MATHEMATICS

STUDENT TEXTBOOK

GRADE 9

FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA

MINISTRY OF EDUCATION

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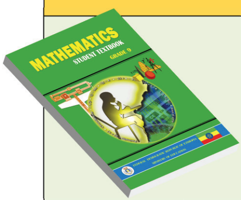

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MATHEMATICS

STUDENT TEXTBOOK

GRADE 9

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STAR

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Egyptian Hieroglyphic	I	II	III	IIIII	Λ	ΛΛ	IΛΛ	ϑ
Greek Herodianic	Ι	ΙΙ	ΙΙΙ	Γ	Δ	ΔΔ	ΔΔΙ	Η
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Unit 1

THE NUMBER SYSTEM

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts and important facts about real numbers.*
- *justify methods and procedures in computation with real numbers.*
- *solve mathematical problems involving real numbers.*

Main Contents

1.1 Revision on the set of rational numbers

1.2 The real number system

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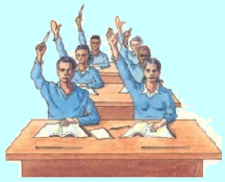
INTRODUCTION

In earlier grades, you have learnt about rational numbers, their properties, and basic mathematical operations upon them. After a review of your knowledge about rational numbers, you will continue studying the number systems in the present unit. Here, you will learn about irrational numbers and real numbers, their properties and basic operations upon them. Also, you will discuss some related concepts such as approximation, accuracy, and scientific notation.

1.1 REVISION ON THE SET OF RATIONAL NUMBERS

ACTIVITY 1.1



The diagram below shows the relationships between the sets of Natural numbers, Whole numbers, Integers and Rational numbers. Use this diagram to answer Questions 1 and 2 given below. Justify your answers.

1 To which set(s) of numbers does each of the following numbers belong?

a 27 **b** -17 **c** $-7\frac{2}{3}$

d 0.625 **e** $0.\overline{615}$

2 **i** Define the set of:

a Natural numbers

b Whole numbers

c Integers

d Rational numbers

ii What relations do these sets have?

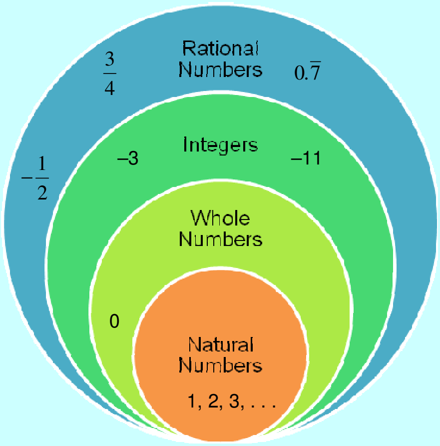


Figure 1.1

1.1.1 Natural Numbers, Integers, Prime Numbers and Composite Numbers

In this subsection, you will revise important facts about the sets of natural numbers, prime numbers, composite numbers and integers. You have learnt several facts about these sets in previous grades, in Grade 7 in particular. Working through Activity 1.2 below will refresh your memory!

ACTIVITY 1.2

1 For each of the following statements write 'true' if the statement is correct or 'false' otherwise. If your answer is 'false', justify by giving a counter example or reason.

- **a** The set {1, 2, 3, . . .} describes the set of natural numbers.
- **b** The set $\{1, 2, 3, \ldots\} \cup \{\ldots -3, -2, -1\}$ describes the set of integers.
- **c** 57 is a composite number.
- **d** $\{1\} \cap \{\text{Prime numbers}\} = \varnothing$.
- **e** $\{\text{Prime numbers}\} \cup \{\text{Composite number}\} = \{1, 2, 3, \ldots\}$.
- **f** $\{\text{Odd numbers}\} \cap \{\text{Composite numbers}\} \neq \varnothing$.
- **g** 48 is a multiple of 12.
- **h** 5 is a factor of 72.
- **i** 621 is divisible by 3.
- **j** $\{\text{Factors of } 24\} \cap \{\text{Factors of } 87\} = \{1, 2, 3\}$.
- **k** $\{\text{Multiples of } 6\} \cap \{\text{Multiples of } 4\} = \{12, 24\}$.
- **l** $2^2 \times 3^2 \times 5$ is the prime factorization of 180.

2 Given two natural numbers a and b, what is meant by:

a a is a factor of b **b** a is divisible by b **c** a is a multiple of b

From your lower grade mathematics, recall that;

✓ The set of natural numbers, denoted by $\mathbb{N}$, is described by $\mathbb{N} = \{1, 2, 3,\ldots\}$

✓ The set of whole numbers, denoted by $\mathbb{W}$, is described by $\mathbb{W} = \{0, 1, 2, 3,\ldots\}$

✓ The set of integers, denoted by $\mathbb{Z}$, is described by $\mathbb{Z} = \{\ldots,-3, -2, -1, 0, 1, 2, 3,\ldots\}$

✓ Given two natural numbers m and p, m is called a multiple of p if there is a natural number q such that

$$m = p \times q.$$

In this case, p is called a factor or divisor of m. We also say m is divisible by p. Similarly, q is also a factor or divisor of m, and m is divisible by q.

For example, 621 is a multiple of 3 because $621 = 3 \times 207$.

Definition 1.1 Prime numbers and composite numbers

- A natural number that has exactly two distinct factors, namely 1 and itself, is called a prime number.
- A natural number that has more than two factors is called a composite number.

Note: 1 is neither prime nor composite.

Group Work 1.1

1. List all factors of 24. How many factors did you find?
2. The area of a rectangle is 432 sq. units. The measurements of the length and width of the rectangle are expressed by natural numbers.

   Find all the possible dimensions (length and width) of the rectangle.
3. Find the prime factorization of 360.

The following rules can help you to determine whether a number is divisible by 2, 3, 4, 5, 6, 8, 9 or 10.

Divisibility test

A number is divisible by:

- ✓ 2, if its unit's digit is divisible by 2.
- ✓ 3, if the sum of its digits is divisible by 3.
- ✓ 4, if the number formed by its last two digits is divisible by 4.
- ✓ 5, if its unit's digit is either 0 or 5.
- ✓ 6, if it is divisible by 2 and 3.
- ✓ 8, if the number formed by its last three digits is divisible by 8.
- ✓ 9, if the sum of its digits is divisible by 9.
- ✓ 10, if its unit's digit is 0.

Observe that divisibility test for 7 is not stated here as it is beyond the scope of your present level.

Example 1 Use the divisibility test to determine whether 2,416 is divisible by 2, 3, 4, 5, 6, 8, 9 and 10.

Solution:
- 2, 416 is divisible by 2 because the unit's digit 6 is divisible by 2.
- 2,416 is divisible by 4 because 16 (the number formed by the last two digits) is divisible by 4.
- 2,416 is divisible by 8 because the number formed by the last three digits (416) is divisible by 8.
- 2,416 is not divisible by 5 because the unit's digit is not 0 or 5.
- Similarly you can check that 2,416 is not divisible by 3, 6, 9, and 10.

Therefore, 2,416 is divisible by 2, 4 and 8 but not by 3, 5, 6, 9 and 10.

A factor of a composite number is called a prime factor, if it is a prime number. For instance, 2 and 5 are both prime factors of 20.

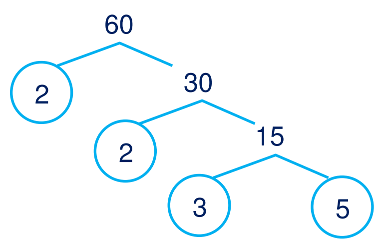
Every composite number can be written as a product of prime numbers. To find the prime factors of any composite number, begin by expressing the number as a product of two factors where at least one of the factors is prime. Then, continue to factorize each resulting composite factor until all the factors are prime numbers.

When a number is expressed as a product of its prime factors, the expression is called the prime factorization of the number.

For example, the prime factorization of 60 is

$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5.$$

The prime factorization of 60 is also found by using a factoring tree.



Note that the set $\{2, 3, 5\}$ is a set of prime factors of 60. Is this set unique? This property leads us to state the Fundamental Theorem of Arithmetic.

Theorem 1.1 Fundamental theorem of arithmetic

Every composite number can be expressed (factorized) as a product of primes. This factorization is unique, apart from the order in which the prime factors occur.

You can use the divisibility tests to check whether or not a prime number divides a given number.

Example 2 Find the prime factorization of 1,530.

Solution: Start dividing 1,530 by its smallest prime factor. If the quotient is a composite number, find a prime factor of the quotient in the same way.

Repeat the procedure until the quotient is a prime number as shown below.

Prime factors

$$\downarrow$$

$$1{,}530 \div 2 = 765$$
$$765 \div 3 = 255$$
$$255 \div 3 = 85$$
$$85 \div 5 = 17; \text{ and 17 is a prime number.}$$

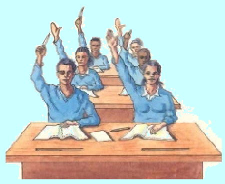
Therefore, $1{,}530 = 2 \times 3^2 \times 5 \times 17$.

1.1.2 Common Factors and Common Multiples

In this subsection, you will revise the concepts of common factors and common multiples of two or more natural numbers. Related to this, you will also revise the greatest common factor and the least common multiple of two or more natural numbers.

A Common factors and the greatest common factor

ACTIVITY 1.3



1 Given the numbers 30 and 45,

a find the common factors of the two numbers.

b find the greatest common factor of the two numbers.

2 Given the numbers 36, 42 and 48,

a find the common factors of the three numbers.

b find the greatest common factor of the three numbers.

Given two or more natural numbers, a number which is a factor of all of them is called a common factor. Numbers may have more than one common factor. The greatest of the common factors is called the greatest common factor (GCF) or the highest common factor (HCF) of the numbers.

➢ The greatest common factor of two numbers a and b is denoted by GCF (a, b).

Example 1 Find the greatest common factor of:

a 36 and 60. **b** 32 and 27.

Solution:

a First, make lists of the factors of 36 and 60, using sets.

Let F_{36} and F_{60} be the sets of factors of 36 and 60, respectively. Then,

$F_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$F_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

You can use the diagram to summarize the information. Notice that the common factors are shaded in *green*. They are 1, 2, 3, 4, 6 and 12 and the greatest is 12.

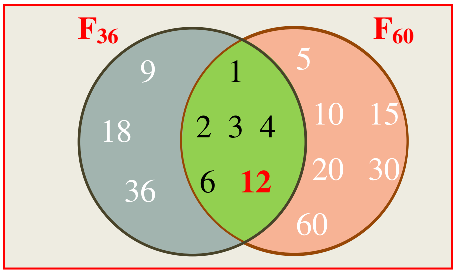


Figure 1.2

i.e., $F_{36} \cap F_{60} = \{1, 2, 3, 4, 6, 12\}$

Therefore, GCF (36, 60) = 12.

b Similarly,

$F_{32} = \{1, 2, 4, 8, 16, 32\}$ and

$F_{27} = \{1, 3, 9, 27\}$

Therefore, $F_{32} \cap F_{27} = \{1\}$

Thus, GCF (32, 27) = 1

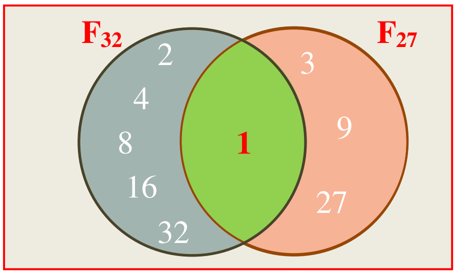


Figure 1.3

➢ Two or more natural numbers that have a GCF of 1 are called **relatively prime.**

Definition 1.2

The greatest common factor (GCF) of two or more natural numbers is the greatest natural number that is a factor of all of the given numbers.

Group Work 1.2

Let $a = 1800$ and $b = 756$

1 Write:

a the prime factorization of a and b

b the prime factors that are common to both a and b.

Now look at these common prime factors; the lowest powers of them (in the two prime factorizations) should be 2^2 and 3^2.

c What is the product of these lowest powers?

d Write down the highest powers of the common prime factors.

e What is the product of these highest powers?

2 **a** Compare the result of 1c with the GCF of the given numbers.

Are they the same?

b Compare the result of 1e with the GCF of the given numbers.

Are they the same?

The above Group Work leads you to another alternative method to find the GCF of numbers. This method (which is a quicker way to find the GCF) is called the prime factorization method. In this method, the GCF of a given set of numbers is the product of their common prime factors, each power to the smallest number of times it appears in the prime factorization of any of the numbers.

Example 2 Use the prime factorization method to find GCF (180, 216, 540).

Solution:

Step 1 Express the numbers 180, 216 and 540 in their prime factorization.

$180 = 2^2 \times 3^2 \times 5$; $216 = 2^3 \times 3^3$; $540 = 2^2 \times 3^3 \times 5$

Step 2 As you see from the prime factorizations of 180, 216 and 540, the numbers 2 and 3 are common prime factors.

So, GCF (180, 216, 540) is the product of these common prime factors with the smallest respective exponents in any of the numbers.

$\therefore$ GCF (180, 216, 540) $= 2^2 \times 3^2 = 36$.

B Common multiples and the least common multiple

Group Work 1.3

For this group work, you need 2 coloured pencils.

Work with a partner

Try this:

* List the natural numbers from 1 to 100 on a sheet of paper.
* Cross out all the multiples of 10.
* Using a different colour, cross out all the multiples of 8.

Discussion:

1. Which numbers were crossed out by both colours?
2. How would you describe these numbers?
3. What is the least number crossed out by both colours? What do you call this number?

Definition 1.3

For any two natural numbers a and b, the least common multiple of a and b denoted by LCM (a, b), is the smallest multiple of both a and b.

Example 3 Find LCM (8, 9).

Solution: Let M_8 and M_9 be the sets of multiples of 8 and 9 respectively.

$$M_8 = \{8, 16, 24, 32, 40, 48, 56, 64, \boxed{72}, 80, 88, \ldots\}$$

$$M_9 = \{9, 18, 27, 36, 45, 54, 63, \boxed{72}, 81, 90, \ldots\}$$

Therefore LCM (8, 9) = 72

Prime factorization can also be used to find the LCM of a set of two or more than two numbers. A common multiple contains all the prime factors of each number in the set. The LCM is the product of each of these prime factors to the greatest number of times it appears in the prime factorization of the numbers.

Example 4 Use the prime factorization method to find LCM (9, 21, 24).

Solution:

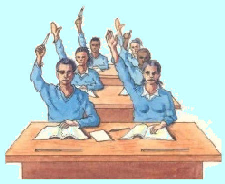
$$\left.\begin{aligned} 9 &= 3 \times 3 = 3^2 \\ 21 &= 3 \times 7 \\ 24 &= 2 \times 2 \times 2 \times 3 = 2^3 \times 3 \end{aligned}\right\}$$

The prime factors that appear in these factorizations are 2, 3 and 7.

Considering the greatest number of times each prime factor appears, we can get 2^3, 3^2 and 7, respectively.

Therefore, LCM (9, 21, 24) = $2^3 \times 3^2 \times 7 = 504$.

ACTIVITY 1.4



1 Find:

- **a** The GCF and LCM of 36 and 48
- **b** GCF (36, 48) × LCM (36, 48)
- **c** 36 × 48

2 Discuss and generalize your results.

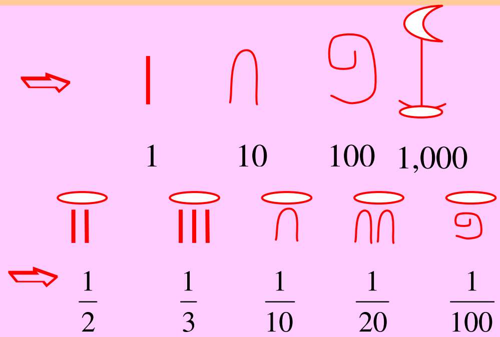
➢ For any natural numbers a and b, GCF (a, b) × LCM $(a, b) = a \times b$.

1.1.3 Rational Numbers

Historical Note:

About 5,000 years ago, Egyptians used hieroglyphics to represent numbers.

The Egyptian concept of fractions was mostly limited to fractions with numerator 1. The hieroglyphic was placed under the symbol to indicate the number as a denominator. Study the examples of Egyptian fractions.



Recall that the set of integers is given by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Using the set of integers, we define the set of rational numbers as follows:

Definition 1.4 **Rational number**

Any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, is called a rational number. The set of rational numbers, denoted by $\mathbb{Q}$, is the set described by

$$\mathbb{Q} = \left\{\frac{a}{b} : a \text{ and } b \text{ are integers and } b \neq 0\right\}.$$

Through the following diagram, you can show how sets within rational numbers are related to each other. Note that natural numbers, whole numbers and integers are included in the set of rational numbers. This is because integers such as 4 and -7 can be written as $\frac{4}{1}$ and $\frac{-7}{1}$.

The set of rational numbers also includes terminating and repeating decimal numbers because terminating and repeating decimals can be written as fractions.

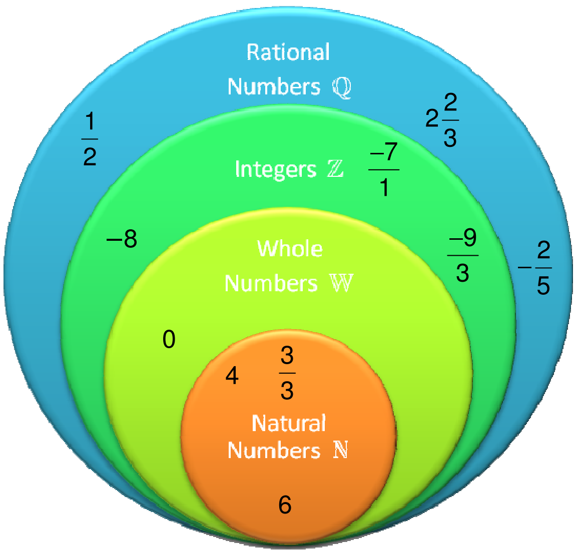


Figure 1.4

For example, -1.3 can be written as $\frac{-13}{10}$ and -0.29 as $\frac{-29}{100}$.

Mixed numbers are also included in the set of rational numbers because any mixed number $\frac{30}{45} = \frac{2\times3\times5}{3\times3\times5} = \frac{2}{3}$. can be written as an improper fraction.

For example, $2\frac{2}{3}$ can be written as $\frac{8}{3}$.

When a rational number is expressed as a fraction, it is often expressed in simplest form (lowest terms). A fraction $\frac{a}{b}$ is in simplest form when GCF $(a, b) = 1$.

Example 1 Write $\frac{30}{45}$ in simplest form.

Solution: $\frac{30}{45} = \frac{2\times3\times5}{3\times3\times5} = \frac{2}{3}$. (by factorization and cancellation)

Hence $\frac{30}{45}$ when expressed in lowest terms (simplest form) is $\frac{2}{3}$.

Exercise 1.1

1 Determine whether each of the following numbers is prime or composite:

a 45 **b** 23 **c** 91 **d** 153

2 Prime numbers that differ by two are called twin primes.

i Which of the following pairs are twin primes?

a 3 and 5 **b** 13 and 17 **c** 5 and 7

ii List all pairs of twin primes that are less than 30.

3 Determine whether each of the following numbers is divisible by 2, 3, 4, 5, 6, 8, 9 or 10:

a 48 **b** 153 **c** 2,470

d 144 **e** 12,357

4 **a** Is 3 a factor of 777? **b** Is 989 divisible by 9?

c Is 2,348 divisible by 4?

5 Find three different ways to write 84 as a product of two natural numbers.

6 Find the prime factorization of:

a 25 **b** 36 **c** 117 **d** 3,825

7 Is the value of $2a + 3b$ prime or composite when $a = 11$ and $b = 7$?

8 Write all the common factors of 30 and 42.

9 Find:

a GCF (24, 36) **b** GCF (35, 49, 84)

10 Find the GCF of $2 \times 3^3 \times 5^2$ and $2^3 \times 3 \times 5^2$.

11 Write three numbers that have a GCF of 7.

12 List the first six multiples of each of the following numbers:

a 7 **b** 5 **c** 14 **d** 25 **e** 150

13 Find:

a LCM (12, 16) **b** LCM (10, 12, 14)

c LCM (15, 18) **d** LCM (7, 10)

14 When will the LCM of two numbers be the product of the numbers?

15 Write each of the following fractions in simplest form:

a $-\frac{3}{9}$ **b** $\frac{24}{120}$ **c** $\frac{48}{72}$ **d** $\frac{72}{98}$

16 How many factors does each of the following numbers have?

a 12 **b** 18 **c** 24 **d** 72

17 Find the value of an odd natural number x if LCM $(x, 40) = 1400$.

18 There are between 50 and 60 eggs in a basket. When Mohammed counts by 3's, there are 2 eggs left over. When he counts by 5's there are 4 left over. How many eggs are there in the basket?

19 The GCF of two numbers is 3 and the LCM is 180. If one of the numbers is 45, what is the other number?

20 **i** Let a, b, c, d be non-zero integers. Show that each of the following is a rational number:

a $\frac{a}{b}+\frac{c}{d}$ **b** $\frac{a}{b}-\frac{c}{d}$ **c** $\frac{a}{b}\times\frac{c}{d}$ **d** $\frac{a}{b}\div\frac{c}{d}$

What do you conclude from these results?

ii Find two rational numbers between $\frac{1}{3}$ and $\frac{3}{4}$.

1.2 THE REAL NUMBER SYSTEM

1.2.1 Representation of Rational Numbers by Decimals

In this subsection, you will learn how to express rational numbers in the form of fractions and decimals.

ACTIVITY 1.5

1 a What do we mean by a 'decimal number'?

b Give some examples of decimal numbers.

2 How do you represent $\frac{3}{4}$ and $\frac{1}{3}$ as decimals?

3 Can you write 0.4 and 1.34 as the ratio (or quotient) of two integers?

Remember that a fraction is another way of writing division of one quantity by another. Any fraction of natural numbers can be expressed as a decimal by dividing the numerator by the denominator.

Example 1 Show that $\frac{3}{8}$ and $\frac{7}{12}$ can each be expressed as a decimal.

Solution: $\frac{3}{8}$ means $3 \div 8$ $\qquad$ $\frac{7}{12}$ means $7 \div 12$

$$\begin{array}{r|l} & 0.375 \\ \hline 8 & 3.000 \\ & \underline{24} \\ & \ \ 60 \\ & \ \ \underline{56} \\ & \ \ \ \ 40 \\ & \ \ \ \ \underline{40} \\ & \ \ \ \ \ \ 0 \end{array} \qquad \begin{array}{r|l} & 0.5833\ldots \\ \hline 12 & 7.0000 \\ & \underline{60} \\ & 100 \\ & \ \ \underline{96} \\ & \ \ \ \ 40 \\ & \ \ \ \ \underline{36} \\ & \ \ \ \ \ \ 40 \\ & \ \ \ \ \ \ \underline{36} \\ & \ \ \ \ \ \ \ \ 4 \end{array}$$

$\therefore \quad \frac{3}{8} = 0.375$ $\qquad$ $\therefore \quad \frac{7}{12} = 0.5833\ldots$

The fraction (rational number) $\frac{3}{8}$ can be expressed as the decimal 0.375. A decimal like 0.375 is called a terminating decimal because the division ends or terminates, when the remainder is zero.

The fraction $\frac{7}{12}$ can be expressed as the decimal 0.58333… (Here, the digit 3 repeats and the division does not terminate.) A decimal like 0.58333... is called a repeating decimal. To show a repeating digit or a block of repeating digits in a repeating decimal number, we put a bar above the repeating digit (or block of digits). For example 0.58333… can be written as $0.58\overline{3}$, and 0.0818181… can be written as $0.0\overline{81}$. This method of writing a repeating decimal is known as bar notation.

The portion of a decimal that repeats is called the repetend. For example,

In 0.583333… $= 0.58\overline{3}$, the repetend is 3.

In 1.777… $= 1.\overline{7}$, the repetend is 7.

In 0.00454545… $= 0.00\overline{45}$, the repetend is 45.

To generalize:

Any rational number $\frac{a}{b}$ can be expressed as a decimal by dividing the numerator a by the denominator b.

When you divide a by b, one of the following two cases will occur.

Case 1 The division process ends or terminates when a remainder of zero is obtained. In this case, the decimal is called a terminating decimal.

Case 2 The division process does not terminate as the remainder never becomes zero. Such a decimal is called a repeating decimal.

Expressing terminating and repeating decimals as fractions

➢ Every terminating decimal can be expressed as a fraction (a ratio of two integers) with a denominator of 10, 100, 1000 and so on.

Example 2 Express each of the following decimals as a fraction in its simplest form (lowest terms):

a 0.85 **b** 1.3456

Solution:

a $0.85 = 0.85 \times \frac{100}{100} = \frac{85}{100} = \frac{17}{20}$ (Why?)

b $1.3456 = 1.3456 \times \frac{10000}{10000} = 1.3456 \times \frac{10^4}{10^4} = \frac{13456}{10000} = \frac{841}{625}$

➢ If d is a terminating decimal number that has n digits after a decimal point, then we rewrite d as

$$d = \frac{10^n \times d}{10^n}$$

The right side of the equation gives the fractional form of d.

For example, if $d = 2.128$, then $n = 3$.

$$\therefore \quad 2.128 = \frac{10^3 \times 2.128}{10^3} = \frac{2128}{1000} = \frac{266}{125}$$

✓ Repeating decimals can also be expressed as fractions (ratios of two integers).

Example 3 Express each of the following decimals as a fraction (ratio of two integers):

a $0.\overline{7}$ **b** $0.\overline{25}$

Solution: a Let $d = 0.\overline{7} = 0.777\ldots$ then,

$10d = 7.777\ldots$ (*multiplying d by* 10 *because* 1 *digit repeats*)

Subtract $d = 0.777\ldots$ (*to eliminate the repeating part* $0.777\ldots$)

$10d = 7.777\ldots$ 1

$1d = 0.777\ldots$ 2 ($d = 1d$)

$9d = 7$ (*subtracting expression* 2 *from expression* 1)

$\therefore d = \frac{7}{9}$ (*dividing both sides by* 9)

Hence $0.\overline{7} = \frac{7}{9}$

b Let $d = 0.\overline{25} = 0.252525\ldots$

Then, $100d = 25.2525\ldots$ (*multiplying d by* 100 *because* 2 *digits repeat*)

$$100d = 25.252525\ldots$$
$$1d = \ \ 0.252525\ldots$$

(subtracting 1d from 100d eliminates the repeating part 0.2525…)

$$99d = 25$$

$$\therefore d = \frac{25}{99}$$

$$\text{So, } 0.\overline{25} = \frac{25}{99}$$

In Example 3a, one digit repeats. So, you multiplied d by 10. In Example 3b, two digits repeat. So you multiplied d by 100.

The algebra used in the above example can be generalized as follows:

➢ In general, if d is a repeating decimal with k non-repeating and p repeating digits after the decimal point, then the formula

$$d = \frac{d\left(10^{k+p} - 10^{k}\right)}{10^{k+p} - 10^{k}}$$

is used to change the decimal to the fractional form of d.

Example 4 Express the decimal $0.3\overline{75}$ as a fraction.

Solution: Let $d = 0.3\overline{75}$, then,

$k = 1$ (*number of non-repeating digits)*

$p = 2$ (*number of repeating digits*) and

$k + p = 1 + 2 = 3.$

$$\Rightarrow d = \frac{d\left(10^{k+p} - 10^{k}\right)}{\left(10^{k+p} - 10^{k}\right)} = \frac{d\left(10^{3} - 10^{1}\right)}{\left(10^{3} - 10^{1}\right)} = \frac{10^{3}d - 10d}{10^{3} - 10}$$

$$= \frac{10^{3} \times 0.3\overline{75} - 10 \times 0.3\overline{75}}{990}$$

$$= \frac{375.\overline{75} - 3.\overline{75}}{990} = \frac{372}{990}$$

From Examples 1, 2, 3 and 4, you conclude the following:

i Every rational number can be expressed as either a terminating decimal or a repeating decimal.

ii Every terminating or repeating decimal represents a rational number.

Exercise 1.2

1 Express each of the following rational numbers as a decimal:

a $\frac{4}{9}$ **b** $\frac{3}{25}$ **c** $\frac{11}{7}$ **d** $-5\frac{2}{3}$ **e** $\frac{3706}{100}$ **f** $\frac{22}{7}$

2 Write each of the following as a decimal and then as a fraction in its lowest term:

a three tenths **b** four thousandths

c twelve hundredths **d** three hundred and sixty nine thousandths.

3 Write each of the following in metres as a fraction and then as a decimal:

a 4 mm **b** 6 cm and 4 mm **c** 56 cm and 4 mm

Hint: Recall that 1 metre(m) = 100 centimetres(cm) = 1000 millimetres(mm).

4 From each of the following fractions, identify those that can be expressed as terminating decimals:

a $\frac{5}{13}$ **b** $\frac{7}{10}$ **c** $\frac{69}{64}$ **d** $\frac{11}{60}$

e $\frac{11}{80}$ **f** $\frac{17}{125}$ **g** $\frac{5}{12}$ **h** $\frac{4}{11}$

Generalize your observation.

5 Express each of the following decimals as a fraction or mixed number in simplest form:

a 0.88 **b** $0.7\overline{7}$ **c** $0.8\overline{3}$ **d** 7.08 **e** $0.52\overline{52}$ **f** $-1.00\overline{3}$

6 Express each of the following decimals using bar notation:

a 0.454545… **b** 0.1345345…

7 Express each of the following decimals without bar notation. (In each case use at least ten digits after the decimal point)

a $0.\overline{13}$ **b** $-0.\overline{305}$ **c** $0.3\overline{81}$

8 Verify each of the following computations by converting the decimals to fractions:

a $0.\overline{275} + 0.\overline{714} = 0.\overline{989}$ **b** $0.\overline{6} - 1.\overline{142857} = -0.\overline{476190}$

1.2.2 Irrational Numbers

Remember that terminating or repeating decimals are rational numbers, since they can be expressed as fractions. The square roots of perfect squares are also rational numbers. For example, $\sqrt{4}$ is a rational number since $\sqrt{4} = 2 = \frac{2}{1}$. Similarly, $\sqrt{0.09}$ is a rational number because, $\sqrt{0.09} = 0.3$ is a rational number.

If $x^2 = 4$, then what do you think is the value of x?

$x = \pm\sqrt{4} = \pm 2$. Therefore x is a rational number. What if $x^2 = 3$?

In Figure 1.4 of Section 1.1.3, where do numbers like $\sqrt{2}$ and $\sqrt{5}$ fit? Notice what happens when you find $\sqrt{2}$ and $\sqrt{5}$ with your calculator:

> **Study Hint**
>
> Most calculators round answers but some truncate answers. i.e., they cut off at a certain point, ignoring subsequent digits.

If you first press the button 2 and then the square-root button, you will find $\sqrt{2}$ *on the display.*

i.e., $\sqrt{2}$: 2 √ = 1.414213562…

$\sqrt{5}$: 5 √ = 2.236067977…

Note that many scientific calculators, such as Casio ones, work the same as the written order, i.e., instead of pressing 2 and then the √ button, you press the √ button and then 2. Before using any calculator, it is always advisable to read the user's manual.

Note that the decimal numbers for $\sqrt{2}$ and $\sqrt{5}$ do not terminate, nor do they have a pattern of repeating digits. Therefore, these numbers are not rational numbers. Such numbers are called **irrational numbers**. In general, if a is a natural number that is not a perfect square, then $\sqrt{a}$ is an irrational number.

Example 1 Determine whether each of the following numbers is rational or irrational.

a 0.16666 . . . **b** 0.16116111611116111116 . . . **c** π

Solution: **a** In 0.16666 . . . the decimal has a repeating pattern. It is a rational number and can be expressed as $\frac{1}{6}$.

b This decimal has a pattern that neither repeats nor terminates. It is an irrational number.

c $\pi = 3.1415926\ldots$ This decimal does not repeat or terminate. It is an irrational number. (*The fraction* $\frac{22}{7}$ *is an approximation to the value of* π. *It is not the exact value!*).

In Example 1, b and c lead us to the following fact:

➢ A decimal number that is neither terminating nor repeating is an irrational number.

1 Locating irrational numbers on the number line

Group Work 1.4

You will need a compass and straight edge to perform the following:

1 To locate $\sqrt{2}$ on the number line:

- Draw a number line. At the point corresponding to 1 on the number line, construct a perpendicular line segment 1 unit long.
- Draw a line segment from the point corresponding to 0 to the top of the 1 unit segment and label it as *c*.

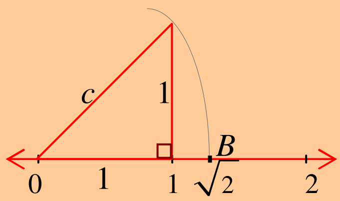


Figure 1.5

- Use the Pythagorean Theorem to show that c is $\sqrt{2}$ unit long.
- Open the compass to the length of *c*. With the tip of the compass at the point corresponding to 0, draw an arc that intersects the number line at B. The distance from the point corresponding to 0 to B is $\sqrt{2}$ units.

2 To locate $\sqrt{5}$ on the number line:

- Find two numbers whose squares have a sum of 5. One pair that works is 1 and 2, since $1^2 + 2^2 = 5$.
- Draw a number line. At the point corresponding to 2, on the number line, construct a perpendicular line segment 1 unit long.
- Draw the line segment shown from the point corresponding to 0 to the top of the 1 unit segment. Label it as *c*.

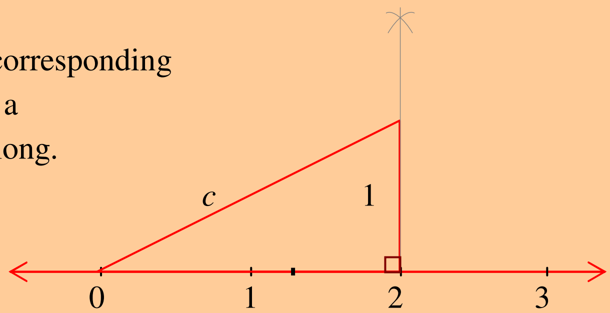


Figure 1.6

- The Pythagorean theorem can be used to show that c is $\sqrt{5}$ units long.

$$c^2 = 1^2 + 2^2 = 5$$

$$c = \sqrt{5}$$

- Open the compass to the length of c. With the tip of the compass at the point corresponding to 0, draw an arc intersecting the number line at B. The distance from the point corresponding to 0 to B is $\sqrt{5}$ units.

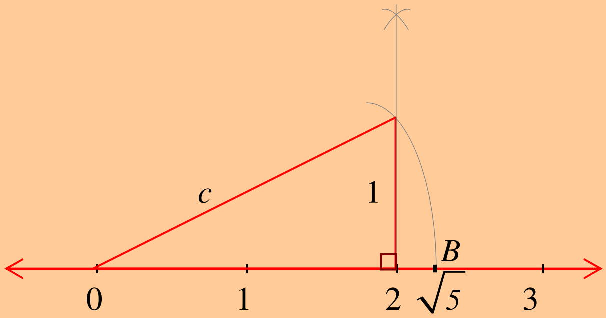
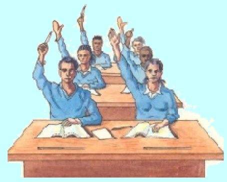


Figure 1.7

Definition 1.5 Irrational number

An irrational number is a number that cannot be expressed as $\frac{a}{b}$, such that a and b are integers and $b \neq 0$.

ACTIVITY 1.6



1 Locate each of the following on the number line, by using geometrical construction:

a $\sqrt{3}$ **b** $-\sqrt{2}$ **c** $\sqrt{6}$

2 Explain how $\sqrt{2}$ can be used to locate:

a $\sqrt{3}$ **b** $\sqrt{6}$

3 Locate each of the following on the number line:

a $1 + \sqrt{2}$ **b** $-2 + \sqrt{2}$ **c** $3 - \sqrt{2}$

Example 2 Show that $3 + \sqrt{2}$ is an irrational number.

Solution: To show that $3 + \sqrt{2}$ is not a rational number, let us begin by assuming that $3 + \sqrt{2}$ is rational. i.e., $3 + \sqrt{2} = \frac{a}{b}$ where a and b are integers, $b \neq 0$.

Then $\sqrt{2} = \frac{a}{b} - 3 = \frac{a-3b}{b}$.

Since $a - 3b$ and b are integers (Why?), $\frac{a-3b}{b}$ is a rational number, meaning that $\sqrt{2} = \left(\frac{a-3b}{b}\right)$ is rational, which is false. As the assumption that $3 + \sqrt{2}$ is rational has led to a false conclusion, the assumption must be false.

Therefore, $3 + \sqrt{2}$ is an irrational number.

ACTIVITY 1.7

Evaluate the following:

1 $0.3030030003\ldots + 0.1414414441\ldots$

2 $0.5757757775\ldots - 0.242442444\ldots$

3 $\left(3+\sqrt{2}\right) \times \left(3-\sqrt{2}\right)$

4 $\sqrt{12} \div \sqrt{3}$

From Example 2 and Activity 1.7, you can generalize the following facts:

i The sum of any rational number and an irrational number is an irrational number.

ii The set of irrational numbers is not closed with respect to addition, subtraction, multiplication and division.

iii If p is a positive integer that is not a perfect square, then $a + b\sqrt{p}$ is irrational where a and b are integers and $b \neq 0$. For example, $3+\sqrt{2}$ and $2-2\sqrt{3}$ are irrational numbers.

Exercise 1.3

1 Identify each of the following numbers as rational or irrational:

a $\frac{5}{6}$ b $2.\overline{34}$ c $-0.1213141516\ldots$

d $\sqrt{0.81}$ e $0.121121112\ldots$ f $\sqrt{5}-\sqrt{2}$

g $\sqrt[3]{72}$ h $1+\sqrt{3}$

2 Give two examples of irrational numbers, one in the form of a radical and the other in the form of a non-terminating decimal.

3 For each of the following, decide whether the statement is 'true' or 'false'. If your answer is 'false', give a counter example to justify.

a The sum of any two irrational numbers is an irrational number.

b The sum of any two rational numbers is a rational number.

c The sum of any two terminating decimals is a terminating decimal.

d The product of a rational number and an irrational number is irrational.

1.2.3 Real Numbers

In Section 1.2.1, you observed that every rational number is either a terminating decimal or a repeating decimal. Conversely, any terminating or repeating decimal is a rational number. Moreover, in Section 1.2.2 you learned that decimals which are neither terminating nor repeating exist. For example, 0.1313313331… Such decimals are defined to be irrational numbers. So a decimal number can be a rational or an irrational number.

It can be shown that every decimal number, be it rational or irrational, can be associated with a unique point on the number line and conversely that every point on the number line can be associated with a unique decimal number, either rational or irrational. This is usually expressed by saying that there exists a one-to-one correspondence between the sets C and D where these sets are defined as follows.

$$C = \{P : P \text{ is a point on the number line}\}$$

$$D = \{d : d \text{ is a decimal number }\}$$

The above discussion leads us to the following definition.

Definition 1.6 Real numbers

A number is called a real number, if and only if it is either a rational number or an irrational number.

The set of real numbers, denoted by $\mathbb{R}$, can be described as the union of the sets of rational and irrational numbers.

$$\mathbb{R} = \{ x : x \text{ is a rational number or an irrational number.}\}$$

The set of real numbers and its subsets are shown in the adjacent diagram.

From the preceding discussion, you can see that there exists a one-to-one correspondence between the set $\mathbb{R}$ and the set C = {P:P is a point on the number line}.

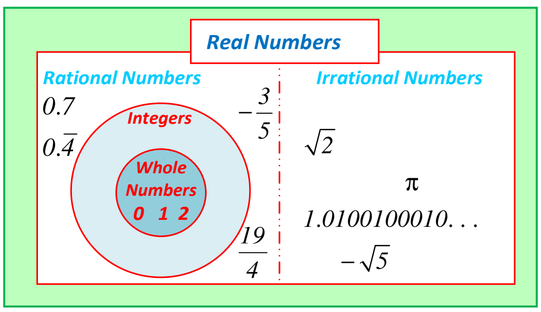


Figure 1.8

It is good to understand and appreciate the existence of a one-to-one correspondence between any two of the following sets.

1 D = $\{x : x$ is a decimal number$\}$

2 P = $\{x : x$ is a point on the number line$\}$

3 $\mathbb{R} = \{x : x$ is a real number$\}$

Since all real numbers can be located on the number line, the number line can be used to compare and order all real numbers. For example, using the number line you can tell that

$$-3 < 0, \quad \sqrt{2} < 2.$$

Example 1 Arrange the following numbers in ascending order:

$$\frac{5}{6},\ 0.8,\ \frac{\sqrt{3}}{2}.$$

Solution: Use a calculator to convert $\frac{5}{6}$ and $\frac{\sqrt{3}}{2}$ to decimals

$5\ \boxed{\div}\ 6\ \boxed{=}\ 0.83333...$ and

$3\ \boxed{\sqrt{\ }}\ \boxed{\div}\ 2\ \boxed{=}\ 0.866025...$

Since $0.8 < 0.8\overline{3} < 0.866025...$, the numbers when arranged in ascending order are

$$0.8,\ \frac{5}{6},\ \frac{\sqrt{3}}{2}.$$

However, there are algebraic methods of comparing and ordering real numbers.

Here are two important properties of order.

1 Trichotomy property

For any two real numbers a and b, one and only one of the following is true

$$a < b \text{ or } a = b \text{ or } a > b.$$

2 Transitive property of order

For any three real numbers a, b and c, if $a < b$ and $b < c$, then, $a < c$.

A third property, stated below, can be derived from the Trichotomy Property and the Transitive Property of Order.

➢ For any two non-negative real numbers a and b, if $a^2 < b^2$, then $a < b$.

You can use this property to compare two numbers without using a calculator.

For example, let us compare $\frac{5}{6}$ and $\frac{\sqrt{3}}{2}$.

$$\left(\frac{5}{6}\right)^2 = \frac{25}{36}, \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} = \frac{27}{36}$$

Since $\left(\frac{5}{6}\right)^2 < \left(\frac{\sqrt{3}}{2}\right)^2$, it follows that $\frac{5}{6} < \frac{\sqrt{3}}{2}$.

Exercise 1.4

1 Compare the numbers a and b using the symbol < or >.

a $a = \frac{\sqrt{6}}{4}, b = 0.\overline{6}$

b $a = 0.432, b = 0.437$

c $a = -0.128, b = -0.123$

2 State whether each set (a – e given below) is closed under each of the following operations:

i addition **ii** subtraction **iii** multiplication **iv** division

a $\mathbb{N}$ the set of natural numbers. **b** $\mathbb{Z}$ the set of integers.

c $\mathbb{Q}$ the set of rational numbers. **d** The set of irrational numbers.

e $\mathbb{R}$ the set of real numbers.

1.2.4 Exponents and Radicals

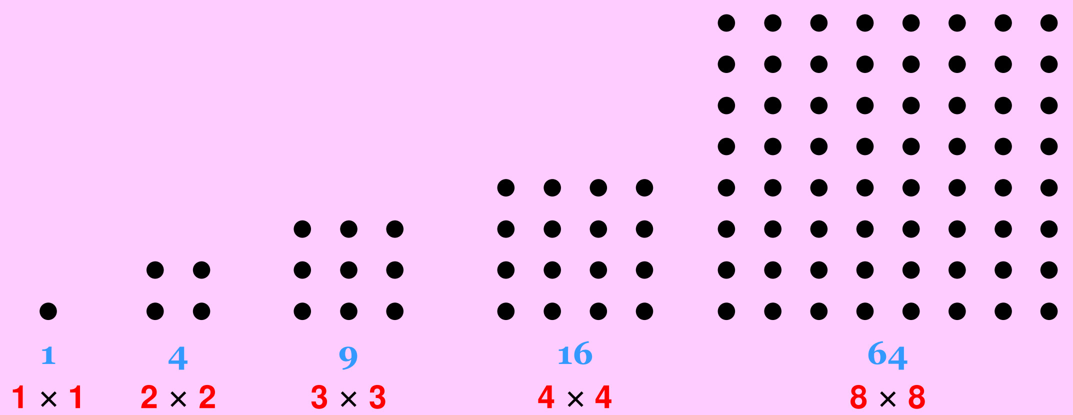
A Roots and radicals

In this subsection, you will define the roots and radicals of numbers and discuss their properties. Computations of expressions involving radicals and fractional exponents are also considered.

Roots

Historical Note:

The Pythagorean School of ancient Greece focused on the study of philosophy, mathematics and natural science. The students, called Pythagoreans, made many advances in these fields. One of their studies was to symbolize numbers. By drawing pictures of various numbers, patterns can be discovered. For example, some whole numbers can be represented by drawing dots arranged in squares.



Numbers that can be pictured in squares of dots are called perfect squares or square numbers. The number of dots in each row or column in the square is a square root of the perfect square. The perfect square 9 has a square root of 3, because there are 3 rows and 3 columns. You say 8 is a square root of 64, because $64 = 8 \times 8$ or 8^2.

Definition 1.7 Square root

For any two real numbers a and b, if $a^2 = b$, then a is a square root of b.

Perfect squares also include decimals and fractions like 0.09 and $\frac{4}{9}$. Since $(0.3)^2 = 0.09$ and $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, it is also true that $(-8)^2 = 64$ and $(-12)^2 = 144$.

So, you may say that -8 is also a square root of 64 and -12 is a square root of 144.

The positive square root of a number is called the principal square root.

The symbol $\sqrt{}$, called a radical sign, is used to indicate the principal square root.

The symbol $\sqrt{25}$ is read as "the principal square root of 25" or just "the square root of 25" and $-\sqrt{25}$ is read as "the negative square root of 25". If b is a positive real number, $\sqrt{b}$ is a positive real number. Negative real numbers do not have square roots in the set of real numbers since $a^2 \geq 0$ for any number a. The square root of zero is zero.

Similarly, since $4^3 = 64$, you say that 64 is the cube of 4 and 4 is the cube root of 64. That is written as $4 = \sqrt[3]{64}$.

The symbol $4 = \sqrt[3]{64}$ is read as "the principal cube root of 64" or just "the cube root of 64".

- Each real number has exactly one cube root.

$(-3)^3 = -27$ so, $\sqrt[3]{-27} = -3$ $\qquad$ $0^3 = 0$ so, $\sqrt[3]{0} = 0$.

You may now generalize as follows:

Definition 1.8 The n^{th} root

For any two real numbers a and b, and positive integer n, if $a^n = b$, then a is called an n^{th} root of b.

Example 1

a -3 is a cube root of -27 because $(-3)^3 = -27$

b 4 is a cube root of 64 because $4^3 = 64$

Definition 1.9 Principal n^{th} root

If b is any real number and n is a positive integer greater than 1, then, the principal n^{th} root of b, denoted by $\sqrt[n]{b}$ is defined as

$$\sqrt[n]{b} = \begin{cases} \text{the positive } n^{th} \text{ root of } b, \text{ if } b > 0. \\ \text{the negative } n^{th} \text{ root of } b, \text{ if } b < 0 \text{ and } n \text{ is odd.} \\ 0, \text{ if } b = 0. \end{cases}$$

i If $b < 0$ and n is even, there is no real n^{th} root of b, because an even power of any real number is a non-negative number.

ii The symbol $\sqrt[n]{}$ is called a radical sign, the expression $\sqrt[n]{b}$ is called a radical, n is called the index and b is called the radicand. When no index is written, the radical sign indicates square root.

Example 2

a $\sqrt[4]{16} = 2$ because $2^4 = 16$

b $\sqrt{0.04} = 0.2$ because $(0.2)^2 = 0.04$

c $\sqrt[3]{-1000} = -10$ because $(-10)^3 = -1000$

Numbers such as $\sqrt{23}$, $\sqrt[3]{35}$ and $\sqrt[3]{10}$ are irrational numbers and cannot be written as terminating or repeating decimals. However, it is possible to approximate irrational numbers as closely as desired using decimals. These rational approximations can be found through successive trials, using a scientific calculator. The method of *successive trials* uses the following property:

➤ For any three positive real numbers a, b and c and a positive integer n

if $a^n < b < c^n$, then $a < \sqrt[n]{b} < c$.

Example 3 Find a rational approximation of $\sqrt{43}$ to the nearest hundredth.

Solution: Use the above property and divide-and-average on a calculator.

Since $6^2 = 36 < 43 < 49 = 7^2$

$6 < \sqrt{43} < 7$

Estimate $\sqrt{43}$ to tenths, $\sqrt{43} \approx 6.5$

Divide 43 by 6.5

$$6.5 \overline{)\,43.000\,}\quad = 6.615$$

Average the divisor and the quotient $\dfrac{6.5 + 6.615}{2} = 6.558$

Divide 43 by 6.558

$$6.558 \overline{)\,43.000\,}\quad = 6.557$$

Now you can check that $(6.557)^2 < 43 < (6.558)^2$. Therefore $\sqrt{43}$ is between 6.557 and 6.558. It is 6.56 to the nearest hundredth.

Example 4 Through successive trials on a calculator, compute $\sqrt[3]{53}$ to the nearest tenth.

Solution:

$$3^3 = 27 < 53 < 64 = 4^3. \text{ That is, } 3^3 < 53 < 4^3. \text{ So } 3 < \sqrt[3]{53} < 4$$

Try 3.5:	$3.5^3 = 42.875$	So $3.5 < \sqrt[3]{53} < 4$
Try 3.7:	$3.7^3 = 50.653$	So $3.7 < \sqrt[3]{53} < 4$
Try 3.8:	$3.8^3 = 54.872$	So $3.7 < \sqrt[3]{53} < 3.8$
Try 3.75:	$3.75^3 = 52.734375$	So $3.75 < \sqrt[3]{53} < 3.8$

Therefore, $\sqrt[3]{53}$ is 3.8 to the nearest tenth.

B Meaning of fractional exponents

ACTIVITY 1.8

1 State another name for $2^{\frac{1}{4}}$.

2 What meaning can you give to $2^{\frac{1}{2}}$ or $2^{0.5}$?

3 Show that there is at most one positive number whose fifth root is 2.

By considering a table of powers of 3 and using a calculator, you can define $3^{\frac{1}{5}}$ as $\sqrt[5]{3}$. This choice would retain the property of exponents by which $\left(3^{\frac{1}{5}}\right)^5 = 3^{\left(\frac{1}{5}\right)\times 5} = 3$.

Similarly, you can define $5^{\frac{1}{n}}$, where n is a positive integer greater than 1, as $\sqrt[n]{5}$. In general, you can define $b^{\frac{1}{n}}$ for any $b \in \mathbb{R}$ and n a positive integer to be $\sqrt[n]{b}$ whenever $\sqrt[n]{b}$ is a real number.

Definition 1.10 The n^{th} power

If $b \in \mathbb{R}$ and n is a positive integer greater than 1, then

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

Example 5 Write the following in exponential form:

a $\sqrt{7}$ **b** $\dfrac{1}{\sqrt[3]{10}}$

Solution:

a $\sqrt{7} = 7^{\frac{1}{2}}$ **b** $\dfrac{1}{\sqrt[3]{10}} = \dfrac{1}{10^{\frac{1}{3}}} = 10^{-\frac{1}{3}}$

Example 6 Simplify:

a $25^{\frac{1}{2}}$ **b** $(-8)^{\frac{1}{3}}$ **c** $64^{\frac{1}{6}}$

Solution:

a $25^{\frac{1}{2}} = \sqrt{25} = 5$ (Since $5^2 = 25$)

b $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$ (Since $(-2)^3 = -8$)

c $64^{\frac{1}{6}} = \sqrt[6]{64} = 2$ (Since $2^6 = 64$)

Group Work 1.5

Simplify:

i **a** $(8\times 27)^{\frac{1}{3}}$ **b** $8^{\frac{1}{3}}\times 27^{\frac{1}{3}}$

ii **a** $\sqrt[3]{8\times 27}$ **b** $\sqrt[3]{8}\times\sqrt[3]{27}$

iii **a** $(36\times 49)^{\frac{1}{2}}$ **b** $36^{\frac{1}{2}}\times 49^{\frac{1}{2}}$

iv **a** $\sqrt{36\times 49}$ **b** $\sqrt{36}\times\sqrt{49}$

What relationship do you observe between a and b in i, ii, iii and iv?

The observations from the above Group Work lead you to think that $5^{\frac{1}{3}}\times 3^{\frac{1}{3}} = (5\times 3)^{\frac{1}{3}}$.

This particular case suggests the following general property (Theorem).

Theorem 1.2

For any two real numbers a and b and for all integers $n \geq 2$, $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$

Example 7 Simplify each of the following.

a $9^{\frac{1}{3}}\times 3^{\frac{1}{3}}$ **b** $\sqrt[5]{16}\times\sqrt[5]{2}$

Solution:

a $9^{\frac{1}{3}}\times 3^{\frac{1}{3}} = (9\times 3)^{\frac{1}{3}}$ (by Theorem 1.2)

$= (27)^{\frac{1}{3}}$ (multiplication)

$= 3$ ($3^3 = 27$)

b $\sqrt[5]{16}\times\sqrt[5]{2} = \sqrt[5]{16\times 2}$

$= \sqrt[5]{32}$

$= 2$

ACTIVITY 1.9

Simplify:

i **a** $\dfrac{64^{\frac{1}{5}}}{2^{\frac{1}{5}}}$ **b** $\left(\dfrac{64}{2}\right)^{\frac{1}{5}}$

ii **a** $\dfrac{8^{\frac{1}{2}}}{2^{\frac{1}{2}}}$ **b** $\left(\dfrac{8}{2}\right)^{\frac{1}{2}}$

iii **a** $\dfrac{27^{\frac{1}{3}}}{729^{\frac{1}{3}}}$ **b** $\left(\dfrac{27}{729}\right)^{\frac{1}{3}}$

What relationship do you observe between a and b in i, ii and iii?

