Quantum Error Correction

EE668: Coding Theory

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Introduction

- Quantum computers offer a new and possibly more powerful computation framework.
- We can use the quantum principle of superposition to design efficient algorithms for problems which have no efficient algorithm in classical setting.
- Qubits (equivalent of bits in classical computers) are susceptible to errors due to interaction with environment and as such an efficient quantum error correction scheme is very important to get reliable results out of quantum computers.

Preliminaries

- Just like we have bits for classical computers, qubits are the fundamental unit of quantum information. A general qubit can be written as

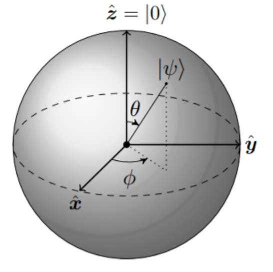
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

- This means the qubit can store information in α and β.
- For a 2 qubit system, the state will be

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

- The state of a n qubit quantum system is described as a vector in 2^n dimensional hilbert space.
- For a 1 qubit system, $\psi \in \text{span}(|0\rangle, |1\rangle) = \mathcal{H}_2$.
- Bloch sphere gives a geometrical interpretation of state of a qubit. State $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ is represented as



- Dictated by laws of quantum mechanics, the operators (analogous to AND, OR etc.) in quantum computers are described by unitary matrices.
- Following are standard Pauli operators for a single qubit.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- These along with I matrix also form basis for 2×2 dimensional matrices.
- Pauli matrices will be useful in our error correction schemes.

- Measurement collapses wave function of qubit.
- Measurements are described by projection matrices that transform the qubit state to it's projection on a plane.
- If we measure $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the standard basis. Then we obtain 0 with a probability $|\alpha|^2$ and 1 with probability $|\beta|^2$.

Quantum Error Correction

1. The state of qubit is continuous (unlike bit) due to which the qubit can have infinite number of errors.
2. No-cloning theorem states that we cannot copy a qubit.
3. Observing the state of a qubit requires measuring it which causes the function to collapse.

We deal with these challenges one by one to obtain a quantum error correction scheme.

- We deal with the first challenge by digitising the quantum error.
- We consider a quantum error that causes qubit to rotate on Bloch sphere. Mathematically,

$$U(\delta\theta,\delta\phi)|\psi\rangle = \cos\frac{\theta+\delta\theta}{2}|0\rangle + e^{i(\phi+\delta\phi)}\sin\frac{\theta+\delta\theta}{2}|1\rangle$$

- Coherent noise such as above is representable in Pauli basis. Thus we have

$$U(\delta\theta,\delta\phi)|\psi\rangle = \alpha_I I|\psi\rangle + \alpha_X X|\psi\rangle + \alpha_Z Z|\psi\rangle + \alpha_{XZ} XZ|\psi\rangle$$

- This shows that any coherent error can be broken down into sum from $\{I, X, Z, XZ\}$ enabling us to treat these errors separately.

From the digitisation, we can represent any coherent error as error type X and Z. While the the X error flips the qubit, Z error introduces a phase shift.

$$X|\psi\rangle = \alpha X|0\rangle + \beta X|1\rangle = \alpha|1\rangle + \beta|0\rangle$$
$$Z|\psi\rangle = \alpha Z|0\rangle + \beta Z|1\rangle = \alpha|0\rangle - \beta|1\rangle$$

Error Detecting Schemes

We add redundancy using *CNOT* gate to encode information of 1 qubit to 2 qubits.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{two-qubit encoder}} |\psi\rangle_L = \alpha|00\rangle + \beta|11\rangle = \alpha|0\rangle_L + \beta|1\rangle_L$$

A two qubit state occupies $\mathcal{H}_4$ and $|\psi\rangle_L \in \operatorname{span}(|00\rangle, |11\rangle) \subset \mathcal{H}_4$. This is called code space.
$\mathcal{F} = \operatorname{span}(|01\rangle, |10\rangle)$ is called error space. A single bit flip transforms code into error space. Since code space and error space are orthogonal, it is possible to distinguish which space the qubits are in without collapsing the wave function.
These measurements are called stabilizer measurements.

