathbf{r} = \lambda_1\mathbf{p} + (1-\lambda_1)\mathbf{q}$$
$$\mathbf{s} = \lambda_2\mathbf{p} + (1-\lambda_2)\mathbf{q}$$

Linear maps preserve the relationship between points on lines, so we also have

$$\gamma(1,\lambda) = \lambda_1\alpha(1,0) + (1-\lambda_1)\beta(0,1) = (\alpha\lambda_1, \beta(1-\lambda_1)) \Rightarrow \lambda = (1-\lambda_1)/\lambda_1$$
$$\delta(1,\mu) = \lambda_2\alpha(1,0) + (1-\lambda_2)\beta(0,1) = (\alpha\lambda_2, \beta(1-\lambda_2)) \Rightarrow \mu = (1-\lambda_2)/\lambda_2$$
$$\left(\frac{\mathbf{p}-\mathbf{r}}{\mathbf{p}-\mathbf{s}}\right)\left(\frac{\mathbf{q}-\mathbf{s}}{\mathbf{q}-\mathbf{r}}\right) = \frac{(1-\lambda_1)}{(1-\lambda_2)}\frac{\lambda_2}{\lambda_1} = \frac{(1-\lambda_1)/\lambda_1}{(1-\lambda_2)/\lambda_2} = \frac{\lambda}{\mu}$$

□

Example: Does there exist a projective linear map taking points $P_1, Q_1, R_1, S_1 = 0, 1, 2, 3$ to points $P_2, Q_2, R_2, S_2 = 0, 1, 7, 8$, where the point $t \in \mathbb{R}$ is interpreted as a point $(t:1)$ in $\mathbb{P}^1$?
Solution: The cross ratios are

$$\{P_1, Q_1; R_1, S_1\} = \left(\frac{0-2}{0-3}\right)\left(\frac{1-3}{1-2}\right) = \frac{4}{3} \neq \{P_2, Q_2; R_2, S_2\} = \left(\frac{0-7}{0-8}\right)\left(\frac{1-8}{1-7}\right) = \frac{7\times 7}{8\times 6}$$

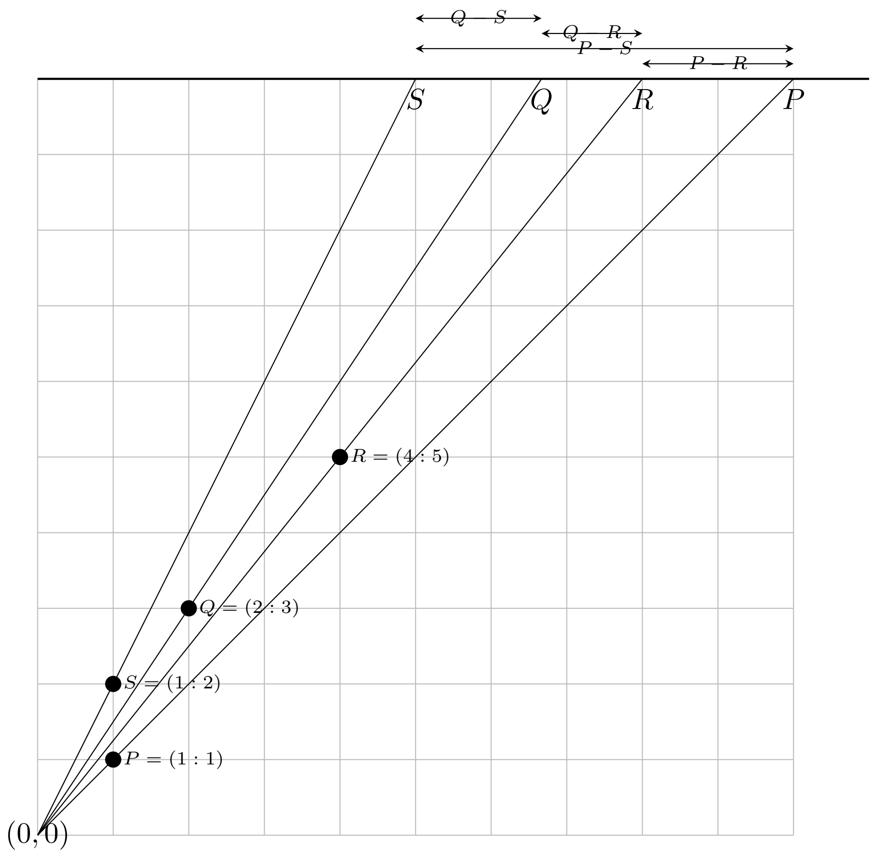
Since these are not equal, there is no such projective linear map.
Example: Does there exist a projective linear map taking points $P_1, Q_1, R_1, S_1 = 0, 1, 2, 5$ to points $P_2, Q_2, R_2, S_2 = 1, 4, 5, 6$, where the point $t \in \mathbb{R}$ is interpreted as a point $(t:1)$ in $\mathbb{P}^1$?
Solution: The cross ratios are

$$\{P_1, Q_1; R_1, S_1\} = \left(\frac{0-2}{0-5}\right)\left(\frac{1-5}{1-2}\right) = \frac{8}{5} = \{P_2, Q_2; R_2, S_2\} = \left(\frac{1-5}{1-6}\right)\left(\frac{4-6}{4-5}\right) = \frac{8}{5}$$

So, there is such a map. Note that to compute the map, we only need to consider the images of the first three points. Given that the cross ratios are equal, the fourth point of one set will automatically get mapped to the fourth point of the other set.

