A Bifurcation Analysis of The Black-Scholes Model

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Contents

1 Introduction

In this paper, we aim to explore the dynamical system analysis of the Black-Scholes (B-S) model. The B-S was the first model in the field of financial mathematics that elegantly priced European call and put options in the 1970's. Published by Economists Fischer Black and Myron Scholes in 1973, and later won them the Nobel Prize in Economics in 1997 [2], the partial differential equation (PDE) describes the theoretical price of the option using variables such as the stock price (S) and time to expiration (t) as well as parameters like risk-free interest rate (r), and the volatility of the underlying asset (σ) [1].

$$C(S_t, K, T, r, \sigma) = S_t N(d_1) - Ke^{-rT} N(d_2) \tag{1}$$

However, in this paper we will examine only the differential form of this equation and limit our focus to the case of the European call option, as in (2)

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \tag{2}$$

By setting each of the partial derivatives to zero, we will effectively transform the partial differential equation (PDE) into an ordinary differential equation (ODE) with two separate cases. To further simplify our analysis, we will apply similar techniques introduced in Section 3.7 of Strogatz's "Nonlinear Dynamics and Chaos." Specifically, we will first represent the stock price, S, as $e^{\wedge}v$, and introduce the variables X and Y for the price of the call option C and its derivative with respect to v. The culmination of these modifications will result in a 2-dimensional linear system, which we will then thoroughly investigate using tools from Section 5.1 and 5.2 in Strogatz's textbook to gain new insights into the behavior and properties of the Black-Scholes equation [3].

2 Transformation and Linearization of PDE to ODE

As described in the Introduction, the Black-Scholes Equation depends on both time and the price of the stock. For the sake of the analysis, in this project we assume that the partial derivative of the call option's price with respect to time is 0. This allows us to transform the equation into an ODE system that is ultimately with respect to the price of the stock only [4].

$$rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

The next step is to apply variable and parameter substitution techniques from Strogatz 3.7 to linearize the equation. Specifically, we have to first rewrite S, the price of the stock, in terms of an exponential.

Let $S = e^v$

$$\begin{aligned}
&\frac{dc}{dv} = \frac{dc}{ds}\frac{ds}{dv} \\
&= \frac{dC}{ds}e^x \\
&= s\frac{dC}{ds} \\
&\implies \frac{d^2c}{dv^2} = \frac{d}{ds}\left(s\frac{dC}{ds}\right)\frac{ds}{dv} \\
&= \frac{dC}{ds}\frac{ds}{dv} + s\frac{d^2c}{ds^2}\frac{ds}{dv} \\
&= s\frac{dC}{ds} + s^2\frac{d^2c}{ds^2}
\end{aligned} \tag{3}$$

By reorganizing the second equation, we then have:

$$\begin{aligned}
s\frac{dc}{ds} &= \frac{dC}{dv} \\
s^2\frac{d^2C}{ds^2} &= \frac{d^2c}{dv^2} - \frac{dC}{dv}
\end{aligned} \tag{4}$$

We could then directly substitute these two equations into the time independent Black-Scholes to obtain

$$r\frac{dC}{dv} + \frac{1}{2}\sigma^2\left(\frac{d^2C}{dv^2} - \frac{dC}{dv}\right) - rC = 0$$

3 Matrix Form

To write the above second order differential equation as a matrix, we introduce variable x and y.

$$x(v) = C(v)$$
$$y(v) = \frac{dC}{dv}$$

The substitution would therefore yield the following 2-D linear ODE system:

$$\frac{dx}{dv} = y$$
$$\frac{dy}{dv} = \frac{2r}{\sigma^2}x + \left(1 - \frac{2r}{\sigma^2}\right)y$$

Which we rewrite as

$$X' = A\vec{X} \tag{5}$$

Where

$$\vec{X}$$

is the differential vector operator (gradient) and A is the matrix coefficients of the system above.
Our analysis is therefore,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2r}{\sigma^2} & 1 - \frac{2r}{\sigma^2} \end{pmatrix}$$
$$\tau = 1 - \frac{2r}{\sigma^2}$$
$$\Delta = -\frac{2r}{\sigma^2}$$

Where the symbols have their usual meanings of trace and determinant.

4 The Eigenvalues

We now proceed to calculate the discriminant of the matrix.

$$\tau^2 - 4\Delta = \left(1 - \frac{2r}{a}\right)^2 + 4\frac{2n}{\sigma^2}$$
$$= \left(1 + \frac{2r}{r}\right)^2$$

Note that this is strictly positive. Assuming the interest rate r is positive (which of course it generally would be), then the trace is negative, which tells us that we have a saddle point equilibrium and two real eigenvalues λ_1 and λ_2 such that

$$\lambda_1 > 0 > \lambda_2$$

Determining the eigenvalues and eigenvectors is as follows.