The observations from the above Activity lead us to the following theorem:

Theorem 1.3

For any two real numbers a and b where $b \neq 0$ and for all integers $n \geq 2$,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}$$

Example 8 Simplify **a** $\dfrac{16^{\frac{1}{3}}}{2^{\frac{1}{3}}}$ **b** $\dfrac{\sqrt[6]{128}}{\sqrt[6]{2}}$

Solution:

a $\dfrac{16^{\frac{1}{3}}}{2^{\frac{1}{3}}} = \left(\dfrac{16}{2}\right)^{\frac{1}{3}}$ (by Theorem 1.3)

$= 8^{\frac{1}{3}} = 2$ (since $2^3 = 8$)

b $\dfrac{\sqrt[6]{128}}{\sqrt[6]{2}} = \sqrt[6]{\dfrac{128}{2}}$

$= \sqrt[6]{64}$

$= 2$ (because $2^6 = 64$)

ACTIVITY 1.10

1 Suggest, with reasons, a meaning for

a $2^{\frac{7}{2}}$ **b** $2^{\frac{9}{2}}$ in terms of $2^{\frac{1}{2}}$

2 Suggest a relation between $5^{\frac{3}{2}}$ and $5^{\frac{1}{2}}$.

Applying the property $(a^m)^n = a^{mn}$, you can write $\left(7^{\frac{1}{10}}\right)^9$ as $7^{\frac{9}{10}}$. In general, you can say $\left(a^{\frac{1}{q}}\right)^p = a^{\frac{p}{q}}$, where p and q are positive integers and $a \geq 0$. Thus, you have the following definition:

Definition 1.11

For $a \geq 0$ and p and q any two positive integers, $a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p$

Exercise 1.5

1 Show that: **a** $64^{\frac{1}{3}} = 4$ **b** $256^{\frac{1}{8}} = 2$ **c** $125^{\frac{1}{3}} = 5$

2 Express each of the following without fractional exponents and without radical signs:

a $81^{\frac{1}{4}}$ **b** $9^{\frac{1}{2}}$ **c** $\sqrt[6]{64}$ **d** $\left(\frac{27}{8}\right)^{\frac{1}{3}}$

e $(0.00032)^{\frac{1}{5}}$ **f** $\sqrt[4]{0.0016}$ **g** $\sqrt[6]{729}$

3 Explain each step of the following:

$$(27\times125)^{\frac{1}{3}} = \left[(3\times3\times3)\times(5\times5\times5)\right]^{\frac{1}{3}} = \left[(3\times5)\times(3\times5)\times(3\times5)\right]^{\frac{1}{3}}$$
$$= 3\times5 = 15$$

4 In the same manner as in Question 3, simplify each of the following:

a $(25\times121)^{\frac{1}{2}}$ **b** $(625\times16)^{\frac{1}{4}}$ **c** $(1024\times243)^{\frac{1}{5}}$

5 Express Theorem 1.2 using radical notation.

6 Show that:

a $7^{\frac{1}{4}}\times5^{\frac{1}{4}} = (7\times5)^{\frac{1}{4}}$ **b** $\sqrt{5}\times\sqrt{3} = \sqrt{5\times3}$

c $\sqrt[3]{7}\times\sqrt[3]{9} = \sqrt[3]{7\times9}$ **d** $11^{\frac{1}{7}}\times6^{\frac{1}{7}} = (11\times6)^{\frac{1}{7}}$

7 Express in the simplest form:

a $32^{\frac{1}{6}} \times 2^{\frac{1}{6}}$ **b** $9^{\frac{1}{3}} \times 3^{\frac{1}{3}}$ **c** $128^{\frac{1}{6}} \times \left(\frac{1}{2}\right)^{\frac{1}{6}}$

d $\sqrt[5]{16} \times \sqrt[5]{2}$ **e** $\sqrt[3]{16} \times \sqrt[3]{4}$ **f** $32^{\frac{1}{7}} \times 4^{\frac{1}{7}}$

g $5^{\frac{1}{8}} \times 27^{\frac{1}{5}} \times \left(\frac{1}{5}\right)^{\frac{1}{8}} \times 9^{\frac{1}{5}}$ **h** $\sqrt[3]{5} \times \sqrt[5]{8} \times \sqrt[3]{\frac{1}{5}} \times \sqrt[5]{4}$

8 Express Theorem 1.3 using radical notation.

9 Simplify:

a $\dfrac{128^{\frac{1}{5}}}{4^{\frac{1}{5}}}$ **b** $\dfrac{9^{\frac{1}{3}}}{243^{\frac{1}{3}}}$ **c** $\dfrac{16^{\frac{1}{4}}}{81^{\frac{1}{4}}}$ **d** $\dfrac{32^{\frac{1}{4}}}{162^{\frac{1}{4}}}$

e $\dfrac{\sqrt[3]{16}}{\sqrt[3]{2}}$ **f** $\dfrac{\sqrt[5]{64}}{\sqrt[5]{2}}$ **g** $\dfrac{\sqrt[6]{512}}{\sqrt[6]{8}}$ **h** $\dfrac{\sqrt[3]{625}}{\sqrt[3]{5}}$

10 Rewrite each of the following in the form $a^{\frac{p}{q}}$:

a $\left(13^{\frac{1}{5}}\right)^{9}$ **b** $\left(12^{\frac{1}{5}}\right)^{11}$ **c** $\left(11^{\frac{1}{6}}\right)^{5}$

11 Rewrite the following in the form $\left(a^{\frac{1}{q}}\right)^{p}$

a $3^{\frac{7}{5}}$ **b** $5^{\frac{6}{3}}$ **c** $64^{\frac{5}{6}}$ **d** $729^{\frac{2}{3}}$

12 Rewrite the expressions in Question 10 using radicals.

13 Rewrite the expressions in Question 11 using radicals.

14 Express the following without fractional exponents or radical sign:

a $\left(27^{\frac{1}{3}}\right)^{5}$ **b** $27^{\frac{5}{3}}$ **c** $8^{\frac{1}{3}}$

15 Simplify each of the following:

a $64^{\frac{1}{6}}$ **b** $81^{\frac{3}{2}}$ **c** $64^{\frac{3}{18}}$

d $81^{\frac{6}{4}}$ **e** $512^{\frac{2}{3}}$ **f** $512^{\frac{6}{9}}$

C Simplification of radicals

ACTIVITY 1.11

1 Evaluate each of the following and discuss your result in groups.

a $\sqrt[3]{(-2)^3}$ **b** $\sqrt{(-3)^2}$ **c** $\sqrt[4]{(-5)^4}$

d $\sqrt[5]{4^5}$ **e** $\sqrt{2^2}$ **f** $\sqrt[7]{(-1)^7}$

2 Does the sign of your result depend on whether the index is odd or even?

Can you give a general rule for the result of $\sqrt[n]{a^n}$ where a is a real number and

i n is an odd integer? **ii** n is an even integer?

To compute and simplify expressions involving radicals, it is often necessary to distinguish between roots with odd indices and those with even indices.

For any real number a and a positive integer n,

$$\sqrt[n]{a^n} = a \text{, if } n \text{ is odd.}$$

$$\sqrt[n]{a^n} = |a| \text{, if } n \text{ is even.}$$

$$\sqrt[5]{(-2)^5} = -2, \quad \sqrt[3]{x^3} = x, \quad \sqrt{(-2)^2} = |-2| = 2$$

$$\sqrt{x^2} = |x|, \quad \sqrt[4]{(-2)^4} = |-2| = 2, \quad \sqrt[4]{x^4} = |x|$$

Example 9 Simplify each of the following:

a $\sqrt{y^2}$ **b** $\sqrt[3]{-27x^3}$ **c** $\sqrt{25x^4}$ **d** $\sqrt[6]{x^6}$ **e** $\sqrt[4]{x^3}$

Solution:

a $\sqrt{y^2} = |y|$ **b** $\sqrt[3]{-27x^3} = \sqrt[3]{(-3x)^3} = -3x$

c $\sqrt{25x^4} = |5x^2| = 5x^2$ **d** $\sqrt[6]{x^6} = |x|$ **e** $\sqrt[4]{x^3} = \left(x^3\right)^{\frac{1}{4}} = x^{\frac{3}{4}}$

A radical $\sqrt[n]{a}$ is in simplest form, if the radicand a contains no factor that can be expressed as an n^{th} power. For example $\sqrt[3]{54}$ is not in simplest form because 3^3 is a factor of 54.

Using this fact and the radical notations of Theorem 1.2 and Theorem 1.3, you can simplify radicals.

Example 10 Simplify each of the following:

a $\sqrt{48}$ **b** $\sqrt[3]{9}\times\sqrt[3]{3}$ **c** $\sqrt[4]{\frac{32}{81}}$

Solution:

a $\sqrt{48}=\sqrt{16\times3}=\sqrt{16}\times\sqrt{3}=4\sqrt{3}$

b $\sqrt[3]{9}\times\sqrt[3]{3}=\sqrt[3]{9\times3}=\sqrt[3]{27}=3$

c $\sqrt[4]{\frac{32}{81}}=\sqrt[4]{\frac{16\times2}{81}}=\sqrt[4]{\frac{16}{81}}\times\sqrt[4]{2}=\frac{\sqrt[4]{16}}{\sqrt[4]{81}}\times\sqrt[4]{2}=\frac{2}{3}\sqrt[4]{2}$

Exercise 1.6

1 Simplify each of the following:

a $\sqrt{8}$ **b** $5\sqrt{32}$ **c** $3\sqrt{8x^2}$ **d** $\sqrt{363}$

e $\sqrt[3]{512}$ **f** $\frac{1}{3}\sqrt{27x^3y^2}$ **g** $\sqrt[4]{405}$

2 Simplify each of the following if possible. State restrictions where necessary.

a $\sqrt{50}$ **b** $2\sqrt{36}$ **c** $\frac{1}{3}\sqrt{72}$ **d** $3\sqrt{8x^2}$ **e** $\sqrt{a^3}$

f $\sqrt{0.27}$ **g** $-\sqrt{63}$ **h** $\frac{\sqrt{180}}{9}$ **i** $\sqrt[3]{16}$ **j** $\sqrt[3]{-54}$

3 Identify the error and write the correct solution each of the following cases:

a A student simplified $\sqrt{28}$ to $\sqrt{25+3}$ and then to $5\sqrt{3}$

b A student simplified $\sqrt{72}$ to $\sqrt{4}\sqrt{18}$ and then to $4\sqrt{3}$

c A student simplified $\sqrt{7x^9}$ and got $x^3\sqrt{7}$

4 Simplify each of the following:

a $8\sqrt{250}$ **b** $\sqrt[3]{16}\times\sqrt[3]{5}$ **c** $\sqrt[4]{5}\times\sqrt[4]{125}$

d $\frac{\sqrt{2}}{7}\times\sqrt{7}\times\sqrt{14}$ **e** $\frac{\sqrt[3]{81}}{\sqrt[3]{3}}$ **f** $\frac{12\sqrt{96}}{3\sqrt{6}}$

g $\frac{2\sqrt{98x^3y^2}}{14\sqrt{xy}}$ $x>0, y>0.$ **h** $4\sqrt{3}\times2\sqrt{18}$

5 The number of units N produced by a company from the use of K units of capital and L units of labour is given by $N = 12\sqrt{LK}$.

a What is the number of units produced, if there are 625 units of labour and 1024 units of capital?

b Discuss the effect on the production, if the units of labour and capital are doubled.

Addition and subtraction of radicals

Which of the following do you think is correct?

1 $\sqrt{2} + \sqrt{8} = \sqrt{10}$ **2** $\sqrt{19} - \sqrt{3} = 4$ **3** $5\sqrt{2} + 7\sqrt{2} = 12\sqrt{2}$

The above problems involve addition and subtraction of radicals. You define below the concept of like radicals which is commonly used for this purpose.

Definition 1.12

Radicals that have the same index and the same radicand are said to be like radicals.

For example,

i $3\sqrt{5},\ -\dfrac{1}{2}\sqrt{5}$ and $\sqrt{5}$ are like radicals.

ii $\sqrt{5}$ and $\sqrt[3]{5}$ are not like radicals.

iii $\sqrt{11}$ and $\sqrt{7}$ are not like radicals.

By treating like radicals as like terms, you can add or subtract like radicals and express them as a single radical. On the other hand, the sum of unlike radicals cannot be expressed as a single radical unless they can be transformed into like radicals.

Example 11 Simplify each of the following:

a $\sqrt{2} + \sqrt{8}$ **b** $3\sqrt{12} - \sqrt{3} + 2\sqrt{\dfrac{1}{3}} + \dfrac{1}{9}\sqrt{27}$

Solution:

a $\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{2\times 4} = \sqrt{2} + \sqrt{4}\sqrt{2} = \sqrt{2} + 2\sqrt{2}$

$= (1+2)\sqrt{2} = 3\sqrt{2}$

b $3\sqrt{12}-\sqrt{3}+2\sqrt{\frac{1}{3}}+\frac{1}{9}\sqrt{27}=3\sqrt{4\times3}-\sqrt{3}+2\sqrt{\frac{1}{3}\times\frac{3}{3}}+\frac{1}{9}\sqrt{9\times3}$

$$= 3\sqrt{4}\times\sqrt{3}-\sqrt{3}+2\frac{\sqrt{3}}{\sqrt{9}}+\frac{1}{9}\sqrt{9}\times\sqrt{3}$$

$$= 6\sqrt{3}-\sqrt{3}+\frac{2}{3}\sqrt{3}+\frac{1}{3}\sqrt{3}$$

$$=\left(6-1+\frac{2}{3}+\frac{1}{3}\right)\sqrt{3}=6\sqrt{3}$$

Exercise 1.7

Simplify each of the following if possible. State restrictions where necessary.

1 **a** $\sqrt{2}\times\sqrt{5}$ **b** $\sqrt{3}\times\sqrt{6}$ **c** $\sqrt{21}\times\sqrt{5}$ **d** $\sqrt{2x}\times\sqrt{8x}$

e $\frac{\sqrt{2}}{\sqrt{2}}$ **f** $\frac{\sqrt{10}}{4\sqrt{3}}$ **g** $\sqrt{50y^3}\div\sqrt{2y}$ **h** $\frac{9\sqrt{40}}{3\sqrt{10}}$

i $4\sqrt[3]{16}\div2\sqrt[3]{2}$ **j** $\frac{9\sqrt{24}\div15\sqrt{75}}{3\sqrt{3}}$

2 **a** $2\sqrt{3}+5\sqrt{3}$ **b** $9\sqrt{2}-5\sqrt{2}$ **c** $\sqrt{3}+\sqrt{12}$

d $\sqrt{63}\;-\;\sqrt{28}$ **e** $\sqrt{75}-\sqrt{48}$ **f** $\sqrt{6}\left(\sqrt{12}-\sqrt{3}\right)$

g $\sqrt{2x^2}-\sqrt{50x^2}$ **h** $5\sqrt[3]{54}\;-\;2\sqrt[3]{2}$ **i** $8\sqrt{24}+\frac{2}{3}\sqrt{54}-2\sqrt{96}$

j $\frac{\sqrt{a+2\sqrt{ab}+b}}{\sqrt{a}+\sqrt{b}}$ **k** $\left(\sqrt{a}-\sqrt{b}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)$

3 **a** Find the square of $7-2\sqrt{10}$.

b Simplify each of the following:

i $\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}$ **ii** $\frac{\sqrt{7+\sqrt{24}}}{2}+\frac{\sqrt{7-\sqrt{24}}}{2}$

iii $\left(\sqrt{p^2+1}-\sqrt{p^2-1}\right)\left(\sqrt{p^2-1}+\sqrt{p^2+1}\right)$

4 Suppose the braking distance d for a given automobile when it is travelling v km/hr is approximated by $d=0.00021\sqrt[3]{v^5}$ m. Approximate the braking distance when the car is travelling 64 km/hr.

1.2.5 The Four Operations on Real Numbers

The following activity is designed to help you revise the four operations on the set of rational numbers which you have done in your previous grades.

ACTIVITY 1.12

1 Apply the properties of the four operations in the set of rational numbers to compute the following (mentally, if possible).

a $\frac{2}{9}+\left(\frac{3}{5}+\frac{7}{9}\right)$ **b** $\frac{3}{7}\times\left(\frac{-11}{21}\right)+\left(\frac{-3}{7}\right)\left(\frac{-11}{21}\right)$

c $\frac{3}{7}+\left(\frac{5}{6}+\frac{-3}{7}\right)$ **d** $\left(\frac{-9}{7}\times\frac{23}{-27}\right)\times\left(\frac{-7}{9}\right)$

2 State a property that justifies each of the following statements.

a $\frac{-2}{3}\left(\frac{3}{2}\times\frac{3}{5}\right)=\left(\frac{-2}{3}\times\frac{3}{2}\right)\times\frac{3}{5}$ **b** $\frac{-7}{9}\left(\frac{3}{2}+\frac{-4}{5}\right)=\frac{-7}{9}\left(\frac{-4}{5}+\frac{3}{2}\right)$

c $\left(\frac{-3}{5}\right)+\left(\frac{-5}{6}\right)<\left(\frac{-1}{5}\right)+\left(\frac{-5}{6}\right)$, since $\frac{-3}{5}<\frac{-1}{5}$

In this section, you will discuss operations on the set of real numbers. The properties you have studied so far will help you to investigate many other properties of the set of real numbers.

Group Work 1.6

Work with a partner

Required:- scientific calculator

1 Try this

Copy and complete the following table. Then use a calculator to find each product and complete the table.

Factors	product	product written as a power
$2^3 \times 2^2$		
$10^1 \times 10^1$		
$\left(\frac{-1}{5}\right)\times\left(\frac{-1}{5}\right)^3$		

2 Try this

Copy the following table. Use a calculator to find each quotient and complete the table.

Division	Quotient	Quotient written as a power
$10^5 \div 10^1$		
$3^5 \div 3^2$		
$\left(\frac{1}{2}\right)^4 \div \left(\frac{1}{2}\right)^2$		

Discuss the two tables:

i a Compare the exponents of the factors to the exponents in the product. What do you observe?

b Write a rule for determining the exponent of the product when you multiply powers. Check your rule by multiplying $3^2 \times 3^3$ using a calculator.

ii a Compare the exponents of the division expressions to the exponents in the quotients. What pattern do you observe?

b Write a rule for determining the exponent in the quotient when you divide powers. Check your rule by dividing 7^5 by 7^3 on a calculator.

3 Indicate whether each statement is false or true. If false, explain:

a Between any two rational numbers, there is always a rational number.

b The set of real numbers is the union of the set of rational numbers and the set of irrational numbers.

c The set of rational numbers is closed under addition, subtraction, multiplication and division excluding division by zero.

d The set of irrational numbers is closed under addition, subtraction, multiplication and division.

4 Give examples to show each of the following:

a The product of two irrational numbers may be rational or irrational.

b The sum of two irrational numbers may be rational or irrational.

c The difference of two irrational numbers may be rational or irrational.

d The quotient of two irrational numbers may be rational or irrational.

5 Demonstrate with an example that the sum of an irrational number and a rational number is irrational.

6 Demonstrate with an example that the product of an irrational number and a non-zero rational number is irrational.

7 Complete the following chart using the words 'yes' or 'no'.

Number	Rational number	Irrational number	Real number
2			
$\sqrt{3}$			
$-\frac{2}{3}$			
$\frac{\sqrt{3}}{2}$			
$1.2\overline{3}$			
$1.20220222\ldots$			
$-\frac{2}{3}\times 1.2\overline{3}$			
$\sqrt{75}+1.2\overline{3}$			
$\sqrt{75}-\sqrt{3}$			
$1.20220222\ldots + 0.13113111\ldots$			

Questions 3, 4, 5 and in particular Question 7 of the above Group Work lead you to conclude that the set of real numbers is closed under addition, subtraction, multiplication and division, excluding division by zero.

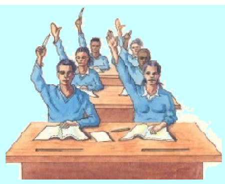
You recall that the set of rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication.

If you add, subtract, multiply or divide (except by 0) two rational numbers, you get a rational number, that is, the set of rational numbers is closed with respect to addition, subtraction, multiplication and division.

From Group work 1.6 you may have realized that the set of irrational numbers is not closed under all the four operations, namely addition, subtraction, multiplication and division.

Do the following activity and discuss your results.

ACTIVITY 1.13



1 Find $a + b$, if

a $a=3+\sqrt{2}$ and $b=3-\sqrt{2}$

b $a=3+\sqrt{3}$ and $b=2+\sqrt{3}$

2 Find $a-b$, if

a $a=\sqrt{3}$ and $b=\sqrt{3}$ **b** $a=\sqrt{5}$ and $b=\sqrt{2}$

3 Find ab, if

a $a=\sqrt{3}-1$ and $b=\sqrt{3}+1$ **b** $a=2\sqrt{3}$ and $b=3\sqrt{2}$

4 Find $a \div b$, if

a $a=5\sqrt{2}$ and $b=3\sqrt{2}$ **b** $a=6\sqrt{6}$ and $b=2\sqrt{3}$

Let us see some examples of the four operations on real numbers.

Example 1 Add $a=2\sqrt{3}+3\sqrt{2}$ and $\sqrt{2}-3\sqrt{3}$

Solution

$$\begin{aligned}\left(2\sqrt{3}+3\sqrt{2}\right)+\left(\sqrt{2}-3\sqrt{3}\right) &= 2\sqrt{3}+3\sqrt{2}+\sqrt{2}-3\sqrt{3}\\ &= \sqrt{3}(2-3)+\sqrt{2}(3+1)\\ &= -\sqrt{3}+4\sqrt{2}\end{aligned}$$

Example 2 Subtract $3\sqrt{2}+\sqrt{5}$ from $3\sqrt{5}-2\sqrt{2}$

Solution:

$$\begin{aligned}\left(3\sqrt{5}-2\sqrt{2}\right)-\left(3\sqrt{2}+\sqrt{5}\right) &= 3\sqrt{5}-2\sqrt{2}-3\sqrt{2}-\sqrt{5}\\ &= \sqrt{5}(3-1)+\sqrt{2}(-2-3)\\ &= 2\sqrt{5}-5\sqrt{2}\end{aligned}$$

Example 3 Multiply

a $2\sqrt{3}$ by $3\sqrt{2}$ **b** $2\sqrt{5}$ by $3\sqrt{5}$

Solution:

a $2\sqrt{3}\times 3\sqrt{2} = 6\sqrt{6}$ **b** $2\sqrt{5}\times 3\sqrt{5} = 2\times 3\times\left(\sqrt{5}\right)^2 = 30$

Example 4 Divide

a $8\sqrt{6}$ by $2\sqrt{3}$ **b** $12\sqrt{6}$ by $\left(\sqrt{2}\times\sqrt{3}\right)$

Solution:

a $8\sqrt{6}\div 2\sqrt{3} = \dfrac{8\sqrt{6}}{2\sqrt{3}} = \dfrac{8}{2}\times\sqrt{\dfrac{6}{3}} = 4\sqrt{2}$

b $12\sqrt{6}\div\left(\sqrt{2}\times\sqrt{3}\right) = \dfrac{12\sqrt{6}}{\sqrt{2}\times\sqrt{3}} = \dfrac{12\sqrt{6}}{\sqrt{6}} = 12$

Rules of exponents hold for real numbers. That is, if a and b are nonzero numbers and m and n are real numbers, then whenever the powers are defined, you have the following laws of exponents.

1 $a^m \times a^n = a^{m+n}$ **2** $(a^m)^n = a^{mn}$ **3** $\dfrac{a^m}{a^n} = a^{m-n}$

4 $a^n \times b^n = (ab)^n$ **5** $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$, $b \neq 0$.

ACTIVITY 1.14

1 Find the additive inverse of each of the following real numbers:

a 5 **b** $-\dfrac{1}{2}$ **c** $\sqrt{2}+1$

d $2.4\overline{5}$ **e** 2.1010010001. . .

2 Find the multiplicative inverse of each of the following real numbers:

a 3 **b** $\sqrt{5}$ **c** $1-\sqrt{3}$ **d** $2^{\frac{1}{6}}$

e 1.71 **f** $\dfrac{\sqrt{2}}{\sqrt{3}}$ **g** $1.\overline{3}$

3 Explain each of the following steps:

$$\begin{aligned}
\left(\sqrt{6}-2\sqrt{15}\right)\times\frac{\sqrt{3}}{3}+\sqrt{20} &= \frac{\sqrt{3}}{3}\times\left(\sqrt{6}-2\sqrt{15}\right)+\sqrt{20}\\
&= \left(\frac{\sqrt{3}}{3}\times\sqrt{6}-\frac{\sqrt{3}}{3}\times 2\sqrt{15}\right)+\sqrt{20}\\
&= \left(\frac{\sqrt{18}}{3}-\frac{2\sqrt{45}}{3}\right)+\sqrt{20}\\
&= \left(\frac{\sqrt{9}\times\sqrt{2}}{3}-\frac{2\sqrt{9}\times\sqrt{5}}{3}\right)+\sqrt{20}\\
&= \left(\frac{3\times\sqrt{2}}{3}-\frac{2\times 3\times\sqrt{5}}{3}\right)+\sqrt{20}\\
&= \left(\sqrt{2}-2\sqrt{5}\right)+\sqrt{20}\\
&= \sqrt{2}+\left[\left(-2\sqrt{5}\right)+2\sqrt{5}\right]\\
&= \sqrt{2}
\end{aligned}$$

Let us now examine the basic properties that govern addition and multiplication of real numbers. You can list these basic properties as follows:

✓ **Closure property:**

The set $\mathbb{R}$ of real numbers is closed under addition and multiplication. This means that the sum and product of two real numbers is a real number; that is, for all $a, b \in \mathbb{R}$,

$a + b \in \mathbb{R}$ and $ab \in \mathbb{R}$

✓ **Addition and multiplication are commutative in $\mathbb{R}$:**

That is, for all $a, b \in \mathbb{R}$,

i $a + b = b + a$

ii $ab = ba$

✓ **Addition and multiplication are associative in $\mathbb{R}$:**

That is, for all, $a, b, c \in \mathbb{R}$,

i $(a + b) + c = a + (b + c)$

ii $(ab)c = a(bc)$

✓ **Existence of additive and multiplicative identities:**

There are real numbers 0 and 1 such that:

i $a + 0 = 0 + a = a$, for all $a \in \mathbb{R}$.

ii $a \cdot 1 = 1 \cdot a = a$, for all $a \in \mathbb{R}$.

✓ **Existence of additive and multiplicative inverses:**

i For each $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0 = (-a) + a$, and $-a$ is called the additive inverse of a.

ii For each non-zero $a \in \mathbb{R}$, there exists $\frac{1}{a} \in \mathbb{R}$ such that $a \times \left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right) \times a$, and $\frac{1}{a}$ is called the multiplicative inverse or reciprocal of a.

✓ **Distributive property:**

Multiplication is distributive over addition; that is, if $a, b, c, \in \mathbb{R}$ then:

i $a(b + c) = ab + ac$

ii $(b + c)a = ba + ca$

Exercise 1.8

1 Find the numerical value of each of the following:

a $(4^{-1})^4 \times 2^5 \times \left(\frac{1}{16}\right)^3 \times (8^{-2})^5 \times (64^2)^3$ **b** $\sqrt{176} - 2\sqrt{275} + \sqrt{1584} - \sqrt{891}$

c $15\sqrt{1.04} - \frac{3}{5}\sqrt{5\frac{5}{9}} + 6\sqrt{\frac{1}{18}} - (5\sqrt{0.02} - \sqrt{300})$

d $\sqrt[4]{0.0001} - \sqrt[5]{0.00032}$ **e** $2\sqrt[3]{0.125} + \sqrt[4]{0.0016}$

2 Simplify each of the following

a $(216)^{\frac{1}{3}}$ **b** $2^{\frac{2}{3}} \times 2^{\frac{3}{5}}$ **c** $\left(3^{\frac{1}{2}}\right)^5$ **d** $\frac{7^{\frac{3}{4}}}{49^{\frac{1}{4}}}$

e $3^{\frac{1}{4}} \times 25^{\frac{1}{8}}$ **f** $16^{\frac{1}{4}} \div 2$ **g** $\sqrt[4]{\sqrt[3]{7}}$ **h** $\frac{\sqrt[5]{32}}{\sqrt[5]{243}}$

3 What should be added to each of the following numbers to make it a rational number? (There are many possible answers. In each case, give two answers.)

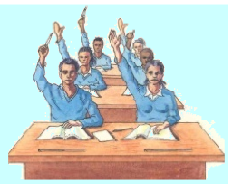
a $5 - \sqrt{3}$ **b** $-2 - \sqrt{5}$ **c** 4.383383338…

d 6.123456… **e** 10.3030003…

1.2.6 Limits of Accuracy

In this subsection, you shall discuss certain concepts such as approximation, accuracy in measurements, significant figures (s.f), decimal places (d.p) and rounding off numbers. In addition to this, you shall discuss how to give appropriate upper and lower bounds for data to a specified accuracy (for example measured lengths).

ACTIVITY 1.15



1 Round off the number 28617 to the nearest

a 10,000 **b** 1000 **c** 100

2 Write the number **i** 7.864 **ii** $6.\overline{437}$ **iii** 4.56556555…

a to one decimal place **b** to two decimal places

3 Write the number 43.25 to

a two significant figures **b** three significant figures

4 The weight of an object is 5.4 kg.

Give the lower and upper bounds within which the weight of the object can lie.

1 Counting and measuring

Counting and measuring are an integral part of our daily life. Most of us do so for various reasons and at various occasions. For example you can count the money you receive from someone, a tailor measures the length of the shirt he/she makes for us, and a carpenter counts the number of screws required to make a desk.

Counting: The process of counting involves finding out the exact number of things. For example, you do counting to find out the number of students in a class. The answer is an exact number and is either correct or, if you have made a mistake, incorrect. On many occasions, just an estimate is sufficient and the exact number is not required or important.

Measuring: If you are finding the length of a football field, the weight of a person or the time it takes to walk down to school, you are measuring. The answers are not exact numbers because there could be errors in measurements.

2 Estimation

In many instances, exact numbers are not necessary or even desirable. In those conditions, approximations are given. The approximations can take several forms. Here you shall deal with the common types of approximations.

A Rounding

If 38,518 people attend a football game this figure can be reported to various levels of accuracy.

To the nearest 10,000 this figure would be rounded up to 40,000.

To the nearest 1000 this figure would be rounded up to 39,000.

To the nearest 100 this figure would be rounded down to 38,500

In this type of situation, it is unlikely that the exact number would be reported.

B Decimal places

A number can also be approximated to a given number of decimal places (d.p). This refers to the number of figures written after a decimal point.

Example 1

a Write 7.864 to 1 d.p. **b** Write 5.574 to 2 d.p.

Solution:

a The answer needs to be written with one number after the decimal point. However, to do this, the second number after the decimal point also needs to be considered. If it is 5 or more, then the first number is rounded up.

That is 7.864 is written as 7.9 to 1 d.p

b The answer here is to be given with two numbers after the decimal point. In this case, the third number after the decimal point needs to be considered. As the third number after the decimal point is less than 5, the second number is not rounded up.

That is 5.574 is written as 5.57 to 2 d.p.

Note that to approximate a number to 1 d.p means to approximate the number to the nearest tenth. Similarly approximating a number to 2 decimal places means to approximate to the nearest hundredth.

C Significant figures

Numbers can also be approximated to a given number of significant figures (s.f). In the number 43.25 the 4 is the most significant figure as it has a value of 40. In contrast, the 5 is the least significant as it only has a value of 5 hundredths. When we desire to use significant figures to indicate the accuracy of approximation, we count the number of digits in the number from left to right, beginning at the first non-zero digit. This is known as the number of significant figures.

Example 2

a Write 43.25 to 3 s.f. **b** Write 0.0043 to 1 s.f.

Solution:

a We want to write only the three most significant digits. However, the fourth digit needs to be considered to see whether the third digit is to be rounded up or not.

That is, 43.25 is written as 43.3 to 3 s.f.

b Notice that in this case 4 and 3 are the only significant digits. The number 4 is the most significant digit and is therefore the only one of the two to be written in the answer.

That is 0.0043 is written as 0.004 to 1 s.f.

3 Accuracy

In the previous lesson, you have studied that numbers can be approximated:

a by rounding up

b by writing to a given number of decimal place and

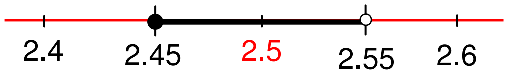
c by expressing to a given number of significant figure.

In this lesson, you will learn how to give appropriate upper and lower bounds for data to a specified accuracy (for example, numbers rounded off or numbers expressed to a given number of significant figures).

Numbers can be written to different degrees of accuracy.

For example, although 2.5, 2.50 and 2.500 may appear to represent the same number, they actually do not. This is because they are written to different degrees of accuracy.

2.5 is rounded to one decimal place (or to the nearest tenths) and therefore any number from 2.45 up to but not including 2.55 would be rounded to 2.5. On the number line this would be represented as



As an inequality, it would be expressed as

$$2.45 \le 2.5 < 2.55$$

2.45 is known as the lower bound of 2.5, while

2.55 is known as the upper bound.

2.50 on the other hand is written to two decimal places and therefore only numbers from 2.495 up to but not including 2.505 would be rounded to 2.50. This, therefore, represents a much smaller range of numbers than that being rounded to 2.5. Similarly, the range of numbers being rounded to 2.500 would be even smaller.

Example 3 A girl's height is given as 162 cm to the nearest centimetre.

i Work out the lower and upper bounds within which her height can lie.

ii Represent this range of numbers on a number line.

iii If the girl's height is h cm, express this range as an inequality.

Solution:

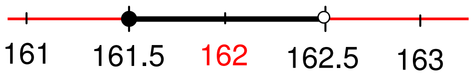
i 162 cm is rounded to the nearest centimetre and therefore any measurement of cm from 161.5 cm up to and not including 162.5 cm would be rounded to 162 cm.

Thus,

lower bound = 161.5 cm

upper bound = 162.5 cm

ii Range of numbers on the number line is represented as



iii When the girl's height h cm is expressed as an inequality, it is given by

$161.5 \le h < 162.5$.

Effect of approximated numbers on calculations

When approximated numbers are added, subtracted and multiplied, their sums, differences and products give a range of possible answers.

Example 4 The length and width of a rectangle are 6.7 cm and 4.4 cm, respectively. Find their sum.

Solution: If the length $l = 6.7$ cm and the width $w = 4.4$ cm

Then $6.65 \leq l < 6.75$ and $4.35 \leq w < 4.45$

The lower bound of the sum is obtained by adding the two lower bounds. Therefore, the minimum sum is $6.65 + 4.35$ that is 11.00.

The upper bound of the sum is obtained by adding the two upper bounds. Therefore, the maximum sum is $6.75 + 4.45$ that is 11.20.

So, the sum lies between 11.00 cm and 11.20 cm.

Example 5 Find the lower and upper bounds for the following product, given that each number is given to 1 decimal place.

$$3.4 \times 7.6$$

Solution:

If $x = 3.4$ and $y = 7.6$ then $3.35 \leq x < 3.45$ and $7.55 \leq y < 7.65$

The lower bound of the product is obtained by multiplying the two lower bounds. Therefore, the minimum product is 3.35×7.55 that is 25.2925

The upper bound of the product is obtained by multiplying the two upper bounds. Therefore, the maximum product is 3.45×7.65 that is 26.3925.

So the product lies between 25.2925 and 26.3925.

Example 6 Calculate the upper and lower bounds to $\frac{54.5}{36.0}$, given that each of the numbers is accurate to 1 decimal place.

Solution: 54.5 lies in the range $54.45 \leq x < 54.55$

36.0 lies in the range $35.95 \leq x < 36.05$

The lower bound of the calculation is obtained by dividing the lower bound of the numerator by the upper bound of the denominator.

So, the minimum value is $54.45 \div 36.05$. i.e., 1.51 (2 decimal places).

The upper bound of the calculation is obtained by dividing the upper bound of the numerator by the lower bound of the denominator.

So, the maximum value is $54.55 \div 35.95$. i.e., 1.52 (2 decimal places).

Exercise 1.9

1 Round the following numbers to the nearest 1000.

a 6856 **b** 74245 **c** 89000 **d** 99500

2 Round the following numbers to the nearest 100.

a 78540 **b** 950 **c** 14099 **d** 2984

3 Round the following numbers to the nearest 10.

a 485 **b** 692 **c** 8847 **d** 4 **e** 83

4 **i** Give the following to 1 d.p.

a 5.58 **b** 4.04 **c** 157.39 **d** 15.045

ii Round the following to the nearest tenth.

a 157.39 **b** 12.049 **c** 0.98 **d** 2.95

iii Give the following to 2 d.p.

a 6.473 **b** 9.587 **c** 0.014 **d** 99.996

iv Round the following to the nearest hundredth.

a 16.476 **b** 3.0037 **c** 9.3048 **d** 12.049

5 Write each of the following to the number of significant figures indicated in brackets.

a 48599 (1 s.f) **b** 48599 (3 s.f) **c** 2.5728 (3 s.f)

d 2045 (2 s.f) **e** 0.08562 (1 s.f) **f** 0.954 (2 s.f)

g 0.00305 (2 s.f) **h** 0.954 (1 s.f)

6 Each of the following numbers is expressed to the nearest whole number.

i Give the upper and lower bounds of each.

ii Using x as the number, express the range in which the number lies as an inequality.

a 6 **b** 83 **c** 151 **d** 1000

7 Each of the following numbers is correct to one decimal place.

i Give the upper and lower bounds of each.

ii Using x as the number, express the range in which the number lies as an inequality.

a 3.8 **b** 15.6 **c** 1.0 **d** 0.3 **e** –0.2

8 Each of the following numbers is correct to two significant figures.

i Give the upper and lower bounds of each.

ii Using x as the number, express the range in which the number lies as an inequality.

a 4.2 **b** 0.84 **c** 420 **d** 5000 **e** 0.045

9 Calculate the upper and lower bounds for the following calculations, if each of the numbers is given to 1 decimal place.

a 9.5×7.6 **b** 11.0×15.6 **c** $\dfrac{46.5}{32.0}$

d $\dfrac{25.4}{8.2}$ **e** $\dfrac{4.9+6.4}{2.6}$

10 The mass of a sack of vegetables is given as 5.4 kg.

a Illustrate the lower and upper bounds of the mass on a number line.

b Using M kg for the mass, express the range of values in which it must lie, as an inequality.

11 The masses to the nearest 0.5 kg of two parcels are 1.5 kg and 2.5 kg. Calculate the lower and upper bounds of their combined mass.

12 Calculate upper and lower bounds for the perimeter of a school football field shown, if its dimensions are correct to 1 decimal place.

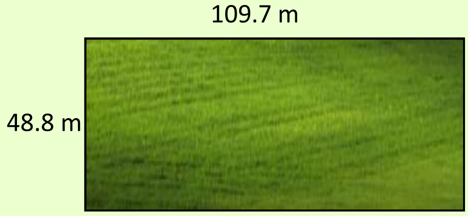


Figure 1.9

13 Calculate upper and lower bounds for the length marked x cm in the rectangle shown. The area and length are both given to 1 decimal place.

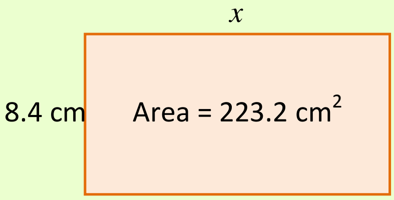


Figure 1.10

1.2.7 Scientific Notation (Standard form)

In science and technology, it is usual to see very large and very small numbers. For instance:

The area of the African continent is about 30,000,000 km^2.

The diameter of a human cell is about 0.0000002 m.

Very large numbers and very small numbers may sometimes be difficult to work with or write. Hence you often write very large or very small numbers in scientific notation, also called standard form.

Example 1 1.86×10^{-6} is written in scientific notation.

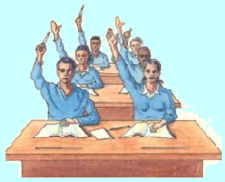
Number from 1 up to but not including 10.

Times 10 to a power.

8.735×10^4 and 7.08×10^{-3} are written in scientific notation.

14.73×10^{-1}, 0.0863×10^4 and 3.86^4 are not written in standard form (scientific notation).

ACTIVITY 1.16



1 By what powers of 10 must you multiply 1.3 to get:

a 13? **b** 130? **c** 1300?

Copy and complete this table.

$13 = 1.3 \times 10^1$
$130 = 1.3 \times 10^2$
$1{,}300 = 1.3 \times$
$13{,}000 =$
$1{,}300{,}000 =$

2 Can you write numbers between 0 and 1 in scientific notation, for example 0.00013?

Copy and complete the following table.

$13.0 = 1.3 \times 10 = 1.3 \times 10^1$
$1.3 = 1.3 \times 1 = 1.3 \times 10^0$
$0.13 = 1.3 \times \frac{1}{10} = 1.3 \times 10^{-1}$
$0.013 = 1.3 \times \frac{1}{100} =$
$0.0013 =$
$0.00013 =$
$0.000013 =$
$0.0000013 =$

Note that if n is a positive integer, multiplying a number by 10^n moves its decimal point n places to the right, and multiplying it by 10^{-n} moves the decimal point n places to the left.

Definition 1.13

A number is said to be in scientific notation (or standard form), if it is written as a product of the form

$$a \times 10^k$$

where $1 \le a < 10$ and k is an integer.

Example 2 Express each of the following numbers in scientific notation:

a 243, 900,000 **b** 0.000000595

Solution:

a $243{,}900{,}000 = 2.439 \times 10^8$.

The decimal point moves 8 places to the left.

b $0.000000595 = 5.95 \times 10^{-7}$.

The decimal point moves 7 places to the right.

Example 3 Express 2.483×10^5 in ordinary decimal notation.

Solution: $2.483 \times 10^5 = 2.483 \times 100{,}000 = 248{,}300$.

Example 4 The diameter of a red blood cell is about 7.4×10^{-4} cm. Write this diameter in ordinary decimal notation.

Solution: $7.4 \times 10^{-4} = 7.4 \times \frac{1}{10^4} = 7.4 \times \frac{1}{10{,}000} = 7.4 \times 0.0001 = 0.00074$.

So, the diameter of a red blood cell is about 0.00074 cm.

Calculators and computers also use scientific notation to display large numbers and small numbers but sometimes only the exponent of 10 is shown. Calculators use a space before the exponent, while computers use the letter E.

➢ The calculator display 5.23 06 means 5.23×10^6. (5,230,000).

The following example shows how to enter a number with too many digits to fit on the display screen into a calculator.

Example 5 Enter 0.00000000627 into a calculator.

Solution: First, write the number in scientific notation.

$0.00000000627 = 6.27 \times 10^{-9}$

Then, enter the number.

6.27 $\boxed{\text{exp}}$ 9 $\boxed{+/-}$ giving 6.27 – 09

Decimal notation	Scientific notation	Calculator display	Computer display
250,000	2.5×10^5	2.5 0.5	2.5 E + 5
0.00047	4.7×10^{-4}	4.7 – 04	4.7 E – 4

Exercise 1.10

1. Express each of the following numbers in scientific notation:

   a 0.00767 b 5,750,000,000 c 0.00083

   d 400,400 e 0.054

2. Express each of the following numbers in ordinary decimal notation:

   a 4.882×10^5 b 1.19×10^{-5} c 2.021×10^2

3. Express the diameter of an electron which is about 0.0000000000004 cm in scientific notation.

1.2.8 Rationalization

ACTIVITY 1.17

Find an approximate value, to two decimal places, for the following:

i $\dfrac{1}{\sqrt{2}}$ ii $\dfrac{\sqrt{2}}{2}$

In calculating this, the first step is to find an approximation of $\sqrt{2}$ in a reference book or other reference material. (It is 1.414214.) In the calculation of $\dfrac{1}{\sqrt{2}}$, 1 is divided by 1.414214… which is a difficult task. However, evaluating $\dfrac{\sqrt{2}}{2}$ as $\dfrac{1.414214}{2} \approx 0.707107$ is easy.

Since $\dfrac{1}{\sqrt{2}}$ is equivalent to $\dfrac{\sqrt{2}}{2}$ (How?), you see that in order to evaluate an expression with a radical in the denominator, first you should transform the expression into an equivalent expression with a rational number in the denominator.

The technique of transferring the radical expression from the denominator to the numerator is called rationalizing the denominator (changing the denominator into a rational number).

The number that can be used as a multiplier to rationalize the denominator is called the rationalizing factor. This is equivalent to 1.

For instance, if $\sqrt{n}$ is an irrational number then $\frac{1}{\sqrt{n}}$ can be rationalized by multiplying it by $\frac{\sqrt{n}}{\sqrt{n}}=1$. So, $\frac{\sqrt{n}}{\sqrt{n}}$ is the rationalizing factor.

Example 1 Rationalize the denominator in each of the following:

a $\frac{5\sqrt{3}}{8\sqrt{5}}$ **b** $\frac{6}{\sqrt{3}}$ **c** $\frac{3}{\sqrt[3]{2}}$

Solution:

a The rationalizing factor is $\frac{\sqrt{5}}{\sqrt{5}}$.

So, $\frac{5\sqrt{3}}{8\sqrt{5}} = \frac{5\sqrt{3}}{8\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{15}}{8\sqrt{25}} = \frac{5\sqrt{15}}{8\sqrt{5^2}} = \frac{5\sqrt{15}}{8\times 5} = \frac{\sqrt{15}}{8}$

b The rationalizing factor is $\frac{\sqrt{3}}{\sqrt{3}}$

So, $\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3^2}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$

c The rationalizing factor is $\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}$ because $\sqrt[3]{2}\times\sqrt[3]{4}=\sqrt[3]{8}=2$

So, $\frac{3}{\sqrt[3]{2}} = \frac{3}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{3\sqrt[3]{4}}{\sqrt[3]{2^3}} = \frac{3\sqrt[3]{4}}{2}$

If a radicand itself is a fraction $\left(\text{for example } \sqrt{\frac{2}{3}}\right)$, then, it can be written in the equivalent form $\frac{\sqrt{2}}{\sqrt{3}}$ so that the procedure described above can be applied to rationalize the denominator. Therefore,

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{\sqrt{6}}{\sqrt{3^2}} = \frac{\sqrt{6}}{3}$$

In general,

For any non-negative integers a, b $(b \neq 0)$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}\sqrt{b}}{\sqrt{b}\sqrt{b}} = \frac{\sqrt{ab}}{b}.$$

Exercise 1.11

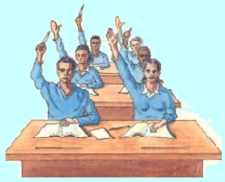
Simplify each of the following. State restrictions where necessary. In each case, state the rationalizing factor you use and express the final result with a rational denominator in its lowest term.

a $\frac{2}{\sqrt{2}}$ **b** $\frac{\sqrt{2}}{\sqrt{6}}$ **c** $\frac{5\sqrt{2}}{4\sqrt{10}}$ **d** $\frac{12}{\sqrt{27}}$ **e** $\sqrt{\frac{5}{18}}$

f $\frac{3}{2\sqrt[3]{3}}$ **g** $\sqrt[3]{\frac{1}{4}}$ **h** $\sqrt{\frac{9}{a^2}}$ **i** $\frac{\sqrt[3]{20}}{\sqrt[3]{4}}$ **j** $\sqrt{\frac{4}{5}}$

More on rationalizations of denominators

ACTIVITY 1.18



Find the product of each of the following:

1 $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$ **2** $\left(5+3\sqrt{2}\right)\left(5-3\sqrt{2}\right)$

3 $\left(\sqrt{5}-\frac{1}{2}\sqrt{3}\right)\left(\sqrt{5}+\frac{1}{2}\sqrt{3}\right)$

You might have observed that the results of all of the above products are rational numbers.

This leads you to the following conclusion:

Using the fact that

$$(a-b)(a+b)=a^2-b^2,$$

you can rationalize the denominators of expressions such as

$\frac{1}{a+\sqrt{b}}, \frac{1}{\sqrt{a}-b}, \frac{1}{\sqrt{a}-\sqrt{b}}$ where $\sqrt{a}, \sqrt{b}$ are irrational numbers as follows.

i $$\frac{1}{a+\sqrt{b}}=\frac{1}{\left(a+\sqrt{b}\right)}\left(\frac{a-\sqrt{b}}{a-\sqrt{b}}\right)=\frac{a-\sqrt{b}}{a^2-\left(\sqrt{b}\right)^2}=\frac{a-\sqrt{b}}{a^2-b}$$

ii $$\frac{1}{\sqrt{a}-b}=\frac{1}{\sqrt{a}-b}\left(\frac{\sqrt{a}+b}{\sqrt{a}+b}\right)=\frac{\sqrt{a}+b}{\left(\sqrt{a}\right)^2-b^2}=\frac{\sqrt{a}+b}{a-b^2}$$

iii $$\frac{1}{\sqrt{a}-\sqrt{b}}=\frac{1}{\left(\sqrt{a}-\sqrt{b}\right)}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)=\frac{\sqrt{a}+\sqrt{b}}{\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2}=\frac{\sqrt{a}+\sqrt{b}}{a-b}$$

Example 2 Rationalize the denominator of each of the following:

a $\dfrac{5}{1-\sqrt{2}}$ **b** $\dfrac{3}{\sqrt{6}+3\sqrt{2}}$

Solution:

a The rationalizing factor is $\dfrac{1+\sqrt{2}}{1+\sqrt{2}}$

So $\dfrac{5}{1-\sqrt{2}} = \dfrac{5(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \dfrac{5+5\sqrt{2}}{1^2-\left(\sqrt{2}\right)^2}$

$$= \frac{5+5\sqrt{2}}{1-2} = -5-5\sqrt{2}$$

b The rationalizing factor is $\dfrac{\sqrt{6}-3\sqrt{2}}{\sqrt{6}-3\sqrt{2}}$

So $\dfrac{3}{\sqrt{6}+3\sqrt{2}} = \dfrac{3}{\left(\sqrt{6}+3\sqrt{2}\right)}\,\dfrac{\sqrt{6}-3\sqrt{2}}{\sqrt{6}-3\sqrt{2}} = \dfrac{3\left(\sqrt{6}-3\sqrt{2}\right)}{\left(\sqrt{6}\right)^2-\left(3\sqrt{2}\right)^2}$

$$= \frac{3\left(\sqrt{6}-3\sqrt{2}\right)}{6-18} = -\frac{1}{4}\left(\sqrt{6}-3\sqrt{2}\right)$$

$$= \frac{3\sqrt{2}-\sqrt{6}}{4}$$

Exercise 1.12

Rationalize the denominator of each of the following:

a $\dfrac{1}{3-\sqrt{5}}$ **b** $\dfrac{\sqrt{18}}{\sqrt{5}-3}$ **c** $\dfrac{2}{\sqrt{5}-\sqrt{3}}$

d $\dfrac{\sqrt{3}+4}{\sqrt{3}-2}$ **e** $\dfrac{10}{\sqrt{7}-\sqrt{2}}$ **f** $\dfrac{3\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$

g $\dfrac{1}{\sqrt{2}+\sqrt{3}-1}$

1.2.9 Euclid's Division Algorithm

A The division algorithm

ACTIVITY 1.19

1. Is the set of non-negative integers (whole numbers) closed under division?
2. Consider any two non-negative integers a and b.
   - **a** What does the statement "a is a multiple of b" mean?
   - **b** Is it always possible to find a non-negative integer c such that $a = bc$?

If a and b are any two non-negative integers, then $a \div b$ $(b \neq 0)$ is some non-negative integer c (if it exists) such that $a = bc$. However, since the set of non-negative integers is not closed under division, it is clear that exact division is not possible for every pair of non-negative integers.

For example, it is not possible to compute $17 \div 5$ in the set of non-negative integers, as

$17 \div 5$ is not a non-negative integer.

$15 = 3 \times 5$ and $20 = 4 \times 5$. Since there is no non-negative integer between 3 and 4, and since 17 lies between 15 and 20, you conclude that there is no non-negative integer c such that $17 = c \times 5$.

You observe, however, that by adding 2 to each side of the equation $15 = 3 \times 5$ you can express it as $17 = (3 \times 5) + 2$. Furthermore, such an equation is useful. For instance it will provide a correct answer to a problem such as: If 5 girls have Birr 17 to share, how many Birr will each girl get? Examples of this sort lead to the following theorem called the Division Algorithm.

Theorem 1.4 Division algorithm

Let a and b be two non-negative integers and $b \neq 0$, then there exist unique non-negative integers q and r, such that,

$$a = (q \times b) + r \quad \text{with } 0 \leq r < b.$$

In the theorem, a is called the dividend, q is called the quotient, b is called the divisor, and r is called the remainder.

Example 1 Write a in the form $b \times q + r$ where $0 \leq r < b$,

a If $a = 47$ and $b = 7$ **b** If $a = 111$ and $b = 3$ **c** If $a = 5$ and $b = 8$

Solution:

a

$$\begin{array}{r|l} & 6 \\ \hline 7 & 47 \\ & 42 \\ \hline & \;\;5 \end{array}$$

$q = 6$ and $r = 5$

$\therefore\ 47 = 7\,(6) + 5$

b

$$\begin{array}{r|l} & 37 \\ \hline 3 & 111 \\ & 9 \\ \hline & 21 \\ & 21 \\ \hline & 0 \end{array}$$

$q = 37$ and $r = 0$

$\therefore\ 111 = 3\,(37) + 0$

c

$$\begin{array}{r|l} & 0 \\ \hline 8 & 5 \\ & 0 \\ \hline & 5 \end{array}$$

$q = 0$ and $r = 5$

$\therefore\ 5 = 8\,(0) + 5.$

Exercise 1.13

For each of the following pairs of numbers, let a be the first number of the pair and b the second number. Find q and r for each pair such that $a = b \times q + r$, where $0 \le r < b$:

a 72, 11	**b** 16, 9	**c** 11, 18
d 106, 13	**e** 176, 21	**f** 25, 39

B The Euclidean algorithm

ACTIVITY 1.20

Given two numbers 60 and 36

1. Find GCF (60, 36).
2. Divide 60 by 36 and find the GCF of 36 and the remainder.
3. Divide 36 by the remainder you got in Step 2. Then, find the GCF of the two remainders, that is, the remainder you got in Step 2 and the one you got in step 3.
4. Compare the three GCFs you got.
5. Generalize your results.

The above Activity leads you to another method for finding the GCF of two numbers, which is called Euclidean algorithm. We state this algorithm as a theorem.

Theorem 1.5 Euclidean algorithm

If a, b, q and r are positive integers such that

$$a = q \times b + r, \text{ then, GCF } (a, b) = \text{GCF } (b, r).$$

Example 2 Find GCF (224, 84).