Example: Determine whether or not there is a projective linear transformation (perspectivity) which takes the points $(1:0), (0:1), (1:1)$ and $(1:2)$ to $(1:1), (2:3), (4:5), (1:2)$, preserving the order of the points. Either prove there is no such map, or find such a map by giving it's matrix.
Solution: Since the first set of points includes a standard frame of reference, it may be easier just to compute the matrix which maps $(1:0), (0:1), (1:1)$ to $(1:1), (2:3), (4:5)$, and see where $(1:2)$ is mapped to. We can see that the required matrix must have the form $A = \begin{pmatrix} \alpha & 2\beta \\ \alpha & 3\beta \end{pmatrix}$, and for $T_A(1:1) = (4:5)$, we must have $A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$. Now we can check that $T_A(1:2) = (6:8) = (3:4) \neq (1:2)$, so there is no projective linear transformation which takes the points $(1:0), (0:1), (1:1)$ and $(1:2)$ to $(1:1), (2:3), (4:5), (1:2)$.

We could also have shown this using the cross ratio. By definition, the cross ratio of the first set of points is $\{(1:0),(0:1);(1:1),(1:2)\} = \frac{1}{2}$ For the second set of points, either we can map the first three to $(1:0),(0:1);(1:1)$, which is acheived by A^{-1}, and we have $T_{A^{-1}}(1:2) = \left[\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right] = (1:-2)$, so $\{(1:1),(2:3);(4:5),(1:2)\} = -\frac{1}{2}$.
We could also compute the cross ratio geometrically, using Proposition 14, as in the following diagram.



$$\{P,Q;R,S\} = \left(\frac{\mathbf{p}-\mathbf{r}}{\mathbf{p}-\mathbf{s}}\right)\left(\frac{\mathbf{q}-\mathbf{s}}{\mathbf{q}-\mathbf{r}}\right) = \left(\frac{2}{5}\right)\left(\frac{-1\frac{2}{3}}{1\frac{1}{3}}\right) = \left(\frac{2}{5}\right)\left(\frac{-5}{4}\right) = -\frac{1}{2}$$

Note, in this computation, I used the line $y = 10$, but any line would give the same result. I scaled all points to have the same $y = 10$ coordinate, e.g., $Q = (2:3) = (6\frac{2}{3}:10)$.

For beginning of lecture 28

Lemma 31. *Projective linear maps preserve collinearity. I.e., if $P, Q, R \in \mathbb{P}^n$ and $A \in PGL(n+1)$, with corresponding map T_A, then P, Q, R are collinear implies $T_A(P), T_A(Q), T_A(R)$ are also collinear.*

Proof. P, Q, R are collinear if and only if the lifts, vectors $\tilde{P}, \tilde{Q}, \tilde{R}$ are coplanar, in the sense that they span a (at most) two dimensional subspace of $\mathbb{R}^{n+1}$.
I.e., assuming they are all distinct, for some μ, λ, $\tilde{R} = \mu\tilde{P} + \lambda\tilde{Q}$. (The case of not being distinct is easy, since 2 points are always on a line.)
Since A defines an invertible linear map on $\mathbb{R}^{n+1}$, we have that $T_A(\tilde{R}) = \mu T_A(\tilde{P}) + \lambda T_A(\tilde{Q})$.
This implies that $\tilde{T_A}(P), \tilde{T_A}(Q), \tilde{T_A}(R)$ span a (at most) two dimensional subspace of $\mathbb{R}^{n+1}$, and so $T_A(P), T_A(Q), T_A(R)$ are collinear.

∎

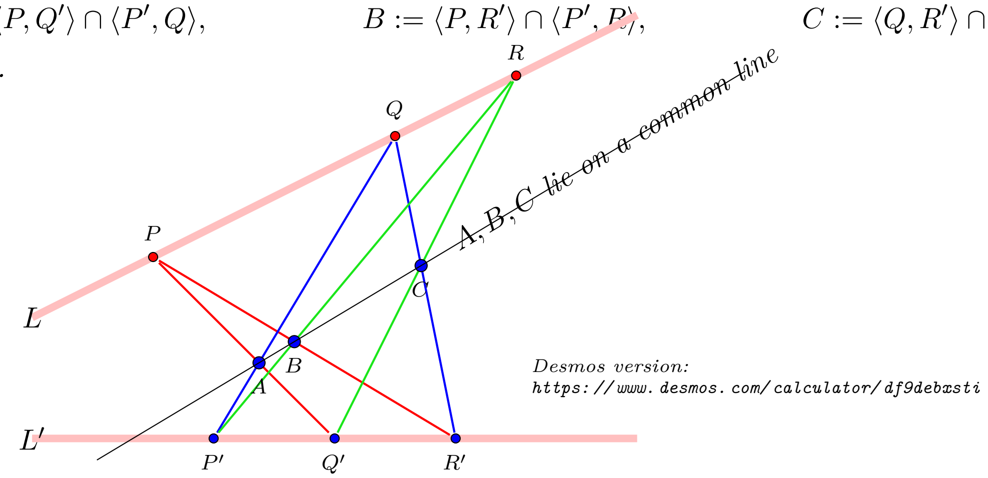
28. Lecture 28: Pappus' Hexagon Theorem

Pappus' Theorem has been known since ancient times. It has been important in the development of algebraic geometry. Pascal's theorem (hexagrammum mysticum) is a generalization. For history, variations and applications see e.g., chapter 1 in "Perspectives on Projective Geometry" by Richter-Gebert, and `https://maa.org/sites/default/files/pdf/upload_library/2/Traves-Monthly-2014.pdf`

