Sahlqvist Formulas

MTH701

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Introduction

- Sahlqvist formula is a special type of modal formula that shows some remarkable properties and so are interest of study.
- Sahlqvist Correspondence theorem states that every Sahlqvist formula corresponds to a class of Kripke frames that are definable in first order logic.
- This means given a sahlqvist formula, we have an algorithm (Sahlqvist-van Benthem algorithm) to find the corresponding class of Kripke frames and the first order condition defining it.

Preliminaries - Frame Language

Frame Definability: Let Γ be a set of modal formulae and K be a class of Kripke frames. We say that Γ defines K if $\mathcal{F} \in K$ iff $\mathcal{F} \Vdash \Gamma$.

First Order Frame Language: For any modal similarity type τ, the first order frame language of τ is the first-order language that has identity symbol $=$ together with $(n+1)$-ary relation symbol R_Δ for each n-ary modal operator Δ in τ.

Second Order Frame Language: Along with all the resources of first order frame language, this language has additionally collection of monadic predicate variables and capable of quantifying on the predicate variables.

Preliminaries - Frame Correspondence

If a class of frames can be defined by a modal formula ϕ and by a formula α from frame language, then ϕ and α are each others frame correspondents.

Examples

- $p \to \diamond p \leftrightarrow \forall x Rxx$
- $\diamond\diamond p \to \diamond p \leftrightarrow \forall xyz \, ((Rxy \wedge Ryz) \to Rxz)$
- $\diamond p \to \Box \diamond p \leftrightarrow \forall xyz \, ((Rxy \wedge Rxz) \to Ryz)$
- $p \wedge (q \Delta_1 r) \to (q \wedge r \Delta_2 p) \Delta_1 r \leftrightarrow \forall xyz \, (R_1 xyz \to R_2 yzx)$

Preliminaries - Frame Definability in Second order logic

In the standard translation, we have a predicate variable P corresponding to each variable p in modal formula. Since a valuation in modal language assigns subset of frame to each propositional variable, quantifying over all valuations is same as quantifying over all subsets of frame. Hence validity of modal formula is inherently a second order concept.

Proposition 1: Let τ be a modal similarity type and ϕ be a τ-formula. Then for any τ-frame $\mathcal{F}$

$$\mathcal{F}, w \Vdash \phi \text{ iff } \mathcal{F} \Vdash \forall P_1 \forall P_2 \ldots \forall P_n \; ST_x(\phi)[w]$$

Examples with no first order frame correspondents

- Löb formula: $\Box(\Box p \to p) \to \Box p$
- McKinsey formula: $\Box\Diamond p \to \Diamond\Box p$

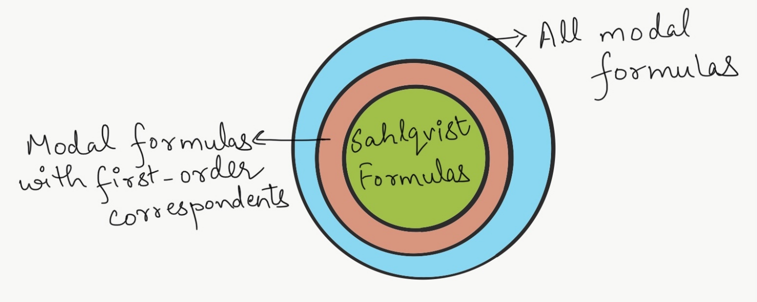
Motivation

Now that we know that modal formulas correspond naturally to second order frame language, the fact that some formulas correspond to first order conditions makes them important to study separately. This motivates us to find answer to following questions -

- Can we identify all the modal formulas that correspond to first order frame conditions?
- Can we find the corresponding first order frame condition once we have identified a formula?

Chagrova's Theorem

- It is undecidable whether an arbitrary basic modal formula has a first-order correspondent.
- Since we can show that a Sahlqvist formula has first-order correspondent, Sahlqvist formulae form a proper subset of set of modal formulae with first-order correspondent. Moreover this set is decidable.



Closed Formulas

A modal formula ϕ is closed iff it contains no proposition variables. A closed formula is built from $\top, \bot$ and any nullary modalities.

Proposition 2: Let ϕ be a closed formula. Then ϕ locally corresponds to a first-order formula $c_\phi(x)$ computable from ϕ.

Proof: Since ϕ has no propositional variable, from proposition 1,

$$\mathcal{F}, w \Vdash \phi \text{ iff } \mathcal{F} \Vdash ST_x(\phi)[w]$$

Thus ϕ corresponds to a first order formula, which can be found by a finite step algorithm.

Positive/Negative Formulas

- An occurrence of propositional variable p is **positive** if it is in the scope of an even number of negation signs; it is a **negative** occurrence if it is in the scope of odd number of negation signs.
- A modal formula ϕ is positive in p (negative in p) if all occurrences of p in ϕ are positive (negative). A formula is called positive/negative if it is positive/negative in all propositional variables occurring in it.

Example: $\Diamond(p \to q)$

Lemma 3: Let ϕ be a modal formula.