Solution: To find GCF (224, 84), you first divide 224 by 84. The divisor and remainder of this division are then used as dividend and divisor, respectively, in a succeeding division. The process is repeated until a remainder 0 is obtained.

The complete process to find GCF (224, 84) is shown below.

Euclidean algorithm

Computation	Division algorithm form	Application of Euclidean Algorithm
$\begin{array}{r\|l} & 2 \\ \hline 84 & 224 \\ & \underline{168} \\ & 56 \end{array}$	$224 = (2 \times 84) + 56$	GCF (224, 84) = GCF (84, 56)
$\begin{array}{r\|l} & 1 \\ \hline 56 & 84 \\ & \underline{56} \\ & 28 \end{array}$	$84 = (1 \times 56) + 28$	GCF (84, 56) = GCF (56, 28)
$\begin{array}{r\|l} & 2 \\ \hline 28 & 56 \\ & \underline{56} \\ & 0 \end{array}$	$56 = (2 \times 28) + 0$	GCF (56, 28) = 28 *(by inspection)*

Conclusion GCF (224, 84) = 28.

Exercise 1.14

1. For the above example, verify directly that

   GCF (224, 84) = GCF (84, 56) = GCF (56, 28).

2. Find the GCF of each of the following pairs of numbers by using the Euclidean Algorithm:

   a 18; 12 **b** 269; 88 **c** 143; 39

   d 1295; 407 **e** 85; 68 **f** 7286; 1684

Key Terms

bar notation	principal n^{th} root
composite number	principal square root
divisible	radical sign
division algorithm	radicand
factor	rational number
fundamental theorem of arithmetic	rationalization
greatest common factor (GCF)	real number
irrational number	repeating decimal
least common multiple (LCM)	repetend
multiple	scientific notation
perfect square	significant digits
prime factorization	significant figures
prime number	terminating decimal

Summary

1 The sets of Natural numbers, Whole numbers, Integers and Rational numbers denoted by $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, and $\mathbb{Q}$, respectively are described by

$\mathbb{N} = \{1, 2, 3, \ldots\}$ $\quad$ $\mathbb{W} = \{0, 1, 2, \ldots\}$ $\quad$ $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{Z},\ b \neq 0\right\}$$

2 **a** A composite number is a natural number that has more than two factors.

b A prime number is a natural number that has exactly two distinct factors, 1 and itself.

c Prime numbers that differ by two are called twin primes.

d When a natural number is expressed as a product of factors that are all prime, then the expression is called the prime factorization of the number.

e Fundamental theorem of arithmetic.

Every composite number can be expressed (factorized) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

3 **a** The greatest common factor (GCF) of two or more numbers is the greatest factor that is common to all numbers.

b The least common multiple (LCM) of two or more numbers is the smallest or least of the common multiples of the numbers.

4 **a** Any rational number can be expressed as a repeating decimal or a terminating decimal.

b Any terminating decimal or repeating decimal is a rational number.

5 Irrational numbers are decimal numbers that neither repeat nor terminate.

6 The set of real numbers denoted by $\mathbb{R}$ is defined by

$$\mathbb{R} = \{x: x \text{ is rational or } x \text{ is irrational}\}$$

7 The set of irrational numbers is not closed under addition, subtraction, multiplication and division.

8 The sum of an irrational and a rational number is always an irrational number.

9 For any real number b and positive integer $n > 1$

$$b^{\frac{1}{n}} = \sqrt[n]{b} \text{ (Whenever } \sqrt[n]{b} \text{ is a real number)}$$

10 For all real numbers a and $b \neq 0$ for which the radicals are defined and for all integers $n \geq 2$:

i $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ **ii** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

11 A number is said to be written in scientific notation (standard notation), if it is written in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer.

12 Let a and b be two non-negative integers and $b \neq 0$, then there exist unique non-negative integers q and r such that $a = (q \times b) + r$ with $0 \leq r < b$.

13 If a, b, q and r are positive integers such that $a = q \times b + r$, then

GCF (a, b) = GCF (b, r).

Review Exercises on Unit 1

1 Determine whether each of the following numbers is divisible by 2, 3, 4, 5, 6, 8, 9 or 10:

a 533 **b** 4, 299 **c** 111

2 Find the prime factorization of:

a 150 **b** 202 **c** 63

3 Find the GCF for each set of numbers given below:

a 16; 64 **b** 160; 320; 480

4 Express each of the following fractions or mixed numbers as a decimal:

a $\frac{5}{8}$ **b** $\frac{16}{33}$ **c** $5\frac{4}{9}$ **d** $3\frac{1}{7}$

5 Express each of the following decimals as a fraction or mixed number in its simplest form:

a 0.65 **b** -0.075 **c** $0.\overline{16}$ **d** $-24.\overline{54}$ **e** $-0.\overline{02}$

6 Arrange each of the following sets of rational numbers in increasing order:

a $\frac{71}{100}, -\frac{3}{2}, \frac{23}{30}$ **b** $3.2, 3.\overline{22}, 3.\overline{23}, 3.2\overline{3}$

c $\frac{2}{3}, \frac{11}{18}, \frac{16}{27}, \frac{67}{100}$

7 Write each of the following expressions in its simplest form:

a $\sqrt{180}$ **b** $\sqrt{\frac{169}{196}}$ **c** $\sqrt[3]{250}$ **d** $2\sqrt{3} + 3\sqrt{2} + \sqrt{180}$

8 Give equivalent expression, containing fractional exponents, for each of the following:

a $\sqrt{15}$ **b** $\sqrt{a+b}$ **c** $\sqrt[3]{x-y}$ **d** $\sqrt[4]{\frac{13}{16}}$

9 Express the following numbers as fractions with rational denominators:

a $\frac{1}{\sqrt{2}+1}$ **b** $\sqrt{\frac{5}{3}}$ **c** $\frac{-5}{\sqrt{3}+\sqrt{7}}$ **d** $\sqrt[4]{\frac{13}{16}}$

10 Simplify

a $\left(3+\sqrt{7}\right)+\left(2\sqrt{7}-12\right)$ **b** $\left(2+\sqrt{5}\right)+\left(2-\sqrt{5}\right)$

c $2\sqrt{6}\div 3\sqrt{54}$ **d** $2\left(3+\sqrt{7}\right)-2\sqrt{7}$

11 If $\sqrt{5}\approx 2.236$ and $\sqrt{10}\approx 3.162$, find the value of $\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}$

12 If $\dfrac{\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=x+\sqrt{6}y$, find the values of x and y.

13 Express each of the following numbers in scientific notation:

a 7,410,00 **b** 0.0000648 **c** 0.002056 **d** 12.4×10^{-6}

14 Simplify each of the following and give the answer in scientific notation:

a $10^{9}\times 10^{-6}\times 27$ **b** $\dfrac{796\times 10^{4}\times 10^{-2}}{10^{-7}}$ **c** 0.00032×0.002

15 The formula $d=3.56\sqrt{h}$ km estimates the distance a person can see to the horizon, where h is the height of the eyes of the person from the ground in metre. Suppose you are in a building such that your eye level is 20 m above the ground. Estimate how far you can see to the horizon.

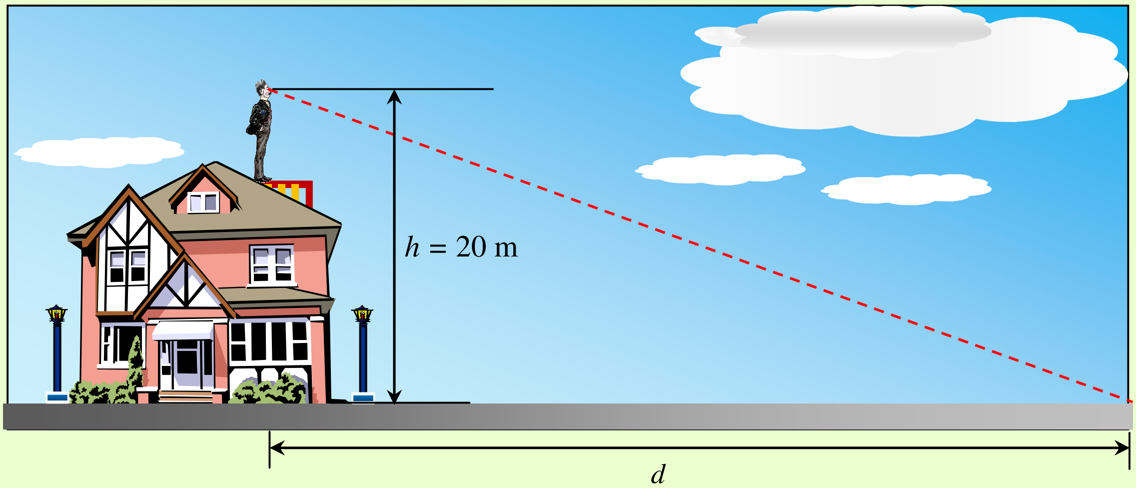
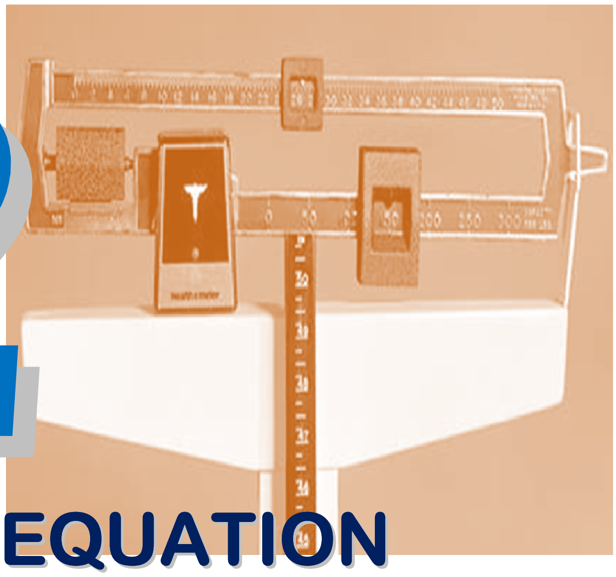


Figure 1.11



Unit 2

SOLUTION OF EQUATION

Unit Outcomes:

After completing this unit, you should be able to:

- *identify equations involving exponents and radicals, systems of two linear equations, equations involving absolute values and quadratic equations.*
- *solve each of these equations.*

Main Contents

2.1 Equations involving exponents and radicals

2.2 Systems of linear equations in two variables

2.3 Equations involving absolute value

2.4 Quadratic equations

Key Terms

Summary

Review Exercises

INTRODUCTION

In earlier grades, you have learnt about algebraic equations and their classification. You also learned about linear equations in one variable and the methods to solve them. In the present unit, we discuss further about equations involving exponents, radicals, and absolute values. You shall also learn about systems of linear equations in two variables, quadratic equations in single variable, and the methods to solve them.

2.1 EQUATIONS INVOLVING EXPONENTS AND RADICALS

Equations are equality of expressions. There are different types of equations that depend on the variable(s) considered. When the variable in use has an exponent other than 1, it is said to be an equation involving exponents.

ACTIVITY 2.1

1 Determine whether or not each of the following is true.

a $2^4 \times 2^5 = 2^{20}$ **b** $(3^2)^3 = 3^6$ **c** $\left(5^2\right)^{\frac{1}{2}} = 5$

d $2^n \times 2^2 = 2^{2n}$ **e** $2^x = 8$ is equivalent to $x = 3$.

2 Express each of the following numbers in power form.

a 8 **b** 27 **c** 625 **d** 343

The above Activity leads you to revise the rules of exponents that you discussed in Unit 1.

Example 1 Solve each of the following equations.

a $\sqrt{x} = 3$ **b** $x^3 = 8$ **c** $2^x = 16$

Solution:

a

$$\sqrt{x} = 3$$

$$\sqrt{x}^2 = 3^2 \qquad \textit{Squaring both sides}$$

$$x = 9$$

Therefore $x = 9$.

b To solve $x^3 = 8$, recall that for any real number a and b, if $a^n = b$, then a is the n^{th} root of b.

$$x^3 = 8$$

$$x = \sqrt[3]{8} = 2$$

c To solve $2^x = 16$, first express $2^x = 16$ as $2^x = 2^4$.

In $2^x = 2^4$, the bases are equal and hence the exponents must be equal.

Therefore $x = 4$ is the solution.

ACTIVITY 2.2

Solve each of the following equations.

a $8^x = 2^{2x+2}$ **b** $4^{x+1} = 2^x$ **c** $\sqrt{5} = 25^{2x}$

The following rule is very useful in solving such equations.

Rule: For $a > 0$, $a^x = a^y$, if and only if $x = y$.

Example 2 Solve $3^{2x+1} = 3^{x-2}$

Solution: By using the rule, since $3 > 0$, $3^{2x+1} = 3^{x-2}$, if and only if the exponents $2x + 1 = x - 2$. From this we can see that the solution is $x = -3$.

Example 3 Solve each of the following equations.

a $8^x = 2^{2x+1}$ **b** $9^{x-3} = 27^{3x}$ **c** $\sqrt[3]{3^x} = 3^{2x+5}$

Solution:

a $8^x = 2^{2x+1}$

$\left(2^3\right)^x = 2^{2x+1}$ *Expressing 8 as a power of 2*

$2^{3x} = 2^{2x+1}$ *Applying laws of exponents*

$3x = 2x + 1$

Therefore $x = 1$.

b $9^{x-3} = 27^{3x}$

$\left(3^2\right)^{x-3} = \left(3^3\right)^{3x}$ *Expressing 9 and 27 as powers of 3*

$3^{2(x-3)} = 3^{3(3x)}$ *Applying laws of exponents*

$3^{2x-6} = 3^{9x}$

$2x - 6 = 9x$

$7x = -6$

Therefore $x = -\frac{6}{7}$.

c $\sqrt[3]{3^x} = 3^{2x+5}$

$\left(3^x\right)^{\frac{1}{3}} = 3^{2x+5}$ *Applying laws of exponents*

$$3^{\frac{x}{3}} = 3^{2x+5}$$

$$\frac{x}{3} = 2x + 5$$

$$x = 3(2x + 5)$$

$$x = 6x + 15$$

$$-5x = 15$$

Therefore $x = -3$.

Exercise 2.1

1 Solve each of the following equations.

a $3^x = 27$ **b** $\left(\frac{1}{4}\right)^x = 16$ **c** $\left(\frac{1}{16}\right)^{3x-1} = 32$

d $81^{5x+2} = \frac{1}{243}$ **e** $9^{2x} = 27^{2x+1}$ **f** $16^{x+4} = 2^{3x}$

g $(3x+1)^3 = 64$ **h** $\sqrt[3]{81^{2x-1}} = 3^x$

2 Solve $(2x+3)^2 = (3x-1)^2$.

3 Solve each of the following equations.

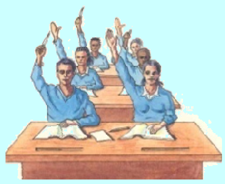
a $9^{2x}27^{1-x} = 81^{2x+1}$ **b** $9^{2x+2}\left(\frac{1}{81}\right)^{x+2} = 243^{-3x-2}$

c $16^{3x+4} = 2^{3x}64^{-4x+1}$

2.2 SYSTEMS OF LINEAR EQUATIONS IN TWO VARIABLES

Recall that, for real numbers a and b, any equation of the form $ax + b = 0$, where $a \neq 0$ is called a linear equation. The numbers a and b are called coefficients of the equation.

ACTIVITY 2.3



1 Solve each of the following linear equations.

a $x - 2 = 7$ **b** $x + 7 = 3$

c $2x = 4$ **d** $2x - 5 = 7$ **e** $3x + 5 = 14$

2 How many solutions do you get for each equation?

Observe that each equation has exactly one solution. In general, any linear equation in one variable has one solution.

Definition 2.1

Any equation that can be reduced to the form $ax + b = 0$, where $a, b \in \mathbb{R}$ and $a \neq 0$, is called a linear equation in one variable.

Group Work 2.1

Form a group and do the following.

1 Solve each of the following equations.

a $7x - 3 = 2(3x + 2)$ **b** $-3(2x + 4) = 2(-3x - 6)$

c $2x + 4 = 2(x + 5)$

2 How many solutions do you get for each equation?

3 What can you conclude about number of solutions?

From the Group Work, observe that such equations can have one solution, infinite solutions or no solution.

Linear equations in two variables

We discussed how we solve equations with one variable that can be reduced to the form $ax + b = 0$. What do you think the solution is, if the equation is given as $y = ax + b$?

ACTIVITY 2.4

1 Which of the following are linear equations in two variables?

a $2x - y = 5$ **b** $-x + 7 = y$ **c** $2x+3 = 4$

d $2x - y^2 = 7$ **e** $\frac{1}{x} + \frac{1}{y} = 6$

2 How many solutions are there for each of the linear equations in two variables?

3 A house was rented for Birr 2,000 per month plus Birr 2 for water consumption per m^3.

a Write an equation for the total cost of x-years rent and 200 m^3 of water used.

b If the total cost for x-years rent and y m^3 of water used is Birr 106,000 write an equation.

Note that $ax+b=0$, is a particular case of $y=ax+b$ when y = 0. This means, for different values of y there will be different equations with their own solutions.

An equation of the type $cx + dy = e$, where c, d and e are arbitrary constants and $c \neq 0$, $d \neq 0$, is called a linear equation in two variables. An equation in two variables of the form $cx + dy = e$ can be reduced to the form $y = ax + b$.

Example 1

a Give solutions to $y = 2x + 1$ where y assumes values 0, 1, 2 and 3.

b Plot some of the ordered pairs that make $y = 2x + 1$ true on the xy-coordinate system.

Solution:

a Let us consider $y = 2x + 1$.

When $y = 0$, the equation becomes $2x + 1 = 0$ and its solution is $x = -\frac{1}{2}$.

When $y = 1$, the equation becomes $2x + 1 = 1$ and its solution is $x = 0$.

When $y = 2$, the equation becomes $2x + 1 = 2$ and its solution is $x = \frac{1}{2}$.

When $y = 3$, the equation becomes $2x + 1 = 3$ and its solution is $x = 1$.

Observe that for each value of y, there is one corresponding value of x. This relation is represented by an ordered pair (x, y). The set of all those ordered pairs that satisfy the equation $y = 2x + 1$ is the solution to the equation $y = 2x + 1$.

b From the four particular cases considered above for $y = 2x + 1$, where y assumes values 0, 1, 2 and 3, we can see that the solution is

$$\left\{\left(-\frac{1}{2}, 0\right), (0, 1), \left(\frac{1}{2}, 2\right), (1, 3)\right\}.$$

Now let us plot these points on the xy-coordinate system. See that there is a line that passes through them.

In general, since y can have any value, there are infinite ordered pairs that make the equation $y = ax + b$ true. The plot of these ordered pairs makes a straight line.

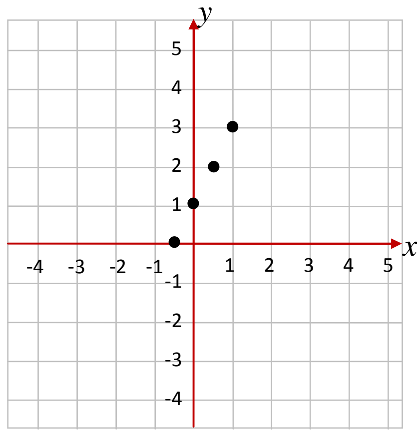


Figure 2.1

System of linear equations and their solutions

You have discussed solutions to a linear equation in two variables and observed that there are infinite solutions. Now you will see the joint consideration of two or more linear equations in two variables.

ACTIVITY 2.5

Consider the equations $y = x + 1$ and $y = -x + 1$.

1. Determine the values of y for each equation when the value of x is $-2, -1, 0, 1$ and 2.
2. Plot the ordered pairs on the xy-coordinate system.
3. What do you observe from the plots of each pair?
4. Discuss what the pair $(0, 1)$ is.

Definition 2.2

A set of two or more linear equations is called a system of linear equations. Systems of two linear equations in two variables are equations that can be represented as

$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$, where a_1, a_2, b_1, b_2, c_1 and c_2 are the parameters of the system whose specific values characterize the system and $a_1 \neq 0$ or $b_1 \neq 0$, $a_2 \neq 0$ or $b_2 \neq 0$.

Example 2 The following are examples of systems of linear equations in two variables.

a $\begin{cases} 2x+3y=1 \\ x-2y=3 \end{cases}$ **b** $\begin{cases} 3x-2y=2 \\ 9x-6y=5 \end{cases}$ **c** $\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$

We now discuss how to solve systems of linear equations.

Definition 2.3

A solution to a system of linear equations in two variables means the set of ordered pairs (x, y) that satisfy both equations.

Example 3 Determine the solution of the following system of linear equations.

$$\begin{cases} 2x+3y=8 \\ 5x-2y=1 \end{cases}$$

Solution: The set $\left\{\left(0,\frac{8}{3}\right),(1,2),\left(2,\frac{4}{3}\right),\left(3,\frac{2}{3}\right),(4,0)\right\}$ contains some of the solutions to the linear equation $2x + 3y = 8$.

The set $\left\{\left(0,-\frac{1}{2}\right),(1,2),\left(2,\frac{9}{2}\right),(3,7)\left(4,\frac{19}{2}\right)\right\}$ contains some of the solutions to the linear equation $5x - 2y = 1$.

From the definition given above, the solution to the given system of linear equations should satisfy both equations $2x + 3y = 8$ and $5x - 2y = 1$.

Therefore, the solution is (1, 2) and it satisfies both equations.

Solution to a system of linear equations in two variables

You saw in Example 3 above that a solution to a system of linear equations is an ordered pair that satisfies both equations in the system. We obtained it by listing some ordered pairs that satisfy each of the component equations and selecting the common one. But it is not easy to list such solutions. So we need to look for another approach to solving systems of linear equations. These include the graphical method, substitution method and elimination method.

Group Work 2.2

1 Draw the line of each component equation in the following systems.

a $\begin{cases} x+y=1 \\ 2x-2y=4 \end{cases}$ b $\begin{cases} 2x-y=2 \\ 4x-2y=5 \end{cases}$ c $\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$

2 Do each pair of lines intersect?

3 What can you conclude from these lines and the solutions of each system?

4 In a certain area, the underage marriage rate decreases from 5% to 0.05% in 12 years. By considering the year 1990 as 0, the linear equation $y = mx + b$ is used to model the underage marriage rate.

a Write the equation of the straight line and determine the year in which under age marriage rate in that area is 0.001% or below.

b Discuss how to model such cases in your kebele.

When we draw the lines of each of the component equations in a system of two linear equations, we can observe three possibilities.

1 The two lines intersect at one point, in which case the system has one solution.

2 The two lines are parallel and never intersect. In this case, we say the system does not have a solution.

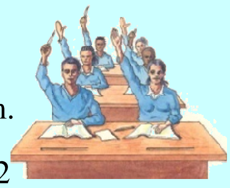
3 The two lines coincide (fit one over the other). In this case, there are infinite solutions.

We now discuss a few graphical and algebraic methods to solve a system of linear equations in two variables: a graphical method, the substitution method, and the elimination method.

Solving system of linear equations by a graphical method

In this method, we need to draw the line of each component equation using the same coordinate system. If the lines intersect, there is one solution, that is the point of their intersection. If the lines are parallel, the system has no solution. If the lines coincide, then there are infinite solutions to the system, since every point (ordered pair) on the line satisfies both equations in the system.

ACTIVITY 2.6



Solve each system by drawing the graph of each equation in the system.

a $\begin{cases} y = x + 1 \\ y = x + 2 \end{cases}$ **b** $\begin{cases} y = x + 2 \\ y = -x - 2 \end{cases}$ **c** $\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$

Example 4 Solve each of the following systems of linear equations.

a $\begin{cases} 2x - 2y = 4 \\ 3x + 4y = 6 \end{cases}$ **b** $\begin{cases} x + 2y = 4 \\ 3x + 6y = 6 \end{cases}$ **c** $\begin{cases} 3x - y = 5 \\ 6x - 2y = 10 \end{cases}$

Solution:

a First, draw the graph of each equation.

In the graph, observe that the two lines are intersecting at (2, 0). Thus, the system has one solution which is (2, 0).

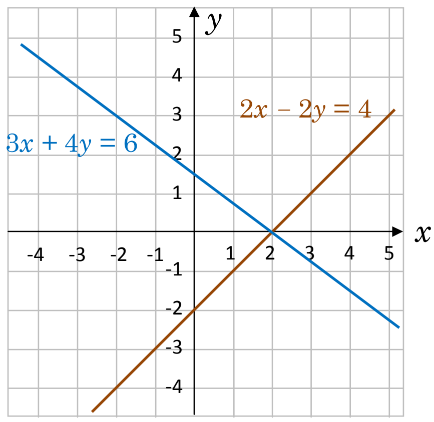


Figure 2.2

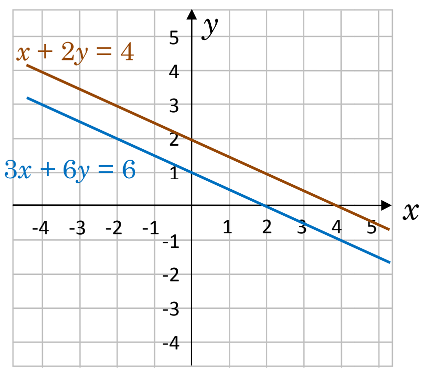


Figure 2.3

b When we draw the line of each component equation, we see that the lines are parallel. This means the lines do not intersect. Hence the system does not have a solution.

c When we draw the line of each component equation, we see that the lines coincide one over the other, which shows that the system has infinite solutions. That is, all points (ordered pairs) on the line are solutions of the system.

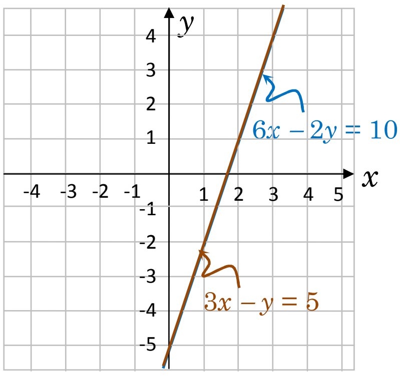


Figure 2.4

Group Work 2.3

Form a group and do the following.

Consider the following systems of linear equations in two variables.

a $\begin{cases} x+4y=2 \\ 3x-4y=6 \end{cases}$ **b** $\begin{cases} -x+2y=4 \\ 3x-y=3 \end{cases}$

1 Solve each by using substitution method.

2 Solve each by using elimination method.

Solving systems of linear equations by the substitution method

To solve a system of two linear equations by the substitution method, you follow the following steps.

1. Take one of the linear equations from the system and write one of the variables in terms of the other.
2. Substitute your result into the other equation and solve for the second variable.
3. Substitute this result into one of the equations and solve for the first variable.

Example 5 Solve the system of linear equations given by $\begin{cases} 2x-3y=5 \\ 5x+3y=9 \end{cases}$

Solution:

Step 1 Take $2x - 3y = 5$ and solve for y in terms of x.

$2x - 3y = 5$ becomes $3y = 2x - 5$

Hence $y = \frac{2}{3}x - \frac{5}{3}$.

Step 2 Substitute $y=\frac{2}{3}x-\frac{5}{3}$ in $5x + 3y = 9$ and solve for x.

$$5x+3\left(\frac{2}{3}x-\frac{5}{3}\right)=9$$

$$5x + 2x - 5 = 9$$

$$7x-5=9$$

$$7x = 14$$

$$x = 2$$

Step 3 Substitute $x = 2$ again into one of the equations and solve for the remaining variable y.

Choosing $2x - 3y = 5$, when we substitute $x = 2$, we get $2\ (2) - 3y = 5$

which becomes $4 - 3y = 5$

$$-3y = 1$$

$$y=-\frac{1}{3}$$

Therefore the solution is $\left(2,-\frac{1}{3}\right)$.

Example 6 Solve each of the following systems of linear equations.

a $\begin{cases} 2x-4y=5 \\ -6x+12y=-15 \end{cases}$ **b** $\begin{cases} 2x-y=1 \\ 3x-2y=-4 \end{cases}$ **c** $\begin{cases} 4x+3y=8 \\ -2x-\frac{3}{2}y=-6 \end{cases}$

Solution:

a $\begin{cases} 2x-4y=5 \\ -6x+12y=-15 \end{cases}$

From $2x - 4y = 5$

$$-4y = -2x + 5$$

$$y=\frac{1}{2}x-\frac{5}{4}$$

Substituting $y=\frac{1}{2}x-\frac{5}{4}$ in $-6x + 12y = -15$, we get

$$-6x+12\left(\frac{1}{2}x-\frac{5}{4}\right)=-15$$

$$-6x + 6x - 15 = -15$$

$-15 = -15$ which is always true.

Therefore, the system has infinite solutions.

b $\begin{cases} 2x - y = 1 \\ 3x - 2y = -4 \end{cases}$

From $2x - y = 1$, we find $y = 2x - 1$

Substituting: $3x - 2(2x - 1) = -4$

$3x - 4x + 2 = -4$

$-x = -6$

Therefore $x = 6$.

Substituting $x = 6$ in $2x - y = 1$ gives

$12 - y = 1$

$y = 11$

So the solution is (6, 11).

c $\begin{cases} 4x + 3y = 8 \\ -2x - \frac{3}{2}y = -6 \end{cases}$

From $4x + 3y = 8$

$3y = -4x + 8$

$y = -\frac{4}{3}x + \frac{8}{3}$

Substituting $y = -\frac{4}{3}x + \frac{8}{3}$ in $-2x - \frac{3}{2}y = -6$ gives $-2x - \frac{3}{2}\left(-\frac{4}{3}x + \frac{8}{3}\right) = -6$

$-2x + 2x - 4 = -6$

$-4 = -6$ which is always false.

Therefore, the system has no solution.

Solving systems of linear equations by the elimination method

To solve a system of two linear equations by the elimination method, you follow the following steps.

1. Select one of the variables and make the coefficients of the selected variable equal but opposite in sign in the two equations.
2. Add the two equations to eliminate the selected variable and solve for the resulting variable.
3. Substitute this result again into one of the equations and solve for the remaining variable.

Example 7 Solve the system of linear equations given by

$$\begin{cases} 2x - y = 5 \\ 2x + 3y = 9 \end{cases}$$

Solution:

Step 1 Select one of the variables, say y and make the coefficients of y opposite to one another by multiplying the first equation by 3.

$$\begin{cases} 2x - y = 5 \\ 2x + 3y = 9 \end{cases} \text{ is equivalent with } \begin{cases} 6x - 3y = 15 \\ 2x + 3y = 9 \end{cases}$$

Step 2 Add the two equations in the system:

$$\begin{cases} 6x - 3y = 15 \\ 2x + 3y = 9 \end{cases}$$ giving $6x - 3y + 2x + 3y = 15 + 9$ which becomes

$8x = 24$.

Therefore $x = 3$.

Step 3 Substitute $x = 3$ into one of the original equations and solve for y.

Choosing $2x - y = 5$ and replacing $x = 3$, get $2\,(3) - y = 5$ from which

$-y = 5 - 6$

$-y = -1$ which is the same as $y = 1$.

Therefore the solution is (3, 1).

Example 8 Solve each of the following systems of linear equations.

a $\begin{cases} 7x + 5y = 11 \\ -3x + 3y = -3 \end{cases}$ **b** $\begin{cases} 2x - 4y = 8 \\ x - 2y = 4 \end{cases}$ **c** $\begin{cases} 2x - 7y = 9 \\ -6x + 21y = 6 \end{cases}$

Solution:

a $\begin{cases} 7x + 5y = 11 \\ -3x + 3y = -3 \end{cases}$

Multiply the first equation by 3 and the second equation by 7 to make the coefficients of the variable x opposite.

We get $\begin{cases} 21x + 15y = 33 \\ -21x + 21y = -21 \end{cases}$

Adding the two equations

$21x + 15y - 21x + 21y = 33 - 21$

which becomes $36y = 12$

$$y = \frac{1}{3}$$

Substituting $y = \frac{1}{3}$ in one of the equations, say $7x + 5y = 11$, we get

$$7x + 5\left(\frac{1}{3}\right) = 11$$

$$7x = 11 - \frac{5}{3}$$

$$7x = \frac{28}{3}$$

$$x = \frac{28}{21} = \frac{4}{3}$$

Therefore the solution is $\left(\frac{4}{3}, \frac{1}{3}\right)$.

b $\begin{cases} 2x - 4y = 8 \\ x - 2y = 4 \end{cases}$

Multiplying the second equation by –2, we get,

$$\begin{cases} 2x - 4y = 8 \\ -2x + 4y = -8 \end{cases}$$

Adding the two equations $2x - 4y - 2x + 4y = 8 - 8$

We get $0 = 0$ which is always true.

Therefore, the system has infinite solutions.

c $\begin{cases} 2x - 7y = 9 \\ -6x + 21y = 6 \end{cases}$

Multiply the first equation by 3 to make the coefficients of the variables opposite.

We get $\begin{cases} 6x - 21y = 27 \\ -6x + 21y = 6 \end{cases}$

Adding the two equations $6x - 21y - 6x + 21y = 27 + 6$, we get that

$0 = 33$ which is always false.

Therefore, the system has no solution.

Solutions of a system of linear equations in two variables and quotients of coefficients

ACTIVITY 2.7

1 Discuss the solution set to each of the following systems.

a $\begin{cases} 3x+y=2 \\ x-2y=3 \end{cases}$ **b** $\begin{cases} x-2y=3 \\ 2x-4y=5 \end{cases}$ **c** $\begin{cases} 2x+3y=1 \\ 4x+6y=2 \end{cases}$

2 Divide each pair of corresponding coefficients as $\frac{3}{1}, \frac{1}{-2}$ and $\frac{2}{3}$ (say for 1a) for each system.

3 Discuss the relationship between the number of solutions and the quotients of coefficients.

4 Solve the given system of two linear equations

$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$; $a_2, b_2, c_2 \neq 0$ in terms of the given coefficients.

From Question 4 of the above Activity, you can reach at the following conclusion.

1 If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ the system has infinite solutions. In this case, every ordered pair that satisfies one of the component equations also satisfies the second. Such a system is said to be **dependent**.

2 If $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ the system has no solutions. This means the two component equations do not have a common solution. In this case, the system is said to be **inconsistent**.

3 If $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ the system has one solution. This means there is only one ordered pair that satisfies both equations. In this case, the system is said to be **independent**.

Example 9 Consider the following systems of linear equations.

a $\begin{cases} 2x+3y=1 \\ x-2y=3 \end{cases}$ **b** $\begin{cases} 3x-2y=2 \\ 9x-6y=5 \end{cases}$ **c** $\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$

By considering the ratio of the coefficients you can determine whether each system has a solution or not.

a The ratio of the coefficients gives $\frac{2}{1} \neq \frac{3}{-2}$.

Therefore, the system has one solution.

b The ratio of the coefficients gives $\frac{3}{9} = \frac{-2}{-6} \neq \frac{2}{5}$.

Therefore, the system has no solution.

c The ratio of the coefficients gives $\frac{1}{2} = \frac{1}{2} = \frac{3}{6}$.

Therefore, the system has infinite solutions.

Remark: Before trying to solve a system of linear equations, it is a good idea to check whether the system has a solution or not.

Word problems leading to a system of linear equations

Systems of linear equations have many real life applications. The real life problems need to be constructed in a mathematical form as a system of linear equations which will be solved by the techniques discussed earlier. Here are some examples.

Group Work 2.4

1 Teshome bought 6 pencils and 2 rubber erasers from a shop and paid a total of Birr 3. Meskerem also paid a total of Birr 3 for 4 pencils and 3 rubber erasers.

2 A company has two brands of fertilizers A and B for sell. A cooperative bought 10 quintals of brand A and 27 quintals of brand B fertilizers and paid a total of Birr 20,000.

Tolosa a successful farm owner, bought 15 quintals of brand A and 9 quintals of brand B fertilizers from the same company and paid a total of Birr 14,250.

i Represent variables for the cost of:

- **a** Each pencil and each rubber eraser in Question 1.
- **b** Each quintal of fertilizer of brand A and each quintal of fertilizer of brand B in Question 2.

ii Formulate the mathematical equations representing each of the situations in Questions 1 and 2 as a system of two linear equations.

iii Solve each system and determine the cost of,

- **a** Each pencil and each rubber eraser in Question 1.
- **b** Each quintal of fertilizer of brand A and each quintal of brand B in Question 2.

Example 10 A farmer collected a total of Birr 11,000 by selling 3 cows and 5 sheep. Another farmer collected Birr 7,000 by selling one cow and 10 sheep. What is the price for a cow and a sheep? (Assume all cows have the same price and also the price of every sheep is the same).

Solution: Let x represent the price of a cow and y the price of a sheep.

Farmer I sold 3 cows for $3x$ and 5 sheep for $5y$ collecting a total of Birr 11,000.

Which means, $3x + 5y = 11{,}000$

Farmer II sold 1 cow for x and 10 sheep for $10y$ collecting a total of Birr 7,000.

Which means, $x + 10y = 7{,}000$

When we consider these equations simultaneously, we get the following system of equations.

$$\begin{cases} 3x+5y=11{,}000 \\ x+10y=7{,}000 \end{cases}$$

Multiplying the first equation by –2 to make the coefficients of y opposite

$$\begin{cases} -6x-10y=-22{,}000 \\ \quad x+10y=7{,}000 \end{cases}$$

Adding the equations we get $-6x + x - 10y + 10y = -22{,}000 + 7{,}000$

$$-5x = -22{,}000 + 7{,}000$$

$$-5x = -15{,}000$$

$$x = 3{,}000$$

Substituting $x = 3{,}000$ in one of the equations, say $x + 10y = 7{,}000$, we get,

$$3{,}000 + 10y = 7{,}000$$

$$10y = 4{,}000$$

$$y = 400$$

Therefore the solution is (3000, 400) showing that the price for a cow is Birr 3,000 and the price for a sheep is Birr 400.

Example 11 Simon has twin younger brothers. The sum of the ages of the three brothers is 48 and the difference between his age and the age of one of his younger brothers is 3. How old is Simon?

Solution: Let x be the age of Simon and y be the age of each of his younger brothers.

The sum of the ages of the three brothers is 48.

So $x + y + y = 48$

$x + 2y = 48.$

The difference between his age and the age of one of his younger brothers is 3 implying

$$x - y = 3.$$

To find Simon's age, we need to solve the system $\begin{cases} x + 2y = 48 \\ x - y = 3 \end{cases}$

Multiplying the second equation by 2 to make the coefficients of y opposite

$$\begin{cases} x + 2y = 48 \\ 2x - 2y = 6 \end{cases}$$

Adding the equations, we get

$$x + 2x + 2y - 2y = 48 + 6$$

$$3x = 54$$

$$x = \frac{54}{3} = 18$$

Therefore, Simon is 18 years old.

Exercise 2.2

1 Which of the following are linear equations in two variables?

a $5x + 5y = 7$ **b** $x + 3xy + 2y = 1$ **c** $x = 2y - 7$

d $y = x^2$ **e** $\frac{4}{x} - \frac{3}{y} = 2$

2 The sum of two numbers is 64. Twice the larger number plus five times the smaller number is 20. Find the two numbers.

3 In a two-digit number, the sum of the digits is 14. Twice the tens digit exceeds the units digit by one. Find the numbers.

4 Determine whether each of the following systems of equations has one solution, infinite solutions or no solution.

a $\begin{cases} 3x - y = 7 \\ -3x + 3y = -1 \end{cases}$ **b** $\begin{cases} 2x + 5y = 12 \\ x - \frac{5}{2}y = 4 \end{cases}$

c $\begin{cases} 3x - y = 7 \\ 2x + 3y = 12 \end{cases}$ **d** $\begin{cases} 4x - 3y = 6 \\ 2x + 3y = 12 \end{cases}$

5 Solve each of the following systems of equations by using a graphical method.

a $\begin{cases} 3x+5y-11=0 \\ 4x-2y=4 \end{cases}$ **b** $\begin{cases} -3x+y=5 \\ 3x-y=5 \end{cases}$ **c** $\begin{cases} \frac{2}{3}x+y=6 \\ -x-\frac{3}{2}y=12 \end{cases}$

d $\begin{cases} x-2y=1 \\ 7x+4y=16 \end{cases}$ **e** $\begin{cases} 0.5x+0.25y=1 \\ x+y=2 \end{cases}$

6 Solve each of the following systems of equations by the substitution method.

a $\begin{cases} 2x+7y=14 \\ x+\frac{7}{2}y=4 \end{cases}$ **b** $\begin{cases} y=x-5 \\ x=y \end{cases}$ **c** $\begin{cases} \frac{2}{3}x-\frac{1}{3}y=2 \\ -x+\frac{1}{2}y=-3 \end{cases}$

d $\begin{cases} -2x+2y=3 \\ 7x+4y=17 \end{cases}$ **e** $\begin{cases} x+3y=1 \\ 2x+5y=2 \end{cases}$

7 Solve each of the following systems of equations by the elimination method.

a $\begin{cases} -3x+y=5 \\ 3x+y=5 \end{cases}$ **b** $\begin{cases} 4x-3y=6 \\ 2x+3y=12 \end{cases}$ **c** $\begin{cases} \frac{2}{3}x-\frac{1}{3}y=2 \\ -x+\frac{1}{3}y=-3 \end{cases}$

d $\begin{cases} \frac{1}{2}x-2y=5 \\ 7x+4y=6 \end{cases}$ **e** $\begin{cases} x+3y=1 \\ 2x+5y=2 \end{cases}$

8 Solve

a $\begin{cases} 3x-0.5y=6 \\ -2x+y=4+2y \end{cases}$ **b** $\begin{cases} \frac{2}{x}+\frac{3}{y}=-2 \\ \frac{4}{x}-\frac{5}{y}=1 \end{cases}$

Hint: Let $a=\frac{1}{x}$ and $b=\frac{1}{y}$

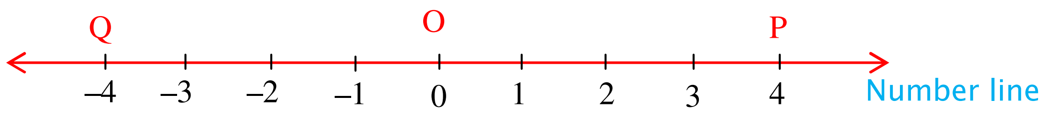
9 Find b and c given that the graph of $y = x^2 + bx + c$ passes through (3, 14) and (–4, 7).

10 A student in a chemistry laboratory has access to two acid solutions. The first solution is 20% acid and the second solution is 45% acid. (The percentages are by volume). How many millilitres of each solution should the student mix together to obtain 100 ml of a 30% acid solution?

2.3 EQUATIONS INVOLVING ABSOLUTE VALUE

In previous sections, you worked with equations having variables x or y that can assume any value. But sometimes it becomes necessary to consider only non-negative values. For example, if you consider distance, it is always non-negative. The distance a number x is located on the real line from the origin is a positive number.

From unit one, recall that the set of real numbers can be represented on a line as follows.



From this, it is possible to determine the distance of each point, representing a number, located far away from the origin or the point representing 0.

Example 1 Let P and Q be points on a number line with coordinates 4 and –4, respectively. How far are the points P and Q from the origin?

Solution: The distance of P and Q from the origin is the same on the real line.

Note: If X is a point on a number line with coordinate a real number x, then the distance of X from the origin is called the **absolute value** of x and is denoted by $|x|$.

Example 2 The points represented by numbers 2 and –2 are located on the number line at an equal distance from the origin. Hence, $|2| = |-2| = 2$.

Example 3 Find the absolute value of each of the following.

a -5 **b** 7 **c** -0.5

Solution:

a $|-5| = 5$ **b** $|7| = 7$ **c** $|-0.5| = 0.5$

In general, the definition of an absolute value is given as follows.

Definition 2.4

The absolute value of a number x, denoted by $|x|$, is defined as follows.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Example 4 Using the definition, determine the absolute value of each of the following.

a 3 **b** -2 **c** -0.4

Solution:

a Since $3 > 0$, $|3| = 3$ **b** Since $-2 < 0$, $|-2| = -(-2) = 2$

c $-0.4 < 0$, and thus $|-0.4| = -(-0.4) = 0.4$

Note: **1** For any real number x, $|x| = |-x|$.

2 For any real number x, $|x|$ is always non-negative.

We considered absolute value as a distance of a point (representing a number) from the origin, or the distance between the location of the number and the origin. However, it is also possible to consider the distance between any other two points on the real line.

Example 5 Find the distance between the points represented by the numbers 3 and 9.

Solution: The distance between the points represented by numbers 3 and 9 is given as

$$|3-9| = |-6| = 6 \text{ or } |9-3| = |6| = 6.$$

The distance between the location of any two real numbers x and y is $|x-y|$ or $|y-x|$.

Note that $|x-y| = |y-x|$.

Example 6 $|5-3| = |2| = 2$ or $|3-5| = |-2| = 2$.

Example 7 Evaluate each of the following.

a $|2-5|$ **b** $|-3-4|$ **c** $|8-3|$ **d** $|2-(-5)|$

Solution:

a $|2-5| = |-3| = 3$ **b** $|-3-4| = |-7| = 7$

c $|8-3| = |5| = 5$ **d** $|2-(-5)| = |2+5| = |7| = 7$

Next, we will discuss equations that involve absolute values and their solutions. Previously, we saw $|3| = |-3| = 3$. So for the equation $|x| = 3$, it is apparent that $x = 3$ or $x = -3$.

Note: For any non-negative number a;

$$|x| = a \text{ means } x = a \text{ or } x = -a.$$

Example 8

a $|x-2| = 3$ means $x - 2 = 3$ or $x - 2 = -3$

$x = 5$ or $x = -1$

b $|x+4| = 5$ means $x + 4 = 5$ or $x + 4 = -5$

$x = 1$ or $x = -9$

This concept of absolute value is essential in solving various problems. Here we see how we can solve equations involving absolute values.

Example 9 Solve $|2x-3|=5$

Solution: Following the definition $|2x-3|=5$ means $2x-3=5$ or $2x-3=-5$,

Solving these linear equations, $x=4$ or $x=-1$.

Example 10 Determine the value of the variable x in each of the following absolute value equations.

a $|x|=4$ **b** $|x-1|=5$ **c** $|-2x+3|=4$

d $|x|=-5$ **e** $|2x+3|=-3$

Solution:

a $|x|=4$ means $x=4$ or $x=-4$

b $|x-1|=5$ means $x-1=5$ or $x-1=-5$

Therefore $x=6$ or $x=-4$.

c $|-2x+3|=4$ means $-2x+3=4$ or $-2x+3=-4$

$-2x=1$ or $-2x=-7$

Therefore $x=\frac{-1}{2}$ or $x=\frac{7}{2}$.

d Since $|x|$ is always non-negative, $|x|=-5$ has no solution.

e Since $|x|$ is always non-negative, $|2x+3|=-3$ has no solution.

Note: For any real number a; $|x|=|a|$ means $x=a$ or $x=-a$.

Example 11 Solve each of the following equations.

a $|x-1|=|2x+1|$ **b** $|3x+2|=|2x-1|$

Solution: a $|x-1|=|2x+1|$ means $x-1=2x+1$ or $x-1=-(2x+1)$

$x-2x=1+1$ or $x+2x=-1+1$

$-x=2$ or $3x=0$

Therefore $x=-2$ or $x=0$.

b $|3x+2|=|2x-1|$ means $3x+2=2x-1$ or $3x+2=-(2x-1)$

$3x-2x=-1-2$ or $3x+2x=1-2$

$x=-3$ or $5x=-1$

Therefore $x=-3$ or $x=-\frac{1}{5}$.

Example 12 Solve each of the following equations.

a $|x-1|=|x+1|$ **b** $|2x+2|=|2x-1|$

Solution:

a $|x-1| = |x+1|$ means $x-1 = x+1$ or $x-1 = -(x+1)$

$$x - x = 1 + 1 \text{ or } x + x = -1 + 1$$

$$0 = 2 \quad \text{or} \quad 2x = 0$$

But $0 = 2$ is impossible.

Therefore $x = 0$.

b $|2x+2| = |2x-1|$ means $2x + 2 = 2x - 1$ or $2x + 2 = -(2x - 1)$

$$2x - 2x = -1 - 2 \text{ or } 2x + 2x = 1 - 2$$

$$0 = -3, \quad \text{or} \quad 4x = -1 .$$

But $0 = -3$ is not possible.

Therefore $x = -\frac{1}{4}$.

Properties of absolute value

For any real numbers x and y;

1 $x \le |x|$.

2 $|xy| = |x||y|$.

3 $\sqrt{x^2} = |x|$.

4 $|x+y| \le |x| + |y|$ (This is called the triangle inequality).

  a If x and y are both non-positive or both non-negative, $|x+y| = |x| + |y|$.

  b If one of x or y is positive and the other is negative, $|x+y| < |x| + |y|$

5 If $y \ne 0$ then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

6 $-|x| \le x \le |x|$.

Exercise 2.3

1 Evaluate each of the following.

a $|2-(-3)|$ **b** $|-4+9|$ **c** $|-5-2|$ **d** $|8| - |3-7|$

2 Solve each of the following equations.

a $|x-5| = -5$ **b** $|x-5| = 5$ **c** $|-(2x-3)| = 7$

d $|3-4x| = 8$ **e** $|x-(3+2x)| = 6$ **f** $|12-(x+7)| = 3$

3 Solve each of the following equations.

a $|5 - x| = |3x - 7|$ **b** $|3x - 2| = |3x - 7|$

c $|5 - 4x| = |7 + 3x|$ **d** $|3x + 4| - |x + 7| = 0$

e $|7 - (x + 3)| + |3x - 3| = 0$

4 Solve each of the following equations.

a $|x - 3| + |x - 3| = 9$ **b** $|3x + 2| - |x - 3| = 5$

c $|-(2x - 3)| + |x| = 12$ **d** $|4x - 2| = 8 + |x - 3|$

e $|5x - (1 - 2x)| - |3 - 2x| = 8$ **f** $|12 - (x + 7)| + |x - 3| = 3$

Hint: Here, for $|x + a| + |x + b| = c$, notice that $|x + a|$ takes either $x + a$ or $-(x + a)$ and also $|x + b|$ takes either $x + b$ or $-(x + b)$, depending on whether they are greater than 0 or less than 0. Therefore, you need to consider four cases to solve such problems!

5 Verify each of the following.

a $|y - x| \le |x| + |y|$ when $x = -2$ and $y = 3$.

b $\sqrt{(3x-7)^2} = |3x-7|$, when $x = 5$.

2.4 QUADRATIC EQUATIONS

Recall that for real numbers a and b, any equation that can be reduced to the form

$ax + b = 0$, where $a \neq 0$ is called a linear equation.

Following the same analogy, for real numbers a, b and c, any equation that can be reduced to the form

$ax^2 + bx + c = 0$, where $a \neq 0$ is called a quadratic equation.

$x^2 + 3x - 2 = 0$, $2x^2 - 5x = 3$, $3x^2 - 6x = 0$, $(x + 3)(x + 2) = 7$ etc, are examples of quadratic equations.

In this section, you will study solving quadratic equations. You will discuss three major approaches to solve quadratic equations, namely, the method of factorization, the method of completing the square, and the general formula. Before you proceed to solve quadratic equations, you will first discuss the concept of factorization.

Expressions

Expressions are combinations of various terms that are represented as a product of variables or numbers and variables.

Example 1 $x^2 + 2x$, $2x^2 + 4x + 2$, $(x + 1)x^2 + 6x$, etc. are expressions.

x^2 and $2x$ are the terms in $x^2 + 2x$ and $2x^2$, $4x$, and 2 are the terms in $2x^2 + 4x + 2$.

Factorizing expressions

ACTIVITY 2.8

1 Multiply each of the following.

a $x(x + 9)$ **b** $(x + 3)(x - 3)$ **c** $(x + 2)(x + 3)$

2 How would it be possible to go back from products to factors? Factorize each of the following.

a $x^2 - 9$ **b** $x^2 + 9x$ **c** $x^2 + 5x + 6$

Factorizing an expression is expressing it as a product of its simplest factors.

Example 2 Factorize $2x^2 - 9x$.

Solution: The two terms in this expression, $2x^2$ and $-9x$, have x as a common factor. Hence $2x^2 - 9x$ can be factorized as $x(2x-9)$.

So $2x^2 - 9x = x(2x - 9)$.

Example 3 Factorize $4x^2 + 12x$.

Solution: $4x^2 + 12x = (4x)x + 3(4x) = (4x)(x + 3)$

Example 4 Factorize $(2x - 1)(3x) + 2(2x - 1)$.

Solution: $(2x - 1)(3x) + 2(2x - 1) = (2x - 1)(3x + 2)$ since $(2x - 1)$ is a common factor.

Factorizing the difference of two squares

If we multiply $(x + 2)$ and $(x - 2)$, we see that $(x + 2)(x - 2) = x^2 - 4 = x^2 - 2^2$.

ACTIVITY 2.9

1 What is $75^2 - 25^2$? How would you compute this?

2 What is $200^2 - 100^2$?

In general,

$$x^2 - a^2 = (x - a)(x + a).$$

Example 5 Factorize $x^2 - 9$.

Solution: $x^2 - 9 = x^2 - 3^2 = (x - 3)(x + 3)$

Example 6 Factorize $4x^2 - 16$.

Solution: $4x^2 - 16 = (2x)^2 - 16 = (2x)^2 - 4^2 = (2x - 4)(2x + 4)$

Factorizing trinomials

You saw how to factorize expressions that have common factors. You also saw factorizing the difference of two squares. Now you will see how to factorize a trinomial $ax^2 + bx + c$ by grouping terms, if you are able to find two numbers p and q such that $p + q = b$ and $pq = ac$.

Example 7 Factorize $x^2 + 5x + 6$.

Solution: Two numbers whose sum is 5 and product 6 are 2 and 3.

So, in the expression, we write $2x + 3x$ instead of $5x$:

$x^2 + 5x + 6 = x^2 + (2x + 3x) + 6$ because $2x + 3x = 5x$.

$= (x^2 + 2x) + (3x + 6)$ (*grouping into two parts*)

$= x(x+2)+3(x+2)\ldots$ (*factorizing each part*)

$=(x+2)(x+3)$ because $(x+2)$ is a common factor.