Theorem 14 (Pappus' Theorem). *Let $L, L' \subset \mathbb{P}^2$ be distinct projective lines.*
Let $P, Q, R \in L \setminus L'$, $P', Q', R' \in L' \setminus L$ be distinct points.
Then the intersection points

$$A := \langle P, Q' \rangle \cap \langle P', Q \rangle, \qquad B := \langle P, R' \rangle \cap \langle P', R \rangle, \qquad C := \langle Q, R' \rangle \cap \langle Q', R \rangle$$

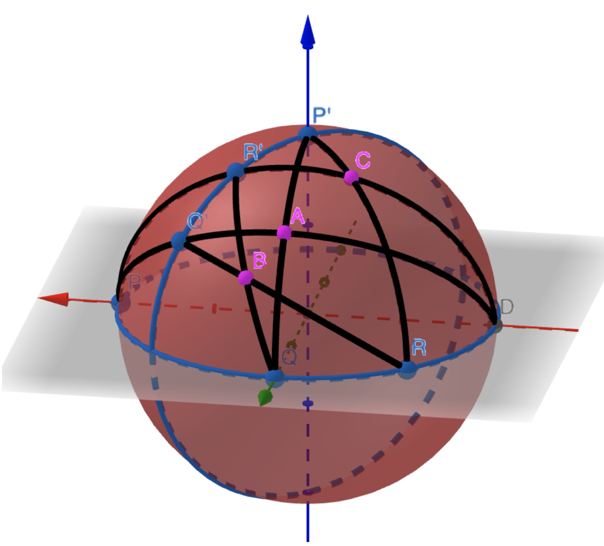
are collinear.



Proof. Since P, Q, P', Q' are distinct and on two distinct lines, these points must be a projective frame of reference, and so by Theorem 13 there is a projective transformation which takes these points to the standard frame of reference for $\mathbb{P}^2$.
Since a projective linear map preserves lines and intersection points, we can now assume

$$P = (1:0:0), \qquad Q = (0:1:0), \qquad P' = (0:0:1), \qquad Q' = (1:1:1).$$

The picture, from `https://www.geogebra.org/3d/zhwkackg` shows these points on the projective plane projected to a sphere. The line through P, Q, R is the "line at infinity", where the sphere intersects the $z = 0$ plane.



Since $R \in \langle P, Q \rangle$, R, being a linear combination of P and Q, has the form $R = (1 : \alpha : 0)$, where we may assume the first coordinate is 1, since it can not be zero, since $R \neq Q$. If the first coordinate is non-zero, by scaling we can set it equal to 1. Since R' is on $\langle P', Q' \rangle$, R' has the form $R' = (1 : 1 : \beta)$, where again, the first coordinates can not be zero, since $R' \neq P'$.
Now compute the lines and intersection points:

$$\left.\begin{aligned} \langle P, Q' \rangle &= \{(\lambda + \mu : \mu : \mu)\} \\ \langle P', Q \rangle &= \{(0 : \lambda_2 : \mu_2)\} \end{aligned}\right\} A = (0:1:1)$$

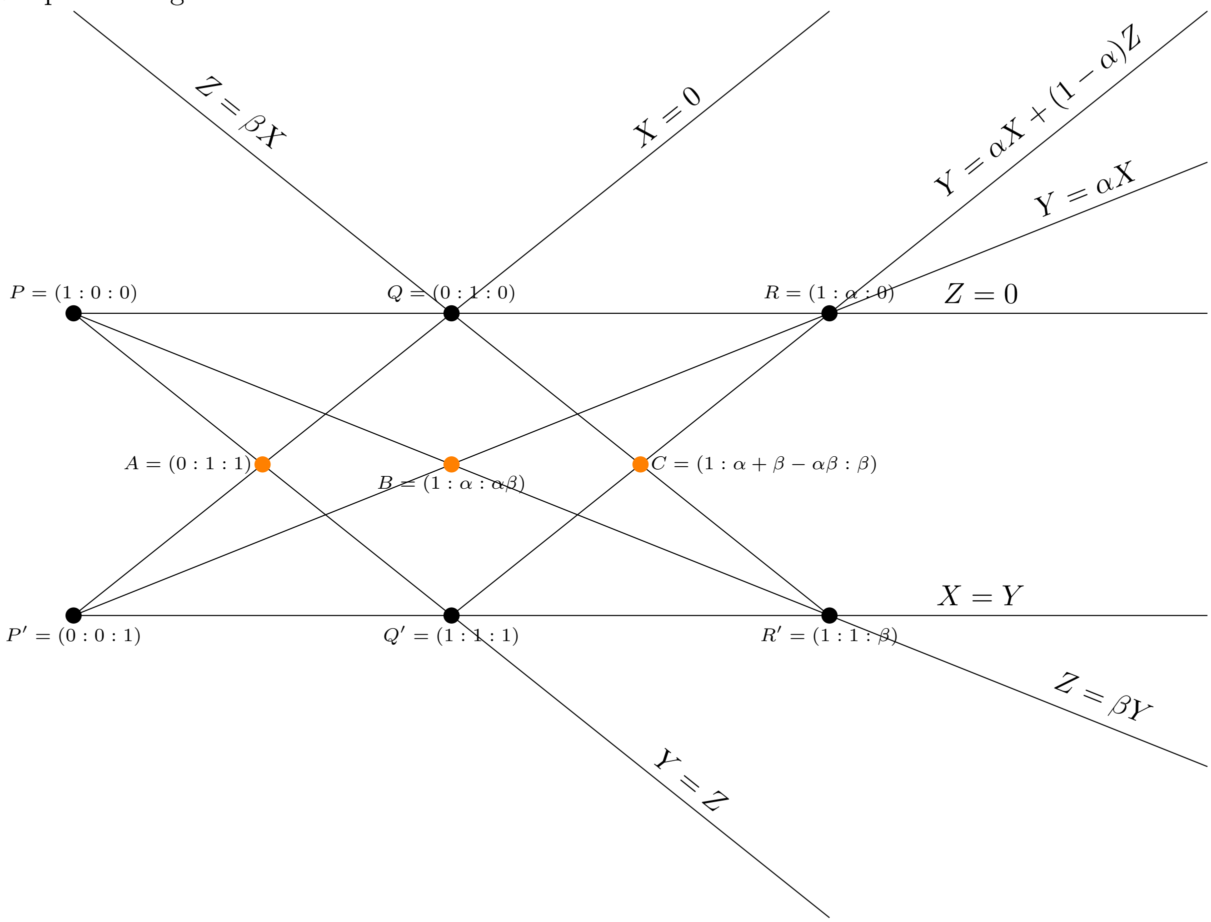
$$\left.\begin{aligned} \langle Q, R' \rangle &= \{(\mu_2 : \mu_2 + \lambda_2 : \mu_2 \beta)\} \\ \langle R, Q' \rangle &= \{(\mu + \lambda : \mu\alpha + \lambda : \lambda)\} \end{aligned}\right\} B = (1 : \alpha + \beta - \alpha\beta : \beta)$$