- For the purpose of distinguishing, we use Z_1Z_2 as

$$Z_1Z_2|\psi\rangle_L = Z_1Z_2(\alpha|00\rangle + \beta|11\rangle) = (+1)|\psi\rangle_L$$

- Therefore, we have the syndrome extraction stage

$$E|\psi\rangle_L|0\rangle_A \overset{\text{syndrome}}{\longrightarrow} \frac{1}{2}\left(I_1I_2 + Z_1Z_2\right)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}\left(I_1I_2 - Z_1Z_2\right)E|\psi\rangle_L|1\rangle_A$$

- For a more general case the error will be

$$\mathcal{E} = \alpha_I I + \alpha_X X$$
$$E = \mathcal{E}_1 \otimes \mathcal{E}_2 = \alpha_I^2 I_1I_2 + \alpha_I\alpha_X\left(X_1 + X_2\right) + \alpha_X^2 X_1X_2$$

- Therefore, in syndrome extraction, we have

$$E|\psi\rangle_L|0\rangle_A \xrightarrow{\text{syndrome}} \left(\alpha_I^2 I_1 I_2 + \alpha_X^2 X_1 X_2\right)|\psi\rangle_L|0\rangle_A + \alpha_I\alpha_X\left(X_1 + X_2\right)|\psi\rangle_L|1\rangle_A$$

- If the measure ancillary bit is 1 then we know that one of the bits have been flipped.
- Let the probability of bit flip error is p. Then if measure is 0, the wavelength collapses to

$$\frac{\left(\alpha_I^2 I_1 I_2 + \alpha_X^2 X_1 X_2\right)}{\sqrt{\left|\alpha_I^2\right|^2 + \left|\alpha_X^2\right|^2}}|\psi\rangle_L|0\rangle_A$$

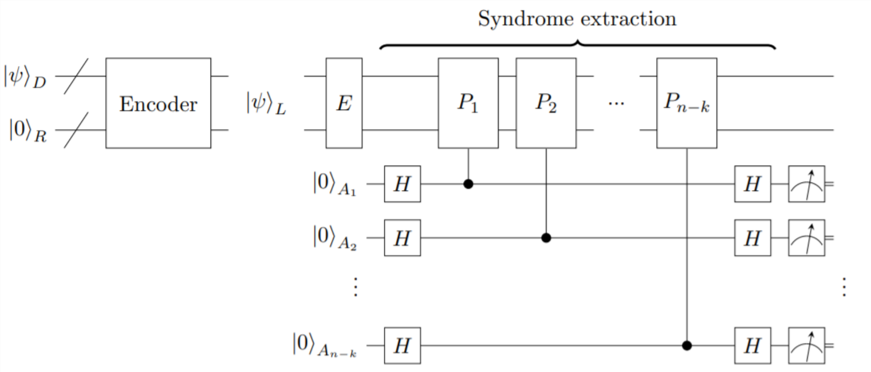
- Hence the probability that a undetected error occured is

$$p_L = \left|\frac{\alpha_X^2}{\sqrt{\left|\alpha_I^2\right|^2 + \left|\alpha_X^2\right|^2}}\right|^2 = \frac{p^2}{(1-p)^2 + p^2} \approx p^2$$

- Analogous to the classical case, we define quantum code distance as the minimum number of errors that can go undetected.
- Consier a $(3, 1)$ code similar to the previous example.
- If the qubits were susceptible to only X errors then minimum errors that go undeteced would be 3 as $X_1X_2X_3|0\rangle_L = |1\rangle_L$.
- However there are also Z errors. Consider

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$|+\rangle_L = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad \text{and} \quad |-\rangle_L = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$$

- The distance of this code will be 1 as $Z|+\rangle_L = |-\rangle_L$


Syndrome extraction
$|\psi\rangle_D$
$|0\rangle_R$
Encoder
$|\psi\rangle_L$
E
P_1
P_2
P_{n-k}
$|0\rangle_{A_1}$
$|0\rangle_{A_2}$
$|0\rangle_{A_{n-k}}$
H