Example 8 Factorize $x^2 + 4x + 4$.

Solution: Two numbers whose sum is 4 and product 4 are 2 and 2. So take $2x + 2x$ instead of $4x$:

$x^2 + 4x + 4 = x^2 + (2x + 2x) + 4$ because $2x + 2x = 4x$

$=\left(x^2+2x\right)+\left(2x+4\right)$....(*grouping*)

$=x\left(x+2\right)+2\left(x+2\right)$......(*take out the common factor for each group*)

$=\left(x+2\right)\left(x+2\right) = \left(x+2\right)^2$.

Such expressions are called perfect squares.

Example 9 Factorize $3x^2 - 14x - 5$.

Solution: Do you have numbers whose sum is -14 and whose product is $3 \times -5 = -15$?

$-15 + 1 = -14$ and $-15 \times 1 = -15$. This means you can use -15 and 1 for grouping, giving

$$3x^2-14x-5=3x^2-15x+x-5$$

$$=(3x^2-15x)+(x-5)$$

$$=3x(x-5)+1(x-5)$$

$$=(3x+1)(x-5)$$

So $3x^2 - 14x - 5 = (3x+1)(x-5)$.

ACTIVITY 2.10

Factorize each of the following.

a $2x^2 + 10x + 12$ **b** $2x^2 - x - 21$ **c** $5x^2 + 14x + 9$

Solving quadratic equations using the method of factorization

Let $ax^2 + bx + c = 0$ be a quadratic equation and let the quadratic polynomial $ax^2 + bx + c$ be expressible as a product of two linear factors, say $(dx + e)$ and $(fx + g)$ where d, e, f, g are real numbers such that $d \neq 0$ and $f \neq 0$.

Then $ax^2 + bx + c = 0$ becomes

$$(dx + e)(fx + g) = 0$$

So, $dx + e = 0$ or $fx + g = 0$ which gives $x = \frac{-e}{d}$ or $x = \frac{-g}{f}$.

Therefore $x = \frac{-e}{d}$ and $x = \frac{-g}{f}$ are possible roots of the quadratic equation $ax^2 + bx + c = 0$.

For example, the equation $x^2 - 5x + 6 = 0$ can be expressed as:

$$(x-2)(x-3) = 0$$

$$x - 2 = 0 \text{ or } x - 3 = 0$$

$$x = 2 \text{ or } x = 3$$

Therefore the solutions of the equation $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$.

In order to solve a quadratic equation by factorization, go through the following steps:

i Clear all fractions and square roots (if any).

ii Write the equation in the form $p(x) = 0$.

iii Factorize the left hand side into a product of two linear factors.

iv Use the *zero-product rule* to solve the resulting equation.

Zero-product rule: If a and b are two numbers or expressions and if $ab = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$.

Example 10 Solve each of the following quadratic equations.

a $4x^2-16=0$ **b** $x^2+9x+8=0$ **c** $2x^2-6x+7=3$

Solution:

a $4x^2-16=0$ is the same as $(2x)^2-4^2=0$

$$(2x-4)(2x+4)=0$$

$$(2x-4)=0 \text{ or } (2x+4)=0$$

Therefore, $x = 2$ or $x = -2$.

b

$$x^2+9x+8=0$$

$$x^2+x+8x+8=0$$

$$(x^2+x)+(8x+8)=0$$

$$x(x+1)+8(x+1)=0$$

$$(x+1)(x+8)=0$$

$$(x+1)=0 \text{ or } (x+8)=0$$

Therefore, $x = -1$ or $x = -8$.

c $2x^2-6x+7=3$ is the same as $2x^2-6x+4=0$

$2x^2-6x+4=0$ can be expressed as

$2x^2-2x-4x+4=0$; (-2 and -4 have sum $= -6$ and product $= 8$).

$$(2x^2-2x)-(4x-4)=0$$

$$2x(x-1)-4(x-1)=0$$

$$(2x-4)(x-1)=0$$

$$(2x-4)=0 \text{ or } (x-1)=0$$

Therefore, $x = 2$ or $x = 1$.

Exercise 2.4

1 Solve each of the following equations.

a $(x-3)(x+4)=0$ **b** $2x^2-6x=0$ **c** $x^2-3x+4=4$

d $2x^2-8=0$ **e** $5x^2=6x$ **f** $x^2-2x-12=7x-12$

g $-x^2-4=0$ **h** $2x^2+8=0$

2 Solve each of the following equations.

a $x^2 - 6x + 5 = 0$ **b** $3x^2 - 2x - 5 = 0$ **c** $x^2 + 7x = 18$

d $-x^2 = 8x - 9$ **e** $5y^2 - 6y + 1 = 0$ **f** $3z^2 + 10z = 8$

3 Find the solution set of each of the following.

a $2x^2 + \frac{3}{2}x + \frac{1}{4} = 0$ **b** $x^2 = -2.5x + \frac{25}{16}$

c $-(6 + 2x^2) + 8x = 0$

Solving quadratic equations by completing the square

Group Work 2.5

Considering $2x^2 + 5x - 4 = 0$, form a group and do the following.

1 Divide each coefficient by 2.

2 Shift the constant term to the right hand side (RHS).

3 Add the square of half of the middle term to both sides.

4 Do we have any perfect square? Why or why not?

5 Do you observe that $\left(x + \frac{5}{4}\right)^2 = \frac{57}{16}$?

6 Discuss the solution.

In many cases, it is not convenient to solve a quadratic equation by factorization method. For example, consider the equation $x^2 + 8x + 4 = 0$. If you want to factorize the left hand side of the equation, i.e., the polynomial $x^2 + 8x + 4$, using the method of splitting the middle term, you need to find two integers whose sum is 8 and product is 4. But this is not possible. In such cases, an alternative method as demonstrated below is convenient.

$$x^2 + 8x + 4 = 0$$

$$x^2 + 8x = -4$$

$$x^2 + 8x + (4)^2 = -4 + (4)^2 \qquad \left(\text{Adding } \left(\frac{1}{2}\text{Coefficient of } x\right)^2 \text{ on both sides}\right)$$

$$(x + 4)^2 = -4 + 16 = 12 \qquad (x^2 + 8x + 16 = (x + 4)^2)$$

$$x + 4 = \pm\sqrt{12} \qquad \text{(Taking square root of both sides)}$$

Therefore $x = -4 + \sqrt{12}$ and $x = -4 - \sqrt{12}$ are the required solutions.

This method is known as the method of completing the square.

In general, go through the following steps in order to solve a quadratic equation by the method of completing the square:

i Write the given quadratic equation in the standard form.

ii Make the coefficient of x^2 unity, if it is not.

iii Shift the constant term to R.H.S.(Right Hand Side)

iv Add $\left(\frac{1}{2}\text{coefficient of } x\right)^2$ on both sides.

v Express L.H.S.(Left Hand Side) as the perfect square of a suitable binomial expression and simplify the R.H.S.

vi Take square root of both the sides.

vii Obtain the values of x by shifting the constant term from L.H.S. to R.H.S.

Note: The number we need to add (or subtract) to construct a perfect square is determined by using the following product formulas:

$$x^2 + 2ax + a^2 = (x + a)^2$$

$$x^2 - 2ax + a^2 = (x - a)^2$$

Note that the last term, a^2, on the left side of the formulae is the square of one-half of the coefficient of x and the coefficient of x^2 is +1. So, we should add (or subtract) a suitable number to get this form.

Example 11 Solve $x^2 + 5x - 3 = 0$.

Solution: Note that $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

Hence, we add this number to get a perfect square.

$$x^2 + 5x - 3 = 0$$

$$x^2 + 5x = 3$$

$$x^2 + 5x + \frac{25}{4} = 3 + \frac{25}{4}$$

$$x^2 + 5x + \frac{25}{4} = \frac{37}{4}; \quad \left(x^2 + 5x + \frac{25}{4} \text{ is a perfet squre.}\right)$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{37}{4}$$

$$\left(x+\frac{5}{2}\right)=\sqrt{\frac{37}{4}} \quad \text{or} \quad \left(x+\frac{5}{2}\right)=-\sqrt{\frac{37}{4}}$$

$$x=-\frac{5}{2}+\sqrt{\frac{37}{4}} \quad \text{or} \quad x=-\frac{5}{2}-\sqrt{\frac{37}{4}}$$

Therefore $x=\dfrac{-5+\sqrt{37}}{2}$ or $x=\dfrac{-5-\sqrt{37}}{2}$.

Example 12 Solve $3x^2 + 12x + 6 = 0$.

Solution: First divide all terms by 3 so that the coefficient of x^2 is + 1.

$3x^2 + 12x + 6 = 0$ becomes $x^2 + 4x + 2 = 0$

$x^2 + 4x = -2$ (*Shifting the constant term to the right side*)

$x^2 + 4x + 4 = -2 + 4$ (*half of 4 is 2 and its square is 4*)

$(x + 2)^2 = 2$ ($x^2 + 4x + 4 = (x + 2)^2$, *a perfect square*)

$(x + 2) = \pm\sqrt{2}$

$x = -2 \pm\sqrt{2}$

Therefore $x=-2-\sqrt{2}$ or $x=-2+\sqrt{2}$.

Example 13 Solve $3x^2 + 12x + 15 = 0$.

Solution: First divide all terms by 3 so that the coefficient of x^2 is + 1.

$3x^2 + 12x + 15 = 0$ becomes $x^2 + 4x + 5 = 0$

$x^2 + 4x = -5$ (*Shifting the constant term to the right side*)

$x^2 + 4x + 4 = -5 + 4$ (*half of 4 is 2 and its square is 4*)

$(x + 2)^2 = -1$ ($x^2 + 4x + 4 = (x + 2)^2$, *a perfect square*)

$(x+2)=\pm\sqrt{-1}$

Since $\sqrt{-1}$ is not a real number, we conclude that the quadratic equation does not have a real solution.

Example 14 Solve $2x^2 + 4x + 2 = 0$.

Solution: $2x^2 + 4x + 2 = 0$ becomes

$x^2 + 2x + 1 = 0$ (*Dividing all terms by 2*)

$(x + 1)^2 = 0$ ($x^2 + 2x + 1 = (x + 1)^2$ *is a perfect square*)

$(x + 1) = 0$

Therefore $x = -1$ is the only solution.

Exercise 2.5

1 Solve each of the following quadratic equations by using the method of completing the square.

a $x^2 - 6x + 10 = 0$ b $x^2 - 12x + 20 = 0$ c $2x^2 - x - 6 = 0$

d $2x^2 + 3x - 2 = 0$ e $3x^2 - 6x + 12 = 0$ f $x^2 - x + 1 = 0$

2 Find the solution set for each of the following equations.

a $20x^2 + 10x - 8 = 0$ b $x^2 - 8x + 15 = 0$ c $6x^2 - x - 2 = 0$

d $14x^2 + 43x + 20 = 0$ e $x^2 + 11x + 30 = 0$ f $2x^2 + 8x - 1 = 0$

3 Reduce these equations into the form $ax^2 + bx + c = 0$ and solve.

a $x^2 = 5x + 7$ b $3x^2 - 8x = 15 - 2x + 2x^2$

c $x(x - 6) = 6x^2 - x - 2$ d $8x^2 + 9x + 2 = 3(2x^2 + 6x) + 2(x - 1)$

e $x^2 + 11x + 30 = 2 + 11x(x + 3)$

Solving quadratic equations using the quadratic formula

Following the method of completing the square, you next develop a general formula that can serve for checking the existence of a solution to a quadratic equation, and for solving quadratic equations.

To derive the general formula for solving $ax^2 + bx + c = 0$, $a \neq 0$, we proceed using the method of completing the square.

The following Group Work will help you to find the solution formula of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, by using the completing the square method.

Group Work 2.6

Consider $ax^2 + bx + c = 0$, $a \neq 0$

1 First divide each term by a.

2 Shift the constant term $\frac{c}{a}$ to the right.

3 Add the square of half of the middle term to both sides.

4 Do you have a perfect square?

5 Solve for x by using completing the square.

6 Do you observe that $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$?

7 What will be the roots of the quadratic equation $ax^2 + bx + c = 0$?

For a general quadratic equation of type $ax^2 + bx + c = 0$, $a \neq 0$, by applying the method of completing the square, you can conclude that the roots are $r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$.

Therefore, the solution set is $\left\{\frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right\}$.

From the above discussions, what do you observe about $b^2 - 4ac$ in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$?

ACTIVITY 2.11

In $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, discuss the possible conditions for x when,

a $b^2 - 4ac > 0$ **b** $b^2 - 4ac = 0$ **c** $b^2 - 4ac < 0$

Note: If any quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ has a solution, then the solution is determined by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and

1. if $b^2 - 4ac > 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ represents two numbers, namely $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$.

   Therefore, the equation has two solutions.

2. if $b^2 - 4ac = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$ is the only solution.

   Therefore, the equation has only one solution.

3. if $b^2 - 4ac < 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is not defined in $\mathbb{R}$.

   Therefore, the equation does not have any real solution.

The expression $b^2 - 4ac$ is called the discriminant or discriminator. It helps to determine the existence of solutions.

Example 15 Using the discriminant, check to see if the following equations have solution(s), and solve if there is a solution.

a $3x^2 - 5x + 2 = 0$ **b** $x^2 - 8x + 16 = 0$ **c** $-2x^2 - 4x - 9 = 0$

Solution:

a $3x^2 - 5x + 2 = 0$; $a = 3$, $b = -5$ and $c = 2$.

So $b^2 - 4ac = (-5)^2 - 4(3)(2) = 1 > 0$

Therefore, the equation $3x^2 - 5x + 2 = 0$ has two solutions.

Using the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-5) - \sqrt{(-5)^2 - 4(3)(2)}}{2(3)} \text{ or } x = \frac{-(-5) + \sqrt{(-5)^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{5 - \sqrt{25 - 24}}{6} \text{ or } x = \frac{5 + \sqrt{25 - 24}}{6}$$

$$x = \frac{5 - \sqrt{1}}{6} \text{ or } x = \frac{5 + \sqrt{1}}{6}$$

$$x = \frac{5 - 1}{6} \text{ or } x = \frac{5 + 1}{6}$$

$$x = \frac{4}{6} \text{ or } x = \frac{6}{6}$$

Therefore $x = \dfrac{2}{3}$ or $x = 1$.

b In $x^2 - 8x + 16 = 0$, $a = 1$, $b = -8$ and $c = 16$

So $b^2 - 4ac = (-8)^2 - 4(1)(16) = 0$

Therefore, the equation $x^2 - 8x + 16 = 0$ has only one solution.

Using the quadratic solution formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b}{2a}$

$$x = \frac{-(-8)}{2(1)} = 4$$

Therefore the solution is $x = 4$.

c In $-2x^2 - 4x - 9 = 0$, $a = -2$, $b = -4$ and $c = -9$

So $b^2 - 4ac = (-4)^2 - 4(-2)(-9) = -56 < 0$

Therefore the equation $-2x^2 - 4x - 9 = 0$ does not have any real solution.

Exercise 2.6

1 Solve each of the following quadratic equations by using the quadratic solution formula.

a $x^2+8x+15=0$ **b** $3x^2-12x+2=0$ **c** $4x^2-4x-1=0$

d $x^2+3x-2=0$ **e** $5x^2+15x+45=0$ **f** $3x^2-4x-2=0$

2 Find the solution set for each of the following equations.

a $x^2+6x+8=0$ **b** $9+30x+25x^2=0$ **c** $9x^2+15-3x=0$

d $4x^2-36x+81=0$ **e** $x^2+2x+8=0$ **f** $2x^2+8x+1=0$

3 Reduce the equations into the form $ax^2+bx+c=0$ and solve.

a $3x^2=5x+7-x^2$ **b** $x^2=8+2x+2x^2$

c $x^2-2(x-6)=6-x$ **d** $x^2-4+x(1+6x)+2(x-1)=4x-3$

e $4-8x^2+6x=2x(x+3)+2x$

4 A school community had planned to reduce the number of grade 9 students per class room by constructing additional class rooms. However, they constructed 4 less rooms than they planned. As the result, the number of students per class was 10 more than they planned. If there are 1200 grade 9 students in the school, determine the current number of class rooms and the number of students per class.

The relationship between the coefficients of a quadratic equation and its roots

You have learned how to solve quadratic equations. The solutions to a quadratic equation are sometimes called roots. The general quadratic equation

$ax^2+bx+c=0,\ a\neq 0$ has roots (solutions)

$$r_1=\frac{-b-\sqrt{b^2-4ac}}{2a} \text{ and } r_2=\frac{-b+\sqrt{b^2-4ac}}{2a}.$$

ACTIVITY 2.12

1 If $r_1=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $r_2=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ are roots of the quadratic equation $ax^2+bx+c=0,\ a\neq 0$ then

a Find the sum of the roots (r_1+r_2).

b Find the product of the roots (r_1r_2).

2 What relationship do you observe between the sum and product of the roots with respect to the quotients of the coefficients of $ax^2 + bx + c = 0$, namely a, b and c?

3 Test your answer on the quadratic equation $2x^2 - 7x + 5 = 0$.

The relationship between the sum and product of the roots of a quadratic equation and its coefficients is stated below and it is called Viete's theorem.

Theorem 2.1 Viete's theorem

If the roots of $ax^2 + bx + c = 0$, $a \neq 0$ are $r_1 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $r_2 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, then $r_1 + r_2 = \dfrac{-b}{a}$ and $r_1 \times r_2 = \dfrac{c}{a}$

You can check Viete's Theorem as follows:

The roots of $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} \text{ and } r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

Their sum is $r_1 + r_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a} + \dfrac{-b+\sqrt{b^2-4ac}}{2a}$

$$= \frac{\left(-b-\sqrt{b^2-4ac}\right)+\left(-b+\sqrt{b^2-4ac}\right)}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$$

and their product is $r_1 \times r_2 = \left(\dfrac{-b-\sqrt{b^2-4ac}}{2a}\right)\left(\dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)$

$$= \left(\frac{b^2-(b^2-4ac)}{(2a)^2}\right) = \left(\frac{4ac}{4a^2}\right) = \frac{c}{a}$$

So the sum of the roots is $\dfrac{-b}{a}$ and the product of the roots is $\dfrac{c}{a}$.

Example 16 If $3x^2 + 8x + 5 = 0$, then find

a The sum of its roots. **b** The product of its roots.

Solution: In $3x^2 + 8x + 5 = 0$, $a = 3$, $b = 8$ and $c = 5$.

Sum of the roots $= \dfrac{-b}{a} = \dfrac{-8}{3}$ and the product of the roots is $\dfrac{c}{a} = \dfrac{5}{3}$.

Exercise 2.7

1 Determine the sum of the roots of the following equations without solving them.

a $x^2 - 9x + 1 = 0$ **b** $4x^2 + 11x - 4 = 0$ **c** $-3x^2 - 9x - 16 = 0$

2 Determine the product of the roots of the following equations without solving them.

a $-x^2 + 2x + 9 = 0$ **b** $2x^2 + 7x - 3 = 0$ **c** $-3x^2 + 8x + 1 = 0$

3 If the sum of the roots of the equation $3x^2 + kx + 1 = 0$ is 7, then what is the value of k?

4 If the product of the roots of the equation $kx^2 + 8x + 3 = 0$ is 1, then what is the value of k?

5 If one of the roots of the equation $x^2 - 4x + k = 0$ exceeds the other by 2, then find the roots and determine the value of k.

6 Determine the value of k so that the equation $x^2 + kx + k - 1 = 0$ has exactly one real root.

Word problems leading to quadratic equations

Quadratic equations can be successfully used for solving a number of problems related to our day-to-day activities.

The following working rule could be useful in solving such problems.

Step 1 Read the given problem carefully and identify the unknown quantity.

Step 2 Define the unknown quantity as the variable x (say).

Step 3 Using the variable x, translate the given problem into a mathematical statement, i.e., a quadratic equation.

Step 4 Solve the quadratic equation thus formed.

Step 5 Interpret the solution of the quadratic equation, i.e., translate the result into the language of the given problem.

Remark:

i At times it may happen that, out of the two roots of the quadratic equation, only one has a meaning for the problem. In such cases, the other root, which does not satisfy the conditions of the given problem, must be rejected.

ii In case there is a problem involving two or more than two unknown quantities, we define only one of them as the variable x. The remaining ones can always be expressed in terms of x, using the condition(s) given in the problem.

Example 17 The sum of two numbers is 11 and their product is 28. Find the numbers.

Solution: Let x and y be the numbers.

You are given two conditions, $x + y = 11$ and $xy = 28$

From $xy = 28$ you can express y in terms of x, giving $y = \frac{28}{x}$

Replace $y = \frac{28}{x}$ in $x + y = 11$ to get $x + \frac{28}{x} = 11$

Now proceed to solve for x from $x + \frac{28}{x} = 11$ which becomes

$$\frac{x^2 + 28}{x} = 11$$

$$x^2 + 28 = 11x$$

$x^2 - 11x + 28 = 0$, which is a quadratic equation.

Then solving this quadratic equation, you get $x = 4$ or $x = 7$.

If $x = 4$ then from $x + y = 11$ you get $4 + y = 11 \Rightarrow y = 7$

If $x = 7$ then from $x + y = 11$ you get $7 + y = 11 \Rightarrow y = 4$

Therefore, the numbers are 4 and 7.

Example 18 Two different squares have a total area of 274 cm^2 and the sum of their perimeters is 88 cm. Find the lengths of the sides of the squares.

Solution: Let the squares be as given below.

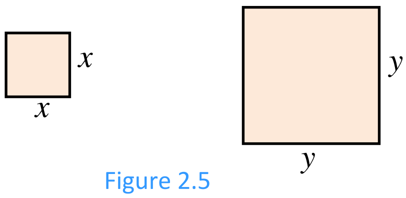


Figure 2.5

Recall, the area of the smaller square is x^2 and area of the bigger square is y^2.

The perimeter of the smaller square is $4x$ and that of the bigger square is $4y$.

So the total area is $x^2 + y^2 = 274$ and the sum of their perimeters is $4x + 4y = 88$.

From $4x + 4y = 88$ you solve for y and get $y = 22 - x$.

Substitute $y = 22 - x$ in $x^2 + y^2 = 274$ and get $x^2 + (22 - x)^2 = 274$.

This equation is $x^2 + 484 - 44x + x^2 = 274$ which becomes the quadratic equation $2x^2 - 44x + 210 = 0$.

Solving this quadratic equation, you get $x = 15$ or $x = 7$.

Therefore, the side of the smaller square is 7 cm and the side of the bigger square is 15 cm.

Exercise 2.8

1. The area of a rectangle is 21 cm^2. If one side exceeds the other by 4 cm, find the dimensions of the rectangle.
2. The perimeter of an equilateral triangle is numerically equal to its area. Find the length of the side of the equilateral triangle.
3. Divide 29 into two parts so that the sum of the squares of the parts is 425. Find the value of each part.
4. The sum of the squares of two consecutive natural numbers is 313. Find the numbers.
5. A piece of cloth costs Birr 200. If the piece was 5 m longer, and the cost of each metre of cloth was Birr 2 less, the cost of the piece would have remained unchanged. How long is the piece and what is its original price per metre?
6. Birr 6,500 were divided equally among a certain number of persons. Had there been 15 more persons, each would have got Birr 30 less. Find the original number of persons.
7. A person on tour has Birr 360 for his daily expenses. If he extends his tour for 4 days, he has to cut down his daily expense by Birr 3. Find the original duration of the tour.
8. In a flight of 600 km, an aircraft was slowed down due to bad weather. Its average speed for the trip was reduced to 200 km/hr and the time increased by 30 minutes. Find the duration of the flight.
9. An express train makes a run of 240 km at a certain speed. Another train whose speed is 12 km/hr less takes an hour longer to cover the same distance. Find the speed of the express train in km/hr.

Key Terms

absolute value	exponents	quadratic equations
completing the square	factorization	quadratic formula
discriminant	graphical method	radicals
elimination method	linear equations	substitution method

Summary

1 Equations are equality of expressions.

2 For $a > 0$, $a^x = a^y$, if and only if $x = y$.

3 An equation of the type $cx + dy = e$, where c and d are arbitrary constants and $d \neq 0$, $c \neq 0$ is called a linear equation in two variables, and its solution is a line (infinite points).

4 A system of linear equations is a set of two or more linear equations, and a system of two linear equations in two variables are equations that can be represented as

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

5 A solution to a system of linear equation in two variables means the set of ordered pairs (x, y) that satisfy both the linear equations.

a $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ implies the system has infinite solutions.

b $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ implies the system has no solutions.

c $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ implies the system has one solution.

6 Geometrically,

a If two lines intersect at one point, the system has one solution.

b If two lines are parallel, and never intersect, the system does not have a solution.

c If the two lines coincide (fit one over the other), the system has infinite solutions.

7 A system of linear equation in two variables can be solved in any of the following ways: graphically, by substitution or by elimination.

8 For any real number x, $|x| = |-x|$.

9 For any real number x, $|x|$ is always non-negative.

10 For any non-negative number a $(a \geq 0)$; $|x| = a$ means $x = a$ or $x = -a$.

11 For any non-negative number a $(a \geq 0)$; $|x| = |a|$ means $x = a$ or $x = -a$.

12 For real numbers a, b and c, any equation that can be reduced to the form $ax^2 + bx + c$, where $a \neq 0$ is called a quadratic equation.

13 Writing an expression as a product of its simplest factors is called factorizing.

14 For real numbers a, b and c, to solve $ax^2 + bx + c$, where $a \neq 0$, the following methods can be used: factorization, completing the square, or the quadratic formula.

15 If the roots of $ax^2 + bx + c$ are $x_1 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$

then $x_1 + x_2 = -\dfrac{b}{a}$ and $x_1 \times x_2 = \dfrac{c}{a}$.

Review Exercises on Unit 2

1 Solve each of the following.

a $(x-3)^3 = 27$ **b** $(2x+1)^2 = 16$ **c** $9^{3x} = 81$

d $\sqrt[3]{(2x)^3} = 14$ **e** $(x-3)^3 = 27(2x-1)^3$

2 Solve each of the following linear equations.

a $2(3x-2) = 3 - x$ **b** $4(3-2x) = 2(3x-2)$

c $(3x-2) - 3(2x+1) = 4(4x-3)$ **d** $4 - 3x = 2\left(1 - \dfrac{3}{2}x\right)$

e $2(1-4x) = -4\left(-\dfrac{1}{2} + 2x\right)$

3 Without solving, determine the number of solutions to each of the following systems of linear equations.

a $\begin{cases} 3x - 4y = 5 \\ 2x + 3y = 3 \end{cases}$ **b** $\begin{cases} 6x + 9y = 7 \\ 2x + 3y = 13 \end{cases}$ **c** $\begin{cases} -x + 4y = 7 \\ 2x - 8y = -14 \end{cases}$

4 Applying all the methods for solving systems of linear equations, solve each of the following.

a $\begin{cases} -2x - 3y = 5 \\ 2x + 3y = -5 \end{cases}$ **b** $\begin{cases} \dfrac{3}{2}x = 5 - 2y \\ x - 3y = 5 \end{cases}$ **c** $\begin{cases} 0.3x - 0.4y = 1 \\ 0.2x + y = 3 \end{cases}$

5 Solve each of the following equations that involve absolute value.

a $|2x-3| = 3$ **b** $3|x-1| = 7$ **c** $\left|\dfrac{1}{2} - 3x\right| = \dfrac{7}{2}$

d $|x+7| = -1$ **e** $|2-0.2x| = 5$ **f** $|2x-3| = 3|1-2x|$

g $|x-5| = |3+2x|$ **h** $|2x-4| = 2|2-x|$ **i** $|x+12| - 2|3x-1| = 0$

j $|5x-12| + |x+2| = 8$ **k** $3|x-7| + 2|1-3x| = 5$

6 Factorize the following expressions.

a $x^2 - 16x$ **b** $4x^2 + 16x + 12$ **c** $1 - 4x^2$

d $12x + 48x^2$ **e** $x^2 + 11x - 42$

7 Solve the following quadratic equations.

a $x^2 - 16x = -64$ **b** $2x^2 + 8x - 8 = 0$

c $4x - 3x^2 - 9 = 10x$ **d** $x^2 + 15x + 31 = 2x - 11$

e $7x^2 + x - 5 = 0$

8 By computing the discriminant $b^2 - 4ac$ for each of the following, determine how many solutions the equation has.

a $x^2 - 16x + 24 = 0$ **b** $2x^2 + 8x - 12 = 0$

c $-4x^2 - x - 2 = 0$ **d** $3x^2 - 6x + 3 = 0$

9 If two roots of a quadratic equation are –2 and 3, determine the quadratic equation.

10 If the sum of two numbers is 13 and their product is 42, determine the numbers.

11 Almaz has taken two tests. Her average score is 7 (out of ten). The product of her scores is 45. What did she score in each test?

12 If a and b are roots of $3x^2 - 6x + 2 = 0$, then find

a $a + b$ **b** ab **c** $\frac{1}{a} + \frac{1}{b}$

d $\frac{1}{a+2} + \frac{1}{b+2}$ **e** $a^2 + b^2$ **f** $a^3 - b^3$

13 Determine the values of p and q for which (–4, –3) will be solution of the system

$$\begin{cases} px + qy = -26 \\ qx - py = 7 \end{cases}.$$

14 An object is thrown vertically upward from a height of h_o ft with an initial speed of v_o ft/sec. Its height h (in feet) after t seconds is given by $h = -16t^2 + v_o t + h_o$. Given this, if it is thrown vertically upward from the ground with an initial speed of 64 ft/sec,

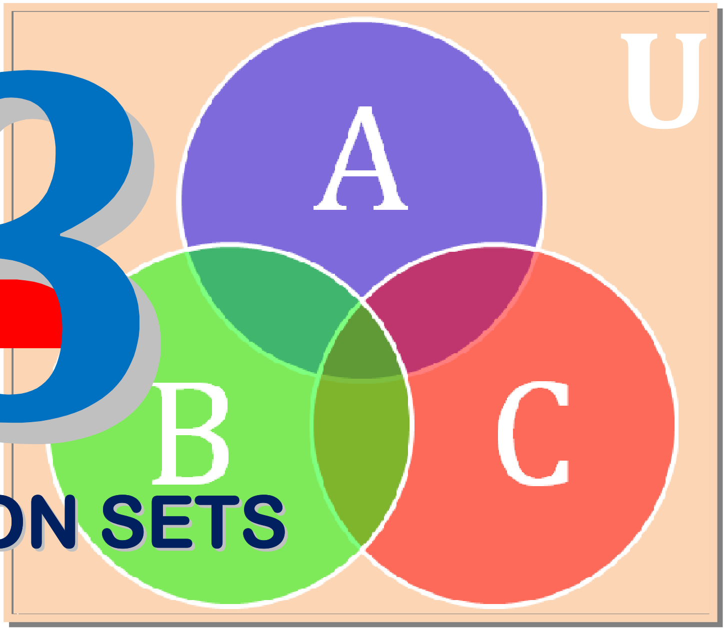
a At what time will the height of the ball be 15 ft? (two answers)

b How long will it take for the ball to reach 63 ft?

15 Determine the value of k so that the quadratic equation $4x^2 - 2x + k^2 - 2k + 1 = 0$ can have exactly one solution.

16 The speed of a boat in still water is 15 km/hr. It needs four more hours to travel 63 km against the current of a river than it needs to travel down the river. Determine the speed of the current of the river.

Unit 3



FURTHER ON SETS

Unit Outcomes:

After completing this unit, you should be able to:

- *understand additional facts and principles about sets.*
- *apply rules of operations on sets and find the result.*
- *demonstrate correct usage of Venn diagrams in set operations.*
- *apply rules and principles of set theory to practical situations.*

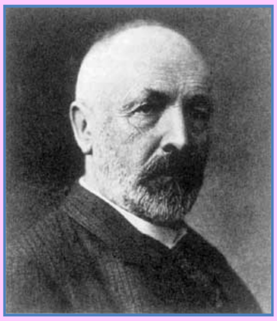
Main Contents

INTRODUCTION

In the present unit, you will learn more about sets. Particularly, you will discuss the different ways to describe sets and their representation through Venn diagrams. Also, you will discuss some operations that, when performed on two sets, give rise to another set. Finally, you will go through some practical problems related to our daily life and try to solve them, using the union and intersection of sets.

HISTORICAL NOTE:

George Cantor (1845-1918)



During the latter part of the 19th century, while working with mathematical entities called infinite series, George Cantor found it helpful to borrow a word from common usage to describe a mathematical idea. The word he borrowed was set. Born in Russia, Cantor moved to Germany at the age of 11 and lived there for the rest of his life. He is known today as the originator of set theory.

3.1 WAYS TO DESCRIBE SETS

ACTIVITY 3.1

1 What is a set? What do we mean when we say an element of a set?

2 Give two members or elements that belong to each of the following sets:

- **a** The set of composite numbers less than 10.
- **b** The set of natural numbers that are less than 50 and divisible by 3.
- **c** The set of whole numbers between 0 and 1.
- **d** The set of real numbers between 0 and 1.
- **e** The set of non-negative integers.
- **f** The set of integers that satisfy $(x - 2)(2x + 1) = 2x^2 - 3x - 2$.

3
- **a** Describe each of the sets in Question 2 by another method.
- **b** State the number of elements that belong to each set in Question 2.
- **c** In how many ways can you describe the sets given in Question 2?

4 Which of the sets in Question 2 have

- **a** no elements?
- **b** a finite number of elements?
- **c** infinitely many elements?

3.1.1 Sets and Elements

Set: A Set is any well-defined collection of objects.

When we say that a set is well-defined, we mean that, given an object, we are able to determine whether the object is in the set or not. For instance, "*The collection of all intelligent people in Africa*" cannot form a well-defined set, since we may not agree on who is an "*intelligent person*" and who is not.

The individual objects in a set are called the elements or the members. Repeating elements in a set does not add new elements to the set.

For example, the set $\{a, a, a\}$ is the same as $\{a\}$.

Notation: Generally, we use capital letters to name sets and small letters to represent elements. The symbol '$\in$' stands for the phrase 'is an element of' (or 'belongs to'). So, $x \in$ A is read as 'x is an element of A' or 'x belongs to A'. We write the statement 'x does not belong to A' as $x \notin$ A.

Since the phrase '*the set of*' occurs so often, we use the symbol called brace (or curly bracket) { }.

For instance, '*the set of all vowels in the English alphabet* is written as

{all vowels in the English alphabet} or $\{a, e, i, o, u\}$.

3.1.2 Description of Sets

A set may be described by three methods:

i Verbal method

We may describe a set in words. For instance,

a The set of all whole numbers less than ten or {all whole numbers less than ten}.

b The set of all natural numbers. This can also be written as {all natural numbers}.

ii The listing method (also called *roster* or *enumeration method*)

If the elements of a set can be listed, then we can describe the set by listing its elements. The elements can be listed completely or partially as illustrated in the following example:

Example 1 Describe (express) each of the following sets using the listing method:

a The set whose elements are a, 2 and 7.

b The set of natural numbers less than 51.

c The set of whole numbers.

d The set of non-positive integers.

e The set of integers.

Solution:

a First let us name the set by A. Then we can describe the set as

$$A = \{a, 2, 7\}$$

b The natural numbers less than 51 are 1, 2, 3, . . ., 50. So, naming the set as B we can express B by the listing method as

$$B = \{1, 2, 3, \ldots, 50\}$$

The three dots after the element 3 (called an ellipsis) indicate that the elements in the set continue in that manner up to and including the last element 50.

c Naming the set of whole numbers by $\mathbb{W}$, we can describe it as

$$\mathbb{W} = \{0, 1, 2, 3, \ldots\}$$

The three dots indicate that the elements continue in the given pattern and there is no last or final element.

d If we name the set by L, then we describe the set as

$$L = \{\ldots, -3, -2, -1, 0\}$$

The three dots that precede the numbers indicate that elements continue from the right to the left in that pattern and there is no beginning element.

e You know that the set of integers is denoted by $\mathbb{Z}$ and is described by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

We use the partial listing method, if listing all elements of a set is difficult or impossible but the elements can be indicated unambiguously by listing a few of them.

Exercise 3.1

1 Describe each of the following sets using a verbal method:

a $A = \{5, 6, 7, 8, 9\}$ **b** $M = \{2, 3, 5, 7, 11, 13\}$

c $G = \{8, 9, 10, \ldots\}$ **d** $E = \{1, 3, 5, \ldots, 99\}$

2 Describe each of the following sets using the listing method (if possible):

a The set of prime factors of 72.

b The set of natural numbers that are less than 113 and divisible by 5.

c The set of non-negative integers.

d The set of rational numbers between $\sqrt{2}$ and $\sqrt{8}$.

e The set of even natural numbers.

f The set of integers divisible by 3.

g The set of real numbers between 1 and 3.

iii The set-builder method (*also known as method of defining property*)

ACTIVITY 3.2

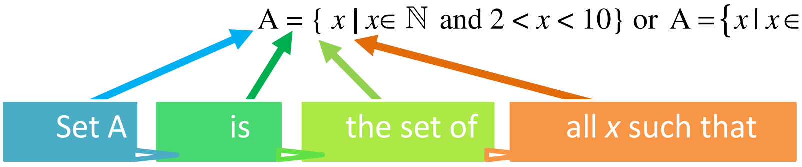
To each description given in column A, match a set that satisfies it from column B.

	A		B
1	$2 < x < 10$ and $x \in \mathbb{N}$	**a**	$\{1, 2, 3, 4, 6, 12\}$
2	$x = 2n$ and $n \in \mathbb{N}$	**b**	$\{-2\}$
3	$2x + 4 = 0$ and x is an integer	**c**	$\{2, 4, 6, \ldots\}$
4	$x \in \mathbb{N}$ and 12 is a multiple of x	**d**	$\{3, 4, 5, 6, 7, 8, 9\}$

The above Activity leads you to the third useful method for describing sets, known as the set-builder method.

For example, A = $\{3, 4, 5, 6, 7, 8, 9\}$ can be described as

$$A = \{x \mid x \in \mathbb{N} \text{ and } 2 < x < 10\} \text{ or } A = \{x \mid x \in \mathbb{N} \text{ and } 2 < x < 10\}$$



Note that "all x such that" may be written as "$x \mid$" or "x:".

Hence we read the above as "set A is the set of all elements x such that x is a natural number between 2 and 10".

Note that in the above set A the properties that characterize the elements of the set are $x \in \mathbb{N}$ and $2 < x < 10$.

Example 2 Express each of the following sets using set-builder method:

a $\mathbb{N} = \{1, 2, 3, \ldots\}$ **b** A = {real numbers between 0 and 1}

c B = {integers divisible by 3} **d** The real solution set of $|x - 1| = 2$

Solution:

a $\mathbb{N} = \{x \mid x \in \mathbb{N}\}$

b $A = \{x \mid x \in \mathbb{R} \text{ and } 0 < x < 1\}$

Note that this set can also be expressed as $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$

c $B = \{x \mid x = 3n, \text{ for some integer } n\}$ or $B = \{3n \mid n \in \mathbb{Z}\}$

d Naming the set by S, we write $S = \{x \mid x \in \mathbb{R} \text{ and } |x - 1| = 2\}$

Exercise 3.2

1 Which of the following collections are well defined? Justify your answer.

a $\{x \mid x \text{ is an interesting bird}\}$. **b** $\{x \mid x \text{ is a good student}\}$.

c The set of natural numbers less than 100. **d** $\{y \mid y \text{ is a factor of } 13\}$.

2 Which of the following are true and which are false?

a $2 \in \{-1, 0, 1\}$ **b** $a \notin \{\{a, c\}\}$ **c** $6 \in \{\text{factors of } 24\}$

3 Describe each of the following sets by

i the listing method. **ii** the set–builder method.

- **a** The set of letters in the word mathematics.
- **b** The set of regional states in Ethiopia.
- **c** The set of whole numbers between 5 and 13.
- **d** The set of even numbers less than 19.
- **e** The set of students in Ethiopia.
- **f** The set of all odd natural numbers.

4 Describe each of the following sets by

i a verbal method. **ii** the set–builder method.

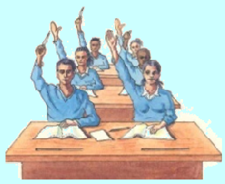
a $\{1, 2, 3,\dots, 10\}$ **b** $\{1, 3, 5, 7,\dots\}$ **c** $\{5, 10, 15, 20,\dots\}$

d {Tuesday, Thursday} **e** $\{2, 3, 5, 7, 11,\dots\}$

3.2 THE NOTION OF SETS

3.2.1 Empty Set, Finite Set, Infinite Set, Subset, Proper Subset

ACTIVITY 3.3



1 How many elements does each of the following sets have?

a $A = \{x \mid x \text{ is a real number whose square is negative one}\}$

b $C = \{x \mid x \in \mathbb{N} \text{ and } 2 < x < 11\}$ **c** $D = \{x \mid x \in \{1, 2, 3\}\}$

d $E = \{x \mid x \text{ is an integer}\}$ **e** $X = \{2, 4, 6, \dots\}$

2 Compare the sets C, D, E and X given in Question 1 above. Which set is contained in another?

Observe from the above Activity that a set may have no elements, a limited number of elements or an unlimited number of elements.

A Empty set

Definition 3.1

A set that contains no elements is called an empty set, or null set.

An empty set is denoted by either $\varnothing$ or $\{\}$.

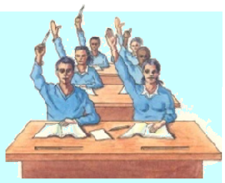
Example 1

a If $A = \{x \mid x \text{ is a real number and } x^2 = -1\}$, $A = \varnothing$ (Why?)

b If $B = \{x \mid x \neq x\}$, $B = \varnothing$. (Why?)

B Finite and infinite sets

ACTIVITY 3.4



Which of the following sets have a finite and which have an infinite number of elements?

1. $A = \{x \mid x \in \mathbb{R} \text{ and } 0 < x < 3\}$
2. $C = \{x \in \mathbb{N} \mid 7 < x < 7^{100}\}$
3. $D = \{x \in \mathbb{N} \mid x \text{ is a multiple of } 3\}$
4. $E = \{x \in \mathbb{Z} \mid 2 < x < 3\}$
5. $M = \{x \in \mathbb{N} \mid x \text{ is divisible by 5 and } x < 101^4\}$

Your observations from the above Activity lead to the following definition:

Definition 3.2

i A set S is called finite, if it contains n elements where n is some non-negative integer.

ii A set S is called infinite, if it is not finite.

Notation: If a set S is finite, then we denote the number of elements of S by n (S).

Example 2 If $S = \{-1, 0, 1\}$, then $n(S) = 3$

Using this notation, we can say that a set S is finite if $n(S) = 0$ or $n(S)$ is a natural number.

Example 3 Find n (S) if:

a $S = \{x \in \mathbb{R} \mid x^2 = -1\}$ **b** $S = \{x \in \mathbb{N} \mid x \text{ is a factor of } 108\}$

Solution:

a n (S) = 0 **b** n (S) = 12

Example 4

a Let E = {2, 4, 6,…}. E is infinite.

b Let T = $\{x \mid x$ is a real number and $0 < x < 1\}$. T is infinite.

C Subsets

ACTIVITY 3.5

What is the relationship between each of the following pairs of sets?

1 M = {all students in your class whose names begin with a vowel};

N = {all students in your class whose names begin with E}

2 A = {1, 3, 5, 7}; B = {1, 2, 3, 4, 5, 6, 7, 8}

3 $E = \{x \in \mathbb{R} \mid (x-2)(x-3) = 0\}$; $F = \{x \in \mathbb{N} \mid 1 < x < 4\}$

Definition 3.3

Set A is a subset of set B, denoted by $A \subseteq B$, if each element of A is an element of B.

Note: If A is not a subset of B, then we denote this by $A \not\subseteq B$.

Example 5 Let $\mathbb{Z} = \{x \mid x$ is an integer$\}$; $\mathbb{Q} = \{x \mid x$ is a rational number$\}$.

Since each element of $\mathbb{Z}$ is also an element of $\mathbb{Q}$, then $\mathbb{Z} \subseteq \mathbb{Q}$

Example 6 Let G = {–1, 0, 1, 2, 3} and H = {0, 1, 2, 3, 4, 5}

$-1 \in G$ but $-1 \notin H$, hence $G \not\subseteq H$.

Note: For any set A

i $\varnothing \subseteq A$ **ii** $A \subseteq A$

Group Work 3.1

Given A = $\{a, b, c\}$

1 List all the subsets of A.

2 How many subsets have you found?

From Group Work 3.1, you can make the following definition.

Definition 3.4

Let A be any set. The power set of A, denoted by P(A), is the set of all subsets of A. That is, $P(A) = \{S \mid S \subseteq A\}$

Example 7 Let $M = \{-1, 1\}$. Then subsets of M are $\varnothing$, $\{-1\}$, $\{1\}$ and M.

Therefore $P(M) = \{\varnothing, \{-1\}, \{1\}, M\}$

D Proper subset

Let $A = \{-1, 0, 1\}$ and $B = \{-2, -1, 0, 1\}$. From these sets, we see that $A \subseteq B$ but $B \not\subseteq A$. This suggests the definition of a proper subset stated below.

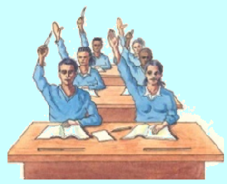
Definition 3.5

Set A is said to be a proper subset of a set B, denoted by $A \subset B$, if A is a subset of B and B is not a subset of A.

That is, $A \subset B$ means $A \subseteq B$ but $B \not\subseteq A$.

Note: For any set A, A is not a proper subset of itself.

ACTIVITY 3.6



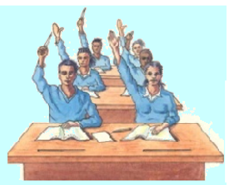
Given $A = \{-1, 0, 1\}$.

i List all proper subsets of A.

ii How many proper subsets of A have you found?

You will now investigate the relationship between the number of elements of a given set and the number of its subsets and proper subsets.

ACTIVITY 3.7



1 Find the number of subsets and proper subsets of each of the following sets:

a $A = \varnothing$ **b** $B = \{0\}$ **c** $C = \{-1, 0\}$ **d** $D = \{-1, 0, 1\}$

2 Copy and complete the following table:

	Set	No. of elements	Subsets	No. of subsets	Proper subsets	No. of proper subsets
a	$\varnothing$	0	$\varnothing$	$1 = 2^0$	-	$0 = 2^0 - 1$
b	$\{0\}$	1	$\varnothing$, $\{0\}$	$2 = 2^1$	$\varnothing$	$1 = 2^1 - 1$
c	$\{-1, 0\}$					
d	$\{-1, 0, 1\}$		$\varnothing$, $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$, $\{0, 1\}$, $\{-1, 1\}$, $\{-1, 0, 1\}$			$7 = 2^3 - 1$

You generalize the result of the above Activity in the form of the following fact.

Fact: If a set A is finite with n elements, then

i The number of subsets of A is 2^n and

ii The number of proper subsets of A is $2^n - 1$.

Exercise 3.3

1 For each set in the left column, choose the sets from the right column that are subsets of it:

i	$\{a, b, c, d\}$	**a**	$\{\ \}$
ii	$\{o, p, k\}$	**b**	$\{1, 4, 8, 9\}$
iii	Set of letters in the word “book”	**c**	$\{o, k\}$
iv	$\{2, 4, 6, 8, 10, 12\}$	**d**	$\{12\}$
		e	$\{6\}$

2 **a** If B = $\{0, 1, 2\}$, find all subsets of B.

b If B = $\{0, \{1, 2\}\}$, find all subsets of B.

3 State whether each of the following statements is true or false. If it is false, justify your answer.

a $\{1, 4, 3\} \subseteq \{3, 4, 1\}$ **b** $\{1, 3, 1, 2, 3, 2\} \not\subseteq \{1, 2, 3\}$

c $\{4\} \subseteq \{\{4\}\}$ **d** $\varnothing \subseteq \{\{4\}\}$

3.2.2 Venn Diagrams, Universal Sets, Equal and Equivalent Sets

A Venn diagrams

ACTIVITY 3.8

1 What is the relationship between the following pairs of sets?

a $\mathbb{W} = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

b $\mathbb{W} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$.

c $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$.

d $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$ and $\mathbb{Q} = \left\{\dfrac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$.

2 Express the relationship between each pair using a diagram.

3 Express the relationship of all the sets, $\mathbb{W}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ using one diagram. Compare your diagram with the one given in Activity 1.1 of Unit 1.

To illustrate various relationships that can arise between sets, it is often helpful to use a pictorial representation called a Venn diagram named after John Venn (1834 – 1883). These diagrams consist of rectangles and closed curves, usually circles. The elements of the sets are written in their respective circles.

For example, the relationship '$A \subset B$' can be illustrated by the following Venn diagram.

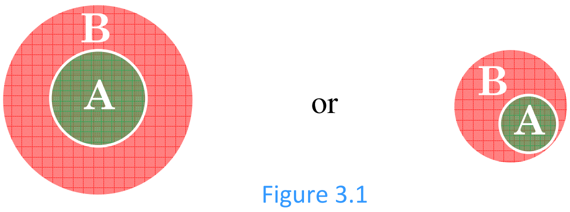


Figure 3.1

Example 1 Represent the following pairs of sets using Venn diagrams:

a $A = \{a, b, c, d\}$; $B = \{a, d\}$

b $C = \{2, 4, 6, 8, \ldots\}$; $D = \{1, 3, 5, 7, \ldots\}$

c $E = \{2^n \mid n \in \mathbb{N}\}$; $F = \{2n \mid n \in \mathbb{N}\}$

d $A = \{1, 3, 5, 7, 9\}$; $B = \{2, 3, 5, 8\}$; $C = \{1, 5, 7\}$

Solution:

a

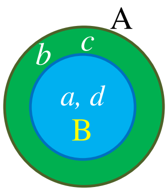


Figure 3.2

b

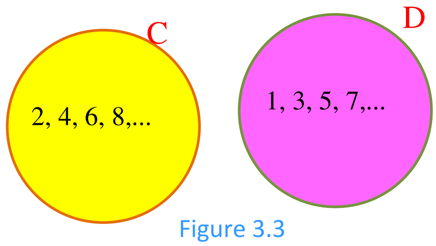


Figure 3.3

c

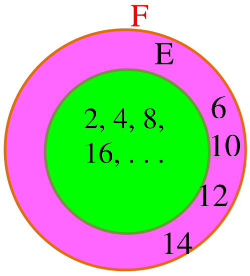


Figure 3.4

d

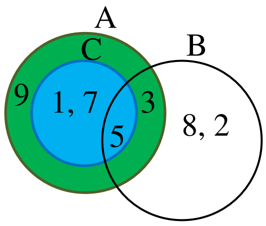


Figure 3.5

B Universal set

Suppose at a school assembly, the following students are asked to stay behind.

G = {all Grade 9 students}.

I = {all students interested in a school play}.

R = {all class representatives of each class}.

Each set G, I and R is a subset of S = {all students in the school}

In this particular example, S is called the universal set.

Similarly, a discussion is limited to a fixed set of objects and if all the elements to be discussed are contained in this set, then this "overall" set is called the universal set. We usually denote the universal set by U. Different people may choose different universal sets for the same problem.

Example 2 Let R={all red coloured cars in East Africa}; T = {all Toyota cars in East Africa}

i Choose a universal set U for R and T.

ii Draw a Venn diagram to represent the sets U, R and T.

Solution:

i There are different possibilities for U. Two of these are:

U = {all cars} or U = {all wheeled vehicles}

ii In both cases, the Venn diagram of the sets U, R and T is

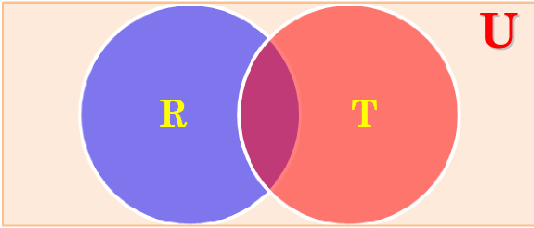


Figure 3.6

Exercise 3.4

1 Draw Venn diagrams to illustrate the relationships between the following pairs of sets:

- **a** A = {1, 9, 2, 7, 4}; L = {4, 9, 8, 2}
- **b** B = {the vowels in the English alphabet}
  M = {the first five letters of the English alphabet}
- **c** C = {1, 2, 3, 4, 5}; M = {6, 9, 10, 8, 7}
- **d** F = {3, 7, 11, 5, 9}; O = {all odd numbers between 2 and 12}

2 For each of the following, draw a Venn diagram to illustrate the relationship between the sets:

- **a** U = {all animals}; C = {all cows}; G ={all goats}
- **b** U = {all people}; M = {all males}; B = {all boys}

C Equal and equivalent sets

ACTIVITY 3.9

From the following pairs of sets identify those:

1 that have the same number of elements.

2 that have exactly the same elements.

- **a** $A = \{1, 2\}$; $B = \{x \in \mathbb{N} \mid x < 3\}$
- **b** $E = \{-1, 3\}$; $F = \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- **c** $R = \{1, 2, 3\}$; $S = \{a, b, c\}$
- **d** $G = \{x \in \mathbb{N} \mid x \text{ is a factor of } 6\}$; $H = \{x \in \mathbb{N} \mid 6 \text{ is a multiple of } x\}$
- **e** $X = \{1, 1, 3, 2, 3, 1\}$; $Y = \{1, 2, 3\}$

i Equality of sets

Let us investigate the relationship between the following two sets;

$$E = \{x \in \mathbb{R} \mid (x-2)(x-3) = 0\} \text{ and } F = \{x \in \mathbb{N} \mid 1 < x < 4\}.$$

By listing completely the elements of each set, we have E = {2, 3} and F = {2, 3}.

We see that E and F have exactly the same elements. So they are equal.

Is $E \subseteq F$? Is $F \subseteq E$?

Definition 3.6

Given two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal. We write this as A = B.

$\therefore$ A = B, if and only if $A \subseteq B$ and $B \subseteq A$.

Example 3 Let A = {1, 2, 3, 4} and B = {1, 4, 2, 3}.

A = B, since these sets contain exactly the same elements.

Note: If A and B are not equal, we write $A \neq B$.