$$\left.\begin{aligned} \langle P, R' \rangle &= \{(\mu_2 + \lambda_2 : \mu_2 : \mu_2 \beta)\} \\ \langle R, P' \rangle &= \{(\mu : \mu\alpha : \lambda)\} \end{aligned}\right\} C = (1 : \alpha : \alpha\beta)$$

Now note that A, B, C all lie on the line $\alpha(1-\beta)X - Y + Z = 0$. (Also, $(\beta - \alpha\beta)\widetilde{A} + \widetilde{C} = \widetilde{B}$ for obvious lifts.) $\square$

Lecture 29: finish off proof of Pappus' theorem

We didn't quite complete the proof of Pappus' theorem.
We found that $A = (0 : 1 : 1), B = (1 : \alpha, \alpha\beta)$.
The remaining details: prove that the intersection point of $\langle Q, R'\rangle$ and $\langle R, Q'\rangle$ is $(1 : \alpha + \beta - \alpha\beta : \beta)$, and check A, B, C are on a line.
Simplified diagram:



$$\begin{aligned}
\langle Q, R'\rangle &= \pi(\mathrm{Span}((0,1,0),(1,1,\beta))) \\
&= \pi(\mathrm{Span}((0,1,0),(1,0,\beta))) \\
&= \pi\left(\left\{(\lambda, \mu, \beta\lambda) \in \mathbb{R}^3 : \lambda, \mu \in \mathbb{R}\right\})\right) \\
&= \{(X : Y : Z) \in \mathbb{P}^2 : Z = \beta X\}
\end{aligned}$$

$$\begin{aligned}
\langle Q', R\rangle &= \pi(\mathrm{Span}((1,1,1),(1,\alpha,0))) \\
&= \pi(\mathrm{Span}((0,1-\alpha,1),(1,\alpha,0))) \\
&= \pi\left(\left\{(\lambda, \lambda\alpha + (1-\alpha)\mu, \mu) \in \mathbb{R}^3 : \lambda, \mu \in \mathbb{R}\right\})\right) \\
&= \{(X : Y : Z) \in \mathbb{P}^2 : Y = \alpha X + (1-\alpha)Z\}
\end{aligned}$$

The point on $\langle Q, R'\rangle \cap \langle Q', R\rangle$ must satisty $Z = \beta X$ and $Y = \alpha X + (1-\alpha)Z = \alpha X + (1-\alpha)\beta X$, so is given by $C = (X : \alpha + \beta(1-\alpha)X : X) = (1 : \alpha + \beta - \alpha\beta : \beta)$.
Finally, A, B, C are all on the same line, since it can be verified that they are on the line

$$\alpha(1-\beta)X - Y + Z = 0$$

$\square$

Lecture 29 continued: Axiomatic projective geometry

Definition 45. *An axiomatic projective plane (P, L, I) consists of a set P, the set of points, and a set L, the set of lines, and a relation $I \subset P \times L$, the incidence relation, i.e., how lines intersect.*
For $p \in P$ and $\ell \in L$, we write

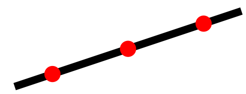
$$p \in \ell \iff (p, \ell) \in I,$$

and say that p is contained in ℓ.
Notation: for $\ell_1, \ell_2 \in L$,

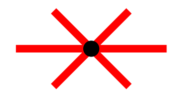
$$\ell_1 \cap \ell_2 = \{p \in P : p \in \ell_1 \text{ and } p \in \ell_2\}$$

These sets must satisfy the following axioms:
(1) Every line contains at least distinct three points:

$$\forall \ell \in L \exists \text{ distinct } x, y, z \in P, \text{ such that } (x, \ell), (y, \ell), (z, \ell) \in I$$



i.e., $x, y, z \in \ell$.
(2) Every point is contained in at least three distinct lines:

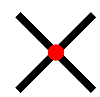


$$\forall x \in P \exists \text{ distinct } l, m, n \in L, \text{ such that } (x, l), (x, m), (x, n) \in I$$

i.e., $x \in \ell \cap m \cap n$
(3) Any two points span a unique line:

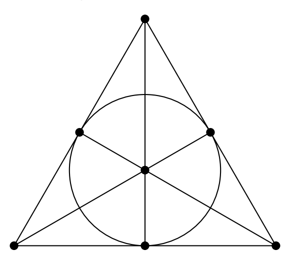
$$\forall x \neq y \in P \exists ! \ell \in L \text{ such that } (x, \ell), (y, \ell) \in I$$

(4) Any two distinct lines intersect at a unique point:



$$\forall l \neq m \in L \exists ! x \in P \text{ such that } (x, l), (x, m) \in I$$

Example $\mathbb{P}^2(\mathbb{F}_2)$ is the smallest axiomatic projective plane, and has 7 points and 7 lines.