1. ϕ is positive in p iff $ST_x(\phi)$ is positive in the corresponding unary predicate P.
2. If ϕ is positive (negative) in p, then $\neg\phi$ is negative (positive) in p.

Upward/Downward Monotone Formulas

Fix a modal language $ML(\tau, \Phi)$.

- A modal formula ϕ is **upward** monotone in p if for every model (W, R, V), $w \in W$ and every valuation V' such that $V(p) \subseteq V'(p)$ and for all $q \neq p$, $V(q) = V'(q)$, following holds

$$\text{if } (W, R, V), w \Vdash \phi, \text{then } (W, R, V'), w \Vdash \phi$$

- A modal formula ϕ is **downward** monotone in p if for every model (W, R, V), $w \in W$ and every valuation V' such that $V'(p) \subseteq V(p)$ and for all $q \neq p$, $V(q) = V'(q)$, following holds

$$\text{if } (W, R, V), w \Vdash \phi, \text{then } (W, R, V'), w \Vdash \phi$$

Upward/Downward Monotone Formulas

Lemma 4: Let ϕ be a modal formula.

1. If ϕ is positive in p, then it is upward monotone in p.
2. If ϕ is negative in p, then it is downward monotone in p.

Proof: Consider $(W, R, V), w$ such that for modal formula ϕ, we have $(W, R, V), w \Vdash \phi$. We use induction on the number of connectives and modal operators.

Base Case:

For $n = 0$, $\phi = p$ is the base case for (1). Take any V' such that $V(p) \subseteq V'(p)$, then $w \in V'(p)$. So $(W, R, V'), w \Vdash p$ (upward monotone).

For $n = 1$, $\phi = \neg p$ is the base case for (2). Since $(W, R, V), w \Vdash \neg p$, for any V' such that $V'(p) \subseteq V(p)$ we have $w \notin V'(p)$. Therefore, $(W, R, V'), w \Vdash \neg p$ (downward monotone).

Proof Continued

Now consider $\phi = \alpha \wedge \beta, \neg\alpha, \Diamond\alpha$. Suppose ϕ is positive in p then

1. $\phi = \alpha \wedge \beta$: Since α and β are positive in p, these are satisfied for any V' such that $V(p) \subseteq V'(p)$ and so is ϕ.
2. $\phi = \neg\alpha$: α is downward monotone. Consider any V' such that $V(p) \subseteq V'(p)$. Then $W \setminus V'(p) \subseteq W \setminus V(p)$. Therefore α is satisfied for $W \setminus V'(p)$ and thus ϕ is satisfied for $V'(p)$.
3. $\phi = \Diamond\alpha$: α is positive. Now consider any V' such that $V(p) \subseteq V'(p)$.
   $$(W, R, V), w \Vdash \Diamond\alpha$$
   $\implies \exists w' \in W$ such that Rww' and $(W, R, V), w' \Vdash \alpha$.
   Therefore $(W, R, V'), w' \Vdash \alpha$ or $(W, R, V'), w \Vdash \Diamond\alpha$.

λ notation

- We use this shorthand notation for substitution. This substitution can be used to denote subsets and thus substituting any occurrence of P is same as assigning P a subset in a valuation.
- For example $\lambda u.u = u$ is true for all u and thus substituting P with it is same as assigning the whole set to P.
  $Py = (\lambda u.u = u)y = (y = y)$ which is true for all y.

Uniform Formulas

A proposition letter p occurs **uniformly** in a modal formula if it occurs only positively or only negatively. A modal formula is **uniform** if all proposition letter it contains occur uniformly.

Theorem 5: If ϕ is a uniform modal formula, then ϕ locally corresponds to a first-order formula $c_\phi(x)$ on frames. Moreover, c_ϕ is effectively computable from ϕ.

Proof: Consider the second-order formula equivalent to ϕ

$$\forall P_1 \ldots \forall P_n ST_x(\phi) \tag{1}$$

We need to show that it has a equivalent first order formula. Now we strip all the second-order quantifiers and substitute every atomic subformula Py with $y = y$(if ϕ is negative in P) else with $y \neq y$. This is equivalent to assigning values to all second-order variables P_i.

Proof Continued

The formula after this substitution is first-order, we denote it by $(*)$. Now we have to show that the second-order formula 1 and $(*)$ are same. Since $(*)$ is an instance of 1, 1 implies $(*)$. Now we have to show

$$\mathcal{M} \Vdash (*) \implies \mathcal{M} \Vdash \forall P_1 \ldots \forall P_n ST_x(\phi)$$

From the assigned values to P_i in $(*)$ and *Lemma 4*, for any choice of $P_1, \ldots, P_n$

$$(\mathcal{M}, P_1, \ldots, P_n) \Vdash ST_x(\phi)$$
$$\text{or } \mathcal{M} \Vdash \forall P_1 \ldots \forall P_n ST_x(\phi)$$

We have also obtained an algorithm for computing first-order equivalent in the proof.