Example 4 Let C = {–1, 3, 1} and D = {–1, 0, 1, 2}.

$C \neq D$, because $2 \in D$, but $2 \notin C$.

ii Equivalence of sets

Consider the sets A = {a, b, c} and B = {2, 3, 4}. Even though these two sets are not equal, they have the same number of elements. So, for each member of set B we can find a partner in set A.

A = {a, b, c}

B = {2, 3, 4}

The double arrow shows how each element of a set is matched with an element of another set. This matching could be done in different ways, for example:

A = {a, b, c}

B = {2, 3, 4}

No matter which way we match the sets, each element of A is matched with exactly one element of B and each element of B is matched with exactly one element of A. We say that there is a one–to–one correspondence between A and B.

Definition 3.7

Two sets A and B are said to be equivalent, written as $A \leftrightarrow B$ (or $A \sim B$), if there is a one-to-one correspondence between them.

Observe that two finite sets A and B are equivalent, if and only if

$n(A) = n(B)$

Example 5 Let $A = \{\sqrt{2}, e, \pi\}$ and $B = \{1, 2, 3\}$.

Since $n(A) = n(B)$, A and B are equivalent sets and we write

$A \leftrightarrow B$.

Note that equal sets are always equivalent since each element can be matched with itself, but equivalent sets are not necessarily equal. For example,

$\{1, 2\} \leftrightarrow \{a, b\}$ but $\{1, 2\} \neq \{a, b\}$.

Exercise 3.5

Which of the following pairs represent equal sets and which of them represent equivalent sets?

1. $\{a, b\}$ and $\{2, 4\}$
2. $\{\varnothing\}$ and $\varnothing$
3. $\{x \in \mathbb{N} \mid x < 5\}$ and $\{2, 3, 4, 5\}$
4. $\{1, \{2, 4\}\}$ and $\{1, 2, 4\}$
5. $\{x \mid x < x\}$ and $\{x \in \mathbb{N} \mid x < 1\}$

3.3 OPERATIONS ON SETS

There are operations on sets as there are operations on numbers. Like the operations of addition and multiplication on numbers, intersection and union are operations on sets.

3.3.1 Union, Intersection and Difference of Sets

A Union of sets

Definition 3.8

The union of two sets A and B, denoted by $A \cup B$ and read "A union B" is the set of all elements that are members of set A or set B or both of the sets. That is, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

The *red* shaded region of the diagram in the figure on the right represents A ∪ B.

An element common to both sets is listed only once in the union. For example, if $A = \{a, b, c, d, e\}$ and $B = \{c, d, e, f, g\}$, then

$A \cup B = \{a, b, c, d, e, f, g\}$.

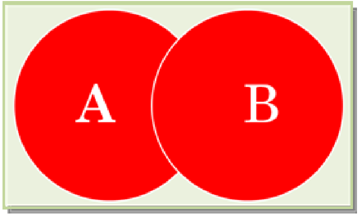


Figure 3.7

Example 1

a $\{a, b\} \cup \{c, d, e\} = \{a, b, c, d, e\}$

b $\{1, 2, 3, 4, 5\} \cup \varnothing = \{1, 2, 3, 4, 5\}$

Properties of the union of sets

ACTIVITY 3.10

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$.

1 Find **a** $A \cup B$ **b** $B \cup A$

What is the relationship between $A \cup B$ and $B \cup A$?

2 Find **a** $A \cup B$ **b** $(A \cup B) \cup C$ **c** $B \cup C$ **d** $A \cup (B \cup C)$

What is the relationship between $(A \cup B) \cup C$ and $A \cup (B \cup C)$?

3 Find $A \cup \varnothing$, what is the relationship between $A \cup \varnothing$ and A?

The above Activity leads you to the following properties:

For any sets A, B and C

1 Commutative property $A \cup B = B \cup A$

2 Associative property $(A \cup B) \cup C = A \cup (B \cup C)$

3 Identity property $A \cup \varnothing = A$

Exercise 3.6

1 Given $A = \{1, 2, \{3\}\}$, $B = \{2, 3\}$ and $C = \{\{3\}, 4\}$, find:

a $A \cup B$ **b** $B \cup C$ **c** $A \cup C$

d $A \cup (B \cup C)$ **e** $(A \cup B) \cup C$

2 State whether each of the following statements is true or false:

a If $x \in A$ and $x \notin B$, then $x \notin (A \cup B)$.

b If $x \in (A \cup B)$ and $x \notin A$, then $x \in B$.

c If $x \notin A$ and $x \notin B$, then $x \notin (A \cup B)$.

d For any set A, $A \cup A = A$.

e For any set A, $A \cup \varnothing = A$.

f If $A \subseteq B$, then $A \cup B = B$.

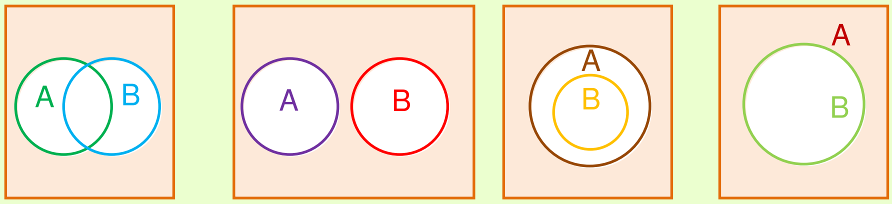
g For any two sets A and B, $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$.

h For any three sets A, B and C, if $A \subseteq B$, and $B \subset C$, then $A \cup B = C$.

i For any three sets A, B and C, if $A \cup B = C$, then $B \subset C$.

j If $A \cup B = \varnothing$, then $A = \varnothing$ and $B = \varnothing$.

3 Using copies of the Venn diagrams below, shade $A \cup B$.



a b c d

Figure 3.8

B Intersection of sets

ACTIVITY 3.11

Consider the two sets G = {2, 4, 6, 8, 10, 12} and H = {1, 2, 3, 4, 5}.

a Draw a Venn diagram that shows the relationship between the two sets.

b Shade the region common to both sets and find their common elements.

Definition 3.9

The intersection of two sets A and B, denoted by $A \cap B$ and read as "*A intersection B*", is the set of all elements common to both set A and set B. That is, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Using the Venn diagram, $A \cap B$ is represented by the *blue* shaded region:

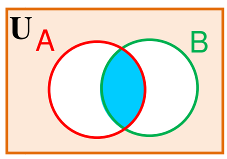


Figure 3.9

Example 2 Let S = {a, b, c, d} and T = {f, b, d, g}. Then $S \cap T$ = {b, d}.

Example 3 Let V = {2, 4, 6,...} (multiples of 2) and

W = {3, 6, 9,...} (multiples of 3).

Then $V \cap W$ = {6, 12, 18,...}, that is, multiples of 6.

Example 4 Let A = {1, 2, 3} and B = {5, 6, 7, 8}, then $A \cap B = \varnothing$.

Definition 3.10

Two or more sets are disjoint if they have no common element.

A and B are disjoint, if and only if $A \cap B = \varnothing$.

In the Venn diagram, the sets A and B are disjoint.

Here $A \cap B = \varnothing$

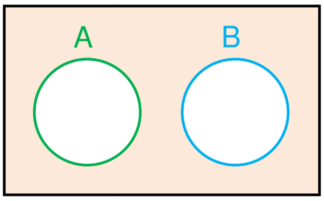


Figure 3.10

Properties of the intersection of sets

ACTIVITY 3.12

Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set and let

$A = \{0, 2, 3, 5, 7\}$, $B = \{0, 2, 4, 6, 8\}$ and

$C = \{x \mid x \text{ is a factor of } 6\}$

1 Find **a** $A\cap C$ **b** $C\cap A$

What is the relationship between $A\cap C$ and $C\cap A$?

2 Find **a** $A\cap B$ **b** $(A\cap B) \cap C$ **c** $B\cap C$ **d** $A\cap(B\cap C)$

What is the relationship between $(A\cap B) \cap C$ and $A\cap(B\cap C)$?

3 Find $A\cap U$. What is the relationship between $A\cap U$ and A?

The above Activity leads you to the following properties:

For any sets A, B and C and the universal set U

1. Commutative Property: $A\cap B = B\cap A$.
2. Associative Property: $(A\cap B)\cap C = A\cap(B\cap C)$.
3. Identity Property: $A\cap U = A$.

Exercise 3.7

1 Given $A = \{a, b, \{c\}\}$, $B = \{b, c\}$ and $C = \{\{c\}, d\}$, find:

a $A\cap B$ **b** $A\cap C$ **c** $B\cap C$ **d** $A\cap(B\cap C)$

2 State whether each of the following statements is true or false:

a If $x\in A$ and $x\notin B$, then $x\in (A\cap B)$. **b** If $x\in (A\cap B)$, then $x\in A$ and $x\in B$.

c If $x\notin A$ and $x\in B$, then $x\in (A\cap B)$. **d** For any set A, $A\cap A = A$.

e If $A \subseteq B$, then $A\cap B = A$.

f For any two sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

g If $A \cap B = \varnothing$, then $A = \varnothing$ or $B = \varnothing$.

h If $(A \cup B) \subseteq A$, then $B \subseteq A$.

i If $A \subseteq B$, then $A \cap B = B$.

j If $A \subseteq B$, then $A \cap B = \varnothing$.

k If $A \subseteq B$, then $B' \subseteq A'$.

3 In each Venn diagram below, shade $(A \cap B) \cap C$.

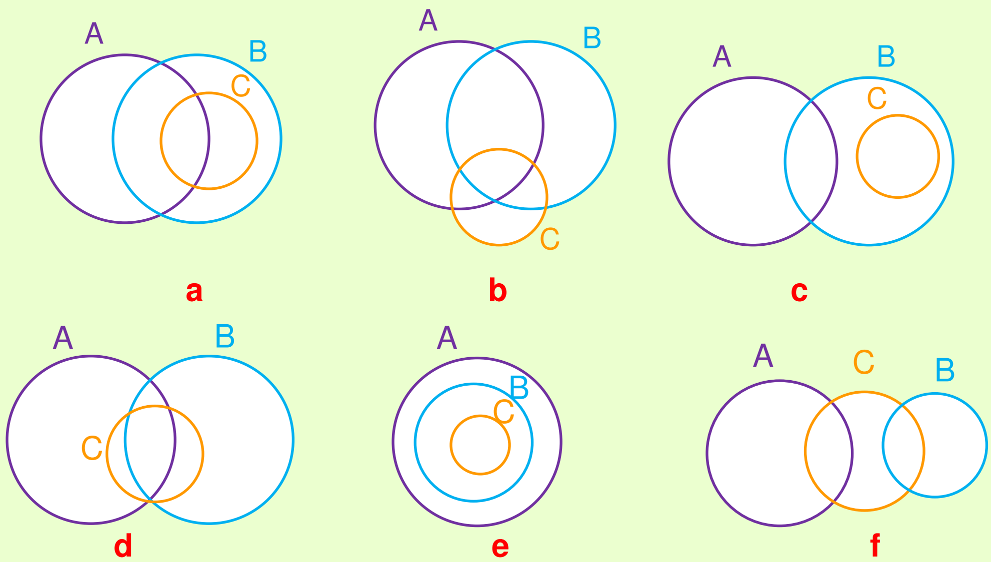


Figure 3.11

C Difference and symmetric difference of sets

i The relative complement (or difference) of two sets

Given two sets A and B, the complement of B relative to A (or the difference between A and B) is defined as follows.

> **Definition 3.11**
>
> The **relative complement** of a set B with respect to a set A (or the **difference** between A and B), denoted by $A - B$, read as "A difference B", is the set of all elements in A that are not in B.
>
> That is, $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

Note: $A - B$ is sometimes denoted by $A \backslash B$. (read as "A less B")

$A - B$ and $A \backslash B$ are used interchangeably.

Using a Venn diagram, A\B can be represented by shading the region in A which is not part of B.

A\B is shaded in *light green*.

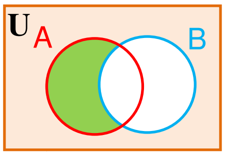


Figure 3.12

Example 5 If A = {x, y, z, w} and B = { a, b, x, y}, then find:

a the complement of B relative to A **b** B\ A **c** B\B

Solution:

a Note that finding "the complement of B relative to A" is the same as finding "the relative complement of B with respect to A". That is A\B.

So, A\B = {z, w}.

b B\A = { a, b}.

c B\B = ∅.

ACTIVITY 3.13

Let A = {0, 2, 3, 5, 7}, B = {0, 2, 4, 6, 8} and C = {1, 2, 3, 6}. Find:

a A\B **b** B\A **c** (A\B)\C **d** A\(B\C)

From the results of the above Activity, we can conclude that the relative complement of sets is neither commutative nor associative.

ii The complement of a set

Let U = {all human beings} and F = {all females}

The Venn diagram of these two sets is as shown. The yellow shaded region (in **U** but outside F) is called the complement of F, denoted by F'.

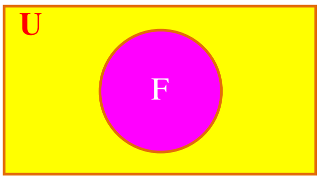


Figure 3.13

It represents all human beings who are not female. The members of F ' are all those members of U that are not members of F.

Definition 3.12

Let A be a subset of a universal set U. The complement (or absolute complement) of A, denoted by A', is defined to be the set of all elements of U that are not in A.

i.e., $A' = \{x \mid x \in U \text{ and } x \notin A\}$.

Using a Venn diagram, we can represent A' by the shaded region as shown in Figure 3.14.

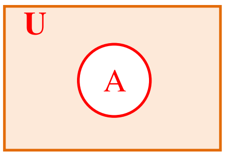


Figure 3.14

Note that for any set A and universal set U,

$A' = U \backslash A$

Example 6 In copies of the Venn diagram on the right, shade

a A\B **b** (A∩B)'

c A∩B' **d** A'∪B'

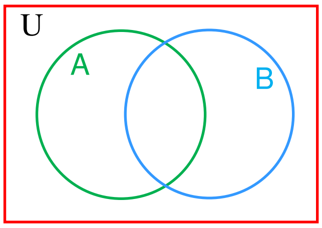


Figure 3.15

Solution:

a Since A\B is the set of all elements in A that are not in B, we shade the region in A that is not part of B (*shaded in green in* Figure 3.16).

A – B is shaded

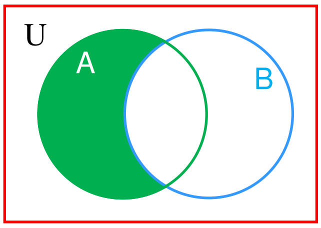
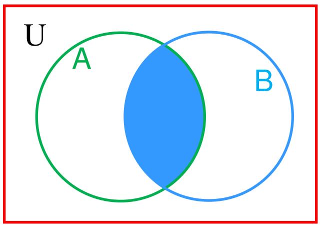


Figure 3.16

b First we shade the region A∩B; then (A∩B)' is the region outside A∩B.

i 

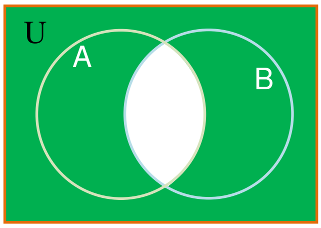
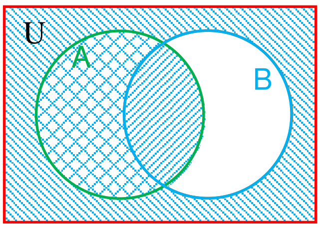
ii 

Figure 3.17

A∩B is the shaded (*blue*) region. (A∩B)' is the *green* shaded region.

c First we shade A with strokes that slant upward to the right (////) and shade B' with strokes that slant downward to the right (\\\\).

Then A∩B' is the cross-hatched region.

i 

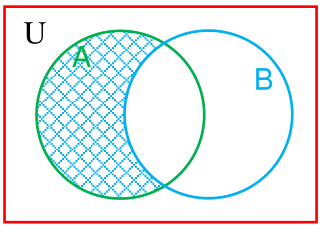
ii 

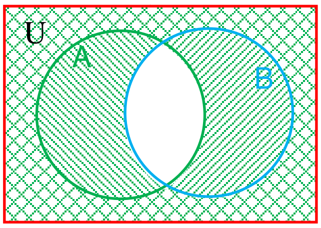
Figure 3.18

A and B' are shaded A∩B' is shaded

Note that the region of A\B is the same as the region of A∩B'.

d First we shade A', the region outside A, with strokes that slant upward to the right (////) and then shade B' with strokes that slant downward to the right (\\\\).

Then A'∪B' is the total shaded region.



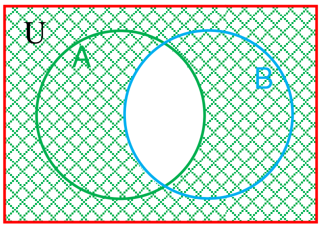


Figure 3.19

A' or B' are shaded A'∪B' is shaded

Note that the region of (A∩B)' is the same as the region A'∪B'.

Note: When we draw two overlapping circles within a universal set, four regions are formed. Every element of the universal set U is in exactly one of the following regions.

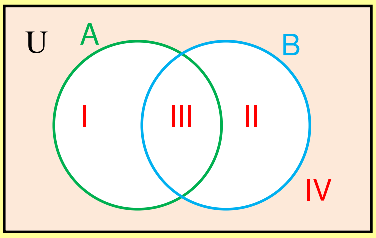


Figure 3.20

I in A and not in B (A\B)

II in B and not in A (B\A)

III in both A and B (A ∩B)

IV in neither A nor B ((A∪B)')

From Activity 3.13 and the above examples, you generalize as follows:

For any two sets A and B, the following properties hold true:

1 A\B = A ∩ B' **2** (A ∩ B)' = A' ∪ B' **3** (A\B)∪B = A∪B

ACTIVITY 3.14

1 In copies of the same Venn diagram used in Example 6, shade

a (A∪B)' **b** A'∩B'

2 Generalize the result you got from Question 1.

HISTORICAL NOTE:

Augustus De Morgan (1806-1871)

Augustus De Morgan was the first professor of mathematics at University College London and a cofounder of the London Mathematical Society.

De Morgan formulated his laws during his study of symbolic logic. De Morgan's laws have applications in the areas of set theory, mathematical logic and the design of electrical circuits.

Group Work 3.2

1. Copy Figure 3.21 and shade the region that represents each of the following
   - **a** $(A \cup B)'$
   - **b** $A' \cup B'$
   - **c** $(A \cap B)'$
   - **d** $A' \cap B'$

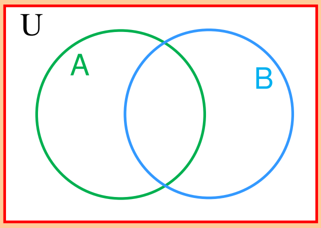


Figure 3.21

2. Discuss what you have observed from Question 1

The above Group Work leads you to the following law called De Morgan's law.

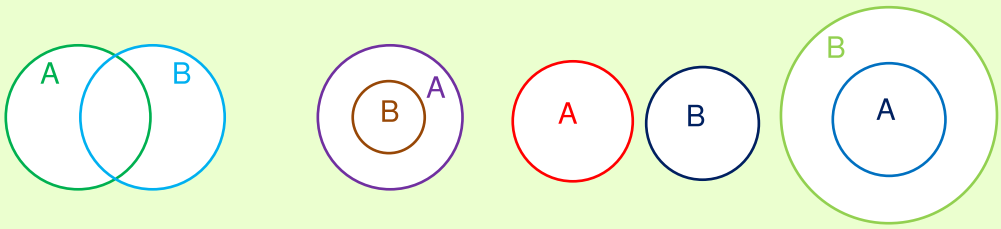
Theorem 3.1 De Morgan's law

For any two sets A and B

1 $(A \cap B)' = A' \cup B'$ **2** $(A \cup B)' = A' \cap B'$

Exercise 3.8

1. Given $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$ find:
   - **a** the relative complement of A with respect to B.
   - **b** the complement of B relative to A.
   - **c** the complement of A relative to B.
2. In each of the Venn diagrams given below, shade $A \backslash B$.



a b c d

Figure 3.22

3. Determine whether each of the following statements is true or false:
   - **a** If $x \in A$ and $x \notin B$ then $x \in (B \backslash A)$
   - **b** If $x \in (A \backslash B)$ then $x \in A$
   - **c** $B \backslash A \subseteq B$, for any two sets A and B

d $(A \backslash B) \cap (A \cap B) \cap (B \backslash A) = \varnothing$, for any two sets A and B

e If $A \backslash B = \varnothing$ then $A = \varnothing$ and $B = \varnothing$

f If $A \subseteq B$ then $A \backslash B = \varnothing$

g If $A \cap B = \varnothing$ then $(A \backslash B) = A$

h $(A \backslash B) \cup B = A \cup B$, for any two sets A and B

i $A \cap A' = \varnothing$

4 Let $U = \{1, 2, 3, \ldots, 8, 9\}$ be the universal set and $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. List the elements of each of the following:

a A' **b** B' **c** $(A \cup C)'$ **d** $(A \backslash B)'$

e $A' \cap B'$ **f** $(A \cup B)'$ **g** $(A')'$ **h** $B \backslash C$ **i** $B \cap C'$

iii The symmetric difference between two sets

ACTIVITY 3.15

Let $A = \{a, b, d\}$ and $B = \{b, d, e\}$. Then find:

a $A \cap B$ **b** $A \cup B$ **c** $A \backslash B$

d $B \backslash A$ **e** $(A \cup B) \backslash (A \cap B)$ **f** $(A \backslash B) \cup (B \backslash A)$

Compare the results of e and f.

What can you conclude from your answers?

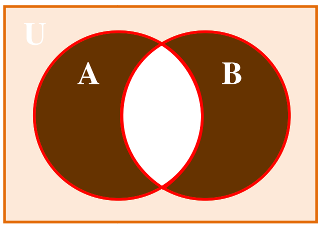
The result of the above Activity leads you to state the following definition.

Definition 3.13

Let A and B be any two sets. The symmetric difference between A and B, denoted by $A \Delta B$, is the set of all elements in $A \cup B$ that are not in $A \cap B$. That is $A \Delta B = \{x \mid x \in (A \cup B) \text{ and } x \notin (A \cap B)\}$

or $A \Delta B = (A \cup B) \backslash (A \cap B)$.

Using a Venn diagram, $A \Delta B$ is illustrated by shading the region in $A \cup B$ that is not part of $A \cap B$ as shown.



$A \Delta B$ is the shaded *dark brown region.*

Figure 3.23

From Activity 3.15 and the above Venn diagram, you observe that

$A \Delta B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$.

Note: If $A \cap B = \varnothing$ then $A \Delta B = A \cup B$.

Example 7 Let $A = \{-1, 0, 1\}$ and $B = \{1, 2\}$. Find $A\Delta B$.

Solution: $A\cup B = \{-1, 0, 1, 2\}$; $A\cap B = \{1\}$

$\therefore\ A\Delta B = (A\cup B)\backslash(A\cap B) = \{-1, 0, 2\}$

Example 8 Let $A = \{a, b, c\}$ and $B = \{d, e\}$. Find $A\Delta B$.

Solution: $A\cup B = \{a, b, c, d, e\}$; $A\cap B = \varnothing$

$\therefore\ A\Delta B = (A\cup B)\backslash\varnothing = A\cup B = \{a, b, c, d, e\}$

Distributivity

Group Work 3.3

1 Given sets A, B and C, shade the region that represents each of the following

a $A\cup(B\cap C)$

b $(A\cup B)\cap(A\cup C)$

c $A\cap(B\cup C)$

d $(A\cap B)\cup(A\cap C)$

2 Discuss what you have observed from Question 1.

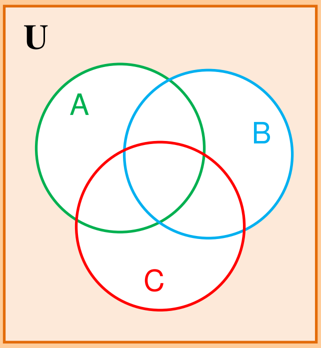


Figure 3.24

As you may have noticed from the above Group Work, the following distributive properties are true:

Distributive properties

For any sets A, B and C

1 Union is distributive over the intersection of sets.

i.e., $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$.

2 Intersection is distributive over the union of sets.

i.e., $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.

Exercise 3.9

1 If $A\cap B = \{1, 0, -1\}$ and $A\cap C = \{0, -1, 2, 3\}$, then find $A\cap(B\cup C)$.

2 Simplify each of the following by using Venn diagram or any other property.

a $A\cap(A\cup B)$ b $P'\cap(P\cup Q)$

c $A\cap(A'\cup B)$ d $P\cup(P\cap Q)$

3.3.2 Cartesian Product of Sets

In this subsection, you will learn how to form a new set of ordered pairs from two given sets by taking the Cartesian product of the sets (named after the mathematician Rene Descartes).

Group Work 3.4

A six-sided die (a cube) has its faces marked with numbers 1,2,3,4,5 and 6 respectively.

Two such dice are thrown and the numbers on the resulting upper faces are recorded. For example, (6, 1) means that the number on the upper face of the first die is 6 and that of the second die is 1. We call these ordered pairs, the outcomes of the throw of our dice.

List the set of all possible outcomes of throws of our two dice such that the two numbers:

- i A: are both even.
- ii B: are both odd.
- iii C: are equal.
- iv D: have a sum equal to 8.
- v E: have a sum equal to 14.
- vi F: have an even sum.
- vii G: have the first number 1 and the second number odd.
- viii H: have a sum less than 12.

For example, A= {(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)}.

The activity of this Group Work leads you to learn about the sets whose elements are ordered pairs.

Ordered pair

An *ordered pair* is an element (x, y) formed by taking x from one set and y from another set. In (x, y), we say that x is the first element and y is the second element.

Such a pair is ordered in the sense that (x, y) and (y, x) are not equal unless $x = y$.

Equality of ordered pairs

$(a, b) = (c, d)$, if and only if $a = c$ and $b = d$.

Earlier also we have discussed ordered pairs when we represented points in the Cartesian coordinate plane. A point P in the plane corresponds to an ordered pair (a, b) where a is the x-coordinate and b is the y-coordinate of the point P.

Example 1 A weather bureau recorded hourly temperatures as shown in the following table.

Time	9	10	11	12	1	2	3
Temp	61	62	65	69	68	72	76

This data enables us to make seven sentences of the form:

At x o'clock the temperature was y degrees.

That is, using the ordered pair (x, y),the ordered pair (9, 61) means.

At 9 *o'clock the temperature was* 61 *degrees.*

So the set of ordered pairs {(9, 61), (10, 62), (11, 65), (12, 69), (1, 68), (2, 72), (3, 76)} are another form of the data in the table, where the first element of each pair is time and the second element is the temperature recorded at that time.

Definition 3.14

Given two non-empty sets A and B, the set of all ordered pairs (a, b) where $a \in$ A and $b \in$ B is called the Cartesian product of A and B, denoted by A × B (read "A cross B").

i.e., $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Note that the sets A and B in the definition can be the same or different.

Example 2 If A = { 1, 2, 3} and B = { 4, 5}, then

A × B = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}

Example 3 Let A = { a, b}, then form A × A.

Solution: $A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$.

Example 4 Let A = {–1, 0} and B = {–1, 0, 1}.

Find A × B and illustrate it by means of a diagram.

Solution: A × B = {(–1, –1), (–1, 0), (–1, 1), (0, –1), (0, 0), (0, 1)}

The diagram is as shown in Figure 3.25.

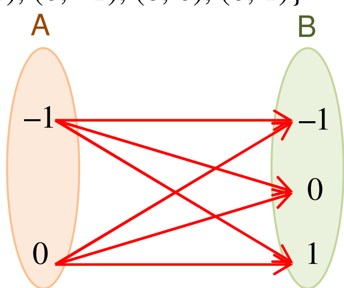
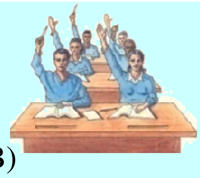


Figure 3.25

Note: $n(A \times B) = n(A) \times n(B)$.

ACTIVITY 3.16



1 Let A = {2, 3} and B = {0, 1, 2}. Find:

a $A \times B$ **b** $B \times A$ **c** $n(A \times B)$

2 Let A = {a, b} B = {c, d, e} and C = {f, e, c}. Find:

a $A \times (B \cap C)$ **b** $A \times (B \cup C)$

c $(A \times B) \cap (A \times C)$ **d** $(A \times B) \cup (A \times C)$

From the result of the Activity, you conclude that:

For any sets A, B and C

i $A \times B \neq B \times A$, for $A \neq B$ *Cartesian product of sets is not commutative.*

ii $n(A \times B) = n(A) \times n(B) = n(B \times A)$. *where A and B are finite sets.*

iii $A \times (B \cap C) = (A \times B) \cap (A \times C)$. *Cartesian product is distributive over intersection.*

iv $A \times (B \cup C) = (A \times B) \cup (A \times C)$. *Cartesian product is distributive over union.*

Exercise 3.10

1 Given A = {2} B = {1, 5} C = {−1, 1} find:

a $A \times B$ **b** $B \times A$ **c** $B \times C$ **d** $A \times (B \cap C)$

e $(A \cup C) \times B$ **f** $(A \times B) \cup (A \times C)$ **g** $B \times B$

2 If $B \times C = \{(1, 1), (1, 2), (1, 3), (4, 1), (4, 2), (4, 3)\}$, find:

a B **b** C **c** $C \times B$

3 If $n(A \times B) = 18$ and $n(A) = 3$ then find $n(B)$.

4 Let U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} be the universal set and A = {0, 2, 4, 6, 8, 9}, B = {1, 3, 6, 8} and C = {0, 2, 3, 4, 5}. Find:

a $A' \times C'$ **b** $B \times A'$ **c** $B \times (A' \setminus C)$

5 If $(2x + 3, 7) = (7, 3y + 1)$, find the values of x and y.

3.3.3 Problems Involving Sets

In this subsection, you will learn how to solve problems that involve sets, in particular the numbers of elements in sets.

The number of elements that are either in set A or set B, denoted by $n(A \cup B)$, may not necessarily be $n(A) + n(B)$ as we can see in the Figure 3.26.

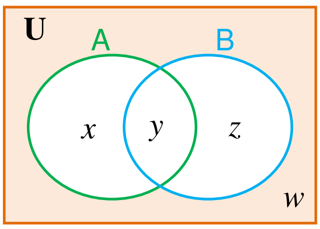


Figure 3.26

In this figure, suppose the number of elements in the closed regions of the Venn diagram are denoted by x, y, z and w.

n (A) = $x + y$ and n (B) = $y + z$.

So, n (A) + n (B) = $x + y + y + z$.

n (A ∪B) = $x + y + z$ = n (A) +n (B) −y

i.e., n (A∪B)= n (A) + n (B) – n (A∩B).

Number of elements in (A∪B)

For any finite sets A and B, the number of elements that are in A∪B is

n (A∪B) = n (A) + n (B) – n (A∩B).

Note: If A∩B =∅, then n (A∪B) = n (A) + n (B).

Example 1 Explain why n (A – B) = n (A) – n (A∩B).

Solution: From Figure 3.26 above, n (A) = $x + y$, n (A∩B) = y

n (A) – n (A∩B) = $(x + y) – y = x$,

x is the number of elements in A that are not in B. So, n (A – B) = x.

∴ n (A – B) = x = n (A) – n (A∩B).

For any finite sets A and B,

n (A\B) = n (A) – n (A∩B)

Example 2 Among 1500 students in a school, 13 students failed in English, 12 students failed in mathematics and 7 students failed in both English and Mathematics.

i How many students failed in either English or in Mathematics?

ii How many students passed both in English and in Mathematics?

Solution: Let E be the set of students who failed in English, M be the set of students who failed in mathematics and U be the set of all students in the school.

Then, n (E) = 13, n (M) = 12, n (E∩M) = 7 and n (U) = 1500.

i n (E∪M) = n (E) + n (M) – n (E∩M) = 13 + 12 – 7 = 18.

ii The set of all students who passed in both subjects is U\(E∪M).

n (U\(E∪M)) = n (U) – n (E∪M)= 1500 – 18 = 1482.

Exercise 3.11

1 For A = {2, 3, . . . 6} and B = {6, 7, . . . 10} show that:

a n (A∪B) = n (A) + n (B) – n (A∩B) **b** n (A × B) = n (A) × n (B)

c n (A × A) = n (A) × n (A)

2 If n (C ∩ D) = 8 and n (C\D) = 6 then find n (C).

3 Using a Venn diagram, or a formula, answer each of the following:

a Given n (Q\P) = 4, n (P\Q) = 5 and n (P) = 7 find n (Q).

b If n (R' ∩ S') + n (R'∩S) = 3, n (R∩S) = 4 and n (S'∩R) = 7, find n (U).

4 Indicate whether the statements below are true or false for all finite sets A and B. If a statement is false give a counter example.

a n (A∪ B) = n (A) + n (B) **b** n (A∩B) = n (A) – n (B)

c If n (A) = n (B) then A = B **d** If A = B then n(A) = n (B)

e n (A × B) = n (A) · n (B) **f** n (A) + n (B) = n (A∪B) – n (A∩B)

g n (A'∪B') = n ((A∪B)') **h** n(A∩B) = n(A∪B) – n(A∩B') – n(A'∩B)

i n(A) + n(A') = n(U)

5 Suppose A and B are sets such that n(A) = 10, n(B) = 23 and n(A∩B) = 4, then find:

a n(A ∪B) **b** n(A\B) **c** n(AΔB) **d** n(B\A)

6 If A = {x | x is a non-negative integer and $x^3 = x$}, then how many proper subsets does A have?

7 Of 100 students, 65 are members of a mathematics club and 40 are members of a physics club. If 10 are members of neither club, then how many students are members of:

a both clubs? **b** only the mathematics club?

c only the physics club?

8 The following Venn diagram shows two sets A and B. If n(A) = 13, n(B) = 8, then find:

a n(A∪B) **b** n(U)

c n(B\A) **d** n(A∩B')

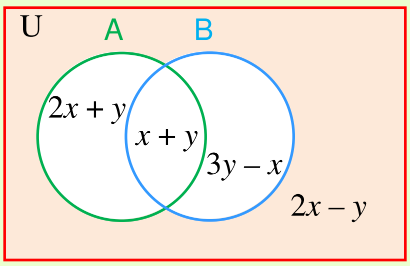


Figure 3.27

Key Terms

complement	infinite set	set
disjoint sets	intersection of sets	subset
element	power set	symmetric difference between sets
empty set	proper subset	union of sets
finite set	relative complement	universal set

Summary

1. A set is a well-defined collection of objects. The objects of a set are called its elements (or members).
2. Sets are described in the following ways:
   - a Verbal method
   - b Listing method
     - i Partial listing method
     - ii Complete listing method
   - c Set-builder method
3. The universal set is a set that contains all elements under consideration in a discussion.
4. The complement of a set A is the set of all elements that are found in the universal set but not in A.
5. A set S is called finite if and only if it is the empty set or has exactly n elements, where n is a natural number. Otherwise, it is called infinite.
6. A set A is a subset of B if and only if each element of A is in set B.
7. i P (A), the power set of a set A, is the set of all subsets of A.

   ii If n (A) = n, then the number of subsets of A is 2^n.
8. Two sets A and B are said to be equal if and only if A $\subseteq$ B and B $\subseteq$ A.
9. Two sets A and B are said to be equivalent if and only if there is a one-to-one correspondence between their elements.
10. i A set A is a proper subset of set B, denoted by A $\subset$ B, if and only if A $\subseteq$ B and B $\not\subseteq$ A.

    ii If n (A) = n, then the number of proper subsets of A is $2^n - 1$.

11 Operations on sets; for any sets A and B,

i $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

ii $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

iii $A - B$ (or $A \backslash B$) $= \{x \mid x \in A \text{ and } x \notin B\}$.

iv $A \Delta B = \{x \mid x \in (A \cup B) \text{ and } x \notin (A \cap B)\}$.

v $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

12 Properties of union, intersection, symmetric difference and Cartesian product:

For all sets A, B and C:

i Commutative properties

a $A \cup B = B \cup A$ **b** $A \cap B = B \cap A$ **c** $A \Delta B = B \Delta A$

ii Associative properties

a $A \cup (B \cup C) = (A \cup B) \cup C$ **c** $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

b $A \cap (B \cap C) = (A \cap B) \cap C$

iii Identity properties

a $A \cup \varnothing = A$ **b** $A \cap U = A$ (U is a universal set)

iv Distributive properties

a $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

c $A \times (B \cup C) = (A \times B) \cup (A \times C)$

d $A \times (B \cap C) = (A \times B) \cap (A \times C)$

v De Morgan's Law

a $(A \cup B)' = A' \cap B'$ **b** $(A \cap B)' = A' \cup B'$

vi For any set A

a $A \cup A' = U$ **b** $(A')' = A$

c $A \cap A' = \varnothing$ **d** $A \times \varnothing = \varnothing$

Review Exercises on Unit 3

1 Which of the following are sets?

a The collection of all tall students in your class.

b The collection of all natural numbers divisible by 3.

c The collection of all students in your school.

d The collection of all intelligent students in Ethiopia.

e The collection of all subsets of the set $\{1, 2, 3, 4, 5\}$.

2 Rewrite the following statements, using the correct notation:

a B is a set whose elements are x, y, z and w.

b 3 is not an element of set B.

c D is the set of all rational numbers between $\sqrt{2}$ and $\sqrt{5}$.

d H is the set of all positive multiples of 3.

3 Which of the following pairs of sets are equivalent?

a $\{1, 2, 3, 4, 5\}$ and $\{m, n, o, p, q\}$

b $\{x \mid x$ is a letter in the word mathematics$\}$ and $\{y \in \mathbb{N} \mid 1 \leq y \leq 11\}$

c $\{a, b, c, d, e, f, \ldots m\}$ and $\{1, 2, 3, 4, 5, \ldots 13\}$

4 Which of the following represent equal sets?

$A = \{a, b, c, d\}$ $B = \{x, y, z, w\}$

$C = \{x \mid x$ is one of the first four letters in the English alphabet$\}$

$D = \varnothing$ $E = \{0\}$ $F = \{x \mid x \neq x\}$ $G = \{x \in \mathbb{Z} \mid -1 < x < 1\}$

5 If $U = \{a, b, c, d, e, f, g, h\}$, $A = \{b, d, f, h\}$ and $B = \{a, b, e, f, g, h\}$, find the following:

a A' **b** B' **c** $A \cap B$ **d** $(A \cap B)'$ **e** $A' \cap B'$

6 In the Venn diagram given below, write the region labelled by I, II, III and IV in terms of A and B.

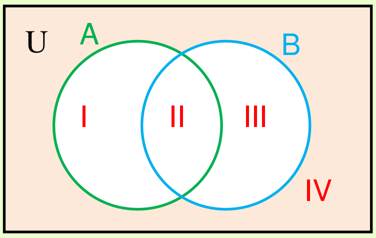


Figure 3.28

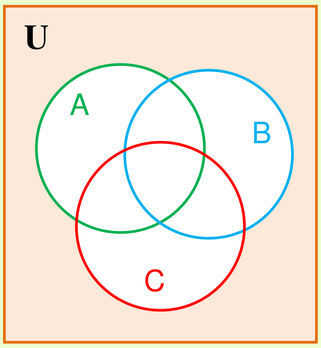


Figure 3.29

7 For each of Questions a, b and c, copy the following Venn diagram and shade the regions that represent:

a $A \cap (B \cap C)$. **b** $A \setminus (B \cap C)$.

c $A \cup (B \setminus C)$.

8 Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{0, 2, 1, 6, 8\}$ and $C = \{3, 6, 9\}$. Then find:

a A' **b** $B \setminus A$ **c** $A \cap C'$

d $C \times (A \cap B)$ **e** $(B \setminus A) \times C$

9 Suppose B is a proper subset of C,

a If n (C) = 8, what is the maximum number of elements in B?

b What is the least possible number of elements in B?

10 If n (U) = 16, n (A) = 7 and n (B) = 12, find:

a n (A') **b** n (B')

c greatest n (A∩B) **d** least n (A∪B)

11 In a class of 31 students, 22 students study physics, 20 students study chemistry and 5 students study neither. Calculate the number of students who study both subjects.

12 Suppose A and B are sets such that A∪B has 20 elements, A∩B has 7 elements, and the number of elements in B is twice that of A. What is the number of elements in:

a A? **b** B?

13 State whether each of the following is finite or infinite:

a $\{x \mid x$ is an integer less than 5$\}$

b $\{x \mid x$ is a rational number between 0 and 1$\}$

c $\{x \mid x$ is the number of points on a 1 cm-long line segment$\}$

d The set of trees found in Addis Ababa.

e The set of "teff" in 1,000 quintals.

f The set of students in this class who are 10 years old.

14 How many letters in the English alphabet precede the letter v? (Think of a shortcut method).

15 Of 100 staff members of a school, 48 drink coffee, 25 drink both tea and coffee and everyone drinks either coffee or tea. How many staff members drink tea?

16 Given that set A has 15 elements and set B has 12 elements, determine each of the following:

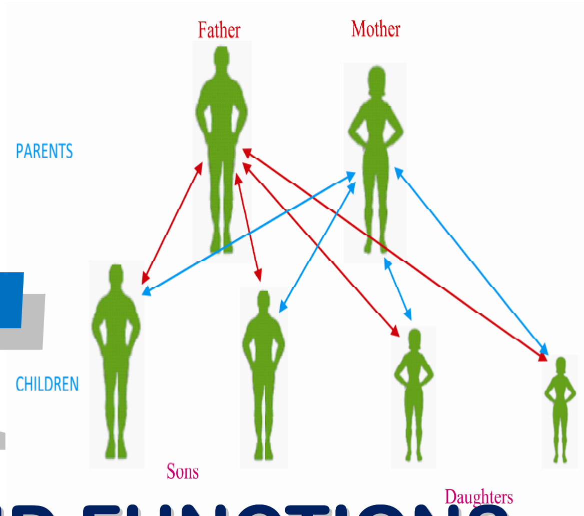
a The maximum possible number of elements in A∪B.

b The minimum possible number of elements in A∪B.

c The maximum possible number of elements in A∩B.

d The minimum possible number of elements in A∩B.

Unit 4



RELATIONS AND FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know specific facts about relation and function.*
- *understand the basic concepts and principles about combination of functions.*
- *sketch graphs of relations and functions (i.e. of linear and quadratic functions).*

Main Contents

INTRODUCTION

In our daily life, we come across many patterns that characterize relations such as brother and sister, teacher and student, etc. Similarly, in mathematics also, we come across different relations such as number a is less than number b, angle α is greater than angle β, set A is subset of set B, and so on. In all these cases, we find that a relation involves pairs of objects in some specific order. In this unit, you will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. You also learn here about special relations which will qualify to be functions.

4.1 RELATIONS

Group Work 4.1

Form a group and do the following.

1 Explain and discuss the meaning of "relation" in your daily life.

2 Give some examples of relations from your daily life.

3 How do you understand relations in mathematical language?

In our daily life we usually talk about relations between various things. For example, we say someone is the father of another person, 5 is greater than 3, Addis Ababa is the capital city of Ethiopia, Wallia Ibex is endemic to Ethiopia, etc.

The Cartesian product of sets is one of the useful ways to describe relations mathematically. For example, let A = {Addis Ababa, Jimma, Nairobi} and

B = {Ethiopia, Kenya, Sudan}. If x and y in the ordered pair (x, y), where $x \in$ A and $y \in$ B, are related by the phrase "x is the capital city of y", then the relation can be described by the set of ordered pairs; {(Addis Ababa, Ethiopia), (Nairobi, Kenya)}. Hence, the given relation is a subset of A $\times$ B.

4.1.1 The Notion of a Relation

In the previous sub-unit, you saw relations in a generalized sense, as relationships between any two things with some relating phrase. The following Activity will help you to realize the mathematical definition of a relation.

ACTIVITY 4.1

1 Let A = { 1, 2, 4, 6, 7} and B = { 5, 12, 7, 9, 8, 3}

List all ordered pairs (x, y) which satisfy each of the following sentences where $x \in$ A, $y \in$ B.

a x is greater than y **b** y is a multiple of x

c The sum of x and y is odd **d** x is half of y

2 Let A = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

List all ordered pairs (x, y) which satisfy each of the following sentences, where $x, y \in A$.

a y is a multiple of x **b** x is the square of y

c x is less than y **d** x is a prime factor of y

3 Let U = {x: x is a student in your class}

i In each of the following, list all ordered pairs (x, y) which satisfy the given sentence where $x, y \in$ U.

a x is taller than y **b** x is younger than y

ii Discuss other ways that you can relate the students in your class.

As you have noticed from the above Activity, each sentence involves what is intuitively understood to be a relationship. Expressions of the type "is greater than", "is multiple of", "is a factor of", "is taller than", etc. which express the relation are referred to as relating phrases.

From Activity 4.1 you might have observed the following:

i In considering relations between objects, order is often important.

ii A relation establishes a pairing between objects.

Therefore, from a mathematical stand point, the meaning of a relation is more precisely defined as follows.

Definition 4.1

Let A and B be non-empty sets. A relation R from A to B is any subset of A × B. In other words, R is a relation from A to B if and only if $R \subseteq (A \times B)$.

Example 1 Let A = { 1, 2, 3, 4} and B = { 1, 3, 5}

i $R_1 = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$ is a relation from A to B because $R_1 \subseteq (A \times B)$. Is R_1 a relation from B to A? Justify.

Notice that we can represent R_1 in the set builder method as

$$R_1 = \{ (x, y) \mid x \in A, y \in B, x < y\}$$

ii $R_2 = \{(1, 1), (2, 1), (3, 1), (3, 3), (4, 1), (4, 3)\}$ is a relation from A to B because $R_2 \subseteq (A \times B)$.

In the set builder method, R_2 is represented by $R_2 = \{ (x, y) \mid x \in A, y \in B, x \geq y\}$

Example 2 Let A = {1, 2, 3} then observe that

$R_1 = \{(1, 2), (1, 3), (2, 3)\}$, $R_2 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

and $R_3 = \{(x, y) \mid x, y \in A, x + y \text{ is odd}\}$ are relations on A.

Exercise 4.1

1 For each of the following relations, determine the relating phrase:

a $R = \{(x, y): x \text{ is taller than } y\}$

b $R = \{(x, y): y \text{ is the square root of } x\}$

c $R = \{(x, y): y = 2x\}$

2 Let A = {2, 4, 6} and B = {1, 3, 5}

a R = {(2, 2), (4, 4), (6, 6)} is a relation on A. Express the relation using set builder method.

b Is R = {(2, 1), (2, 3), (2, 5), (1, 2), (3, 4), (5, 6)} a relation from A to B? Give the reason for your answer.

c If R is a relation from A to B given by $R = \{(x, y): y = x - 1\}$, then list the elements of R.

3 If $R = \{(x, y): y = 2x + 1\}$ is a relation on A, where A = {1, 2, 3, 4, 5, 6}, then list the elements of R.

4 Write some ordered pairs that belong to the relation given by

$R = \{(x, y): y < 2x;\ x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$

4.1.2 Domain and Range

ACTIVITY 4.2

Let A = {1, 2, 4, 6, 7} and B = {5, 12, 7, 9, 8, 3}

Let R_1 and R_2 be relations given by:

$R_1 = \{(x, y) \mid x \in A, y \in B, x > y\}$ and $R_2 = \left\{(x, y): x \in A, y \in B, x = \frac{1}{2}y\right\}$

Represent each of the following sets using complete listing method.

a $D = \{x : (x, y) \in R_1\}$

b $D = \{x : (x, y) \in R_2\}$

c $R = \{y : (x, y) \in R_1\}$

d $R = \{y : (x, y) \in R_2\}$

Observe that in each case, the sets represented by D contain the first coordinates and sets represented by R contain the second coordinates of the respective relations.

In the above discussion the set of all the first coordinates of the ordered pairs of a relation R is called the domain of R and the set of all second coordinates of the ordered pairs of R is called the range of R.

We give the definition of domain and range formally as follows.

Definition 4.2

Let R be a relation from a set A to a set B. Then

i Domain of R = $\{ x : (x, y)$ belongs to R for some $y\}$

ii Range of R = $\{ y : (x, y)$ belongs to R for some $x\}$

Example 1 Given the relation R = {(1, 3), (2, 5), (7, 1), (4, 3)}, find the domain and range of the relation R.

Solution: Since the domain contains the first coordinates, domain = {1, 2, 7, 4} and the range contains the second coordinates, range = {3, 5, 1}

Example 2 Given A = {1, 2, 4, 6, 7} and B = {5, 12, 7, 9, 8, 3}

Find the domain and range of the relation R = $\{(x, y): x\in A, y\in B, x > y\}$

Solution: If we describe R by complete listing method, we will find

R = {(4, 3), (6, 3), (7, 3), (6, 5), (7, 5)}.

This shows that the domain of R = {4, 6, 7} and the range of R = {3, 5}

Exercise 4.2

1 For the relation given by the set of ordered pairs {(5, 3), (–2, 4), (5, 2), (–2, 3)} determine the domain and the range.

2 Let A = {1, 2, 3, 4} and R = $\{(x, y): y = x + 1; x, y \in A\}$ List the ordered pairs that satisfy the relation and determine the domain and the range of R.

3 Find the domain and the range of each of the following relations:

a $R = \left\{(x, y): y = \sqrt{x}\right\}$ **b** $R = \left\{(x, y) : y = x^2\right\}$

c R = $\{(x, y) : y$ is a mathematics teacher in section $9x\}$

4 Let A = $\{x: 1 \le x < 10\}$ and B = {2, 4, 6, 8}. If R is a relation from A to B given by R = $\{(x, y): x + y = 12\}$, then find the domain and the range of R.

4.1.3 Graphs of Relations

By now, you have understood what a relation is and how it can be described using sets. You will now see how relations can be represented through graphs.

You may graphically represent a relation R from A to B by locating the ordered pairs in a coordinate system or by using arrows in a diagram displaying the members of both sets, or as a region on a coordinate system.

ACTIVITY 4.3

Discuss the following.

a A coordinate system (or *xy*-coordinate system).

b A point on a coordinate system.

c A region on a coordinate system.

From Unit 3, recall that $\mathbb{R} \times \mathbb{R} = \{(x, y): x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ is represented by a set of points in the *xy*-coordinate system.

Example 1 Let A = {2, 3, 5} and B = {6, 7, 10} and the relation from A to B be "*x* is a factor of *y*."

Elements of R = {(2, 6), (2, 10), (3, 6), (5, 10)} with Domain x = {2, 3, 5} and Range y = {6, 10}.

This relation can be graphically represented as shown in the adjacent figure.

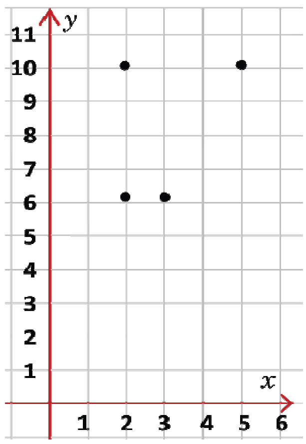


Figure 4.1

Alternatively, we use arrows in a diagram displaying the relation between the members of both sets as shown below.

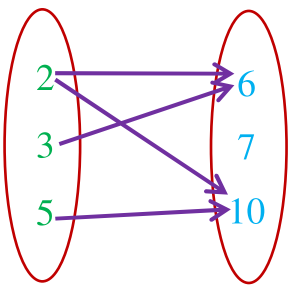


Figure 4.2

Group Work 4.2

Form a group and perform each of the following to sketch the graph of the relation R = $\{(x, y): x < y$, where x and y are real numbers$\}$

1. Draw the graph of the line $y = x$ on the coordinate system using a broken line.
2. Choose arbitrary ordered pairs, one from one side and the other from another side of the line(s) and determine which of the pairs satisfy the relation.
3. What do you think will the region that contains the ordered pair satisfying the relation be?
4. Shade the region which contains points representing the ordered pair satisfying the relation.
5. Determine the domain and the range of the relation.

In general, to sketch graphs of relations involving inequalities, do the following:

1. Draw the graph of a line(s) in the relation on the xy-coordinate system.
2. If the relating inequality is $\leq$ or $\geq$, use a solid line; if it is $<$ or $>$, use a broken line.
3. Then take arbitrary ordered pairs represented by points, one from one side and the other from another side of the line(s), and determine which of the pairs satisfy the relation.
4. The region that contains points representing the ordered pair satisfying the relation will be the graph of the relation.

Note: A graph of a relation when the relating phrase is an inequality is a region on the coordinate system.

Example 2 Sketch the graph of the relation

R = $\{(x, y): y > x$, where x and y are real numbers$\}$.

Solution: To sketch the graph,

1. Draw the graph of the line $y = x$.
2. Since the relation involves $y > x$, use a broken line.
3. Take points representing ordered pairs, say (0, 4) and (3, –2) from above and below the line $y = x$.
4. The ordered pair (0, 4) satisfies the relation. Hence, the region above the line $y = x$, where the point representing (0, 4) is contained, is the graph of the relation R.

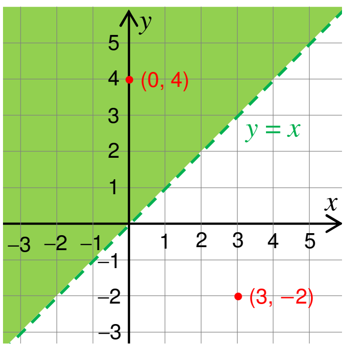
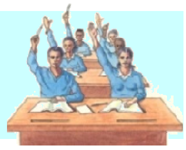


Figure 4.3

ACTIVITY 4.4



Sketch the graph of the relation R = $\{(x, y): y \leq 2x; x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

Example 3 Sketch the graph of the relation R = $\{(x, y): y \geq x + 1; x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

Solution:

1 Draw the graph of the line $y = x + 1$.

2 Since the relating inequality is $\geq$ use solid line.

3 Select two points one from one side and another from the other side of the line. For example points with coordinates (0, 5) and (2, 0). Obviously, (0, 5) satisfies the relation

$$R = \{(x, y): y \geq x + 1\}, \text{ as } 5 \geq 0 + 1.$$

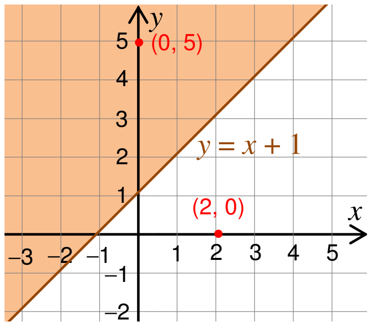


Figure 4.4

4 Shade the region containing the point with coordinates (0, 5). So the graph of the relation R = $\{(x, y): y \geq x + 1\}$ is as shown by the shaded region.