Quiz: Which of the following is an axiomatic projective plane (with usual concepts of points and lines, and standard definitions of intersections etc):

$$\mathbb{P}^2(\mathbb{R}), \mathbb{P}^3(\mathbb{R}), \mathbb{R}^2, \mathbb{R}^3, S^2, S^1, \mathbb{R}, \mathcal{H}^2?$$

28.1. Projective planes over finite fields. The simplest way to obtain a finite axiomatic projective plane is to take $\mathbb{P}^2(\mathbb{F}_q)$. Here, $\mathbb{F}_q$ is a finite field with q elements, which exists for all q which is a power of a prime, $q = p^n$, for a prime p. In the case $q = p$, then $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, integers mod p. For $n > 1$, $\mathbb{F}_q = \mathbb{F}_p[x]/(f(x))$ where $f(x)$ is an irreducible polynomial of degree n. I.e., the quotient of the polynomial ring $\mathbb{F}_p[x]$ by a prime ideal generated by $f(x)$. If you've not seen this in algebra, then just work with the case $\mathbb{F}_p$.

Proposition 15. *The projective plane $\mathbb{P}^2(\mathbb{F}_q)$ for a field $\mathbb{F}_q$ with q elements*

- *has $q^2 + q + 1$ points.*
- *There are $q^2 + q + 1$ lines.*
- *Each line contains $q + 1$ points.*
- *Each point is on $q + 1$ lines.*
- *$\mathbb{P}^2(\mathbb{F}_q)$ satisfies the axioms of an axiomatic projective plane.*

PROOF.
- There are a total of q^3-1 points in $(\mathbb{F}_q^3)^*$. Each line in $(\mathbb{F}_q^3)^*$ contains $q-1$ points (all the points on the line in $\mathbb{F}_q^3$ except $\mathbf{0}$). Lines in $\mathbb{F}_q^3$ only intersect at $\mathbf{0}$, so there are
$$\frac{q^3-1}{q-1} = q^2+q+1$$
lines in $\mathbb{F}_q^3$, and so this is the number of points in $\mathbb{P}^2(\mathbb{F}_q)$.
- The number of lines is equal to the number of points because any line ℓ in $\mathbb{P}^2(\mathbb{F}_q)$ corresponds to a plane V through the origin in $\mathbb{F}_q^3$, that is, $\ell = \pi(V)$, and $V = \widetilde{\ell}$.

  Any plane through the origin, V in $\mathbb{F}_q^3$ corresponds to the line which is orthogonal to the plane. That is, for some $\mathbf{w}$, we have $V = (\mathbf{w}\mathbb{F}_q)^\perp$, so there is a correspondance, $[\mathbf{w}] \leftrightarrow \pi((\mathbf{w}\mathbb{F}_q)^\perp)$, and so the number of lines in $\mathbb{P}^2(\mathbb{F}_q)$ equals the number of points. This is part of a more general duality between points and lines in $\mathbb{P}^2$.
- Each plane V through the origin in $(\mathbb{F}_q)^3$ contains q^2-1 non-zero points, since if $\mathbf{v},\mathbf{u}$ is a basis for the plane, the points are
$$V = \{a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{F}_q\}.$$
Within this plane, any line $\mathbf{w}\mathbb{F}_q = \{a\mathbf{w} : a \in \mathbb{F}_q\}$ through the origin contains $q-1$ non-zero points. Any two lines in $(\mathbb{F}_q)^3$ only intersect at $\mathbf{0}$. So there are $q+1 = (q^2-1)/(q-1)$ lines through the origin in each plane V through the originin $\mathbb{F}_q^3$, which after projectivising become $q+1$ points (of the form $[\mathbf{w}]$ for $\mathbf{w} \in V$) on each line $\ell = \pi(V)$.
- Each point $[\mathbf{w}]$ is contained in $q+1$ lines. This is because the plane $V = (\mathbf{w}\mathbb{F}_q)^\perp$ in $\mathbb{F}_q^3$ contains $q+1$ lines through the origin. For each of these lines, $\mathbf{v}\mathbb{F}_q$, we get a plane $V = \mathrm{Span}(\mathbf{v}, \mathbf{w})$. Note that since by definition $\mathbf{w}$ and $\mathbf{v}$ are orthogonal, they must not lie on the same line. So V is a plane. Since $\mathbf{w} \in V$, we get $[\mathbf{w}] = \pi(\mathbf{w}) \in \ell := \pi(V)$. So $[\mathbf{w}]$ lies on $q+1$ points.
- Since $q \geq 1$, we have that axioms (1) and (2) hold for $\mathbb{P}^2(\mathbb{F}_q)$. So we just need to check axioms (3) and (4).
- Axiom (3): If $[\mathbf{u}]$ and $[\mathbf{v}]$ are two distinct points in $\mathbb{P}^2(\mathbb{F}_q)$, then the line containing them both is given by $\pi(V)$ where
$$V = \{a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{F}_q\}.$$
V is a 2-dimensional subspace, since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, (otherwise $[\mathbf{u}] = [\mathbf{v}]$), so $\pi(V)$ is a line. Any line must have the form $\pi(V)$ for some 2-dimensional subspace V, and any such space V must contain $\mathbf{u}$ and $\mathbf{v}$ if $\pi(V)$ contains $[\mathbf{u}]$ and $[\mathbf{v}]$. So we have $\mathrm{Span}(\mathbf{u},\mathbf{v}) \subset V$, but since $\mathbf{u}$ and $\mathbf{v}$ span a 2-dimensional space, and V is 2-dimensional, we know from linear algebra that $\mathrm{Span}(\mathbf{u},\mathbf{v}) = V$. Hence the line containing $\mathbf{u}$ and $\mathbf{v}$ is unique.
- Axiom (4): If ℓ_1 and ℓ_2 are two distinct lines, then these must have the form $\ell_1 = \pi((\mathbf{v}_1\mathbb{F}_q)^\perp)$ and $\ell_2 = \pi((\mathbf{v}_2\mathbb{F}_q)^\perp)$ for some vectors $\mathbf{v}_1, \mathbf{v}_2$. Then the point given by $[\mathbf{v}_1 \times vv_2]$ is contained in both these lines. Any such point $[\mathbf{w}]$ in ℓ_1 and ℓ_2 must be determined by the equations $\mathbf{w}\cdot\mathbf{v}_1 = \mathbf{w}\cdot\mathbf{v}_2 = 0$. Since ℓ_1 and ℓ_2 are distinct, $\mathbf{v}_1$ and $\mathbf{v}_2$ must be linearly independent, so the two equations defined by $\mathbf{w}\cdot\mathbf{v}_1 = \mathbf{w}\cdot\mathbf{v}_2 = 0$ are linearly independent. Since $\mathbb{F}_q^3$ is 3 dimensional, by the rank nullity theorem, the set of solutions to $\mathbf{w}\cdot\mathbf{v}_1 = \mathbf{w}\cdot\mathbf{v}_2 = 0$ is 1 dimensional, spanned by some vector $\mathbf{w}$, which defines the unique point $[\mathbf{w}]$ on ℓ_1 and ℓ_2, unique by the 1-dimensionality of this set of solutions.