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2r}{\sigma^2} & 1 - \frac{2r}{\sigma^2} \end{pmatrix}$$

For simplicity, let $K = \frac{2r}{\sigma^2}$

$$\implies A = \begin{pmatrix} 0 & 1 \\ K & 1 - K \end{pmatrix} \tag{6}$$

Since A is a 2×2 matrix, we can consider the negative of the standard determinant to find the eigenvalues.

$$\det(A - \lambda I) = \det(\lambda I - A)$$
$$= \begin{vmatrix} \lambda & -1 \\ -k & -1 + k + \lambda \end{vmatrix}$$
$$= \lambda(\lambda + k - 1) - k$$
$$= \lambda^2 + \lambda(k - 1) - k$$
$$= (\lambda - 1)(\lambda + k) = 0$$
$$\Rightarrow \lambda_1 = 1$$
$$\lambda_2 = -k = \lambda_2 = \frac{-2r}{\sigma^2}$$

5 The Eigenvectors

Our modal matrix is given as

$$M(\lambda_i) = \begin{pmatrix} \lambda_i & -1 \\ -k & \lambda_i + k - 1 \end{pmatrix} \tag{7}$$

where K is defined as previously and $i = 1, 2$.

Eigenvector 1

$$m\vec{v}_1 = 0 \tag{8}$$

As usual, we consider the kernel where $\vec{V}_1$ is a 2D vector in real space (because we have real eigenvalues). Of course, we expect eigenvectors going in opposite directions to create the saddle point.

$$\begin{aligned} &\Rightarrow x_1 - y_1 = 0 \\ &\Rightarrow x_1 = y_1 \\ &\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Eigenvector 2

$$\begin{aligned} &M(\lambda_2) = \begin{pmatrix} \lambda_2 & -1 \\ -k & \lambda_2 + k - 1 \end{pmatrix} \\ &\Rightarrow m(\lambda_2 = -k) = \begin{pmatrix} -k & -1 \\ -k & -1 \end{pmatrix} \\ &\Rightarrow m\vec{v}_2 = 0 \\ &\Rightarrow \begin{pmatrix} -k & -1 \\ -k & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\ &- kx - y = 0 \\ &\Rightarrow y = kx \\ &\overrightarrow{v_2} = \begin{pmatrix} 1 \\ k \end{pmatrix} \end{aligned}$$

6 General Solution

For a 2D system of differential equations with independent variable v we have the general solution

$$\vec{x}(v) = c_1 e^{\lambda_1 v}\vec{v}_1 + c_2 e^{\lambda_2 v}\vec{v}_2 \tag{9}$$

where C_1 and C_2 are real constants determined by the initial conditions. In our case,

$$\Rightarrow \vec{x}(s) = c_1 e^{v}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-\frac{2r}{\sigma^2}v}\begin{pmatrix} 1 \\ \frac{2r}{\sigma^2} \end{pmatrix} \tag{10}$$

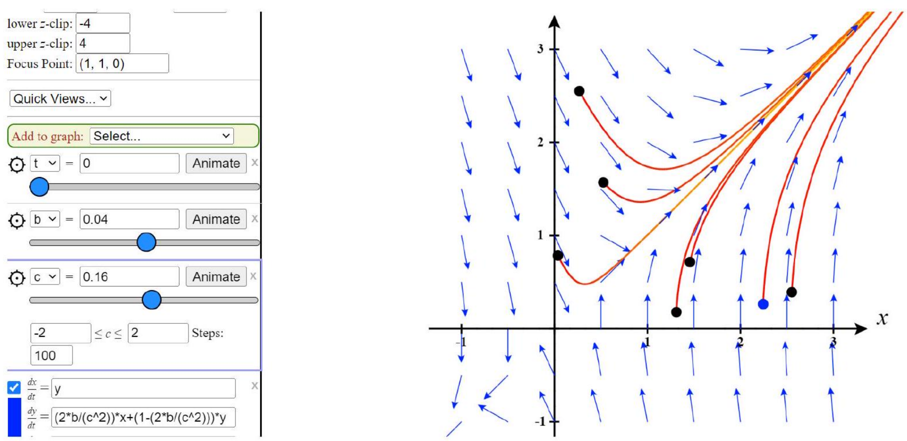
To obtain the solution to our original problem, we transform the equation back with the reverse transformation as follows.

