Assignment 10:

1. **A chicken lays n eggs. Each egg independently does or doesn't hatch, with probability p of hatching. For each egg that hatches, the chick does or doesn't survive (independently of the other eggs), with probability s of survival. Let N ← Bin(n, p) be the number of eggs which hatch, X be the number of chicks which survive, and Y be the number of chicks which hatch but don't survive (so X + Y = N). Find the marginal PMF of X, and the joint PMF of X and Y. Are they independent?**

Marginally we have $X \sim \text{Bin}(n, ps)$, as shown on a previous homework problem using a story proof (the eggs can be thought of as independent Bernoulli trials with probability ps of success for each). Here X and Y are *not* independent, unlike in the chicken-egg problem from class (where N was Poisson). This follows immediately from thinking about an *extreme case*: if $X = n$, then clearly $Y = 0$. So they are not independent: $P(Y = 0) < 1$, while $P(Y = 0|X = n) = 1$.

To find the joint distribution, condition on N and note that only the $N = i+j$ term is nonzero: for any nonnegative integers i, j with $i + j \leq n$,

$$\begin{aligned}
P(X = i, Y = j) &= P(X = i, Y = j|N = i + j)P(N = i + j) \\
&= P(X = i|N = i + j)P(N = i + j) \\
&= \binom{i+j}{i} s^i (1-s)^j \binom{n}{i+j} p^{i+j}(1-p)^{n-i-j} \\
&= \frac{n!}{i!j!(n-i-j)!}(ps)^i(p(1-s))^j(1-p)^{n-i-j}.
\end{aligned}$$

If we let Z be the number of eggs which don't hatch, then from the above we have that (X, Y, Z) has a Multinomial$(n, (ps, p(1-s), (1-p))$ distribution, which makes sense intuitively since each egg independently falls into 1 of 3 categories: hatch-and-survive, hatch-and-don't-survive, and don't-hatch, with probabilities $ps, p(1-s), 1-p$ respectively.