$\square$

Note, in the class we only proved the first point; we may skip the rest due to lack of time. However, you are expected to know this material.

Dobble

The game Dobble was invented based on a finite projective plane. Junior Dobble is isomorphic to $\mathbb{P}^2(\mathbb{F}_5)$, and corresponds to a finite projective plane of order 6. Each line contains 6 points, each point is on 6 lines. In the game as sold, one card is missing.

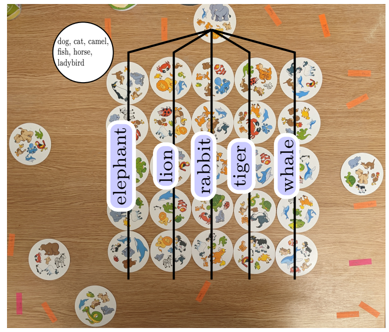
The missing card is:
{dog, cat, camel, fish, ladybird, horse}.

Each line corresponds to a different animal.

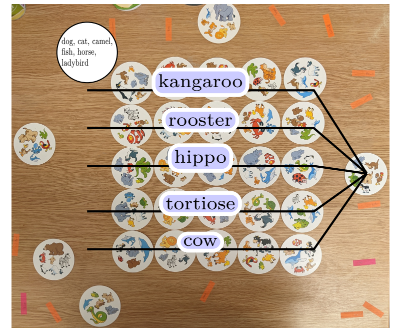
There are 31 different animal symbols, one for each line in the projective space $\mathbb{P}^2(\mathbb{F}_5)$.
I have made the following choice of animal to point of $\mathbb{P}^2(\mathbb{F}_5)$. A different choice would correspond to an element of $PGL(3, \mathbb{F}_5)$, which has order 372000.

animal	orthogonal vector	animal	orthogonal vector
dog	(0:0:1)	duck	(1:1:1)
elephant	(1:0:0)	parrot	(1:1:2)
lion	(4:0:1)	crocodile	(1:1:3)
rabbit	(2:0:1)	octopus	(1:1:4)
tiger	(3:0:1)	shark	(1:1:0)
whale	(1:0:1)	penguin	(3:1:1)
kangaroo	(0:1:1)	zebra	(3:1:2)
rooster	(0:3:1)	crab	(3:1:3)
hippo	(0:2:1)	sheep	(3:1:4)
tortoise	(0:4:1)	bear	(3:1:0)
cow	(0:0:1)	horse	(2:1:1)
dolphin	(4:1:0)	fish	(2:1:2)
gorilla	(4:1:1)	cat	(2:1:3)
snake	(4:1:2)	camel	(2:1:4)
frog	(4:1:3)	ladybird	(2:1:0)
owl	(4:1:1)		

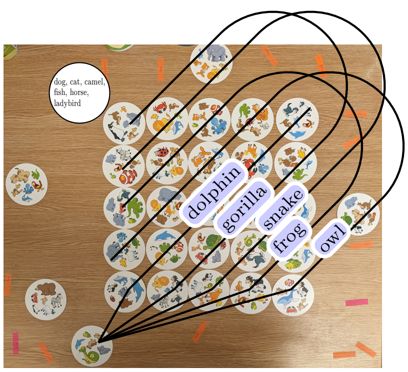
E.g. the "cat" line, consists of points $(x : y : z)$ with $y = 2z - 2x$, i.e., $(x, y, z) \cdot (2, 1, -2) = 0$ meaning "cat" is the line orthogonal to $(2 : 1 : -2) \equiv (2 : 1 : 3) \equiv (4 : 2 : 1)$.
You can work out the point on $\mathbb{P}^2(\mathbb{F}_5)$ corresponding to your card by just considering two of the animals on it. E.g., the card containing gorilla $(4 : 1 : 1)$ and cat $(2 : 1 : 3)$ must be a point $(x : y : z)$ with $4x + y + z \equiv 2x + y + 3z \equiv 0 \mod 5$. From this we get $x \equiv y + z$ (from first condition) and then $2y + 2z + y + 3z \equiv 3y \equiv 0$ (sub in second) so $y \equiv 0$, and $x \equiv z$, so (up to scaling) the point is $(1 : 0 : 1)$.