Example 4 Sketch the graph of the relation R = $\{(x, y): y \geq x^2\}$.

1 Draw the graph of $y = x^2$ using solid curve.

2 Select two points from inside and outside the curve, say the point with coordinates (0, 2) from inside of the curve and (3, 0) from outside of the curve. Clearly, (0, 2) satisfies the relation since $2 \geq 0^2$ is true.

Hence, the graph of the relation is the shaded part in the figure (containing the point with coordinates (0, 2)).

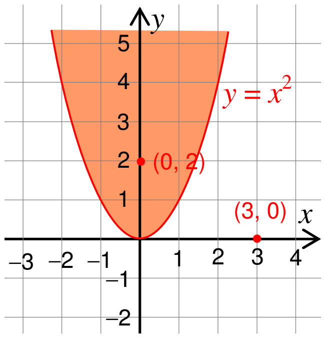


Figure 4.5

We have discussed how to sketch graphs of relations involving one inequality. It is also possible to sketch a graph of a relation with two or more relating inequalities. The approach to sketching the graphs is similar, except that, in such cases we consider the intersection of regions. If a relation has the connective "or", we use union instead of intersection.

ACTIVITY 4.5

1 Sketch the graph of each of the following relations

a $R_1 = \{(x, y): x \geq 0; x, y \in \mathbb{R}\}$ **b** $R_2 = \{(x, y): y \geq 0; x, y \in \mathbb{R}\}$

c $R_3 = \{(x, y): x \geq 0 \text{ and } y \geq 0; x, y \in \mathbb{R}\}$

2 What relation did you observe among the graphs of the relations R_1, R_2 and R_3?

To sketch the graph of a relation with two or more inequalities,

i Using the same coordinate system, sketch the regions of each inequality.

ii Determine the intersection of the regions.

Example 5 Sketch the graph of the relation

$R = \{(x, y): y \geq x + 2 \text{ and } y > -x, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

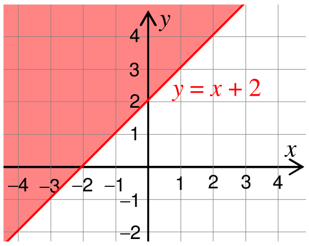


Figure 4.6

Solution:

First sketch the graph of the relation

$R = \{(x, y): y \geq x + 2, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

Next, on the same diagram, sketch the graph of

$$R = \{(x, y): y > -x, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$$

The two shaded regions have some overlap. The intersection of the two regions is the graph of the relation.

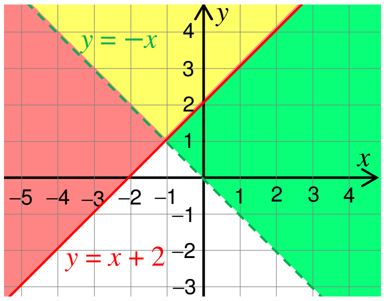


Figure 4.7

So, taking only the common region, we obtain the graph of the relation

$R = \{(x, y): y \geq x + 2 \text{ and } y > -x, x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

as shown in Figure 4.8.

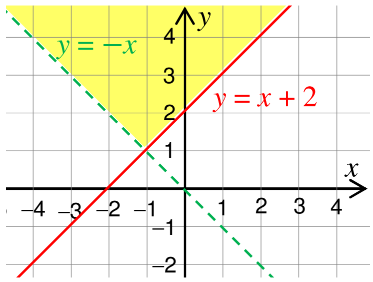


Figure 4.8

Group Work 4.3

1 Discuss how you can determine the domain and range of a relation from its graph.

2 Is there any simple way of finding the domain and range from the graph of a relation?

It is possible to determine the domain and range of a relation from its graph. The domain of a relation is the x-coordinate of the set of points through which a vertical line meets the graph of the relation and the range of a relation is the y-coordinate of the set of points through which a horizontal line meets the graph of the relation.

Example 6 Find the domain and the range of the relation

$R = \{(x, y): y \geq x + 2 \text{ and } y > -x;\ x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

From the graph sketched above, since any vertical line meets the graph, the domain of the relation is the set of real numbers, $\mathbb{R}$.

That is, domain of R = $\mathbb{R}$. But not all horizontal lines meet the graph, only those that pass through y: $y > 1$. Hence, the range of the relation is the set $\{y: y > 1\}$.

Example 7 Sketch the graph of the following relation and determine its domain and range.

$R = \{(x, y): y < 2x \text{ and } y > -x\}$.

Solution: Sketch the graphs of $y < 2x$ and $y > -x$ on same coordinate system.

Note that these two lines divide the coordinate system into four regions.

Take any points one from each region and check if they satisfy the relation. Say, (3, 0), (0, 4), (–1, 0) and (0,–2).

(3, 0) satisfies both inequalities of the relation. So the graph of the relation is the region that contains (3, 0).

Hence, Domain of R = $\{x \in \mathbb{R}: x > 0\}$

Range of R = $\{y: y \in \mathbb{R}\}$.

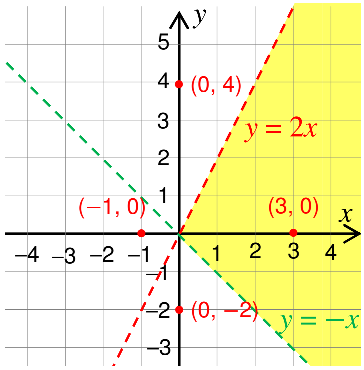


Figure 4.9

Exercise 4.3

1 Let A = {2, 3, 5} and B = {6, 10, 15}and R: A → B

a If $R = \{(x, y): y = 2x + 5\}$, then plot the points of R on a coordinate system, and determine the domain and range of the relation.

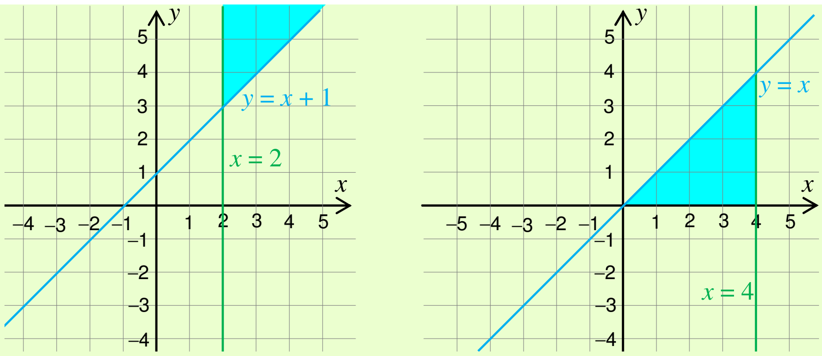
b Let $R = \{(x, y): x \text{ is a divisor of } y\}$. Plot the points of R on a coordinate system, and determine the domain and range of the relation.

2 For each of the following relations, sketch the graph and determine the domain and the range.

a $R = \{(x, y): y \geq 3x - 2\}$ **b** $R = \{(x, y): y \geq 2x - 1 \text{ and } y \leq -2x + 1\}$

c $R = \{(x, y): y \geq 2x - 1 \text{ and } y \leq 2x - 1\}$

3 From the graph of each of the following relations, represented by the shaded region, specify the relation and determine the domain and the range:



a b

Figure 4.10

4.2 FUNCTIONS

In this section, you shall learn about particular types of relations which are called functions, the domain and range of a function, and combinations of functions. Remember that the concept of functions is one of the most important in mathematics. There are many terms such as ‘map’ or ‘mapping’ used to denote a function.

4.2.1 Functions

Group Work 4.4

1 Consider the following relations

$R_1 = \{(1, 2), (3, 4), (2, 5), (5, 6), (4, 7)\}$

$R_2 = \{(1, 2), (3, 2), (2, 5), (6, 5), (4, 7)\}$

$R_3 = \{(1, 2), (1, 4), (2, 5), (2, 6), (4, 7)\}$

a What differences do you see between these relations?

b How are the first elements of the coordinates paired with the second elements of the coordinates?

c In each relation, are there ordered pairs with the same first coordinate?

2 Let $R_1 = \{(x, y)$: x and y are persons in your kebele where y is the father of $x\}$

$R_2 = \{(x, y)$: x and y are persons in your kebele where x is the father of $y\}$

Discuss the difference between these two relations R_1 and R_2.

Definition 4.3

A function is a relation such that no two ordered pairs have the same *first*-coordinates and different *second*-coordinates.

Example 1 Consider the relation R = {(1, 2), (7, 8), (4, 3), (7, 6)}

Since 7 is paired with both 8 and 6 the relation R is not a function.

Example 2 Let R = {(1, 2), (7, 8), (4, 3)}. This relation is a function because no *first*-coordinate is paired (mapped) with more than one element of the *second*-coordinate.

Example 3 Consider the following arrow diagrams.

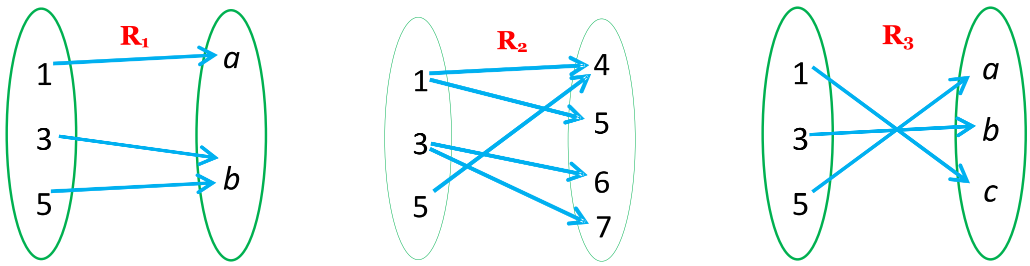


Figure 4.11

Which of these relations are functions?

Solution: R_1 is a function. (Why?)

R_2 is not a function because 1 and 3 are both mapped onto two numbers.

R_3 is a function. (Why?)

Example 4 The relation R = {(x, y): y is the father of x} is a function because no child has more than one father.

Example 5 Consider the relation R = {(x, y): y is a grandmother of x}.

This relation is not a function since everybody (x) has two grandmothers.

Domain and range of a function

In Section 4.1.2 you learnt about the domain and range of a relation. As a function is a special type of a relation, the domain and range of a function are determined in exactly the same way.

Example 6 For each of the following functions, determine the domain and the range.

a F = {(2, –1), (4, 3), (0, 1)} **b** F = {(2, –1), (4, 3), (0, –1), (3, 4)}

Solution:

a Domain D = {0, 2, 4} and range R = {–1, 1, 3 }

b Domain D = {0, 2, 3, 4} and range R = {–1, 3, 4}

You will now consider some functions that are defined by a formula.

Example 7 Is the relation R = $\{(x, y): x = y^2\}$ a function?

Solution: This is not a function because numbers for x are paired with more than one number in y. For example, (9, –3) and (9, 3) satisfy the relation with 9 being mapped to both –3 and 3.

Example 8 Is $R = \{(x, y): y = |x|\}$ a function?

Solution: Since for every number there is unique absolute value, each number x is mapped to one and only one number y, so the relation $R = \{(x, y): y = |x|\}$ is a function.

Notation: If x is an element in the domain of a function f, then the element in the range that is associated with x is denoted by $f(x)$ and is called the image of x under the function f. This means $f = \{(x, y): y = f(x)\}$

The notation $f(x)$ is called function notation. Read $f(x)$ as "f of x".

Note: f, g and h are the most common letters used to designate a function. But, any letter of the alphabet can be used.

A function from A to B can sometimes be denoted as f: A→B, where the domain of f is A and the range of f is a subset of B, in which case we say B contains the images of the elements of A under the function f.

Example 9 Consider the function $R = \{(x, y): y = |x|\}$. Here the rule $y = |x|$ can be written as $f(x) = |x|$ As a result of which, $f(0) = |0| = 0$, $f(-2) = |-2| = 2$ and $f(3) = |3| = 3$.

Example 10 If R = $\{(x, y)$: y is twice $x\}$, then we can denote this function by $f(x) = 2x$.

ACTIVITY 4.6

1 Consider the following arrow diagram of a function f.

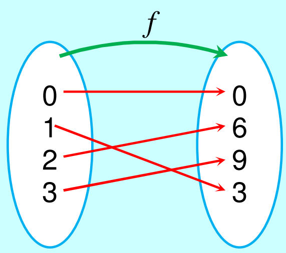


Figure 4.12

Find an algebraic rule for f.

2 For each of the following functions find the domain and the range:

a $f(x) = 5x - 1$ **b** $f(x) = x^2$ **c** $f(x) = \sqrt{x^2 - 3}$

Observe that the domain of a function is the set on which the given function is defined.

Example 11 Consider $f(x) = 2x + 2$.

Since $f(x) = 2x + 2$ is defined for all $x \in \mathbb{R}$, the domain of the function is the set of all real numbers. The range is also $\mathbb{R}$ since every real number y has a real number x such that $y = f(x) = 2x + 2$.

Example 12 Let $f(x) = \sqrt{x-3}$

Since the expression in the radical must be non-negative, $x - 3 \geq 0$.

This implies $x \geq 3$. So the domain is the set $D = \{x: x \geq 3\}$.

Since the value of $\sqrt{x-3}$ is always non-negative, the range is the set

$R = \{y: y \geq 0\}$.

Example 13 Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 7, 9\}$

If $f: A \to B$ is the function given by $f(x) = 2x + 1$, then find the domain and the range of f.

Solution: Since $f(1) = 3 \in B, f(2) = 5 \in B, f(3) = 7 \in B$ and $f(4) = 9 \in B$, the domain of f is $D = \{1, 2, 3, 4\}$ and the range of f is $R = \{3, 5, 7, 9\}$.

Remark: If $f: A \to B$ is a function, then, for any $x \in A$ the image of x under f, $f(x)$ is called the **functional value of f at x**. For example, if $f(x) = x - 3$, then the functional value of f at $x = 5$ is $f(5) = 5 - 3 = 2$. Finding the functional value of f at x is also called **evaluating the function** at x.

Example 14 Take $f(x) = \sqrt{x-3}$ and evaluate:

a $f(3)$ **b** $f(12)$

Solution:

a $f(3) = \sqrt{3-3} = \sqrt{0} = 0$ **b** $f(x) = \sqrt{12-3} = \sqrt{9} = 3$

Example 15 For the function $f(x) = 1 - x^2$

a Find the domain and the range **b** Evaluate $f(2)$ and $f(-1)$

Solution:

a The domain of the function is $D = \{x: x \in \mathbb{R}\}$, since it is defined for all real numbers. The range is $R = \{y : y \leq 1\}$

b $f(2) = 1 - (2)^2 = 1 - 4 = -3$ and $f(-1) = 1 - (-1)^2 = 1 - 1 = 0$.

Exercise 4.4

1. Determine whether each of the following relations is a function or not, and give reasons for those that are not functions.
   - **a** R = {(–1, 2), (1, 3), (–1, 3)}
   - **b** R = {(1, 1), (1, 3), (–1, 3), (2, 1)}
   - **c** R = {(x, y): y is the area of triangle x}
   - **d** R = {(x, y): x is the area of triangle y}
   - **e** R = {(x, y): y is a multiple of x}
   - **f** R = {(x, y): $y = x^2 + 3$}
   - **g** R = {(x, y): $y < x$}
   - **h** R = {(x, y): x is the son of y}
2. Is every function a relation? Explain your answer.
3. Find the domain and the range of each of the following functions:
   - **a** $f(x) = 3$ **b** $f(x) = 1 - 3x$
   - **c** $f(x) = \sqrt{x+4}$ **d** $f(x) = |x| - 1$ **e** $f(x) = \dfrac{1}{2x}$
4. If $f(x) = 2x + \sqrt{x+4}$, evaluate each of the following:
   - **a** $f(-4)$ **b** $f(5)$
5. Match each of the functions in column A with its corresponding domain in column B:

	A		B
1	$f(x) = \sqrt{2-x}$	a	$\{x: x \geq 3\}$
2	$f(x) = 2x - 1$	b	$\{x: x \leq 2\}$
3	$f(x) = \sqrt{x-3}$	c	$\{x: x \in \mathbb{R}\}$

6. Match each of the functions in column A with its corresponding range in column B.

	A		B
1	$f(x) = \sqrt{2-x}$	a	$\{y: y \geq 0\}$
2	$f(x) = 2x - 1$	b	$\{y: y \in \mathbb{R}\}$
3	$f(x) = \sqrt{x-3}$	c	$\{y: y \geq 10\}$

4.2.2 Combinations of Functions

In this sub-section, you will learn how to find the sum, difference, product and quotient of two functions, all known as combinations of functions.

Group Work 4.5

1 Consider the functions $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{10-x}$

a Find $f + g$; $f - g$; fg and $\dfrac{f}{g}$.

b Determine the domain and the range of each function.

c Is the domain of f and g the same as the domain of $f + g$? Is this always true?

A Sum of functions

Suppose f and g are two functions. The sum of these functions is a function which is defined as $f + g$, where $(f + g)(x) = f(x) + g(x)$.

Example 1 If $f(x) = 2 - x$ and $g(x) = 3x + 2$ then the sum of these functions is given by

$(f + g)(x) = (2 - x) + (3x + 2) = 2x + 4$, which is also a function.

The domain of $f = \mathbb{R}$ and the domain of $g = \mathbb{R}$.

The function $(f + g)(x) = 2x + 4$ has also domain $= \mathbb{R}$.

Example 2 Let $f(x) = 2x$ and $g(x) = \sqrt{2x}$. Determine

a the sum $f + g$

b the domain of $(f + g)$

Solution:

a $(f + g)(x) = f(x) + g(x) = 2x + \sqrt{2x}$

b Domain of $f + g = \{x: x \geq 0\}$.

B Difference of functions

Suppose f and g are two functions. The difference of these functions is also a function, defined as $f - g$, where $(f - g)(x) = f(x) - g(x)$.

Example 3 If $f(x) = 3x + 2$ and $g(x) = x - 4$, then the difference of these functions is

$(f - g)(x) = f(x) - g(x) = (3x + 2) - (x - 4) = 2x + 6$ and

the domain of $f - g = \mathbb{R}$.

Example 4 Let $f(x) = 2x$ and $g(x) = \sqrt{1-x}$. Determine:

a the difference $f - g$

b the domain of $f - g$

Solution:

a $(f-g)(x) = f(x) - g(x) = 2x - \sqrt{1-x}$

b Domain of $f - g = \{x: x \leq 1\}$.

C Product of functions

Suppose f and g are two functions. The product of these functions is also a function, defined as fg, $(fg)(x) = f(x)g(x)$. Again,

Example 5 If $f(x) = 2x$ and $g(x) = 3 - x$ then the product of these functions

$(fg)(x) = f(x)\, g(x) = (2x)(3-x) = 6x - 2x^2$ and

the domain of $fg = \mathbb{R}$.

Note: The domain of the sum, difference and product of functions f and g is the intersection of the domain of f and of the domain of g.

D Quotients of functions

Suppose f and g are two functions with $g \neq 0$. The quotient of these functions is also a function, defined as $\frac{f}{g}$ where $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

Example 6 If $f(x) = 3$ and $g(x) = 2 + x$ then the quotient of these functions

$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3}{2+x}$ and the domain of $\frac{f}{g} = \mathbb{R} \backslash \{-2\}$.

Example 7 Let $f(x) = \frac{x}{x-2}$ and $g(x) = \frac{x-3}{2x}$.

1 Find **a** $f+g$ **b** $f-g$ **c** fg **d** $\frac{f}{g}$ and

2 Determine the domain of each function.

Solution:

1 **a** $(f+g)(x) = f(x) + g(x) = \frac{x}{x-2} + \frac{x-3}{2x} = \frac{3x^2 - 5x + 6}{2x(x-2)}$

b $(f-g)(x) = f(x) - g(x) = \frac{x}{x-2} - \frac{x-3}{2x} = \frac{x^2 + 5x - 6}{2x(x-2)}$

c $(fg)(x) = f(x)\, g(x) = \left(\frac{x}{x-2}\right)\left(\frac{x-3}{2x}\right) = \frac{x(x-3)}{2x(x-2)} = \frac{x-3}{2(x-2)}$

d $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\dfrac{x}{x-2}}{\dfrac{x-3}{2x}} = \left(\frac{x}{x-2}\right)\left(\frac{2x}{x-3}\right) = \frac{2x^2}{x^2-5x+6}$$

2 Domain of $f+g$ = Domain of $f-g$ = Domain of fg

$= \mathbb{R}\backslash\{0, 2\}$ or $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

Domain of $\frac{f}{g} = \mathbb{R}\backslash\{0, 2, 3\}$ or $(-\infty, 0) \cup (0, 2) \cup (2, 3) \cup (3, \infty)$.

Example 8 Let $f(x) = 8 - 3x$ and $g(x) = -x - 5$. Determine:

a $2f + g$ **b** $3g - 2f$ **c** $(3f)g$ **d** $\frac{4g}{3f}$

Solution:

a $2f(x) + g(x) = 2(8-3x) + (-x-5) = 11 - 7x$

b $3g(x) - 2f(x) = 3(-x-5) - 2(8-3x) = -3x - 15 - 16 + 6x = 3x - 31$

c $(3f(x))g(x) = 3(8-3x)(-x-5) = 9x^2 + 21x - 120$

d $\frac{4g(x)}{3f(x)} = \frac{4(-x-5)}{3(8-3x)} = \frac{-4x-20}{24-9x}$

Through the above examples, you have seen how to determine the combination of functions. Now, you shall discuss how to evaluate functional values of combined functions for given values in the domains in the examples that follow.

Example 9 Let $f(x) = 2 - 3x$ and $g(x) = x - 3$. Evaluate $\frac{f}{g}(4)$ and $(f+g)(4)$

Solution: $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{2-3x}{x-3}$. So $\frac{f}{g}(4) = \frac{2-3(4)}{4-3} = -10$

$(f+g)(x) = f(x) + g(x) = -2x - 1$. So $(f+g)(4) = -2(4) - 1 = -9$.

Example 10 Let $f(x) = x - 1$ and $g(x) = 3x$. Determine:

a $(2f+3g)(1)$ **b** $\frac{f}{2g}(3)$

Solution:

a $(2f + 3g)(1) = 2(1-1) + 3(3(1)) = 9$ **b** $\frac{f}{2g}(3) = \frac{3-1}{2(3)(3)} = \frac{2}{18} = \frac{1}{9}$

Exercise 4.5

1 If $f = \{(1, 2), (-3, 2), (2, 5)\}$ and $g = \{(2, 4), (1, 5), (3, 2)\}$. Find:

a $f + g$ and $f - g$ **b** the domain of $(f + g)$

2 Let $f = \{(2, 3), (4, 9), (3, -8)\}$ and $g = \{(1, 2), (2, 5), (3, 10), (4, 17)\}$. Determine:

a $-2f$ **b** fg **c** $fg(2)$ **d** g^2

3 Write down the domain of each function in question number 2.

4 Let $f(x) = \dfrac{2}{x-1}$ and $g(x) = \dfrac{2x-2}{3x+3}$. Find:

a $f + g$ **b** fg **c** domain of $(f + g)$ and fg

5 Let $f(x) = 3x - 3$ and $g(x) = \dfrac{2}{x-1}$. Evaluate:

a $2fg(2)$ **b** $\left(\dfrac{f}{g} - 2f\right)(3)$ **c** $(f - g)(4)$

6 Is it always possible to deduce the domain of

i $f + g$ **ii** $f - g$ **iii** $f \cdot g$ **iv** $\dfrac{f}{g}$

from the domain of f and g? If your answer is yes, how?

4.3 GRAPHS OF FUNCTIONS

In this section, you will learn how to draw graphs of functions, with special emphasis on linear and quadratic functions. You will also study some of the important properties of these graphs.

4.3.1 Graphs of Linear Functions

Definition 4.4

If a and b are fixed real numbers, $a \neq 0$, then $f(x) = ax + b$ for $x \in \mathbb{R}$ is called a linear function. If $a = 0$, then $f(x) = b$ is called a constant function. Sometimes linear functions are written as $y = ax + b$.

Example 1 $f(x) = 2x + 1$ is a linear function with $a = 2$ and $b = 1$

Example 2 $f(x) = 3$ is a constant function.

From Section 4.2.1 recall that functions are special types of relations. Hence, a linear function is also a relation. From the description we used for relations, linear functions can also be described as

$$R = \{(x, y): y = ax + b; x, y \in \mathbb{R}\}; \text{ or } R = \{(x, f(x)): f(x) = ax + b; x, y \in \mathbb{R}\}$$

What are the properties of linear functions? What do a and b stand for?

Drawing graphs of linear functions will help us to answer these questions. Let us recall how to evaluate functions:

Example 3 If $f(x) = 3x - 1$, then $f(2) = 3(2) - 1 = 6 - 1 = 5$.

You will now evaluate functions at selected points from the domain and then use these points to draw graphs of linear functions.

Example 4 Consider the linear function $f(x) = 2x + 3$.

Evaluate the values of the function for the x values in the table below.

x	–3	–2	–1	0	1	2	3
$f(x)$							

At $x = -3$, $f(-3) = 2(-3) + 3 = -3$ and at $x = -2$, $f(-2) = 2(-2) + 3 = -1$.

In the same way, $f(-1) = 1$; $f(0) = 3$; $f(1) = 5$; $f(2) = 7$; and $f(3) = 9$. So the table becomes

x	–3	–2	–1	0	1	2	3
$f(x)$	–3	–1	1	3	5	7	9

This table is pairing the values of x and $f(x)$. This is taken as a representative of

$R = \{(-3, -3), (-2, -1), (-1, 1), (0, 3), (1, 5), (2, 7), (3, 9)\}$

Now you can plot these points in a coordinate system to draw the graph of the given function.

Example 5 Draw the graph of the linear function $f(x) = -2x + 3$.

Solution:

a First you construct a table of values from the domain.

x	–3	–2	–1	0	1	2	3
$f(x)$	9	7	5	3	1	–1	–3

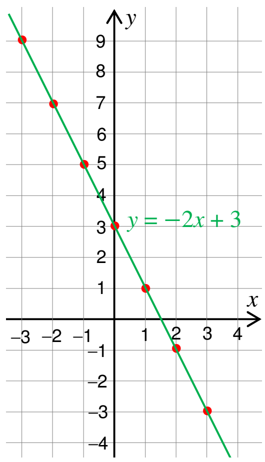


Figure 4.13

b Now you plot these points on a coordinate system and draw a line through these points. This line is the graph of the linear function $f(x) = -2x + 3$. (*see* Figure 4.12).

Example 6 Draw the graph of the constant function

$f(x) = 2$.

Solution: You construct a table of values of the function, plot the ordered pairs and draw a line through the points to get the required graph.

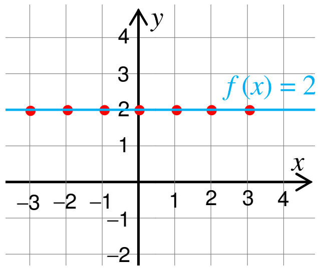


Figure 4.14

x	–3	–2	–1	0	1	2	3
$f(x)$	2	2	2	2	2	2	2

ACTIVITY 4.7

Write down what you observe from the graphs of the linear functions drawn above.

In a linear function $f(x) = ax + b$, a is called the coefficient of x. This a is also the slope of the graph of the linear function. From the graphs given above, you should have noticed that:

i Graphs of linear functions are straight lines.

ii If $a > 0$, then the graph of the linear function $f(x) = ax + b$ is increasing,

iii If $a < 0$, then the graph of the linear function $f(x) = ax + b$ is decreasing,

iv If $a = 0$, then the graph of the constant function $f(x) = b$ is a horizontal line.

v If $x = 0$, then $f(0) = b$. This means $(0, b)$ lies on the graph of the function, and the graph passes through the ordered pair $(0, b)$. This point is called the y-intercept. It is the point at which the graph intersects the y-axis.

vi If $f(x) = 0$, then $0 = ax + b \Rightarrow x = \frac{-b}{a}$. This means $(\frac{-b}{a}, 0)$ lies on the graph of the function and the graph passes through the ordered pair $(\frac{-b}{a}, 0)$. This point is called the x-intercept. It is the point at which the graph intersects the x-axis.

Example 7 For the linear function $f(x) = 7x + 2$, determine the y-intercept and the x-intercept.

Solution: At the y-intercept, $x = 0$ and $f(0) = 2$. So the y-intercept is (0, 2).

At the x-intercept $y = 0$ and $0 = 7x + 2 \Rightarrow x = -\frac{2}{7}$. So the x-intercept is $(-\frac{2}{7}, 0)$.

Example 8 Is the graph of the linear function $f(x) = 2 - 2x$ increasing or decreasing?

Solution: Since $f(x) = 2 - 2x$ is the same as $f(x) = -2x + 2$ and the coefficient of x is -2, the graph is decreasing.

You have learnt how to use table of values of a linear function to draw its graph. It is also possible to draw the graph of a linear function by using the x-intercept and y-intercept.

Example 9 Draw the graph of $f(x) = 4x - 4$.

Solution: The x-intercept is the ordered pair with $y = 0$. That is, (1, 0).

The y-intercept is the ordered pair with $x = 0$. That is, (0, –4).

Plot these intercepts on a coordinate system and draw a line that passes through them.

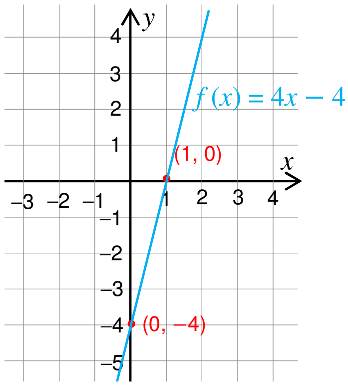


Figure 4.15

You can also use the concept of slope for drawing the graph of linear functions. To draw the graph of a linear function $f(x) = ax + b$, first mark the y-intercept. Then from the y-intercept move a units up (if $a > 0$) or a units down (if $a < 0$) and one unit to the right, and locate a point. Then, draw the line that passes through the y-intercept and this point. This line is the graph of the linear function.

Example 10 Draw the graph of $f(x) = 2x + 1$.

Solution: The slope of the graph of the linear function $f(x) = 2x + 1$ is 2 and the y-intercept is (0, 1). If you move 2 units up from the y-intercept and one unit to the right, you will get the point (1, 3). So the line that passes through (0, 1) and (1, 3) is the graph of the function $f(x) = 2x + 1$.

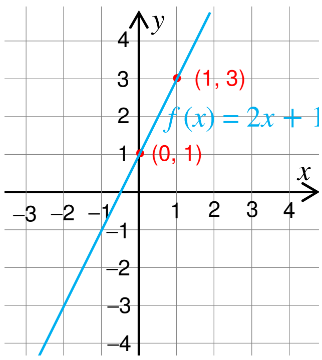


Figure 4.16

Example 11 Draw the graph of the linear function $f(x) = -3x + 1$.

Solution: The slope of the graph of the linear function $f(x) = -3x + 1$ is -3 and the y-intercept is $(0, 1)$.

If you move 3 units down from the y-intercept and one unit to the right, you will get the point $(1, -2)$. Then the line that passes through $(0, 1)$ and $(1, -2)$ is the graph of the function $f(x) = -3x + 1$.

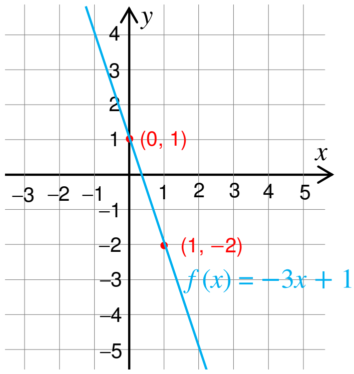


Figure 4.17

Exercise 4.6

1 Determine whether each of the following is a linear function or not.

a $f(x) - 1 = 3x$ **b** $3 = x - 2y$

c $x + y = 1 - 3x$ **d** $2x^2 - 2x = y$

2 Construct tables of values of the following functions for the given domain A:

a $f(x) = 2x - 1$; $A = \{-1, 1, 2, 3\}$ **b** $y = \frac{x}{3} - 1$; $A = \{-6, -3, 0, 3, 6\}$

c $f(x) = 1 - 3x$; $A = \{-3, -2, -1, 0, 1, 2, 3\}$

3 Determine the slope, y-intercept and x-intercept of each of the following linear functions:

a $x + y - 1 = 0$ **b** $f(x) = 3x - 4$

c $y - 3 = x$ **d** $f(x) - 5 = 3x$

4 State if the graph of each of the following linear functions is increasing or decreasing:

a $3x - 2 = 2y$ **b** $y - 2x + 5 = 1$

c $f(x) - 7 = 2$ **d** $f(x) = 4$

5 Draw the graph of each of the following by constructing a table of values for $-3 \le x \le 3$:

a $y - 3x - 5 = 4$ **b** $4 = 4x - 2y$

c $f(x) = 1 - 7x$ **d** $y = 1$

6 Draw the graph of each of the following by using the intercepts:

a $3x - 5 = y$ **b** $4 + 2y = 4x$ **c** $f(x) = 3x - 5$

7 Draw the graph of each of the following by using the value of slope:

a $3y - 3x - 5 = 4$ **b** $f(x) = 4x + 2$ **c** $3x - 4 = 5x - 2y$

4.3.2 Graphs of Quadratic Functions

In the previous sub-section, you have discussed linear functions, their graphs and some important properties. In this sub-section, you will learn about quadratic functions, their graphs and some properties that include the minimum and maximum value of quadratic functions.

Definition 4.5

A function defined by $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$ is called a quadratic function. a is called the leading coefficient.

Example 1 $f(x) = 2x^2 + 3x + 2$ is a quadratic function with $a = 2$, $b = 3$, and $c = 2$.

Note: Any function that can be reduced to the form $f(x) = ax^2 + bx + c$ is also called a quadratic function.

Example 2 $f(x) = (x - 2)(x + 2)$ can be expressed as $f(x) = x^2 - 4$.

So $f(x) = (x - 2)(x + 2)$ is a quadratic function with $a = 1$, $b = 0$, and $c = -4$.

Let us now draw graphs of quadratic functions by constructing tables of values.

ACTIVITY 4.8

1 Construct a table of values for each of the following quadratic functions, for $-3 \leq x \leq 3$:

a $f(x) = x^2$ b $f(x) = x^2 + 3x + 2$ and c $f(x) = -2x^2 + x - 4$

2 Using the tables in Question 1a plot the points (x, x^2) on xy-coordinate systems. Connect those points by smooth curves.

3 Discuss the type of graphs you obtained.

The graph of a quadratic function is a curve known as parabola.

Example 3 Draw the graph of $f(x) = -x^2$.

Solution: The table of values is

x	-2	-1	0	1	2
$f(x)$	-4	-1	0	-1	-4

The graph is as shown in Figure 4.18.

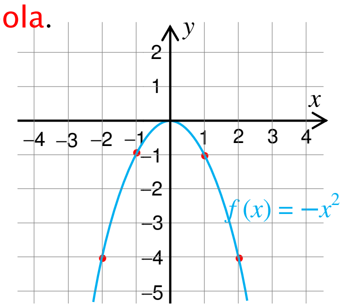


Figure 4.18

ACTIVITY 4.9

Write down what you observe from the graphs of the quadratic functions drawn above.

You may have noticed that:

i The graph of the parabola is opened either upward or downward depending on the sign of the coefficient of x^2.

ii There is a turning point on the graphs.

iii These graphs are symmetrical.

The turning point of the graph of a quadratic function is called the vertex of the parabola and the vertical line that passes through the vertex is called the axis of the parabola.

Example 4 For the quadratic function $f(x) = x^2$, determine the vertex and the axis of the parabola.

Solution: The graph of the quadratic function $f(x) = x^2$ is as given, the vertex of the parabola is (0, 0) and the axis of the parabola is the y-axis.

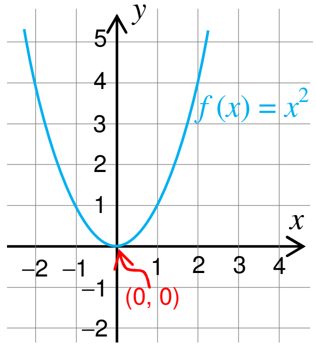


Figure 4.19

Having drawn the graphs of $f(x) = x^2$ and $f(x) = -x^2$, you shall now examine quadratic functions of the type $f(x) = ax^2 + c$ for some $c \in \mathbb{R}$.

Group Work 4.6

1 Using the same coordinate system, sketch the graphs of the following quadratic functions by using table of values:

i a $f(x) = 3x^2$ b $f(x) = 3x^2 - 1$ c $f(x) = 3x^2 + 1$

ii a $f(x) = -3x^2$ b $f(x) = -3x^2 - 1$ c $f(x) = -3x^2 + 1$

2 Write down your observations from the graphs and discuss in groups.

3 Can you sketch these graphs using some other methods? Explain and discuss.

Sketching graphs of quadratic function using a table of values

Example 5 Sketch the graph of $f(x) = 2x^2$.

x	–2	–1	0	1	2
$f(x)$	8	2	0	2	8

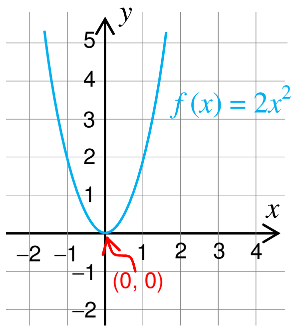


Figure 4.20

Example 6 Sketch the graph of $f(x) = 2x^2 - 3$.

x	–2	–1	0	1	2
$f(x)$	5	–1	–3	–1	5

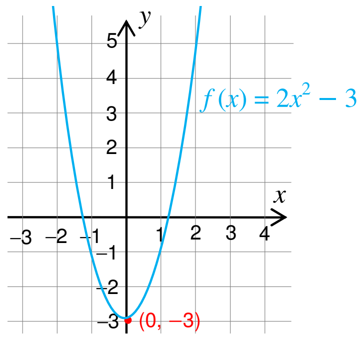


Figure 4.21

Example 7 Sketch the graph of $f(x) = 2x^2 + 3$.

x	–2	–1	0	1	2
$f(x)$	11	5	3	5	11

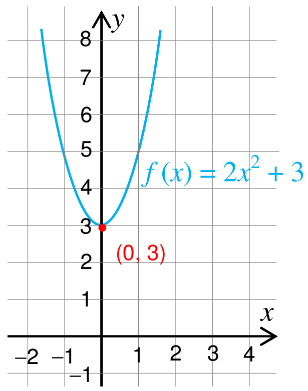


Figure 4.22

Observe that the graphs are all parabolas and they all open upward but their vertices are in different places. Also note that the corresponding values of $f(x) = 2x^2 + 3$ are 3 units more than the values of $f(x) = 2x^2$ and the corresponding values of $f(x) = 2x^2 - 3$ are 3 units less than the values of $f(x) = 2x^2$. Using this, the graphs of the functions of

$$f(x) = 2x^2 - 3 \text{ and } f(x) = 2x^2 + 3 \text{ can be obtained from the graph of } f(x) = 2x^2.$$

This leads us to another way of sketching graphs of quadratic functions.

From graphs of quadratic functions of the form

$f(x) = ax^2$ and $f(x) = ax^2 + c$, $a \neq 0$, $c \in \mathbb{R}$, we can summarize:

Case 1: If $a > 0$,

1. The graph opens upward.
2. The vertex is $(0, 0)$ for $f(x) = ax^2$ and $(0, c)$ for $f(x) = ax^2 + c$.
3. The domain is all real numbers.
4. The range is $\{y: y \geq 0\}$ for $f(x) = ax^2$ and $\{y: y \geq c\}$ for $f(x) = ax^2 + c$.
5. The vertical line that passes through the vertex is the axis of the parabola (or the axis of symmetry).

Case 2: If $a < 0$,

1. The graph opens downward.
2. The vertex is (0, 0) for $f(x) = ax^2$ and $(0, c)$ for $f(x) = ax^2 + c$.
3. The domain is all real numbers.
4. The range is $\{y: y \le 0\}$ for $f(x) = ax^2$ and $\{y: y \le c\}$ for $f(x) = ax^2 + c$.
5. The vertical line that passes through the vertex is the axis of the parabola (or the axis of symmetry).

Sketching graphs of quadratic functions, using the shifting rule

So far we have used tables of values to sketch graphs of quadratic functions. Now we shall see how to use the shifting rule to sketch the graphs of quadratic functions. As you have seen in Examples 5, 6 and 7, you can sketch the graph of $f(x) = 2x^2 + 3$ by shifting the graph of $f(x) = 2x^2$ by 3 units upward, and the graph of $f(x) = 2x^2 - 3$ can be obtained by shifting the graph of $f(x) = 2x^2$ by 3 units downward.

Example 8 Sketch the graph of $f(x) = x^2 - 1$ and $f(x) = x^2 + 1$ by shifting $f(x) = x^2$ and determine the vertex of each graph.

Solution: The graph of $f(x) = x^2$ is as shown in Figure 4.23a.

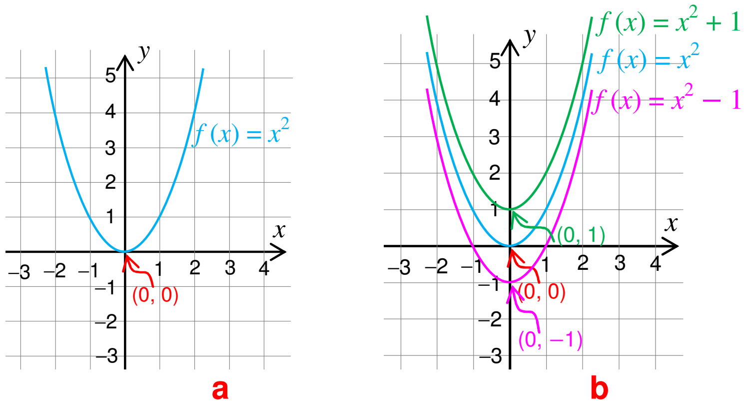


Figure 4.23

The graph of $f(x) = x^2 - 1$ is obtained by shifting the graph of $f(x) = x^2$ by 1 unit downward giving a vertex at (0, –1); that of $f(x) = x^2 + 1$ is obtained by shifting the graph of $f(x) = x^2$ by 1 unit upward, to a vertex at (0, 1). *See* Figure 4.23b.

Example 9 Sketch the graph of

$f(x) = (x - 3)^2$ and contrast it with the graph of $f(x) = x^2$.

Solution: By constructing a table of values, you can draw the graph of

$f(x) = (x - 3)^2$ and see that it is a shifting of the graph of $f(x) = x^2$ by 3 units to the right. The vertex of the graph is (3, 0) (*See* Figure 4.24).

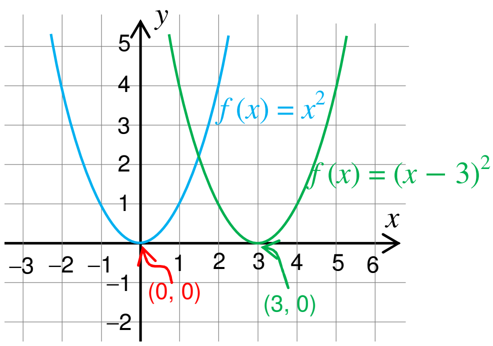


Figure 4.24

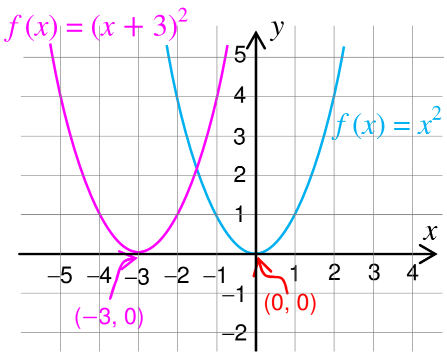


Figure 4.25

Example 10 Sketch the graph of

$f(x) = (x + 3)^2$ and contrast it with the graph of $f(x) = x^2$.

Solution: Using a table of values, you get the graph of $f(x) = (x + 3)^2$ and see that it is a shifting of the graph of $f(x) = x^2$ by 3 units to the left, giving a vertex at (–3, 0) (*See* Figure 4.25).

Let $k > 0$, then the graph of $f(x) = (x - k)^2$ is obtained by shifting the graph of $f(x) = x^2$ by k units to the right and the graph of $f(x) = (x + k)^2$ is obtained by shifting the graph of $f(x) = x^2$ by k units to the left.

By shifting the graph of $f(x) = x^2$ in the x and y directions you can sketch graphs of quadratic functions such as

a $f(x) = (x + 3)^2 + 2$ **b** $f(x) = (x - 3)^2 - 2$ **c** $f(x) = x^2 + 4x + 2$

Example 11 Sketch the graph of $f(x) = (x + 3)^2 + 2$

Solution: First sketch the graph of $f(x) = x^2$. To obtain the graph of

$f(x) = (x + 3)^2$ shift the graph of $f(x) = x^2$ to the left by 3 units.

After this, to obtain the graph of $f(x) = (x + 3)^2 + 2$ shift the graph of $f(x) = (x + 3)^2$ by 2 units upward.

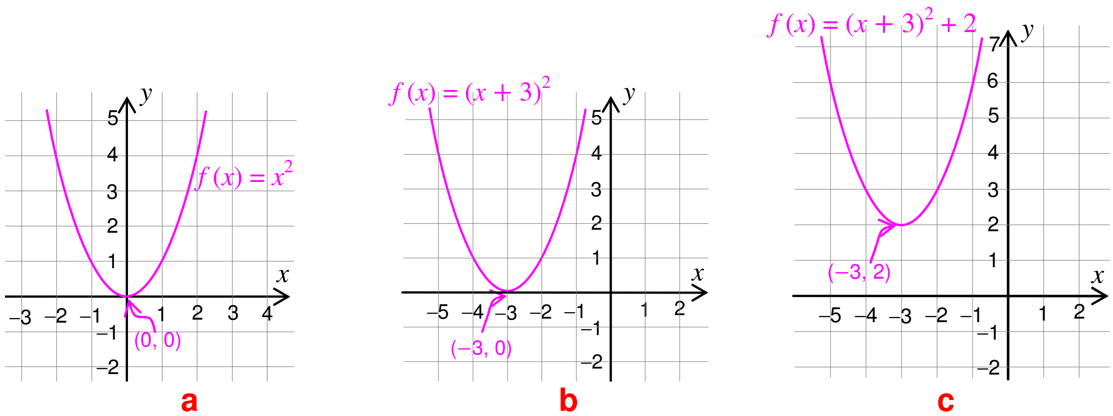
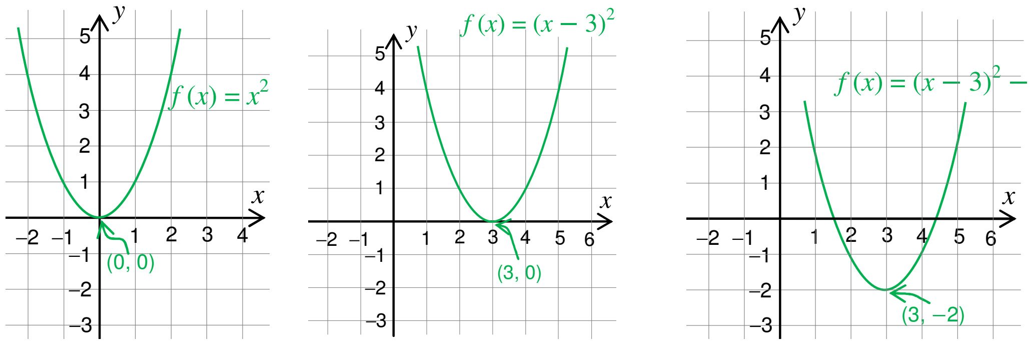


Figure 4.26

Example 12 Sketch the graph of $f(x) = (x - 3)^2 - 2$.

Solution: First sketch the graph of $f(x) = x^2$.

To obtain the graph of $f(x) = (x - 3)^2$ shift the graph of $f(x) = x^2$ to the right by 3 units so that the vertex is at (3, 0). After this, to obtain the graph of $f(x) = (x - 3)^2 - 2$, shift the graph of $f(x) = (x - 3)^2$ by 2 units downward so that the vertex is at (3, –2).



a b c

Figure 4.27

Example 13 Sketch the graph of $f(x) = x^2 + 4x + 2$.

Solution: In order to sketch the graph of $f(x) = x^2+4x+2$, first we need to transform this function into the form of $f(x) = (x + k)^2 + c$ by completing the square.

Therefore $f(x) = x^2 + 4x + 2$ can be expressed

$$f(x) = (x + 2)^2 - 2$$

Now you can sketch the graph of $f(x) = (x + 2)^2 - 2$ as above by shifting the graph of $f(x) = x^2$ by 2 units to the left and then by 2 units downward.

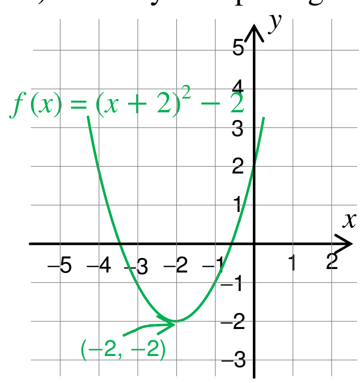


Figure 4.28

Note:

1 The graph of $f(x) = (x + k)^2 + c$ opens upward.

2 The vertex of the graph of $f(x) = (x + k)^2 + c$ is $(-k, c)$ and the vertex of the graph of $f(x) = (x - k)^2 - c$ is $(k, -c)$. Similarly the vertex of the graph of $f(x) = (x + k)^2 - c$ is $(-k, -c)$ and the vertex of the graph of $f(x) = (x - k)^2 + c$ is (k, c).

Minimum and maximum values of quadratic functions

Suppose you throw a stone upward. The stone turns down after it reaches its maximum height. Similarly, a parabola turns after it reaches a maximum or a minimum y value.

Group Work 4.7

1 Let $f(x)$ be a quadratic function. Discuss how to determine the maximum or minimum value of $f(x)$.

2 Justify your conclusion by considering some parabolas.

Recall that if the leading coefficient of the quadratic function $f(x) = ax^2 + bx + c$ is positive ($a > 0$), then the graph of the function opens upward (and if $a < 0$, then the graph opens downward). When the graph of a quadratic function opens upward, the function has a minimum value, whereas if the graph opens downward, it has a maximum value. The minimum or the maximum value of a quadratic function is obtained at the vertex of its graph.

Example 14 The minimum value of a quadratic function expressed as

$$f(x) = (x + k)^2 + c \text{ is } c.$$

Similarly, the maximum value of $f(x) = -(x+k)^2 + c$ is c.

Example 15 Sketch the graph of $f(x) = x^2 + 6x - 5$ and determine the minimum value of $f(x)$.

Solution: $f(x) = x^2 + 6x + 9 - 9 - 5 = (x + 3)^2 - 14.$

Hence the graph can be sketched by shifting the graph of $f(x) = x^2$ by 3 units to the left side and then downward by 14 units.

Hence, the minimum value of f is -14.

In this case, the range of the function is

$\{y: y \geq -14\} = [-14, \infty)$.

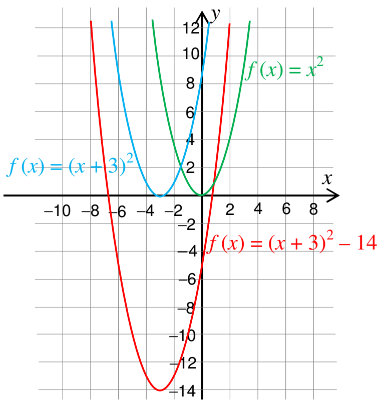


Figure 4.29

Example 16 Find the maximum value of the function $f(x) = -x^2 + 6x - 8$, and sketch its graph.

Solution: $f(x) = -x^2 + 6x - 9 + 9 - 8$

$= -(x^2 - 6x + 9) + 1;$

$f(x) = -(x - 3)^2 + 1.$

The graph of $f(x) = -(x - 3)^2 + 1$ has vertex (3, 1) and hence the maximum value of f is 1.

In this case, the range of the function is

$\{y: y \leq 1\} = (-\infty, 1]$

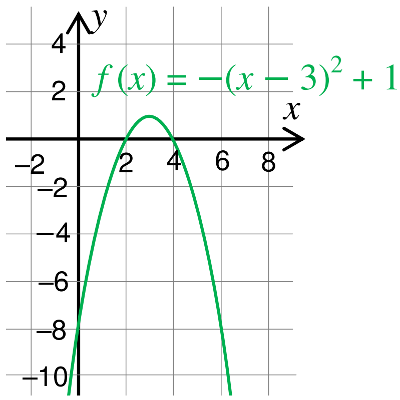


Figure 4.30

Exercise 4.7

1 For each of the following quadratic functions, determine a, b and c:

a $f(x) = 2 + 3x - 2x^2$ b $f(x) = 3x^2 - 4x + 1$ c $f(x) = (x - 3)(2 - x)$

2 For each of the following quadratic functions prepare a table of values in the interval $-3 \leq x \leq 3$.

a $f(x) = -4x^2$ b $f(x) = 3x^2 + 2$ c $f(x) = 2x^2 - 3x + 2$

3 Sketch the graph for each of the following quadratic functions by constructing tables of values:

a $f(x) = -3x^2$ b $f(x) = 7x^2 - 3$ c $f(x) = 2x^2 + 6x + 1$

4 Find the domain and range of each of the following functions:

a $f(x) = 3 + 4x - x^2$ b $f(x) = x^2 + 2x + 1$ c $f(x) = (x - 3)(x - 2)$
d $f(x) = -3x^2 - 2$ e $f(x) = 3x^2 + 2$

5 Sketch the graph of each of the following quadratic functions by using the shifting rule:

a $f(x) = 9x^2 + 1$ b $f(x) = x^2 - 3$ c $f(x) = (x - 5)^2$
d $f(x) = (x - 2)^2 + 13$ e $f(x) = (x + 1)^2 - 7$ f $f(x) = 4x^2 + 7x + 3$

6 Find the vertex and the axis of symmetry of the following functions:

a $f(x) = x^2 - 5x + 8$ b $f(x) = (x - 4)^2 - 3$ c $f(x) = x^2 - 8x + 3$

7 Determine the minimum or the maximum value of each of the following functions and draw the graphs:

a $f(x) = x^2 + 7x - 10$ b $f(x) = x^2 + 4x + 1$ c $f(x) = 2x^2 - 4x + 3$
d $f(x) = 4x^2 + 2x + 4$ e $f(x) = -x^2 - 4x$ f $f(x) = -6 - x^2 - 4x$

Key Terms

axis of symmetry	leading coefficient	turning point
combination of functions	linear functions	vertex
constant function	quadratic function	x-intercept
coordinate system	relation	y-intercept
domain	range	
function	slope	

Summary

1 In a relation, two things are related to each other by a relating phrase.

2 Mathematically, a relation is a set of ordered pairs. If A and B are two non-empty sets, then the relation from A to B is a subset of A × B that satisfies the relating phrase.

