Assignment 3:

1. **For a group of 7 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely. random day of the week**

   Let A_i be the event that there are no birthdays in the ith season. The probability that all seasons occur at least once is $1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Note that $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$. Using the inclusion-exclusion principle and the symmetry of the seasons,

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) &= \sum_{i=1}^{4} P(A_i) - \sum_{i=1}^{3} \sum_{j>i} P(A_i \cap A_j) \\ &\quad + \sum_{i=1}^{3} \sum_{j>i} \sum_{k>j} P(A_i \cap A_j \cap A_k) \\ &= 4P(A_1) - 6P(A_1 \cap A_2) + 4P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

   We have $P(A_1) = (3/4)^7$. Similarly,

$$P(A_1 \cap A_2) = \frac{1}{2^7} \text{ and } P(A_1 \cap A_2 \cap A_3) = \frac{1}{4^7}.$$

   Therefore, $P(A_1 \cup A_2 \cup A_3 \cup A_4) = 4(\frac{3}{4})^7 - \frac{6}{2^7} + \frac{4}{4^7}$. So the probability that all 4 seasons occur at least once is $1 - (4(\frac{3}{4})^7 - \frac{6}{2^7} + \frac{4}{4^7}) \approx 0.513$.

2. **Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday?**

   Solution: We will solve this both by direct counting and using inclusion-exclusion.

   Direct Counting Method: There are two general ways that Alice can have class every day: either she has 2 days with 2 classes and 3 days with 1 class, or she has 1 day with 3 classes, and has 1 class on each of the other 4 days. The number of possibilities for the former is $\binom{5}{2}\binom{6}{2}^2 6^3$ (choose the 2 days when she has 2 classes, and then select 2 classes on those days and 1 class for the other days). The number of possibilities for the latter is $\binom{5}{1}\binom{6}{3}6^4$. So the probability is

$$\frac{\binom{5}{2}\binom{6}{2}^2 6^3 + \binom{5}{1}\binom{6}{3}6^4}{\binom{30}{7}} = \frac{114}{377} \approx 0.302.$$