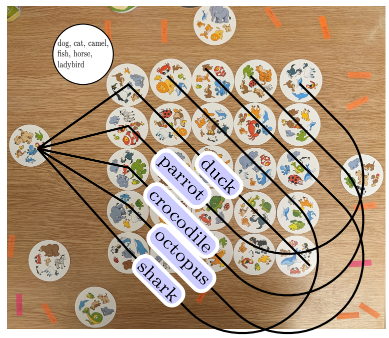
lines of the form $\{(\alpha s : t : s) : (t : s) \in \mathbb{P}^1\}$



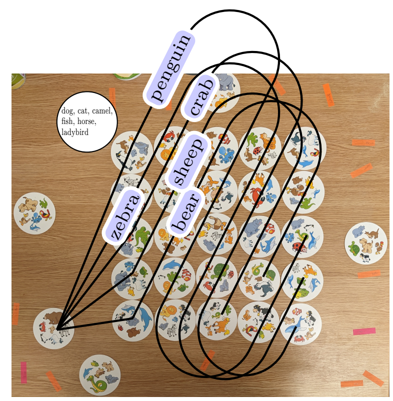
lines of the form $\{(t : \alpha s : s) : (t : s) \in \mathbb{P}^1\}$



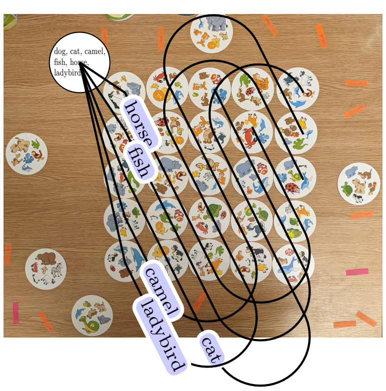
lines of the form $\{(t : t + \alpha s : s) : (t : s) \in \mathbb{P}^1\}$



lines of the form $\{(t : \alpha s - t : s) : (t : s) \in \mathbb{P}^1\}$



lines of the form $\{(t : 2t + \alpha s : s) : (t : s) \in \mathbb{P}^1\}$



lines of the form $\{(t : \alpha s - 2t : s) : (t : s) \in \mathbb{P}^1\}$

			$(0:1:0)$			
$(1:-2:0)$	(0,4)	(1,4)	(2,4)	(3,4)	(4,4)	
	(0,3)	(1,3)	(2,3)	(3,3)	(4,3)	
$(1:-1:0)$	(0,2)	(1,2)	(2,2)	(3,2)	(4,2)	$(1:0:0)$
	(0,1)	(1,1)	(2,1)	(3,1)	(4,1)	
$(1:2:0)$	(0,0)	(1,0)	(2,0)	(3,0)	(4,0)	
$(1:1:0)$						

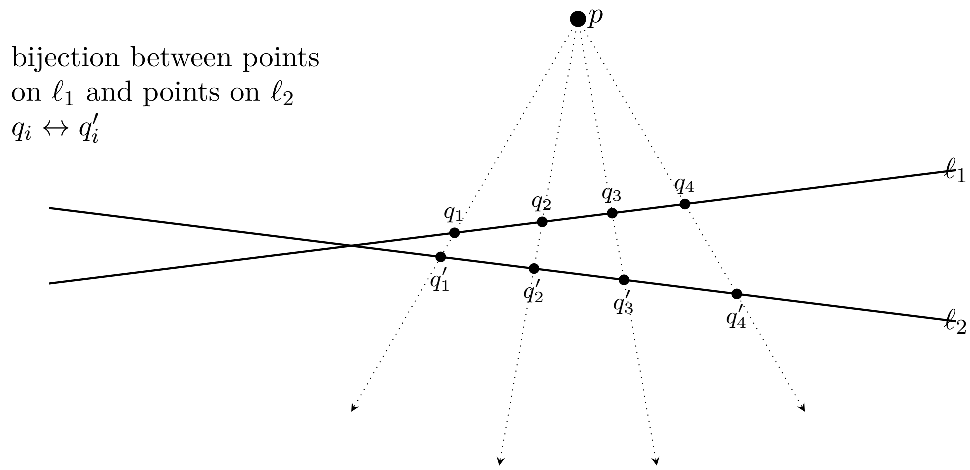
interpret (x, y) as $(x : y : 1)$; all computations mod 5

This shows the 6 families of "parallel" lines. Note that "parallel" here means not meeting in the copy of $\mathbb{F}_5^2 \subset \mathbb{P}^2(\mathbb{F}_5)$ given by $(x, y) \mapsto (x : y : 1)$.

