Assignment 5:

1. **Give an example of 3 events A, B, C which are pairwise independent but not independent. Hint: find an example where whether C occurs is completely determined if we know whether A occurred and whether B occurred, but completely undetermined if we know only one of these things.**

Example 2.5.5 (Pairwise independence doesn't imply independence). Consider two fair, independent coin tosses, and let A be the event that the first is Heads, B the event that the second is Heads, and C the event that both tosses have the same result. Then A, B, and C are pairwise independent but not independent, since $P(A \cap B \cap C) = 1/4$ while $P(A)P(B)P(C) = 1/8$. The point is that just knowing about A or just knowing about B tells us nothing about C, but knowing what happened with *both* A and B gives us information about C (in fact, in this case it gives us perfect information about C). $\square$

On the other hand, $P(A \cap B \cap C) = P(A)P(B)P(C)$ does not imply pairwise independence; this can be seen quickly by looking at the extreme case $P(A) = 0$, when the equation becomes $0 = 0$ and tells us nothing about whether B and C are independent.

We can define independence of any number of events similarly. Intuitively, the idea is that knowing what happened with any particular subset of the events gives us no information about what happened with the events not in that subset.

2. **A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green?**

Solution: Let A be the event that the initial marble is green, B be the event that the removed marble is green, and C be the event that the remaining marble is green. We need to find $P(C|B)$. There are several ways to find this; one natural way is to condition on whether the initial marble is green:

$$P(C|B) = P(C|B,A)P(A|B) + P(C|B,A^c)P(A^c|B) = 1P(A|B) + 0P(A^c|B).$$

To find $P(A|B)$, use Bayes' Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{1/2}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{1/2}{1/2 + 1/4} = \frac{2}{3}.$$

So $P(C|B) = 2/3$.