3 If A and B are any sets and R $\subseteq$ (A × B), we call R a binary relation from A to B or a binary relation between A and B.A relation R $\subseteq$ (A × A) is called a relation in or on A.

4 The set $\{x: (x, y) \in$ R for some $y\}$ is called the domain of the relation R.

The set $\{y: (x, y) \in$ R for some $x\}$ is called the range of the relation R.

5 A function is a special type of a relation in which each x-coordinate is paired with exactly one unique y-coordinate.

6 A function from A to B can sometimes be denoted as f: A→B, where the domain of f is A and the range of f is a subset of B, in which case B contains the images of the elements of A by the function f.

7 Let f and g be functions. We define the sum $f + g$, the difference $f - g$, the product fg, and the quotient $\frac{f}{g}$ as:

$f + g : (f + g)(x) = f(x) + g(x)$ $\qquad$ $fg : (fg)(x) = f(x)\, g(x)$

$f - g : (f - g)(x) = f(x) - g(x)$ $\qquad$ $\frac{f}{g} : \frac{f}{g}(x) = \frac{f(x)}{g(x)};\, g(x) \neq 0$

8 If a and b are fixed real numbers, $a \neq 0$, then $f(x) = ax + b$ for $x \in \mathbb{R}$ is called a linear function. If $a = 0$ then $f(x) = b$ is called a constant function. Sometimes linear functions are written as $y = ax + b$.

9 In $f(x) = ax + b$ for $a \neq 0$, $x \in \mathbb{R}$, a represents the slope, $(0, b)$ represents the y-intercept and $\left(\frac{-b}{a}, 0\right)$ represents the x-intercept.

10 A function defined by $f(x) = ax^2 + bx + c$ ($a, b, c \in \mathbb{R}$ and $a \neq 0$) is called quadratic function. a is called the leading coefficient.

11 We can sketch the graph of a linear function by using either a table of values, or the x- and y-intercepts.

12 We can sketch the graph of a quadratic function by using either a table of values or the shifting rule.

13 The graph of $f(x) = ax^2 + bx + c$ opens upward if $a > 0$ and downward if $a < 0$.

14 The vertex is the point on a coordinate system at which a graph of a quadratic function turns either upward or downward.

15 The axis of a parabola (or axis of symmetry) is a vertical line that passes through the vertex of the parabola.

16 The domain and range of linear functions is the set of all real numbers.

17 The domain of a quadratic function is the set of all real numbers, whereas the range is;

$\{y: y \geq k\}$ if the leading coefficient is positive and k is the value of y at the vertex.

$\{y: y \leq k\}$ if the leading coefficient is negative and k is the value of y at the vertex.

18 The maximum or minimum point (depending on the sign of a) of a quadratic function $f(x) = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Review Exercises on Unit 4

1 For the relation $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$ find the domain and the range.

2 If the domain of the relation $R = \{(x, y): y = x + 3\}$ is $A = \{1, 2, 3, 4\}$ then list all the ordered pairs that are members of the relation and find the range.

3 Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c\}$

- a Find $A \times B$.
- b Determine relations as subsets of $A \times B$ such that:
  - i $R_1 = \{(x, y): x \text{ is odd}\}$
  - ii $R_2 = \{(x, y): 1 \leq x \leq 3\}$

4 Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$

- a If R is a relation from A to B then, is it true that R is also a relation from B to A? Explain your answer.
- b If $R \subseteq (A \times B)$ such that $R = \{(2, 4), (2, 2), (4, 4), (4, 2)\}$, then is R also a relation from B to A?
- c What can we conclude from b?

5 Let R = {(x, y): x is taller than y}.

a Does (x, x) belong to the relation? Explain.

b Is it true that if (x, y) belongs to R, then (y, x) also belongs to R?

c If (x, y) and (y, z) belong to R, then is it true that (x, z) belongs to R?

6 Let R = {(x, y): $y = x$}. Show that each of the statements in Question 5 is true.

7 Find the domain and the range of each of the following relations:

a $R = \{(x, y): y = 2x\}$ **b** $R = \{(x, y): y = |x|\}$

c $R = \{(x, y): x, y \in \{1, 2, 3, 4, 5\} \text{ and } y = 2x - 1\}$

d $R = \{(x, y): y = \sqrt{x^2 - 4}\}$

8 Sketch the graph of each of the following relations and determine the domain and the range:

a $R = \{(x, y): y \geq -2x+3\}$ **b** $R = \{(x, y): y = 2x + 1\}$

c $R = \{(x, y): y < -x + 3\}$ **d** $R = \{(x, y): y \geq |x|\}$

e $R = \{(x, y): y \leq x \text{ and } y \geq 1 - x\}$ **f** $R = \{(x, y): y \leq |x| \text{ and } y \geq 0\}$

g $R = \{(x, y): y = x+1 \text{ and } y = 1 - x\}$

h $R = \{(x, y): y \leq x+1, y \geq 1 - x \text{ and } x \geq 0\}$

i $R = \{(x, y): y > x - 2, y \geq -x - 2 \text{ and } y \leq 4\}$

9 For the following graph, specify the relation and write down the domain and range:

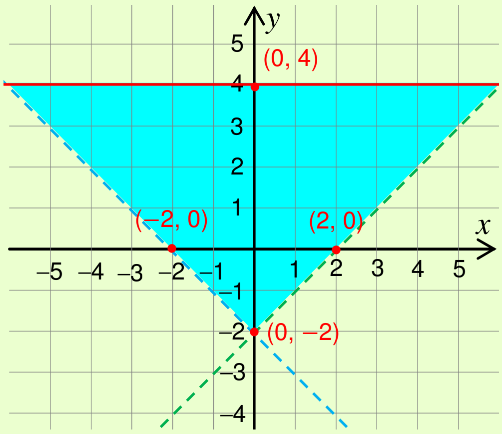


Figure 4.31

10 Determine whether each of the following relations is a function. If it is not, give a reason.

a R = $\{(a, 1), (b, 2), (c, 3)\}$

b R = $\{(1,3), (2, 3), (3, 3), (4, 3), (5, 3)\}$

c R = $\{(1, 4), (1, 5), (1, 6), (5, 4), (5, 5)\}$

11 If A = $\{2, 5, 7\}$ and B = $\{2, 3, 4, 6\}$, then is A×B a function? Explain your answer.

12 Let $f = \{(1, 2), (2, 3), (5, 6), (7, 8)\}$

a Find the domain and range of f

b Evaluate $f(2)$ and $f(5)$

13 Let $f(x) = 2x + 1$ and $g(x) = -3x - 4$

i Determine:

a $f + g$ **b** $f - g$ **c** fg **d** $\dfrac{f}{g}$

ii Evaluate:

a $(2f + 3g)(1)$ **b** $(3fg)(3)$ **c** $\dfrac{3f}{2g}(4)$

iii Find the domain of $\dfrac{f}{g}$:

14 Let $f(x) = \dfrac{x+4}{2x}$ and $g(x) = \dfrac{2x+4}{x+1}$.

i Determine:

a fg **b** $\dfrac{g}{f}$ **c** $2f - \dfrac{f}{g}$

ii Find the domains of

a fg **b** $\dfrac{g}{f}$ **c** $2f - \dfrac{f}{g}$

iii Evaluate

a $(f - g)(1)$ **b** $\dfrac{g}{f}(2)$ **c** $(2f - \dfrac{f}{g})(3)$

15 Construct tables of values and sketch the graph of each of the following:

a $f(x) = 3x + 2$ **b** $x - 2y = 1$

c $f(x) = 2 - 7x$ **d** $f(x) = -3x^2 - 1$

e $f(x) = 3 - 2x + x^2$

16 Sketch the graph of each of the following by using x- and y-intercepts:

a $f(x) = 7 + 2x$ **b** $f(x) = 3x - 5$

c $3x - y = 4$

17 By using shifting rule, sketch the graph of each of the following:

a $f(x) = 4x^2 - 2x$ **b** $f(x) = x^2 - 8x + 7$

c $f(x) = 4x + 6 - 3x^2$

18 For the function $f(x) = 3x^2 - 5x + 7$, determine:

a whether it turns upward or downward

b the vertex

c the axis of symmetry

19 Determine the minimum (or the maximum) value of the following functions:

a $f(x) = (x - 4)^2 - 5$ **b** $f(x) = 2x^2 - 6x + 7$

c $f(x) = 3x^2 - 5x + 8$ **d** $f(x) = -x^2 + 6x - 5$

e $f(x) = -2 + 4x - 2x^2$

20 Determine the range of each of the following functions:

a $f(x) = (x + 5)^2 + 3$ **b** $f(x) = x^2 - 9x + 10$

c $f(x) = -8 - x^2 - 6x$ **d** $f(x) = -x^2 + 2x + 4$

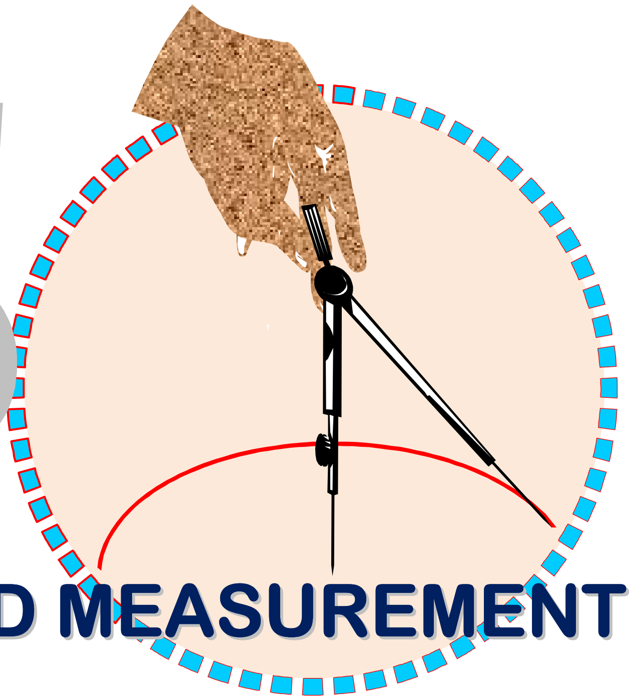
21 A mobile phone technician uses the linear function $c(t) = 2t + 15$ to determine the cost of repair, where t is the time in hours and $c(t)$ the cost in Birr. How much will you pay if it takes him 3 hours to repair your mobile?

22 A real estate sells houses for Birr 200,000 plus Birr 400 per one square metre.

a Find the function that represents the cost of the house that has an area of x m^2.

b Calculate the cost of the house that has an area of 80 m^2.

Unit 5



GEOMETRY AND MEASUREMENT

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about regular polygons.*
- *apply postulates and theorems in order to prove congruence and similarity of triangles.*
- *construct similar figures.*
- *apply the concept of trigonometric ratios to solve problems in practical situations;*
- *know specific facts about circles.*
- *solve problems on areas of triangles and parallelograms.*

Main Contents

INTRODUCTION

You have learnt several concepts, principles and theorems of geometry and measurement in your lower grades. In the present unit, you will learn more about geometry and measurement. Regular polygons and their properties, congruency and similarity of triangles, radian measure of an angle, trigonometrical ratios, properties of circles, perimeter and area of a segment and a sector of a circle, areas of plane figures, and volumes of solid figures are the major topics covered in this unit.

5.1 REGULAR POLYGONS

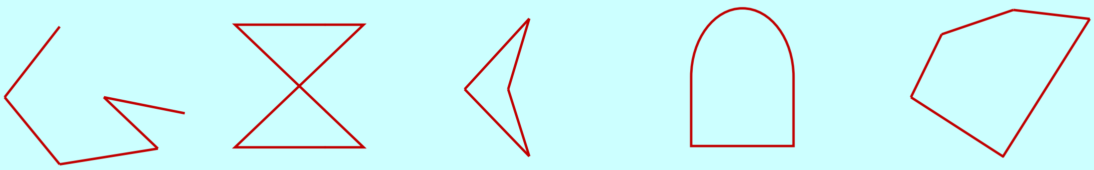
A Revision on polygons

The following Activity might help you recall important facts about polygons that you studied in previous grades.

ACTIVITY 5.1

1. What is a polygon?
2. Discuss the difference between a convex polygon and a concave polygon.
3. Find the sum of the measures of the interior angles of:

   a a triangle. **b** a quadrilateral. **c** a pentagon.
4. Which of the following figures are polygons?



a b c d e

Figure 5.1

Definition 5.1

A polygon is a simple closed curve, formed by the union of three or more line segments, no two of which in succession are collinear. The line segments are called the sides of the polygon and the end points of the sides are called the vertices.

In other words, a polygon is a simple closed plane shape consisting of straight-line segments such that no two successive line segments are collinear.

B Interior and exterior angles of a polygon

When reference is made to the angles of a polygon, we usually mean the interior angles. As the name indicates, an interior angle is an angle in the interior of a polygon at a vertex.

ACTIVITY 5.2

1 Draw a diagram to show what is meant by an interior angle of a polygon.

2 a How many interior angles does an n-sided polygon have?

b How many diagonals from a vertex can an n-sided polygon have?

c Into how many triangles can an n-sided polygon be partitioned by drawing diagonals from one vertex?

3 What relationships are there between the number of sides, the number of vertices and the number of interior angles of a given n-sided polygon?

Note that the number of *vertices*, *angles* and *sides* of a polygon are the same.

Number of sides	Number of interior angles	Name of polygon
3	3	Triangle
4	4	Quadrilateral
5	5	Pentagon
6	6	Hexagon
7	7	Heptagon
8	8	Octagon
9	9	Nonagon
10	10	Decagon

Definition 5.2

An angle at a vertex of a polygon that is supplementary to the interior angle at that vertex is called an exterior angle. It is formed between one side of the polygon and the extended adjacent side.

Example 1 In the polygon ABCD in Figure 5.2, ∠DCB is an interior angle; ∠BCE and ∠DCF are exterior angles of the polygon at the vertex C.

(There are two possible exterior angles at any vertex, which are equal.)

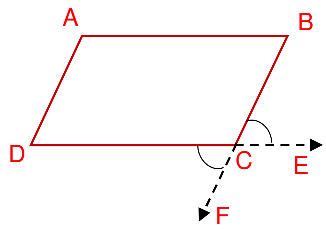


Figure 5.2

C The sum of the measures of the interior angles of a polygon

Let us first consider the sum of the measures of the interior angles of a triangle.

ACTIVITY 5.3

- Draw a fairly large triangle on a sheet of thin cardboard.

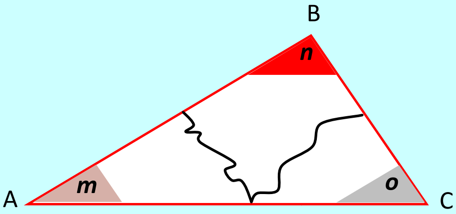


Figure 5.3

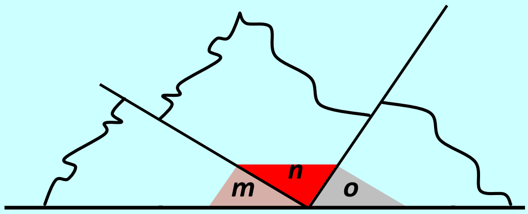


Figure 5.4

- Now tear the triangle into three pieces, making sure each piece contains one corner (angle).
- Fit these three pieces together along a straight line as shown in Figure 5.4.

1. Observe that the sum of the three angles is a straight angle.
2. What is the sum of the measures of the interior angles of ΔABC?
3. Given below are the measures of $\angle A$, $\angle B$ and $\angle C$. Can a triangle ABC be made with the given angles? Explain.

   a. m $(\angle A) = 36^{o}$; m $(\angle B) = 78^{o}$; m $(\angle C) = 66^{o}$.
   b. m $(\angle A) = 124^{o}$; m $(\angle B) = 56^{o}$; m $(\angle C) = 20^{o}$.
   c. m $(\angle A) = 90^{o}$; m $(\angle B) = 74^{o}$; m $(\angle C) = 18^{o}$.

Based on observations from the above Activity, we state the following theorem.

Theorem 5.1 Angle sum theorem

The sum of the measures of the three interior angles of any triangle is 180^{o}.

ACTIVITY 5.4

Partitioning a polygon into triangles as shown in Figure 5.5 can help you to determine the sum of the interior angles of a polygon.

Complete the following table.

Number of sides of the polygon	Number of triangles	Sum of interior angles
3	1	1×180^o
4	2	2×180^o
5	3	3×180^o
6		$\_\_ \times 180^o$
7		
8		
n		$\_\_ \times 180^o$

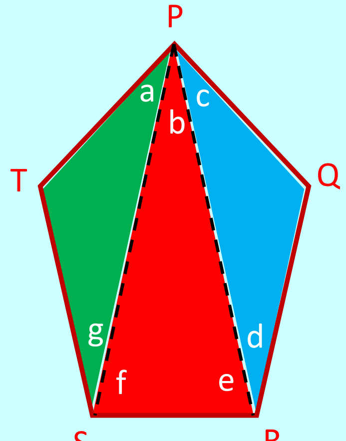


Figure 5.5

From the above Activity, you can generalize the sum of interior angles of a polygon as follows:

Theorem 5.2

If the number of sides of a polygon is n, then the sum of the measures of all its interior angles is equal to $(n - 2) \times 180^o$.

From Activity 5.4 and Theorem 5.2, you can also observe that an n-sided polygon can be divided into $(n - 2)$ triangles. Since the sum of interior angles of a triangle is 180^o, the sum of the angles of the $(n - 2)$ triangles is given by:

$$S = (n - 2) \times 180^o.$$

ACTIVITY 5.5

1 Using Figure 5.6, verify the formula $S = (n - 2) \times 180^o$ given above.

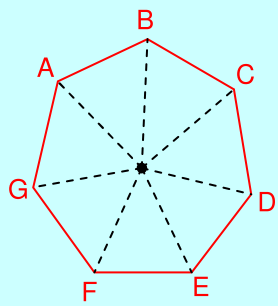


Figure 5.6

Hint: Angles at a point add up to 360^o.

2 By dividing each of the following figures into triangles, show that the formula $S = (n - 2) \times 180^o$ for the sum of the measures of all interior angles of an n-sided polygon is valid for each of the following polygons:

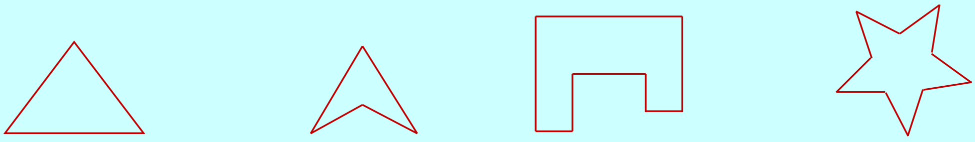


Figure 5.7

3 In a quadrilateral ABCD, if m(∠A) = 80°, m(∠B) = 100° and m(∠D) = 110°, find m(∠C).

4 If the measures of the interior angles of a hexagon are x^o, $2x^o$, 60^o, $(x + 30)^o$, $(x - 10)^o$ and $(x + 40)^o$, find the value of x.

5 **a** Let i_1, i_2, i_3 be the measures of the interior angles of the given triangle, and let e_1, e_2 and e_3 be the measures of the exterior angles, as indicated in Figure 5.8.

Explain each step in the following:

$$i_1 + e_1 = 180^o$$
$$i_2 + e_2 = 180^o$$
$$i_3 + e_3 = 180^o$$

$$(i_1 + e_1) + (i_2 + e_2) + (i_3 + e_3) = 180^o + 180^o + 180^o$$

$$(i_1 + i_2 + i_3) + (e_1 + e_2 + e_3) = 3 \times 180^o$$

$$180^o + e_1 + e_2 + e_3 = 3 \times 180^o$$

$$e_1 + e_2 + e_3 = 3 \times 180^o - 180^o = 2 \times 180^o$$

$$e_1 + e_2 + e_3 = 360^o$$

That is, *the sum of the measures of the exterior angles of a triangle, taking one angle at each vertex, is* 360^o.

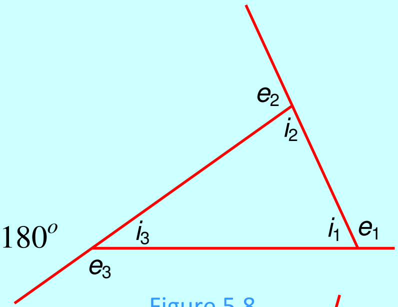


Figure 5.8

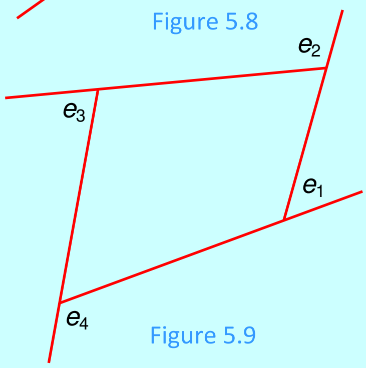


Figure 5.9

b Repeat this for the quadrilateral given in Figure 5.9. Find the sum of the measures of the exterior angles of the quadrilateral. i.e., find $e_1 + e_2 + e_3 + e_4$

c If $e_1, e_2, e_3 \ldots e_n$ are the measures of the exterior angles of an n-sided polygon, then $e_1 + e_2 + e_3 + \ldots + e_n =$ ______.

6 Show that the measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles.

5.1.1 Measures of Angles of a Regular Polygon

Suppose we consider a circle with centre O and radius r, and divide the circle into n equal arcs. (The figure given on the right shows this when $n = 8$.)

For each little arc, we draw the corresponding chord. This gives a polygon with vertices $P_1, P_2, \ldots, P_n$. Since the arcs have equal lengths, the chords (which are the sides of the polygon) are equal. If we draw line segments from O to each vertex of the polygon, we get n isosceles triangles. In each triangle, the degree measure of the central angle O is given by:

$$m(\angle O) = \frac{360^o}{n}.$$

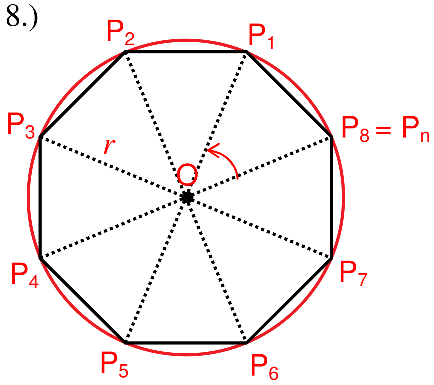


Figure 5.10

Since the vertex angles at O of each isosceles triangle have equal measures, namely $\frac{360^o}{n}$, it follows that all the base angles of all the isosceles triangles are also equal.

From this, it follows that the measures of all the angles of the polygon are equal; the measure of an angle of the polygon is twice the measure of any base angle of any one of the isosceles triangles. So, the polygon has all of its sides equal and all of its angles equal. A polygon of this type is called a regular polygon.

Definition 5.3

A regular polygon is a convex polygon in which the lengths of all of its sides are equal and the measures of all of its angles are equal.

Note that the measure of an interior angle of an n-sided regular polygon is $\frac{S}{n}$, where $S = (n - 2) \times 180^o$ is the sum of the measures of all of its interior angles. Hence, we have the following:

The measure of each interior angle of a regular n-sided polygon is $\frac{(n-2)180^o}{n}$.

A polygon is said to be inscribed in a circle if all of its vertices lie on the circle.

For example, the quadrilateral shown in Figure 5.11 is inscribed in the circle.

Any regular polygon can be inscribed in a circle. Because of this, the centre and the radius of a circle can be taken as the centre and radius of an inscribed regular polygon.

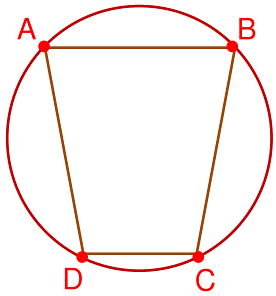


Figure 5.11

Example 1

i Find the measure of each interior angle and each central angle of a regular polygon with:

a 3 sides **b** 5 sides

ii Find the measure of each exterior angle of a regular n-sided polygon.

Solution:

i **a** Since the sum of interior angles of a triangle is 180^o, each interior angle is $\frac{180^o}{3} = 60^o$.

Recall that a 3-sided regular polygon is an equilateral triangle.

To find the measure of a central angle in a regular n-sided polygon, recall that the sum of the measures of angles at a point is 360°. Hence, the sum of the measures of the central angles is 360^o. (Figure 5.10 illustrates this for n = 8.). So, the measure of each central angle in an n-sided regular polygon is $\frac{360^o}{n}$. From this, we conclude that the measure of each central angle of an equilateral triangle is $\frac{360^o}{3} = 120^o$.

b Recall that the sum of all interior angles of a 5-sided polygon is $(5 - 2)\times 180^o = 3 \times 180^o = 540^o$. So, the measure of each interior angle of a regular pentagon is $540^o \div 5 = 108^o$.

Also, the measure of each central angle of a regular pentagon is $\frac{360^o}{5} = 72^o$.

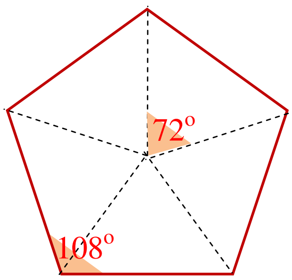


Figure 5.12

ii To find the measure of each exterior angle in a regular n-sided polygon, notice that at each vertex, the sum of an interior angle and an exterior angle is 180^o

(*See* Figure 5.13).

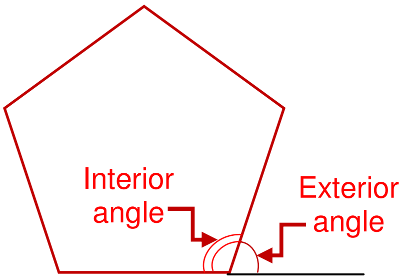


Figure 5.13

Hence each exterior angle will measure

$$180^o - \left[\frac{(n-2)}{n}\,180^o\right] = \frac{n180^o - (n-2)180^o}{n} = \frac{360^o}{n},$$

which is the same as the measure of a central angle.

We can summarize our results about angle measures in regular polygons as follows.

For any regular n-sided polygon:

i Measure of each interior angle $= \dfrac{(n-2)180^o}{n}$.

ii Measure of each central angle $= \dfrac{360^o}{n}$.

iii Measure of each exterior angle $= \dfrac{360^o}{n}$.

Exercise 5.1

1 In Figure 5.14a, no two line segments that are in succession are collinear, and no two segments intersect except at their end points. Yet the figure is not a polygon. Why not?

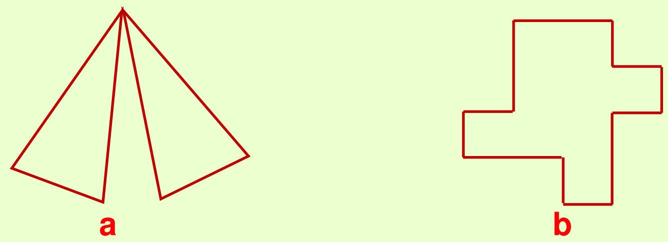


Figure 5.14

2 Is Figure 5.14b a polygon? How many sides does it have? How many vertices does it have? What is the sum of the measures of all of its interior angles?

3 ABCD is a quadrilateral such that the measures of three of its interior angles are given as $m(\angle D) = 112^o$, $m(\angle C) = 75^o$ and $m(\angle B) = 51^o$. Find $m(\angle A)$.

4 Find the measure of an interior angle of a regular polygon with:

a 10 sides **b** 20 sides **c** 12 sides

5 Find the number of sides of a regular polygon, if the measure of each of its interior angles is:

a 150^o **b** 160^o **c** $147.\overline{27}^o$

6 If the measure of a central angle of a regular polygon is 18^o, find the measure of each of its interior angles.

7 **i** Can a regular polygon be drawn such that the measure of each exterior angle is:

a 20^o? **b** 16^o? **c** 15^o?

In each case, if your answer is no, justify it; if yes, find the number of sides.

ii Can a regular polygon be drawn such that the measure of each interior angle is:

a 144^o?   **b** 140^o?   **c** 130^o?

In each case if your answer is yes, find the number of sides; if no, justify it.

8 ABCDEFGH is a regular octagon; the sides $\overline{AB}$ and $\overline{DC}$ are produced to meet at N. Find m(∠AND).

9 Figure 5.15 represents part of a regular polygon of which $\overline{AB}$ and $\overline{BC}$ are sides, and R is the centre of the circle in which the polygon is inscribed. Copy and complete the following table.

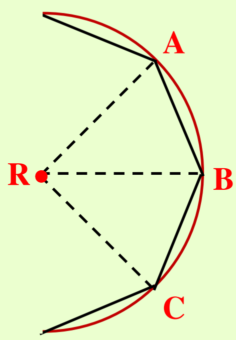


Figure 5.15

Number of sides	m(∠ARB) or m(∠BRC)	m(∠ABR) or m(∠CBR)	m (∠ABC)
3			
4			
5			
6			
	45^o		
9	40^o	70^o	140^o
			144^o
12			
15			
18			
20			

5.1.2 Properties of Regular Polygons

ACTIVITY 5.6

1 Which of the following plane figures can be divided exactly into two identical parts by drawing a line through them? (In other words, which of the following plane figures have a line of symmetry?) Discuss.

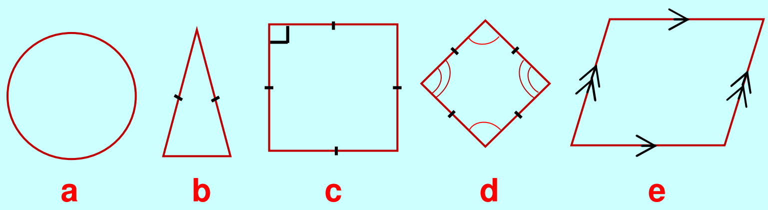


Figure 5.16

2 Which of the above figures have more than one line of symmetry?

3 How many lines of symmetry does a regular n-sided polygon have?

A figure has a line of symmetry, if it can be folded so that one half of the figure coincides with the other half. The Ethiopian flag has a line of symmetry along the black broken line shown. The right half is a reflection of the left half, and the centre line is the line of reflection.

The line of reflection is also called the line of symmetry. A figure that has at least one line of symmetry is called a symmetrical figure.

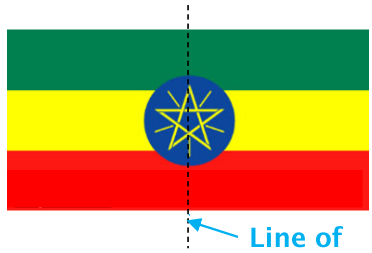


Figure 5.17

Some figures have more than one line of symmetry. In such cases, the lines of symmetry always intersect at one point.

Note that equilateral triangles, squares and regular pentagons have as many lines of symmetry as their sides.

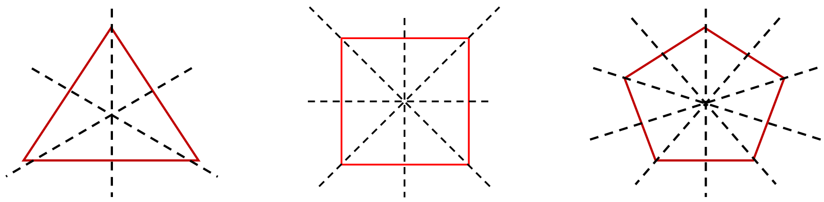


Figure 5.18

To generalize, a regular n-sided polygon always has n lines of symmetry.

Circumscribed regular polygons

A polygon whose sides are tangent to a circle is said to circumscribe the circle. For example, the quadrilateral KLMN *circumscribes* the circle. The circle is *inscribed* in the quadrilateral.

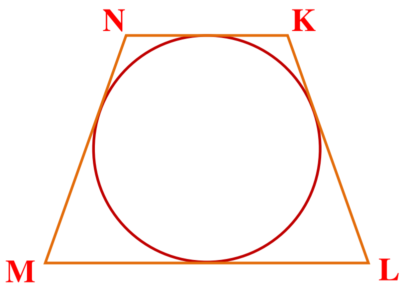


Figure 5.19

It is possible to circumscribe any n-sided regular polygon about a circle. The method is shown by circumscribing a 5-sided polygon, in Figure 5.20.

The idea is that the radii to the points of tangency make 5 congruent angles at the centre whose measures add up to 360°. Using a protractor, we construct five radii, making adjacent central angles of $\frac{360^o}{5} = 72^o$. The radii end at S, R, W, X, Y.

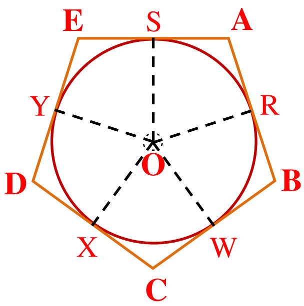


Figure 5.20

Line segments perpendicular to the radii at their endpoints are tangent to the circle and form the circumscribed pentagon as desired.

Can you see how the vertices A, B, C, D, E are determined?

Regular polygons have a special relation to circles. A regular polygon can always be inscribed in or circumscribed about a circle.

This leads us to state the following property about regular polygons:

A circle can always be inscribed in or circumscribed about any given regular polygon.

In Figure 5.20 above, the radius OX of the inscribed circle is the distance from the centre to the side (CD) of the regular polygon. This distance from the centre to any side of the polygon, denoted by a, is the same. This distance a is called the apothem of the regular polygon.

Definition 5.4

The distance a from the centre of a regular polygon to a side of the polygon is called the apothem of the polygon. That is, the apothem a of a regular polygon is the length of the line segment drawn from the centre of the polygon perpendicular to the side of the polygon.

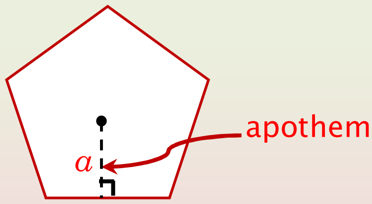


Figure 5.21

Regular Pentagon

The following example illustrates how to find perimeter, area and apothem.

Example 1 In Figure 5.22, the regular pentagon ABCDE is inscribed in a circle with centre O and radius r. Write formulae for the side s, perimeter P, apothem a and area A of the regular pentagon.

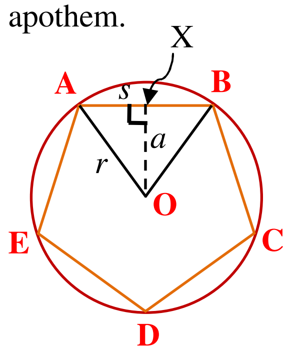


Figure 5.22

Solution: To solve the problem, join O to the vertices A and B as shown so that ΔOAB is formed.

ΔOAB is an isosceles triangle (why?). Draw the perpendicular from O to $\overline{AB}$. It meets $\overline{AB}$ at X. ∠AOB is a central angle of the regular pentagon.

So, $m(\angle AOB) = \dfrac{360^o}{5} = 72^o$.

Now, $\Delta AOX \cong \Delta BOX$ (*verify this*).

Therefore, $\angle AOX \cong \angle BOX$.

Therefore, $m(\angle AOX) = m(\angle BOX) = \dfrac{1}{2} m(\angle AOB) = \dfrac{1}{2}(72^o) = 36^o$.

Let s = AB, the length of the side of the regular pentagon.

Since $\Delta\, AOX \cong \Delta BOX$, we have $\overline{AX} \cong \overline{XB}$. So, $AX = \dfrac{1}{2} AB = \dfrac{1}{2} s$.

Now in the right angled triangle AOX you see that

$$\sin(\angle AOX) = \frac{AX}{AO}. \text{ i.e., } \sin\left(\frac{1}{2}(\angle AOB)\right) = \frac{\frac{1}{2}s}{r}$$

$$\sin 36^o = \frac{\frac{1}{2}s}{r}. \text{ So } \frac{1}{2}s = r\sin 36^o$$

Therefore, $s = 2r\sin 36^{\circ}$. . . . . (1)

Perimeter P of the polygon is

P = AB + BC + CD + DE + EA

But since AB = BC = CD = DE = EA = s, we have $P = s + s + s + s + s = 5s$.

Since from (1) we have $s = 2r\sin 36^{\circ}$, the perimeter of the regular pentagon is

$$P = 5 \times 2r\sin 36^{\circ}$$

$\therefore \quad P = 10r\sin 36^{\circ}$. . . . . (2)

To find a formula for the apothem, a, consider ΔAOX

$$\cos(\angle AOX) = \frac{XO}{AO}$$

Since m (∠AOX) = 36^o, XO = a, and AO = r.

$$\cos(36^o) = \frac{a}{r}$$

So, $a = r\cos 36^{\circ}$. . . . . (3)

To find the area of the regular pentagon, first we find the area of ΔAOB. Taking the height and the base of ΔAOB as OX and AB, respectively, we have,

$$\text{Area of } \Delta \text{ AOB} = \frac{1}{2}\text{AB} \times \text{OX} = \frac{1}{2} \times s \times a = \frac{1}{2}as$$

Now the area of the regular pentagon ABCDE = area of ΔAOB + area of ΔBOC + area of ΔCOD + area of Δ DOE + area of ΔEOA.

Since all these triangles are congruent, the area of each triangle is $\frac{1}{2}as$.

So, the area of the regular pentagon ABCDE $= 5 \times \left(\frac{1}{2}as\right) = \frac{1}{2}a(5s) = \frac{1}{2}aP$ . . . (4)

Since $36^o = \frac{180^o}{5}$, where 5 is the number of sides, we can generalize the above formulae for any n-sided regular polygon by replacing 36^o by $\frac{180^o}{n}$, as follows.

Theorem 5.3

Formulae for the length of side s, apothem a, perimeter P and area A of a regular polygon with n sides and radius r are

1 $s = 2r \sin \frac{180^o}{n}$

2 $a = r \cos \frac{180^o}{n}$

3 $P = 2nr \sin \frac{180^o}{n}$

4 $A = \frac{1}{2} aP$

Example 2

a Find the length of the side of an equilateral triangle if its radius is $\sqrt{12}$ cm.

b Find the area of a regular hexagon whose radius is 5 cm.

c Find the apothem of a square whose radius is $\sqrt{8}$ cm.

Solution:

a By the formula, the length of the side is $s = 2r \sin \frac{180^o}{n}$.

So, replacing r by $\sqrt{12}$ and n by 3, we have,

$$s = 2 \times \sqrt{12} \times \sin \frac{180^o}{3} = 2 \times \sqrt{12} \times \sin 60^o$$

$$= 2 \times \sqrt{12} \times \frac{\sqrt{3}}{2} = \sqrt{12 \times 3} = \sqrt{36} = 6; \left(\sin 60^o = \frac{\sqrt{3}}{2}\right).$$

Therefore, the length of the side of the equilateral triangle is 6 cm.

b To find the area of the regular hexagon, we use the formula

$A = \frac{1}{2} aP$, where a is the apothem and P is the perimeter of the regular hexagon.

Therefore,

$$A = \frac{1}{2} aP = \frac{1}{2}\left(r \cos \frac{180^o}{n}\right)\left(2nr \sin \frac{180^o}{n}\right) \quad (\textit{Substituting formulae for } a \textit{ and } P)$$

$$= \frac{1}{2} \times \left(5 \times \cos \frac{180^o}{6}\right) \times \left(2 \times 6 \times 5 \sin \frac{180^o}{6}\right)$$

$$= \frac{1}{2} \times 5 \times \frac{\sqrt{3}}{2} \times 2 \times 6 \times 5 \times \frac{1}{2}; \ (\cos 30^o = \frac{\sqrt{3}}{2}, \ \sin 30^o = \frac{1}{2})$$

$$= \frac{75\sqrt{3}}{2} \text{cm}^2$$

c To find the apothem of the square, we use the formula $a = r\cos\dfrac{180^o}{n}$..

Replacing r by $\sqrt{8}$ and n by 4, we have

$$a = \sqrt{8}\cos\frac{180^o}{4} = \sqrt{8}\cos 45^o \qquad (\cos 45^o = \frac{\sqrt{2}}{2})$$

$$= \sqrt{8}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{16}}{2} = 2 \text{ cm.}$$

Exercise 5.2

1 Which of the capital letters of the English alphabet are symmetrical?

2 Draw all the lines of symmetry on a diagram of a regular:

a hexagon **b** heptagon **c** octagon

How many lines of symmetry does each one have?

3 If a regular polygon of n sides has every line of symmetry passing through a vertex, what can you say about n?

4 State which of the following statements are true and which are false:

a A parallelogram which has a line of symmetry is a rectangle.

b A rhombus which has a line of symmetry must be a square.

c An isosceles triangle with more than one line of symmetry is an equilateral triangle.

d A pentagon that has more than one line of symmetry must be regular.

5 Show that the length of each side of a regular hexagon is equal to the length of the radius of the hexagon.

6 Show that the area A of a square inscribed in a circle with radius r is $A = 2r^2$.

7 Determine whether each of the following statements is true or false:

a The area of an equilateral triangle with apothem $\sqrt{3}$ cm and side 6 cm is $9\sqrt{3}$ cm^2.

b The area of a square with apothem $\sqrt{2}$ cm and side $2\sqrt{2}$ cm is $8\sqrt{2}$ cm^2.

8 Find the length of a side and the perimeter of a regular nine-sided polygon with radius 5 units.

9 Find the length of a side and the perimeter of a regular twelve-sided polygon with radius 3 cm.

10 Find the ratio of the perimeter of a regular hexagon to its radius and show that the ratio does not depend on the radius.

11 Find the radius of an equilateral triangle with perimeter 24 units.

12 Find the radius of a square with perimeter 32 units.

13 Find the radius of a regular hexagon with perimeter 48 units.

14 The radius of a circle is 12 units. Find the perimeter of a regular inscribed:

a triangle **b** heptagon **c** decagon

5.2 FURTHER ON CONGRUENCY AND SIMILARITY

Congruency

Today, modern industries produce large numbers of products; often many of these are the same size and/or shape. To determine these shapes and sizes, the idea of congruency is very important.

Two plane figures are congruent if they are exact copies of each other.

Group Work 5.1

1 Look carefully at the figures given below. Make a list of pairs that appear to be congruent.

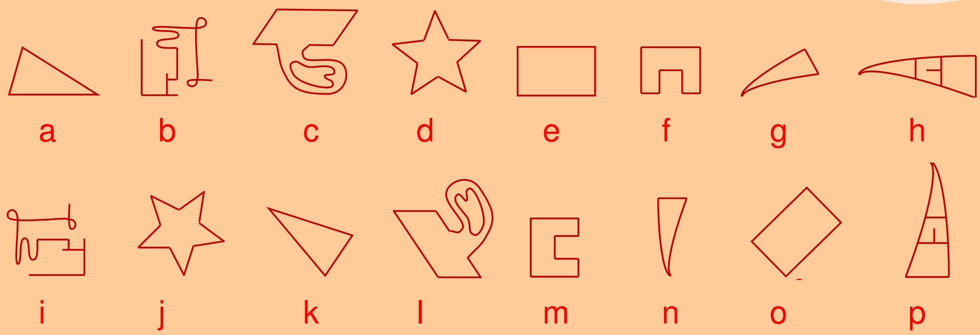


Figure 5.23

2 Test whether each pair is, in fact, congruent by tracing one on a thin transparent paper and placing the tracing on the other.

5.2.1 Congruency of Triangles

Triangles that have the same size and shape are called congruent triangles. That is, the six parts of the triangles (three sides and three angles) are correspondingly congruent. If two triangles, ΔABC and ΔDEF are congruent like those given below, then we denote this as

$$\Delta ABC \cong \Delta DEF.$$

The notation "≅" means "is congruent to".

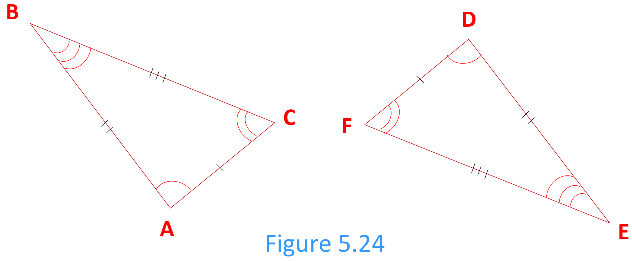


Figure 5.24

Congruent angles	Congruent sides
$\angle A \cong \angle D;\ \angle B \cong \angle E;\ \angle C \cong \angle F$	$\overline{AB} \cong \overline{DE};\ \overline{BC} \cong \overline{EF};\ \overline{AC} \cong \overline{DF}.$

Parts of congruent triangles that "match" are called corresponding parts. For example, in the triangles above, $\angle B$ corresponds to $\angle E$ and $\overline{AC}$ corresponds to $\overline{DF}$.

Two triangles are congruent when all of the corresponding parts are congruent. However, you do not need to know all of the six corresponding parts to conclude that the triangles are congruent. Each of the following Theorems states that three corresponding parts determine the congruence of two triangles.

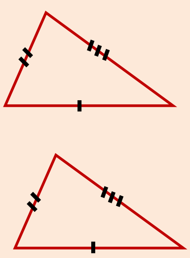
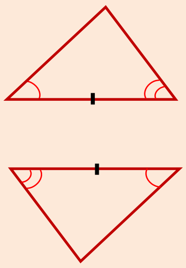
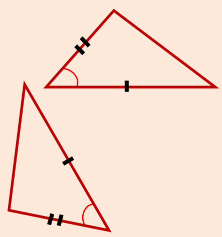
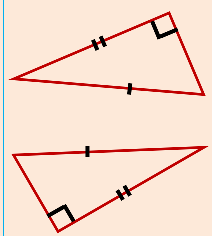
Congruent triangles	Two triangles are congruent if the following corresponding parts of the triangles are congruent.			
	three sides (SSS)	two angles and the included side (ASA)	two sides and the included angle (SAS)	a right angle, hypotenuse and a side (RHS)
				
	a	b	c	d

Figure 5.25

Example1 Determine whether each pair of triangles is congruent. If so, write a congruence statement and state why the triangles are congruent.

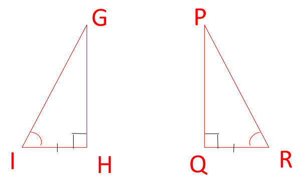


Figure 5.26

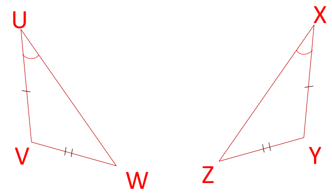


Figure 5.27

Solution:

For the first two triangles:
(Figure 5.26)

Since m(∠H) = 90^o and m(∠Q) = 90^o, ∠H ≅ ∠Q

Also $\overline{IH} \cong \overline{RQ}$ (given)

∠I ≅ ∠R (given)

∴ Δ GHI ≅ Δ PQR (by ASA)

For the last two triangles:
(Figure 5.27)

$\overline{UV} \cong \overline{XY}$ (given)

∠VUW ≅ ∠YXZ (given)

$\overline{VW} \cong \overline{YZ}$ (given)

So two sides and an angle are congruent. But the angle is not included between the sides. So we cannot conclude that the triangles are congruent.

Example 2 In Figure 5.28, PQRS is a square. A and B are points on $\overline{QR}$ and $\overline{SR}$, such that $\overline{QA} \cong \overline{SB}$.

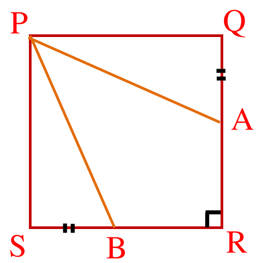


Figure 5.28

Prove that: ∠ PAQ ≅ ∠PBS

Solution: $\overline{PQ} \cong \overline{PS}$ (*sides of a square*)

$\overline{QA} \cong \overline{SB}$ (*given*)

∠Q ≅ ∠S (*right angles*)

Therefore, ΔPQA ≅ ΔPSB (by SAS).

Therefore, ∠PAQ ≅ ∠PBS (corresponding angles of congruent triangles).

Example 3 Given ΔABC ≅ ΔRST.

Find the value of y, if m(∠A) = 40^o and m(∠R) = $(2y + 10)^o$.

Solution: Since ΔABC≅ ΔRST, the corresponding angles are congruent.

So, ∠A ≅ ∠R.

Therefore, m (∠A) = m(∠R).

i.e., $40^o = (2y + 10)^o$. So, $y = 15^o$.

Exercise 5.3

1 Check whether the following four triangles are congruent:

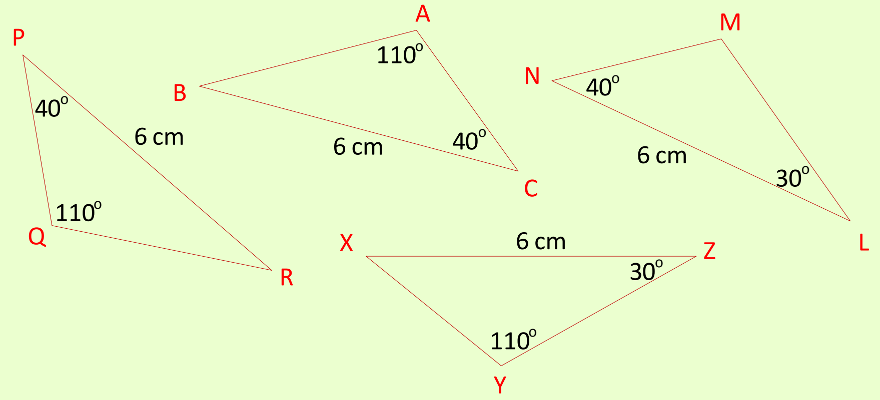


Figure 5.29

2 Which of the triangles are congruent to the blue-shaded triangle? Give reasons for your answer.

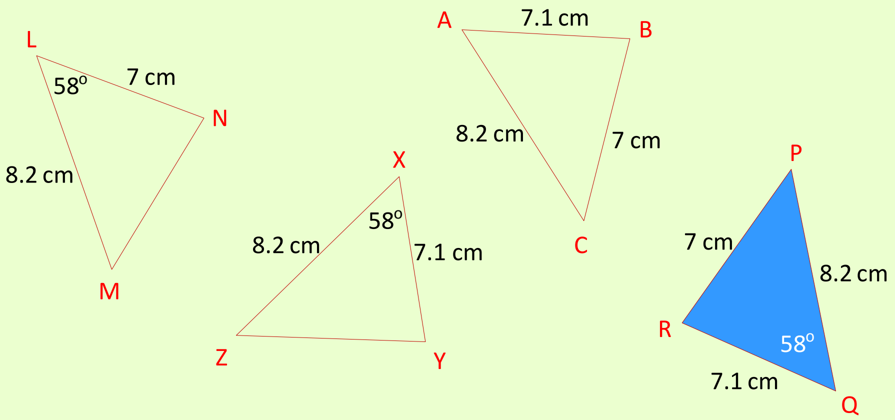


Figure 5.30

3 Which of the following pairs of triangles are congruent? For those that are congruent, state whether the reason is SSS, ASA, SAS or RHS.

a

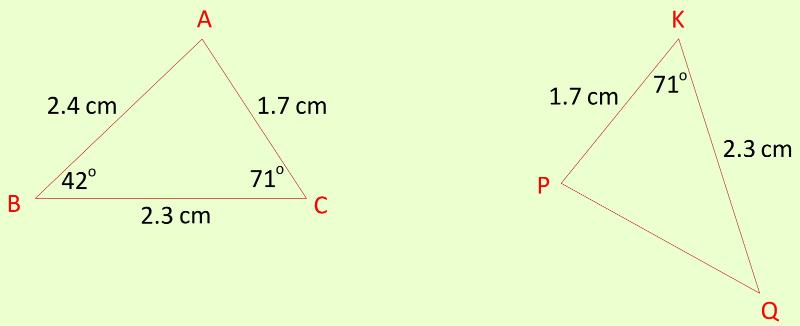


Figure 5.31

b

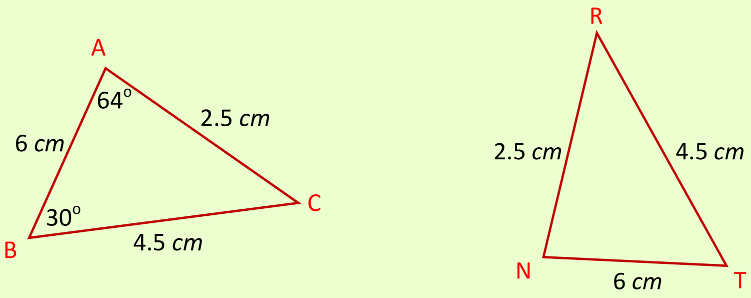


Figure 5.32

c

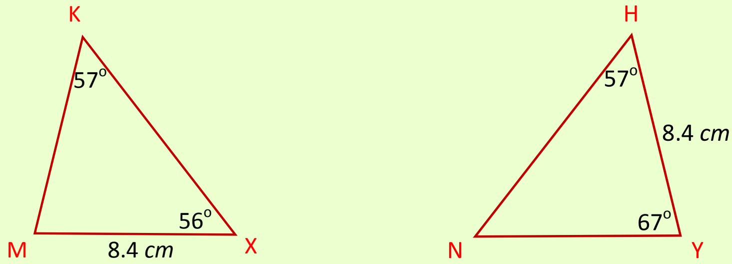


Figure 5.33

4 **a** ABC is an isosceles triangle with $\overline{AB} \cong \overline{AC}$, and M is the midpoint of $\overline{BC}$. Prove that $\angle ABC \cong \angle ACB$.

b In Figure 5.34 below, prove that ΔBDF is equilateral.