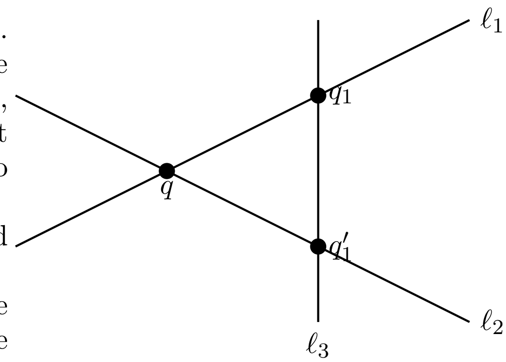
29. Lecture 30: Finite Axiomatic Projective Geometries

Proposition 16 (Proposition/Definition). *A finite axiomatic projective plane has $n^2 + n + 1$ points for some integer n, and has $n+1$ points on each line and $n+1$ lines through each point. Such a projective plane is called a* <u>*projective plane of order n.*</u>

Proof. **Step 1:** Set up a bijection, (by a "perspectivity") between points on any two lines:



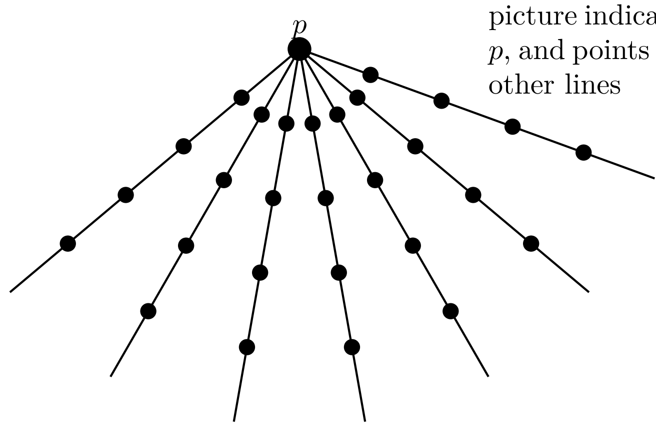
If ℓ_1 and ℓ_2 are two lines, then pick a point p not on ℓ_1 or ℓ_2.
Construct p as follows:



- By axiom (4), ℓ_1 and ℓ_2 intersect at a unique point q.
- By axiom (3), ℓ_1 and ℓ_2 both contain at least three distinct points, so ℓ_1 contains another point, $q_1 \neq q$, ℓ_2 contains a point, $q_1' \neq q$. Since q is the unique point on ℓ_1 and ℓ_2, q_1 is not in ℓ_2 and q_1' is not in ℓ_1, so $q_1 \neq q_1'$.
- By axiom (2) there is a unique line ℓ_3 through q_1 and q_1'.
- By axiom (1) there are at least 3 points on ℓ_3, so there must be another point p apart from q_1 and q_1'. Since ℓ_3 meets ℓ_1 and ℓ_2 in unique points q_1, q_1', the point p is not on ℓ_1 or ℓ_2.

Now the bijection is defined as follows: Let $q_1, \dots, q_N$ be the points on ℓ_1. Then by axiom (3) there is a unique line ℓ_i joining q_i to p. By axiom (4) there is a unique point q_i' on the line ℓ_i and ℓ_2, so we map q_i to q_i'. This map must be injective by the uniqueness property, and surjective, since for any point on ℓ_2 we can obtain the required point on ℓ_2 by the same procedure.
This process works for any pair of lines, so all lines must have the same number of points, N.
step 2: Set up a bijection between points on a line and lines through a point:
For any line ℓ, take a point p not on ℓ, which is possible by the argument in step 1, since there must be at least two different lines, by axiom (2).
Now any line ℓ_i through p intersects ℓ in a unique point q_i, by axiom (4), so we have a map $\ell_i \leftrightarrow q_i$, which is injective by uniqueness, and surjective by axiom (3).
This means that all points are on N lines.

step 3: Compute the total number of points:



picture indicates lines through p, and points on them, but not other lines

Let $n = N - 1$. So every line contains $n + 1$ points, and every point is on $n + 1$ lines.
Now take any point p. Every other point lies on exactly one of the $n + 1$ lines through p, and each of these lines contains n points not counting p, so this gives $(n+1)n$ points not equal to p, so together with p there are $n^2 + n + 1$ points. □

Example: The Fano plane is a finite projective plane of order 2. $\mathbb{P}^2(\mathbb{F}_3)$ is a finite projective plane of order 3.

Definition 46. *We say two axiomatic projective planes (P, L, I) and (P', L', I') are isomorphic if there are bijective maps*

$$f_P : P \to P', \qquad f_L : L \to L', \qquad f_I : I \to I'$$

such that for $p \in P, \ell \in L$

$$p \in \ell \iff f_L(p) \in f_L(\ell)$$

Conjecture: Every finite projective plane has order q, where q is a power of a prime.

There are finite projective fields of order q for all powers of prime q, namely $\mathbb{P}^2(\mathbb{F}_q)$.

However, it's not true that every finite projective plane is isomorphic to $\mathbb{P}^2(\mathbb{F}_q)$ for some finite field $\mathbb{F}_q$ with $q = p^n$ elements for some prime p. There are 4 different finite projective planes of order 9. Just one of these is $\mathbb{P}^2(\mathbb{F}_9)$.

Not very much is known about this problem.
See "A001231: Number of non-isomorphic projective planes of order n." at `https://oeis.org/A001231`, for the list that is known; the number of projective planes of order $n = 2, \ldots, 10$:

n	2	3	4	5	6	7	8	9	10
$n^2 + n + 1$	7	13	21	31	(43)	57	73	91	(111)
number of non-isomorphic planes	1	1	1	1	0	1	1	4	0

I have put 43 and 111 in brackets, because there are no projective planes with this number of points. It is not currently known whether or not there is a projective plane of order 12.
The Bruck–Ryser Theorem says that for a finite projective plane of order q, with $q \equiv 1$ or $2 \mod 4$, q is a sum of two squares.