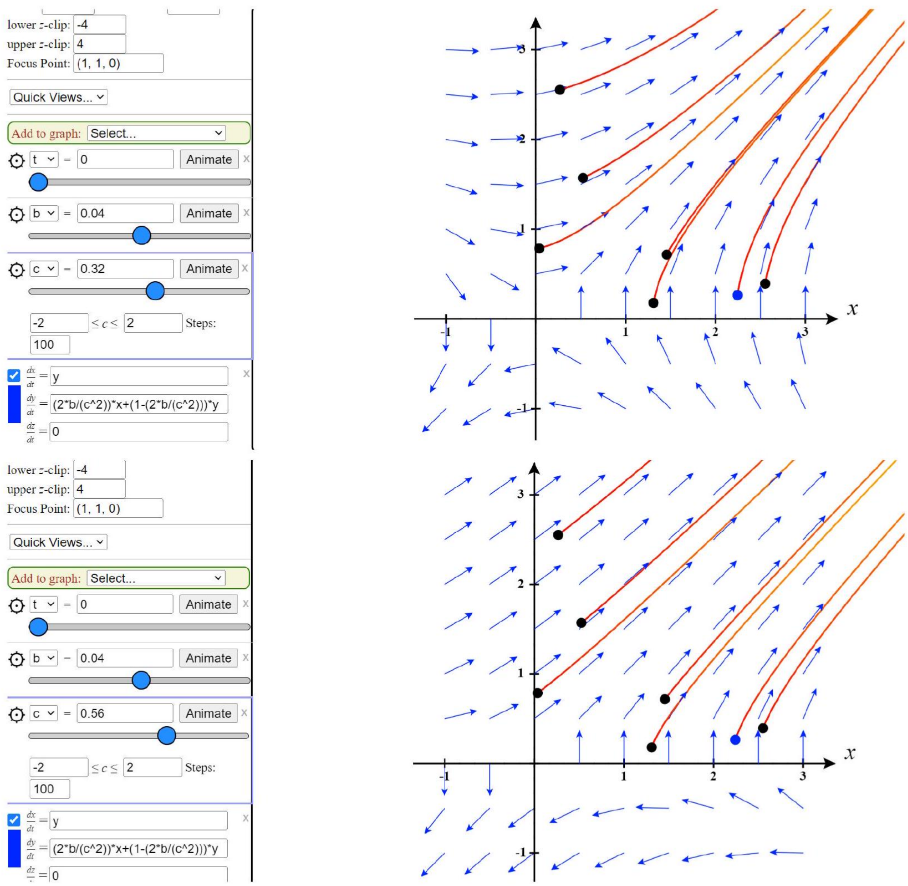
$$\begin{aligned}
&S = e^{v} \\
&\Rightarrow v = \ln s \\
&\Rightarrow \vec{x}(s) = c_1 e^{\ln s}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-\frac{2r}{\sigma^2}\ln s}\begin{pmatrix} 1 \\ \frac{2r}{\sigma^2} \end{pmatrix} \\
&= C_1 S\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{\ln s^{-\frac{2r}{\sigma^2}}}\begin{pmatrix} 1 \\ \frac{2r}{\sigma^2} \end{pmatrix} \\
&= C_1 S\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 s^{-\frac{2r}{\sigma^2}}\begin{pmatrix} 1 \\ \frac{2r}{\sigma^2} \end{pmatrix}
\end{aligned}$$

7 Phase Portrait Analysis

We conclude that our equilibrium is a saddle point. Assuming the interest rate, r, is positive, then our positive discriminant and positive determinant of the matrix indicates that the actual solutions will always approach a dominant vector. In the case of x starting at a positive initial value in the first quadrant which is the realistic assumption in any real world financial model, then according to our general solution, the larger eigenvalue suggest that x which is the price of the call option would eventually increases at the same rate as the y, the derivative of call option's price with respect to v, which is $ln(S)$.

Furthermore, a bifurcation analysis of the matrix gives us insights into the behavior of the solutions if we alter the parameters. In particular, we hold the interest rate, r (input as b in Calplot3D), constant, and use three different typical volatility values for sigma such as Google (0.2), Facebook (0.45) and Tesla (0.3) stock (input as c in Calplot3D) for our study [5]. The corresponding graphs shows that as the volatility of the stock increases, the general solution converges faster to the 1:1 rate mentioned above.





(b is the interest rate r) (c is the volatility σ)

8 Real World Example - Tesla

Although the Black-Scholes is no longer used in industry, and this paper is focused primarily on the qualitative features of the equation itself, we have included an example of pricing a call option in the real world. Assuming the following information for a call option on TSLA stock [6]

Strike price (K): 360.00usd
Option price (C): 0.01usd
Time to expiration (T): 0.0849 years (31 days until April 6, 2023)
Risk-free interest rate (r): 0.05
Underlying stock price (S): 687.20usd
Annualized volatility (σ): 0.6

The Black-Scholes formula can be used to calculate the theoretical value of the call option:

$$C = S\Phi(d_1) - Ke^{-rT}\Phi(d_2) \ d_1 \quad = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \ d_2 = d_1 - \sigma\sqrt{T} \tag{11}$$

Plugging in the values:

$$d_1 = \frac{1}{0.6\sqrt{0.0849}}\left[\ln\left(\frac{687.20}{360.00}\right) + \left(0.05 + \frac{0.6^2}{2}\right)0.0849\right] \tag{12}$$
$$= 1.7907 \tag{13}$$
$$d_2 = 1.7907 - 0.6\sqrt{0.0849} = 1.7096 \tag{14}$$
$$\Phi(d_1) = 0.9636 \tag{15}$$
$$\Phi(d_2) = 0.9477 \tag{16}$$
$$\implies C = 687.20 \times 0.9636 - 360.00 \times e^{-0.05\times 0.0849} \times 0.9477 = 327.29 \tag{17}$$

Therefore, the theoretical value of the call option is 327.29usd.

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10 References

References

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