- Stabilizer codes work by distributing the information present in a quantum system to a larger quantum system.
- A register of k data qubits, $|\psi\rangle_D$, is entangled with $m = n - k$ redundancy qubits $|0\rangle_R$ via an encoding operation to create a logical qubit $|\psi\rangle_L$.
- Stabilizers are used for syndrome extraction. For each stabiliser P_i, the syndrome extraction circuits gives following mapping.

$$E|\psi\rangle_L|0\rangle_{A_i} \longrightarrow \frac{1}{2}\left(I^{\otimes n} + P_i\right) E|\psi\rangle_L|0\rangle_{A_i} + \frac{1}{2}\left(I^{\otimes n} - P_i\right) E|\psi\rangle_L|1\rangle_{A_i}$$

- Shor code is the first code proposed that has the capability of not only detecting but correcting error as well for any single qubit error.
- Shor code uses code concatenation. The two codes concatenated are the three-qubit code for bit-flips and the three-qubit code for phase-flips.

$$
\begin{aligned}
\mathcal{C}_{3b} &= \text{span}\left\{|0\rangle_{3b} = |000\rangle, |1\rangle_{3b} = |111\rangle\right\}, & \mathcal{S}_{3b} &= \langle Z_1Z_2, Z_2Z_3\rangle \\
\mathcal{C}_{3\mathrm{p}} &= \text{span}\left\{|0\rangle_{3p} = |+++\rangle, |1\rangle_{3p} = |---\rangle\right\}, & \mathcal{S}_{3p} &= \langle X_1X_2, X_2X_3\rangle
\end{aligned}
$$

- The output of bit flip code is fed as input to phase flip code obtaining a 9 qubit codeword.

$$\mathcal{C}_{[|9,1,3]]} = \text{span}\left\{ \begin{array}{l} |0\rangle_9 = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ |1\rangle_9 = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \end{array} \right\}$$

$$\mathcal{S}_{[[9,3,3]]} = \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \rangle .$$

Now we measure the 8 bit syndrome using the stabilizers in $\mathcal{S}_{[[9,3,3]]}$

Error	Syndrome, S	Error	Syndrome, S
X_1	10000000	Z_1	00000010
X_2	11000000	Z_2	00000010
X_3	01000000	Z_3	00000010
X_4	00100000	Z_4	00000011
X_5	00110000	Z_5	00000011
X_6	00010000	Z_6	00000011
X_7	00001000	Z_7	00000001
X_8	00001100	Z_8	00000001
X_9	00000100	Z_9	00000001

- Each single bit flip error has a unique syndrome and thus we can find the corresponding operator $\mathcal{R}$ such that $\mathcal{R}E|\psi\rangle_L = |\psi\rangle_L$.
- For example, if syndromes is 10000000, then first qubit has been subject to X error and by choosing $\mathcal{R} = X_1$, we can recover the codeword.
- In contrast, the syndrome of Z error is same if the error occurs in same block of code.
- We can still recover the codeword in following way. Suppose the error is Z_2. We choose $\mathcal{R} = Z_1$ based on the table. Then $\mathcal{R}E|\psi\rangle_L = Z_1Z_2|\psi\rangle_L$. Since Z_1Z_2 is a stabiliser, we recover the codeword correctly.
- Thus, Shor code has the ability to correct all single qubit errors and has $d = 3$.

- To determine recovery operator, we measure syndrome and use lookup table. For $m = n - k$, we have to store and search in a 2^m table which becomes very large as m increases.
- The threshold theorem for stabilizer codes states that increasing the distance of a code will result in a corresponding reduction in the logical error rate p_L, provided the physical error rate p of the individual code qubits is below a threshold $p < p_{th}$.
- This also poses a challenge that stabiliser codes can be useful only if our quantum experiments have error probability lower than threshold.
- We have assumed throughout that our syndrome extraction is error free however that is not the case. To take that into account, we have to increase overhead of coding.

Thank You!