Very Simple Sahlqvist Formulas

- A very simple Sahlqvist antecedent is a formula built up from $\top, \bot$ and propositional variables, using only $\wedge$ and $\Diamond$.
- A very simple Sahlqvist formula is an implication $\phi \to \psi$ in which ψ is positive and ϕ is a very simple Sahlqvist antecedent.

Theorem 6: Let $\chi = \phi \to \psi$ be a very simple Sahlqvist formula. Then χ locally corresponds to a first-order formula. Moreover, c_χ is effectively computable from χ.

Proof: The corresponding second-order formula will be $\forall P_1 \ldots \forall P_n(ST_x(\phi) \to ST_X(\psi))$. We denote $ST_x(\psi)$ with POS.

$$\forall P_1 \ldots \forall P_n(ST_x(\phi) \to POS) \tag{2}$$

We will now give it's reduction to first-order formula.

Proof Continued

Step 1. Pull out existential quantifiers from antecedent.

$$(\exists x_i \alpha(x_i) \wedge \beta) \leftrightarrow \exists x_i(\alpha(x_i) \wedge \beta) \text{ and } (\exists x_i \alpha(x_i) \to \beta) \leftrightarrow \forall x_i(\alpha(x_i) \to \beta)$$

$$\forall P_1 \dots \forall P_n \forall x_1 \dots \forall x_m (REL \wedge AT \to POS) \tag{3}$$

Step 2. Creating instances

We can assume that every P occurring in consequent also occurs in antecedent otherwise we can just use the substitution $\lambda u.u \neq u$ for P. Let P_i be a unary predicate and let $P_i x_{i_1}, \dots, P_i x_{i_k}$ be occurrences of P_i in antecedent. Define

$$\sigma(P_i) := \lambda u.(u = x_{i_1} \vee \dots \vee u = x_{i_k})$$

$\sigma(P_i)$ is the minimal instance that makes the antecedent $REL \wedge AT$ true. Therefore for any model $\mathcal{M}$

$$\mathcal{M} \Vdash AT[ww_1 \dots w_m] \implies \mathcal{M} \Vdash \forall y(\sigma(P_i)(y) \to P_i y)[ww_1 \dots w_m]$$

Proof Continued

Step 3. Substitution

Substitute $\sigma(P_i)$ for each occurrence of P_i which will result in

$$[\sigma(P_1)/P1, \ldots, \sigma(P_n)/P_n]\forall x_1 \ldots \forall x_m(REL \wedge AT \rightarrow POS)$$

By our choice of $\sigma(P_i)$, the formula $[\sigma(P_1)/P1, \ldots, \sigma(P_n)/P_n]AT$ will be trivially true. So the above formula is equivalent to

$$\forall x_1 \ldots \forall x_m(REL \rightarrow [\sigma(P_1)/P1, \ldots, \sigma(P_n)/P_n]POS) \tag{4}$$

First formula implies this because this is just an instant of that formula. To prove that this formula implies first formula, we assume that

$$\mathcal{M} \Vdash \forall x_1 \ldots \forall x_m(REL \rightarrow [\sigma(P_1)/P1, \ldots, \sigma(P_n)/P_n]POS)$$
$$\mathcal{M} \Vdash REL \wedge AT[ww_1 \ldots w_m]$$

and we need to show that $\mathcal{M} \Vdash POS[ww_1 \ldots w_m]$.

Proof Continued

It follows from the assumption that

$$\mathcal{M} \Vdash [\sigma(P_1)/P1, \ldots, \sigma(P_n)/P_n]POS[ww_1 \ldots w_m]$$

Now $\sigma(P_i)$ is minimal instance making AT true. So for any instance of P_i that makes AT true, we will have $\forall y(\sigma(P_i) \to P_i y)[ww_1 \ldots w_m]$. As POS is upward monotone, the instance of P_i will satisfy POS.

Example:

$$p \to \Diamond p$$
$$\forall P(Px \to \exists z(Rxz \wedge Pz))$$
$$(\lambda u.u = x)x \to \exists z(Rxz \wedge (\lambda u.u = x)z))$$
$$x = x \to \exists z(Rxz \wedge z = x)$$
$$Rxx$$

Sahlqvist Formulas

- Let τ be a modal similarity type. A Sahlqvist antecedent over τ is a formula built up from $\top, \bot$, boxed atoms ($\Box_k p$), and negative formulas, using $\wedge, \vee$ and existential modal operators.
- A Sahlqvist implication is an implication $\phi \to \psi$ in which ψ is positive and ϕ is Sahlqvist antecedent.
- A Sahlqvist formula is a formula that is built from Sahlqvist implications by applying boxes and conjunctions, and by applying disjunctions only between formulas that do not share any proposition variables.
- Examples - $\Box(p \to \Diamond p), p \wedge \Diamond\neg p \to \Diamond p$

Sahlqvist Correspondence Theorem: Let τ be a modal similarity type, and let χ be a Sahlqvist formula over τ. Then χ locally corresponds to a first-order formula $c_\chi(x)$ on frames. c_χ is efficiently computable from χ using Sahlqvist-van Benthem algorithm.

References

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