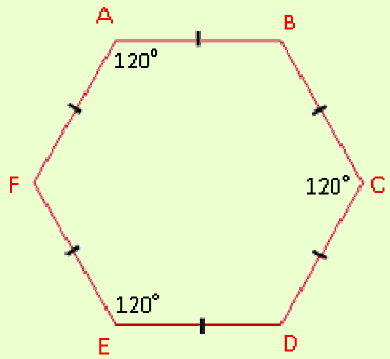


Figure 5.34

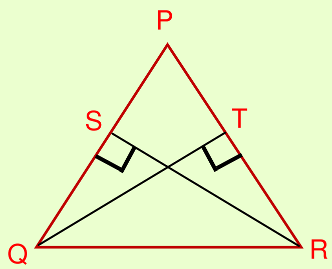


Figure 5.35

c In Figure 5.35, prove: If $\overline{RS} \cong \overline{QT}$ then, $\overline{PQ} \cong \overline{PR}$.

d ABC is an isosceles triangle with $\overline{AB} \cong \overline{AC}$. $\overline{AX}$ is the bisector of $\angle BAC$ meeting BC at X. Prove that X is the midpoint of BC.

e ABCD is a parallelogram. Prove that $\angle ABC \cong \angle ADC$.

Hint: First join AC and use alternate angles.

5.2.2 Definition of Similar Figures

After an architect finishes the plan of a building, it is usual to prepare a model of the building. In different areas of engineering, it is usual to produce models of industrial products before moving to the actual production. What relationships do you see between the model and the actual product?

Figures that have the same shape but that might have different sizes are called similar. Each of the following pairs of figures are similar, with one shape being an enlargement of the other.

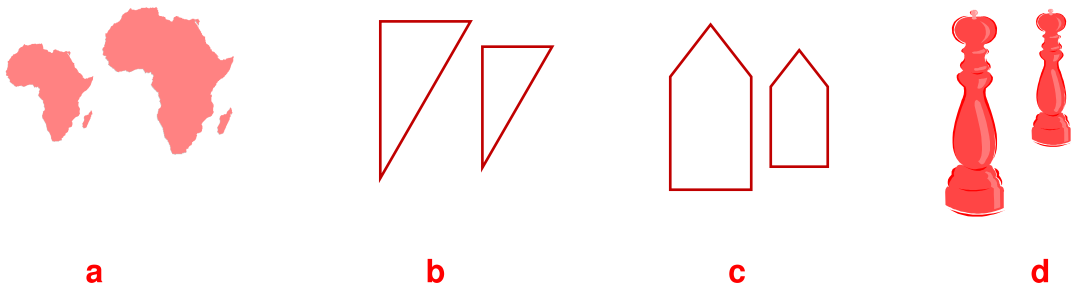


Figure 5.36

From your Grade 8 mathematics, recall that:

An enlargement is a transformation of a plane figure in which each of the points such as A, B, C is mapped onto A', B', C' by the same scale factor, k, from a fixed point O. The distances of A', B', C' from the point O are found by multiplying each of the distances of A, B, C from O by the scale factor k.

$OA' = k \times OA \qquad OB' = k \times OB \qquad OC' = k \times OC$

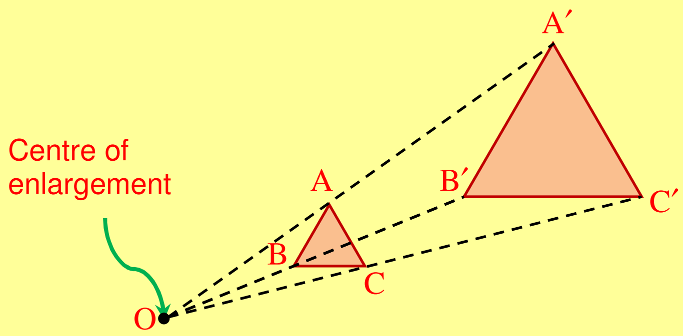


Figure 5.37

Group Work 5.2

1 Answer the following questions based on Figure 5.38.

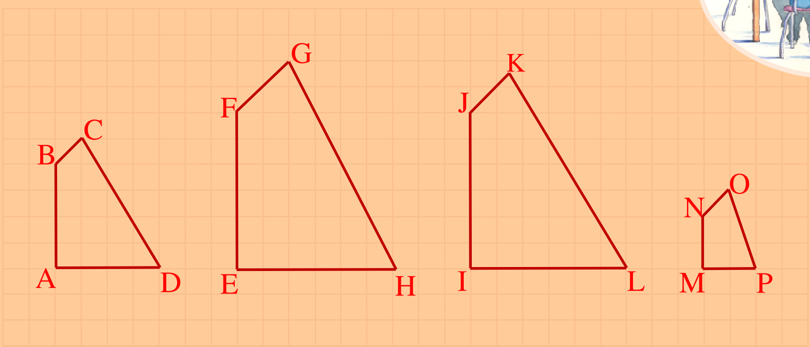


Figure 5.38

i If a figure is enlarged, do you always get a similar figure?

ii Which of the figures above are similar figures? Discuss.

iii What can you say about the angles B and J?

iv Which of the angles are equal to angle C? Which of the angles are equal to angle D?

v What other equal angles can you find? Discuss.

vi What can you say about the angles of two similar polygons? Discuss.

2 Figures ABCDE and FGHIJ are similar and

$$\frac{BC}{GH} = \frac{8}{4} = 2.$$

Find the ratio of other corresponding sides of ABCDE and FGHIJ.

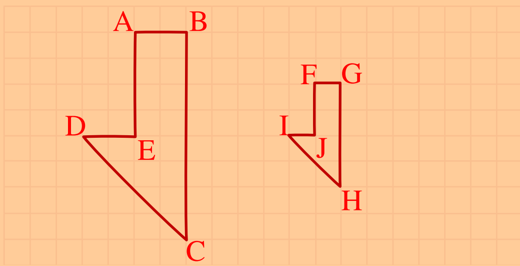


Figure 5.39

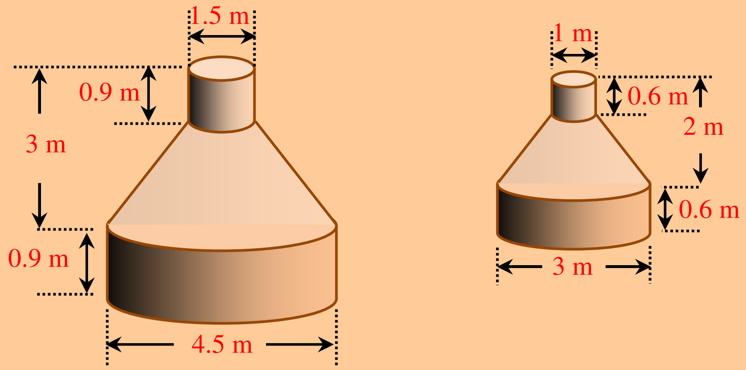


Figure 5.40

3 Two solid figures have the dimensions as shown in Figure 5.40 above. Are the figures similar? How can you make sure? Discuss.

4 **a** Is a rectangle of length 6 cm and width 4 cm similar to a rectangle of length 12 cm and width 18 cm?

b Is a triangle, two of whose angles are 85^o and 72^o similar to a triangle two of whose angles are 23^o and 85^o?

How could you have answered parts a and b of this question without drawing? Discuss.

From the above Group Work, we may conclude the following.

In similar figures:

i One is an enlargement of the other.

ii Angles in corresponding positions are congruent.

iii Corresponding sides have the same ratio.

In the case of a polygon, the above fact can be stated as:

Similar polygons	Two polygons of the same number of sides are similar, if their corresponding angles are congruent and their corresponding sides have the same ratio.

Example 1

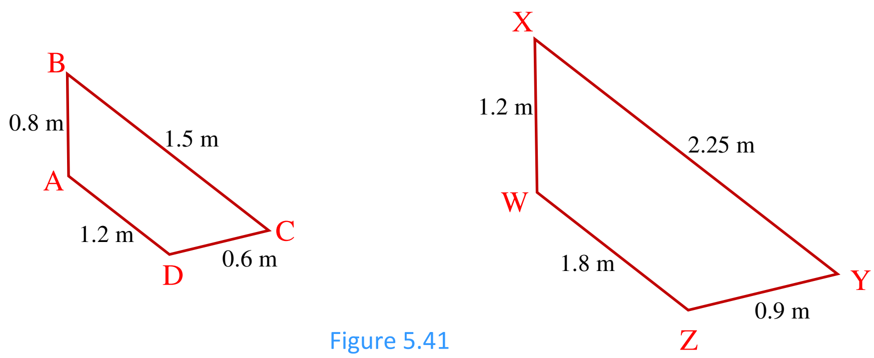


Figure 5.41

If quadrilateral ABCD is similar to quadrilateral WXYZ, we write ABCD ~ WXYZ. (the symbol ~ means "is similar to").

Corresponding angles of similar polygons are congruent. You can use a protractor to make sure the angles have the same measure.

$\angle A \cong \angle W$ $\qquad \angle B \cong \angle X$

$\angle C \cong \angle Y$ $\qquad \angle D \cong \angle Z$

Using corresponding angles as a guide, you can easily identify the corresponding sides.

$\overline{AB} \leftrightarrow \overline{WX}$, $\overline{BC} \leftrightarrow \overline{XY}$

$\overline{CD} \leftrightarrow \overline{YZ}$, $\overline{DA} \leftrightarrow \overline{ZW}$

(the symbol $\leftrightarrow$ means "corresponds to ").

A special relationship also exists between the corresponding sides of the polygons. Compare the ratios of lengths of the corresponding sides:

$$\frac{AB}{WX}=\frac{0.8}{1.2}=\frac{2}{3},\ \frac{BC}{XY}=\frac{1.5}{2.25}=\frac{2}{3},\ \frac{CD}{YZ}=\frac{0.6}{0.9}=\frac{2}{3},\ \frac{DA}{ZW}=\frac{1.2}{1.8}=\frac{2}{3}.$$

You can see the ratios of the lengths of the corresponding sides are all equal to $\frac{2}{3}$.

Example 2 Referring to Figure 5.42, if ABCDE ~ HIJKL, then find the lengths of:

a $\overline{IJ}$ **b** $\overline{CD}$ **c** $\overline{HL}$

Solution: Since ABCDE ~ HIJKL, we have,

$$\frac{AB}{HI}=\frac{BC}{IJ}=\frac{CD}{JK}=\frac{DE}{KL}=\frac{AE}{HL}$$

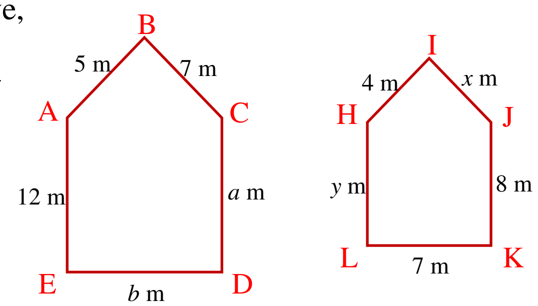


Figure 5.42

a To find the length of $\overline{IJ}$, we use

$$\frac{AB}{HI}=\frac{BC}{IJ}$$

$$\frac{5}{4}=\frac{7}{x}\quad (AB = 5,\ HI = 4,\ BC = 7,\ IJ = x)$$

So, $x=\frac{4\times 7}{5}=5.6$

Therefore, the length of side $\overline{IJ}$ is 5.6 m.

b In the same way,

$$\frac{AB}{HI}=\frac{CD}{JK}$$

$\frac{5}{4}=\frac{a}{8}$. So, $a=10$.

Therefore, CD = 10 m.

c

$\frac{AB}{HI}=\frac{AE}{HL}$. i.e., $\frac{5}{4}=\frac{12}{y}$

So, $y=9.6$.

Therefore, HL = 9.6 m.

Exercise 5.4

1 **a** All of the following polygons are regular. Identify the similar ones.

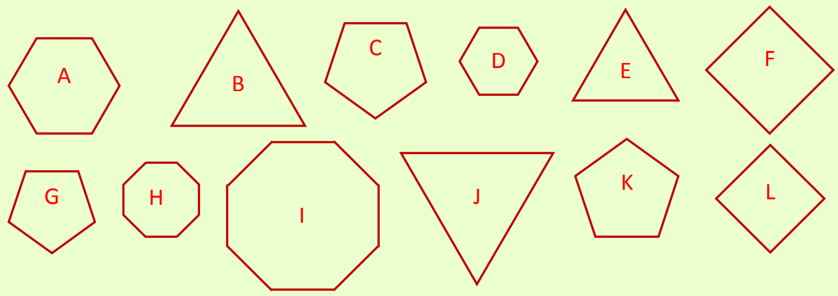


Figure 5.43

b Explain why regular polygons with the same number of sides are always similar.

2 Explain why all circles are similar.

3 Decide whether or not each pair of polygons is similar. Explain your reasoning.

a

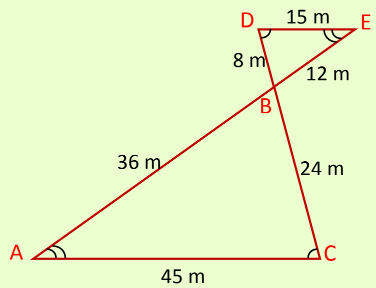


Figure 5.44

b

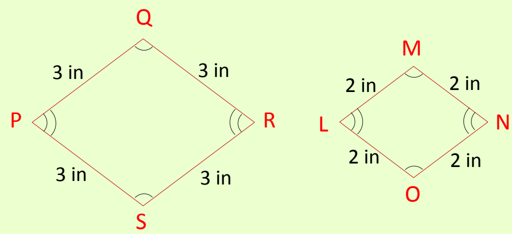


Figure 5.45

c

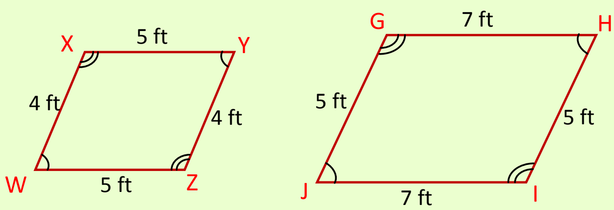


Figure 5.46

5.2.3 Theorems on Similarity of Triangles

You may start this section by recalling the following facts about similarity of triangles.

Definition 5.5

Two triangles are said to be similar, if

1. their corresponding sides are proportional (have equal ratio), and
2. their corresponding angles are congruent.

That is, $\Delta ABC \sim \Delta DEF$ if and only if

$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ and

$\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$

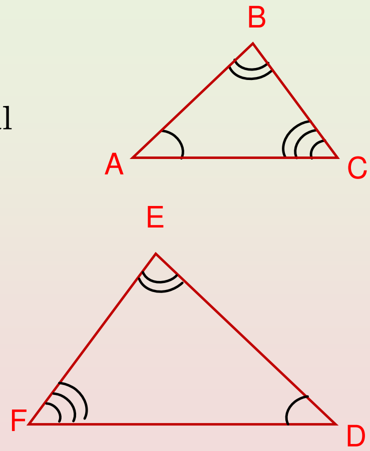


Figure 5.47

The following theorems on similarity of triangles will serve as tests to check whether or not two triangles are similar.

Theorem 5.4 SSS similarity theorem

If the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.

Restatement:

Given ΔABC and ΔDEF. If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$,

then $\Delta ABC \sim \Delta DEF$.

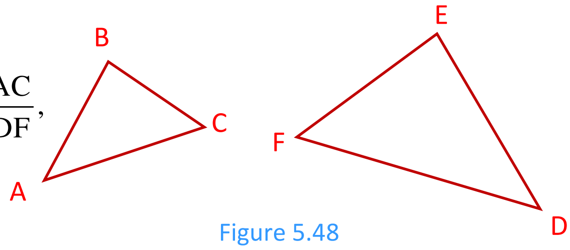


Figure 5.48

Example 1 Are the two triangles in Figure 5.49 similar?

Solution: Since $\frac{PQ}{ST} = \frac{QR}{TU} = \frac{PR}{SU} = \frac{1}{2}$,

$\Delta PQR \sim \Delta STU$ (by SSS similarity theorem).

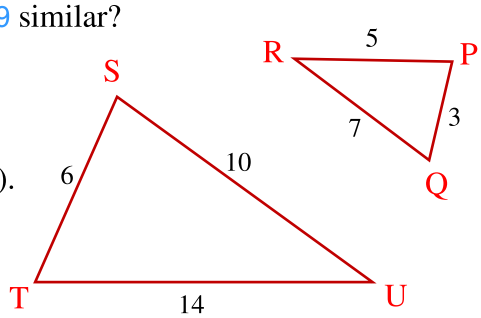


Figure 5.49

Group Work 5.3

1 Investigate if Theorem 5.5 works for two polygons whose number of sides is greater than 3.

2 Consider a square ABCD and a rhombus PQRS.

Are the ratios: $\frac{AB}{PQ}$, $\frac{BC}{QR}$ and $\frac{CD}{RS}$ equal?

We now state the second theorem on similarity of triangles, which is called the side-angle-side (SAS) similarity theorem.

Theorem 5.5 SAS similarity theorem

Two triangles are similar, if two pairs of corresponding sides of the two triangles are proportional and if the included angles between these sides are congruent.

Restatement:

Given two triangles ΔABC and ΔPQR, if

$\frac{AB}{PQ} = \frac{AC}{PR}$ and $\angle A \cong \angle P$, then,

$\Delta ABC \sim \Delta PQR$.

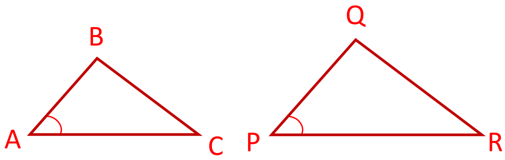


Figure 5.50

Example 2 Use the SAS similarity theorem to check whether the given two triangles are similar.

Solution: Since $\frac{LN}{PR} = \frac{12}{6} = 2$, and also

$\frac{MN}{QR} = \frac{16}{8} = 2$, the corresponding sides have equal ratios (i.e., they are proportional).

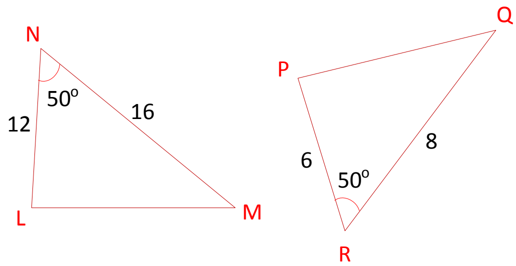


Figure 5.51

Also, since m(∠N) = m(∠R), it follows that the included angles of these proportional sides are congruent.

Therefore, $\Delta LMN \sim \Delta PQR$ by the SAS similarity theorem.

Finally, we state the third theorem on similarity of triangles, which is called the Angle-Angle (AA) similarity theorem.

Theorem 5.6 AA similarity theorem

If two angles of one triangle are congruent to two corresponding angles of another triangle, then the two triangles are similar.

Restatement:

Given two triangles, namely ΔABC and ΔDEF. If ∠A ≅ ∠D and ∠C ≅ ∠F, then ΔABC ~ ΔDEF.

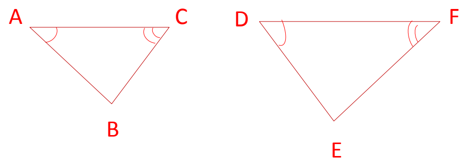


Figure 5.52

Example 3 In Figure 5.53, determine whether the two given triangles are similar.

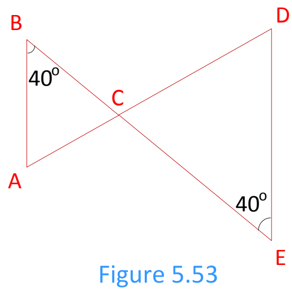


Figure 5.53

Solution: In ΔABC and ΔDEC, m(∠B) = m(∠E) = 40^o.

So, **i** ∠B ≅ ∠E.

ii ∠ACB ≅ ∠DCE (since they are vertically opposite angles).

Therefore, ΔABC ~ ΔDEC by the AA similarity theorem.

Exercise 5.5

1 State whether each of the following statements is true or false.

a If two triangles are similar, then they are congruent.

b If two triangles are congruent, then they are similar.

c All equilateral triangles are congruent.

d All equilateral triangles are similar.

2 Which of the following pairs of triangles are similar? If they are similar, explain why.

a

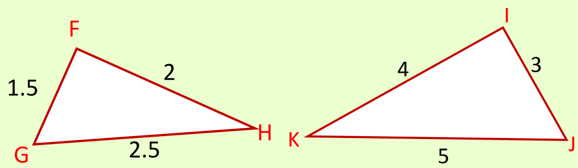


Figure 5.54

b

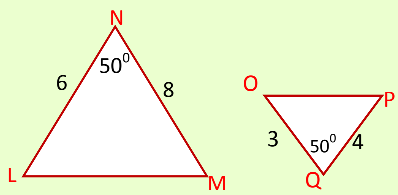


Figure 5.55

3 The pairs of triangles given below are similar. Find the measures of the blank sides.

a

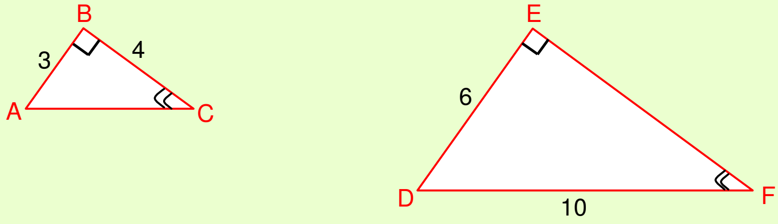


Figure 5.56

b

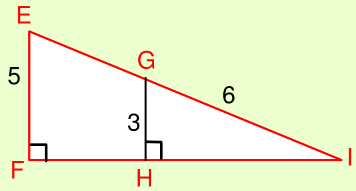


Figure 5.57

4 In Figure 5.58, prove that:

a Δ ADC ~ Δ BEC

b Δ AFE ~ Δ BFD

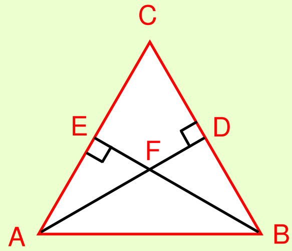


Figure 5.58

5 In Figure 5.59, quadrilateral DEFG is a square and, $\angle$C is a right angle. Prove that:

a $\dfrac{AD}{EF} = \dfrac{DG}{EB}$

b $\dfrac{AD}{CG} = \dfrac{DG}{CF}$

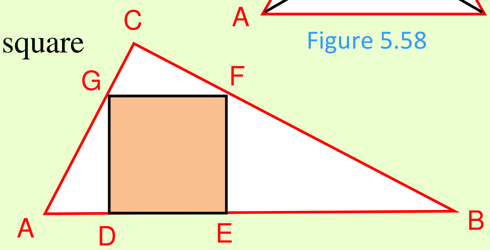


Figure 5.59

5.2.4 Theorems on Similar Plane Figures

Ratio of perimeters and ratio of areas of similar plane figures

ACTIVITY 5.7

Consider Figure 5.60 given below and answer the following questions:

a Show that the two rectangles are similar.

b What is the ratio of the corresponding sides?

c Find the perimeter and the area of each rectangle.

d What is the ratio of the two perimeters?

e What is the ratio of the two areas?

f What is the relationship between the ratio of the corresponding sides and the ratio of the perimeters?

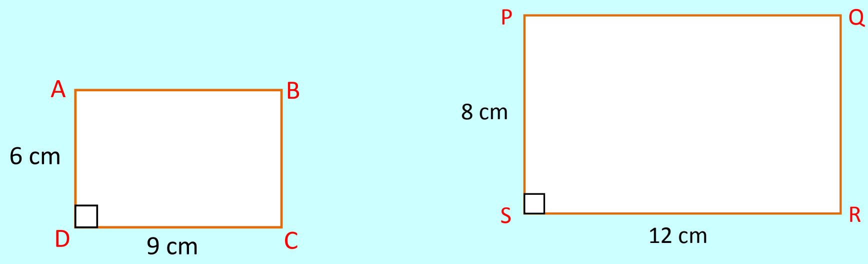


Figure 5.60

g What is the relationship between the ratio of the corresponding sides and ratio of the areas?

The results of the Activity lead you to the following theorem.

Theorem 5.7

If the ratio of the lengths of the corresponding sides of two similar triangles is k, then

i the ratio of their perimeters is k

ii the ratio of their areas is k^2.

Proof:-

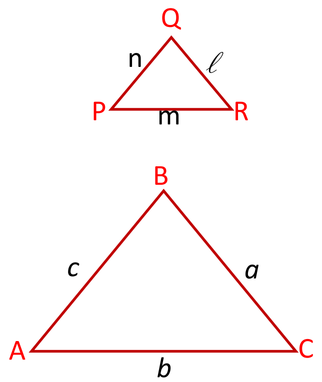


Figure 5.61

i Given $\Delta ABC \sim \Delta PQR$.

Then, $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{AC}{PR}$.

i.e., $\dfrac{c}{n} = \dfrac{a}{l} = \dfrac{b}{m}$.

Since the common value of these ratios is k, we have

$$\frac{c}{n} = \frac{a}{l} = \frac{b}{m} = k.$$

So, $c = kn$, $a = kl$, $b = km$.

Now the perimeter of $\Delta ABC = AB + BC + CA = c + a + b = kn + kl + km$.

From this, we obtain $c + a + b = kn + kl + km = k\,(n + l + m)$

Therefore, $\dfrac{c+a+b}{n+l+m} = k$.

That is, $\dfrac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta PQR} = k$.

This shows that the ratio of their perimeters = the ratio of their corresponding sides.

ii To prove that the ratio of their areas is the square of the ratio of any two corresponding sides:

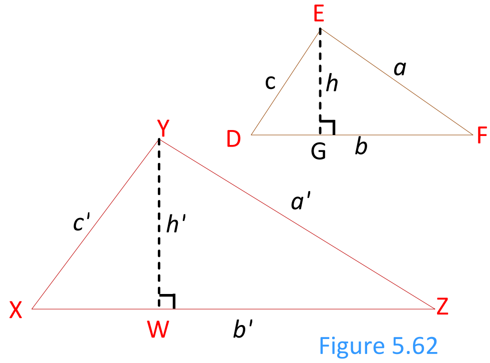


Figure 5.62

Let $\Delta DEF \sim \Delta XYZ$. Then,

$$\frac{DE}{XY} = \frac{EF}{YZ} = \frac{DF}{XZ} = k.$$

That is. $\frac{c}{c'} = \frac{a}{a'} = \frac{b}{b'} = k$.

Let $\overline{EG}$ be the altitude from E to $\overline{DF}$ and $\overline{YW}$ be the altitude from Y to XZ.

Since ΔDEG and ΔXYW are right triangles and $\angle D \cong \angle X$, we have

$\Delta DEG \sim \Delta XYW$ (AA similarity)

Therefore, $\frac{h}{h'} = \frac{c}{c'} = k$.

Now, area of $\Delta DEF = \frac{1}{2}bh$, and area of $\Delta XYZ = \frac{1}{2}b'h'$.

Therefore, $$\frac{\text{area of } \Delta DEF}{\text{area of } \Delta XYZ} = \frac{\frac{1}{2}bh}{\frac{1}{2}b'h'} = \frac{b}{b'} \times \frac{h}{h'} = k \times k = k^2.$$

So, the ratio of their areas is the square of the ratio of their corresponding sides.

Now we state the same fact for any two polygons.

Theorem 5.8

If the ratio of the lengths of any two corresponding sides of two similar polygons is k, then

i the ratio of their perimeters is k.

ii the ratio of their areas is k^2.

Exercise 5.6

1 Let ABCD and EFGH be two quadrilaterals such that ABCD ~ EFGH.

If AB = 15 cm, EF = 18 cm and the perimeter of ABCD is 40 cm, find the perimeter of EFGH.

2 Two triangles are similar. A side of one is 2 units long. The corresponding side of the other is 5 units long. What is the ratio of:

a their perimeters? **b** their areas?

3 Two triangles are similar. The sides of one are three times as long as the sides of the other. What is the ratio of the areas of the smaller to the larger?

4 The areas of two similar triangles are 144 unit^2 and 81 unit^2.

a What is the ratio of their perimeters?

b If a side of the first is 6 units long, what is the length of the corresponding side of the second?

5 The sides of a polygon have lengths 5, 7, 8, 11, and 19 units. The perimeter of a similar polygon is 75 units. Find the lengths of the sides of the larger polygon.

5.2.5 Construction of Similar Figures

Enlargement

Group Work 5.4

Work with a partner

1 Draw a triangle ABC on squared paper as shown below.

2 Take a point O and draw rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$; on these rays mark points A' ,B' and C' such that $OA' = 2\ OA$; $OB' = 2\ OB$; $OC' = 2\ OC$

3 What can you say about $\Delta\ ABC$ and $\Delta\ A'B'C'$?

4 Is $\dfrac{OA'}{OA} = \dfrac{A'B'}{AB}$?

5 What properties have not changed?

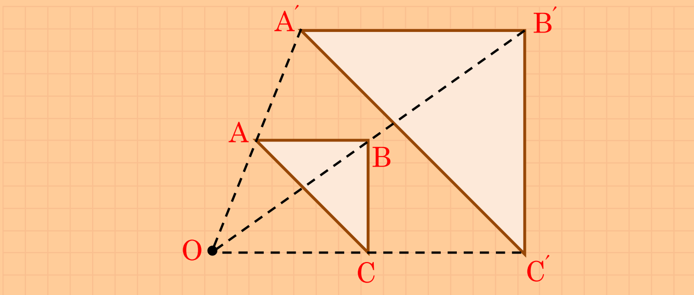


Figure 5.63

Figure 5.63 shows triangle ABC and its image triangle $A'B'C'$ under the transformation enlargement. In the equation $OA' = 2OA$, the factor 2 is called the scale factor and the point O is called the centre of enlargement.

In general,

An enlargement with centre O and scale factor k (where k is a real number) is the transformation that maps each point P to point P' such that

i P' is on the ray $\overrightarrow{OP}$ and **ii** $OP' = k\ OP$

If an object is enlarged, the result is an image that is mathematically similar to the object but of different size. The image can be either larger, if $k > 1$, or smaller if $0 < k < 1$.

Example 1 In Figure 5.64 below, ΔABC is enlarged to form $\Delta A'B'C'$. Find the centre of enlargement.

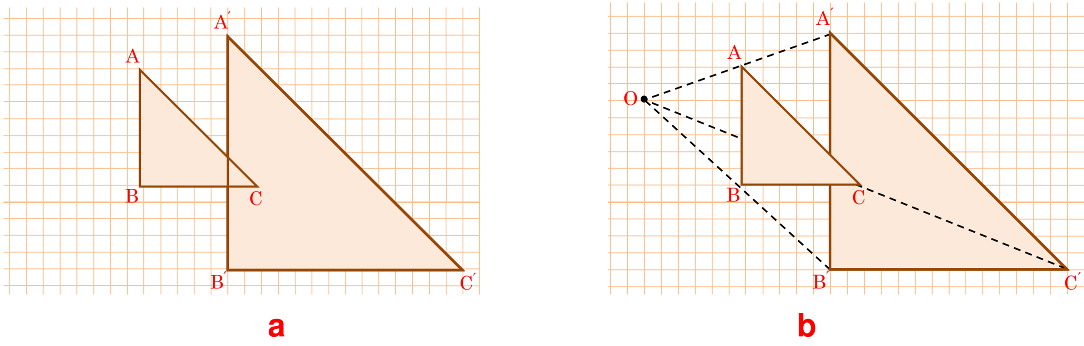


Figure 5.64

Solution: The centre of enlargement is found by joining corresponding points on the object and image with straight lines. These lines are then extended until they meet. The point at which they meet is the centre of enlargement O (*See* Figure 5.64b above).

Example 2 In Figure 5.65 below, the rectangle ABCD undergoes a transformation to form rectangle $A'B'C'D'$.

i Find the centre of enlargement.

ii Calculate the scale factor of enlargement.

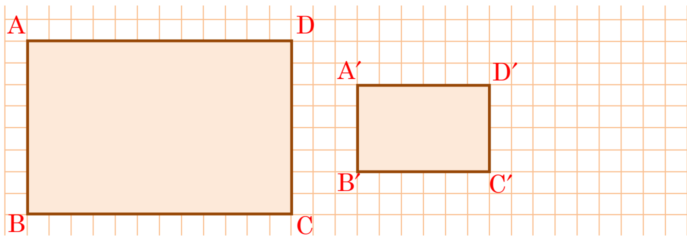


Figure 5.65

Solution:

i By joining corresponding points on both the object and the image, the centre of enlargement is found at O, as shown in Figure 5.66 below.

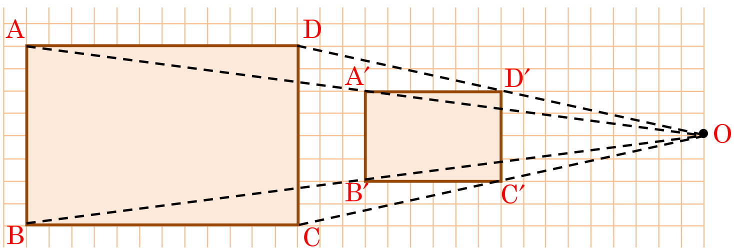


Figure 5.66

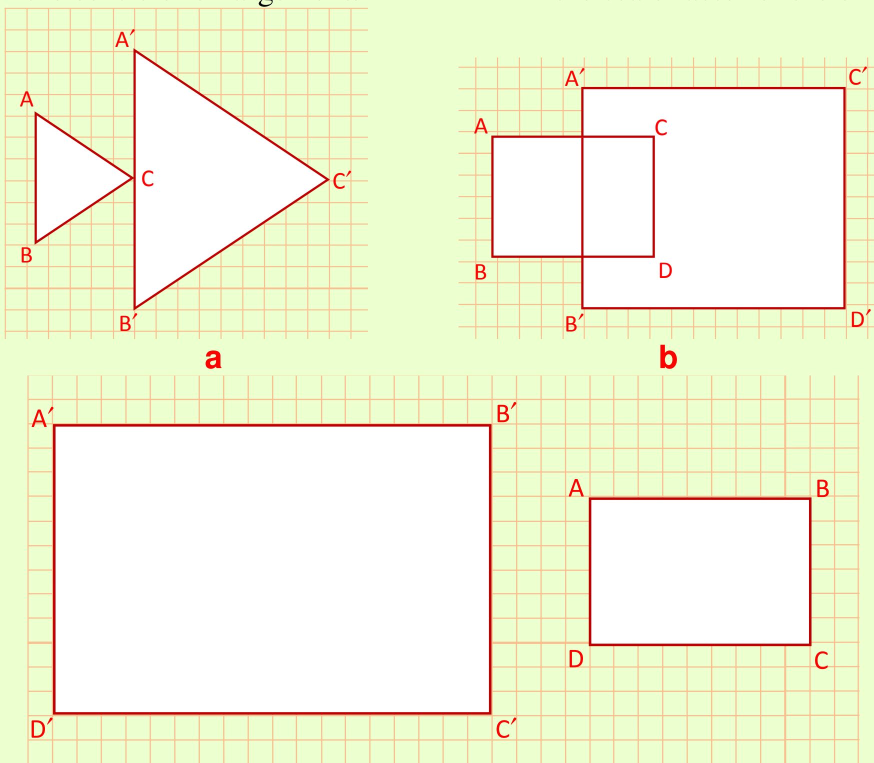
ii The scale factor of enlargement $= \dfrac{A'B'}{AB} = \dfrac{4}{8} = \dfrac{1}{2}$

If the scale factor of enlargement is greater than 1, then the image is larger than the object. If the scale factor lies between 0 and 1 then the resulting image is smaller than the object. In these latter cases, although the image is smaller than the object, the transformation is still known as an enlargement.

Exercise 5.7

1 Copy the following figures and find:

i the centre of enlargement. **ii** the scale factor of the enlargement.



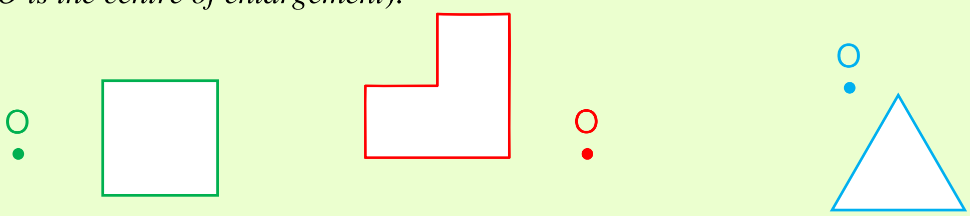
c

Figure 5.67

2 Copy and enlarge each of the following figures by a scale factor of:

i 3 **ii** $\frac{1}{2}$

(*O is the centre of enlargement*).



a **b** **c**

Figure 5.68

5.2.6 Real-Life Problems Using Congruency and Similarity

The properties of congruency and similarity of triangles can be applied to solve some real-life problems and also to prove certain geometric properties. For example, see the following examples.

Example 1 Show that the diagonals of a rectangle are congruent (Figure 5.69).

Solution: Suppose ABCD is a rectangle (Figure 5.69b).

Then, ABCD is a parallelogram (why?) so that the opposite sides of ABCD are congruent. In particular, $\overline{AB} \cong \overline{DC}$. Consider ΔABC and ΔDCB.

Clearly, $\angle ABC \cong \angle DCB$ (both are right angles),

Hence, $\Delta ABC \cong \Delta DCB$ by the SAS congruence property. Consequently, $\overline{AC} \cong \overline{DB}$ as desired.

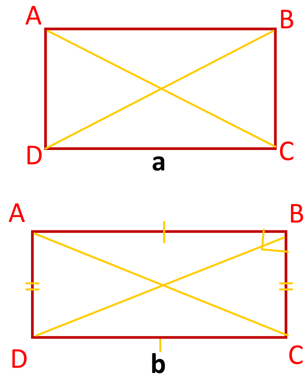


Figure 5.69

Carpenters use the result of Example 1 when framing rectangular shapes. That is, to determine whether a quadrilateral is a rectangle, a carpenter can measure opposite sides to see if they are congruent (if so, the shape is a parallelogram). Then the carpenter can measure the diagonals to see if they are congruent (if so, the shape is a rectangle).

Example 2 When Ali planted a tree 5 m away from point A, the tree just blocked the view of a building 50 m away. If the building was 20 m tall, how tall was the tree?

Solution: Label the figure as shown. Let x be the height of the tree.

We have $\dfrac{BE}{CD} = \dfrac{AE}{AD}$

$$\frac{x}{20} = \frac{5}{50}$$

$$x = 20 \times \frac{5}{50} = 2 \text{ m}$$

$\therefore$ The height of the tree was 2 m.

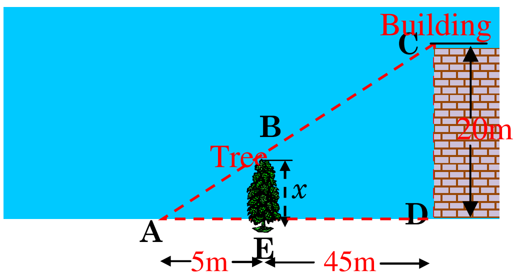


Figure 5.70

Exercise 5.8

1. Aweke took 1 hour to cut the grass in a square field of side 30 m. How long will it take him to cut the grass in a square field of side 120 m?
2. A line from the top of a cliff to the ground just passes over the top of a pole 20 m high. The line meets the ground at a point 15 m from the base of the pole. If it is 120 m away from this point to the base of the cliff, how high is the cliff?
3. A tree casts a shadow of 30 m. At the same time, a 10 m pole casts a shadow of 12 m. Find the height of the tree.

5.3 FURTHER ON TRIGONOMETRY

5.3.1 Radian Measure of an Angle

An angle is the union of two rays with a common end point.

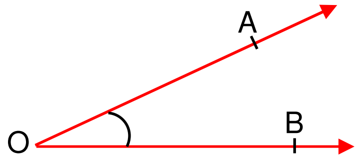


Figure 5.71

In general, we associate each angle with a real number called the measure of the angle. The two measures that are most frequently used are degree and radian.

i Measuring angles in degrees

We know that a right angle contains 90^o, and that a complete rotation can be thought of as an angle of 360^o. In view of this latter fact, we can define a degree as follows.

Definition 5.6

A degree, denoted by (^{o}), is defined as the measure of the central angle subtended by an arc of a circle equal in length to $\frac{1}{360}$ of the circumference of the circle.

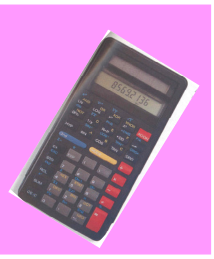
- ✓ A minute which is denoted by ('), is $\frac{1}{60}$ of a degree.
- ✓ A second which is denoted by ("), is $\frac{1}{60}$ of a minute.

So, we have the following relationship.

$$1' = \left(\frac{1}{60}\right)^o,\ 1'' = \left(\frac{1}{60}\right)'\ \text{ that is } 1'' = \left(\frac{1}{3600}\right)^o \quad \text{or} \quad 1^o = 60' \text{ and } 1' = 60''$$

Calculator Tip

Use your calculator to convert 20° 41'16'', which is read as 20 degrees, 41 minutes and 16 seconds, into degrees, (as a decimal).



ii Measuring angles in radians

Another unit used to measure angles is the radian. To understand what is meant by a radian, we again start with a circle. We measure a length equal to the radius r of the circle along the circumference of the circle, so that arc $\overset{\frown}{AB}$ is equal to the radius r. $\angle AOB$ is then an angle of 1 radian. We define this as follows.

Definition 5.7

A radian (rad) is defined as the measure of the central angle subtended by an arc of a circle equal in length to the radius of the circle.

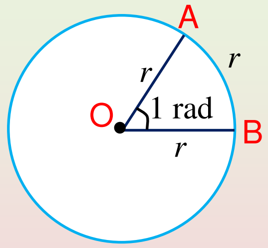


Figure 5.72

You know that the circumference of a circle is equal to $2\pi r$. Since an arc of length r along the circle gives 1 rad, a complete rotation of length $2\pi r$ generates an angle of 2π radians. On the other hand, we know that a complete revolution represents an angle of 360^o. This gives us the following relationship:

$$1 \text{ revolution} = 360^o = 2\pi \text{ radians}$$

i.e., $180^o = \pi$ radians, from which we obtain,

$1 \text{ radian} = \left(\frac{180}{\pi}\right)^o \approx 57.3^o.$ $\qquad 1^o = \frac{\pi}{180}$ radian ≈ 0.0175 radian.

Therefore, we have the following conversion rules for degrees and radians.

To convert radians to degrees, multiply by $\frac{180^o}{\pi}$.

To convert degrees to radians multiply by $\frac{\pi}{180^o}$.

Example 1

i Convert each of the following to radians:

a 30^o **b** 90^o

ii Convert each of the following to degrees:

a $\frac{\pi}{4}$ rad **b** $\frac{\pi}{3}$ rad

Solution:

i **a** $30^o = 30^o \times \frac{\pi}{180^o} = \frac{\pi}{6}$ rad. **b** $90^o = 90^o \times \frac{\pi}{180^o} = \frac{\pi}{2}$ rad.

ii **a** $\frac{\pi}{4}$ rad $= \frac{\pi}{4} \times \frac{180^o}{\pi} = 45^o$. **b** $\frac{\pi}{3}$ rad $= \frac{\pi}{3} \times \frac{180^o}{\pi} = 60^o$.

5.3.2 Trigonometrical Ratios to Solve Right-angled Triangles

ACTIVITY 5.8

1 What is the meaning of a trigonometric ratio?

2 Given right-angled triangles, ΔABC and $\Delta A'B'C'$,

if $m(\angle A) = m(\angle A')$, what can you say about the two triangles?

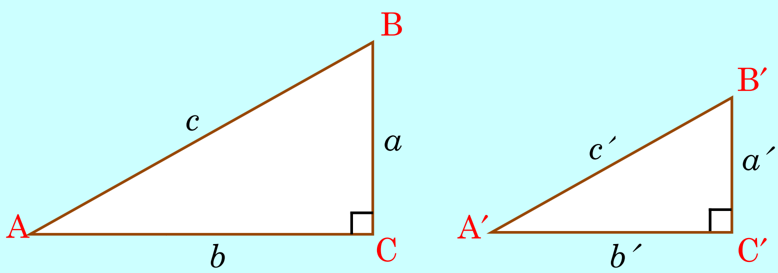


Figure 5.73

The answers to these questions should have lead you to recall the relationships that exist between an angle and the sides of a right-angled triangle, which enable you to solve problems that involve right-angled triangles.

Consider the two triangles in Figure 5.73 above.

Given $m(\angle A) = m(\angle A')$

i $\angle A \cong \angle A'$

ii $\angle C \cong \angle C'$

Therefore, $\Delta ABC \sim \Delta A'B'C'$ (by AA similarity)

This means $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$

From this we get,

1 $\dfrac{BC}{AB} = \dfrac{B'C'}{A'B'}$ 2 $\dfrac{AC}{AB} = \dfrac{A'C'}{A'B'}$ 3 $\dfrac{BC}{AC} = \dfrac{B'C'}{A'C'}$

OR $\dfrac{a}{c} = \dfrac{a'}{c'}$, $\dfrac{b}{c} = \dfrac{b'}{c'}$ and $\dfrac{a}{b} = \dfrac{a'}{b'}$

The fractions or ratios in each of these proportions are called trigonometric ratios.

Sine:- The fractions in proportion 1 above are formed by dividing the opposite side of $\angle A$ (or $\angle A'$) by the hypotenuse of each triangle. This ratio is called the sine of $\angle A$. It is abbreviated to sin A.

Cosine:- The fractions in proportion 2 are formed by dividing the adjacent side to ∠A (or ∠A') by the hypotenuse of each triangle. This ratio is called the cosine of ∠A. It is abbreviated to cos A.

Tangent:- The fractions in proportion 3 are formed by dividing the opposite side of ∠A (or ∠A') by the adjacent side. This ratio is called the tangent of ∠ A. It is abbreviated to tan A.

The following abbreviations are commonly used.

adj = adjacent side hyp = hypotenuse opp = opposite side.

The above discussion can be summarized and expressed as follows.

$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB}$; $\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB}$

$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC}$

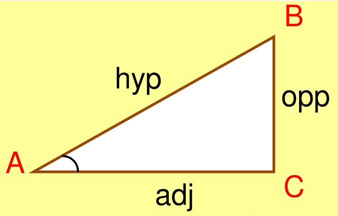


Figure 5.74

Example 1 In the following right triangle, find the values of sine, cosine and tangent of the acute angles.

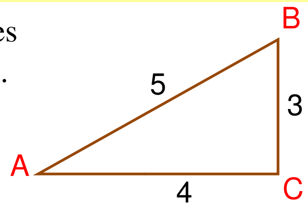


Figure 5.75

Solution:

$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB} = \frac{3}{5}$; $\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB} = \frac{4}{5}$; $\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{3}{4}$

Similarly, $\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{AC}{AB} = \frac{4}{5}$; $\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{BC}{AB} = \frac{3}{5}$

$\tan B = \frac{\text{opp}}{\text{adj}} = \frac{AC}{BC} = \frac{4}{3}$

ACTIVITY 5.9

1 Using ruler and compasses, draw an equilateral triangle ABC in which each side is 4 cm long. Draw the altitude $\overline{AD}$ perpendicular to $\overline{BC}$.

a What is m(∠ABD)? m(∠BAD) ? Give reasons.

b Find the lengths BD and AD (write the answers in simplified radical form).

c Use these to find sin 30°, tan 30°, cos 30°, sin 60°, tan 60°, cos 60°. What do you notice?

2 Draw an isosceles triangle ABC in which $\angle C$ is a right angle and AC = 2 cm.

a What is m($\angle$A)?

b Calculate the lengths AB and BC (leave your answer in radical form).

c Calculate sin 45^o, cos 45^o, tan 45^o.

From the above Activity, you have probably discovered that the values of sine, cosine and tangent of the angles 30^o, 45^o and 60^o are as summarized in the following table.

$\angle$A	30^o	45^o	60^o
sin A	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos A	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan A	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

The angles 30^o, 45^o and 60^o are called special angles, because they have these exact trigonometric ratios.

Example 2 A ladder 6 m long leans against a wall and makes an angle of 60^o with the ground. Find the height of the wall. How far from the wall is the foot of the ladder?

Solution: Consider ΔABC in the figure.

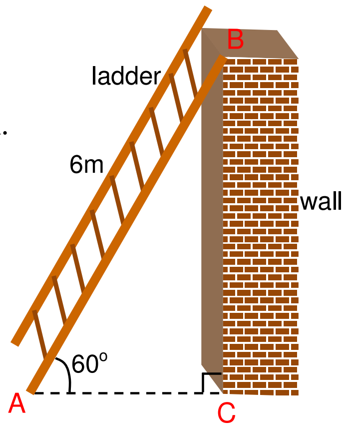


Figure 5.76

m($\angle$A) = 60^o, m($\angle$C) = 90^o, m($\angle$B) = 30^o and AB = 6 m.

We want to find BC and AC.

To find BC, we use $\sin 60^o = \frac{BC}{AB}$. But, $\sin 60^o = \frac{\sqrt{3}}{2}$

So, $\frac{\sqrt{3}}{2} = \frac{BC}{6}$

Therefore, BC = $3\sqrt{3}$ m, which is the height of the wall.

To find the distance between the foot of the ladder and the wall, we use

$\cos 60^o = \frac{AC}{AB}$. $\cos 60^o = \frac{1}{2}$ and AB = 6.

So, $\frac{1}{2} = \frac{AC}{6}$ which implies AC = 3 m.

In the above example, if the angle that the ladder made with the ground were 50^o, how would you solve the problem?

To solve this problem, you would need trigonometric tables, which give you the values of sin 50^o and cos 50^o.

5.3.3 Trigonometrical Values of Angles from Tables

($\sin\theta$, $\cos\theta$ and $\tan\theta$, for $0^o \leq \theta < 180^o$)

In the previous section, we created a table of trigonometric ratios for the special angles (namely 30^o, 45^o and 60^o). Theoretically, by following the same method, a table of trigonometric ratios can be constructed for any angle. There are tables of approximate values of trigonometric ratios of acute angles that have already been constructed by advanced arithmetical processes. One such table is included at the end of this book.

ACTIVITY 5.10

Using the trigonometric table, find the value of each of the following:

a $\cos 50^o$ **b** $\sin 20^o$ **c** $\tan 10^o$ **d** $\sin 80^o$

If you know the value of one of the trigonometric ratios of an angle, you can use a table of trigonometric ratios to find the angle. The procedure is illustrated in the following example.

Example 1 Find the measure of the acute angle A, correct to the nearest degree, if $\sin A^o = 0.521$.

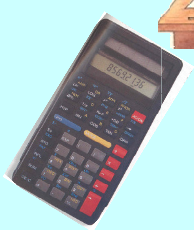
Solution: Referring to the "sine" column of the table, we find that 0.521 does not appear there. The two values in the table closest to 0.521 (one smaller and one larger) are 0.515 and 0.530. These values correspond to 31^o and 32^o, respectively.

Note that 0.521 is closer to 0.515, whose value corresponds to 31^o.

Therefore, $m(\angle A) = 31^o$ (*to the nearest degree*)

ACTIVITY 5.11

1 Use your trigonometric table to find the value of the acute angle A, correct to the nearest degree,.



a $\sin(A) = 0.92$ **d** $\sin(A) = 0.981$

b $\cos(A) = 0.984$ **e** $\cos(A) = 0.422$

c $\tan(A) = 0.3802$ **f** $\tan(A) = 2.410$

2 Use your calculator to find the values.(check your calculator is in degrees mode)

Using trigonometric ratios, you can now solve right-angled triangles and related problems. To solve a right-angled triangle means to find the missing parts of the triangle when some parts are given. For example, if you are given the length of one side and the measure of an angle (other than the right angle), you can use the appropriate trigonometric ratios to find the required parts.

In short, in solving a right-angled triangle, we need to use

a the trigonometric ratios of acute angles.

b Pythagoras theorem which is $a^2 + b^2 = c^2$, where a is the length of the side opposite to $\angle A$, b is the length of the side opposite to $\angle B$ and c is the length of the hypotenuse.

Example 2 Find the lengths of the sides indicated by the small letters.

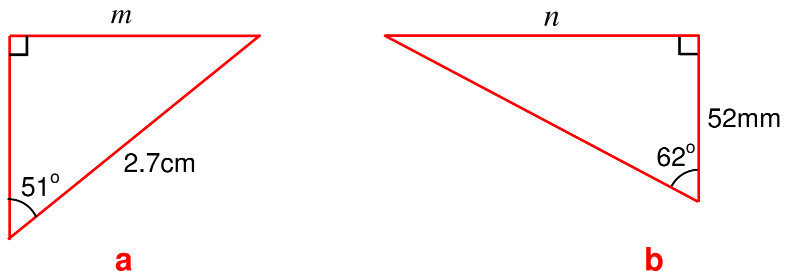


Figure 5.77

Solution:

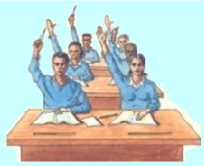
a $\sin 51^o = \dfrac{m}{2.7}$.

So, $m = 2.7 \sin 51^o = 2.7 \times 0.777 \approx 2.1$ cm (*1 decimal place*)

b $\tan 62^o = \dfrac{n}{52}$.

So, $n = 52 \tan 62^o = 52 \times 1.881 \approx 98$ mm (*to the nearest mm*)

ACTIVITY 5.12



Using Figure 5.78, write each of the following in terms of the lengths a, b, c.

1 **a** $\sin(\angle A)$ **b** $\cos(\angle A)$

c $\tan(\angle A)$ **d** $\dfrac{\sin(\angle A)}{\cos(\angle A)}$

e $\sin(\angle B)$ **f** $\cos(\angle B)$

g $\tan(\angle B)$ **h** $\dfrac{\sin(\angle B)}{\cos(\angle B)}$

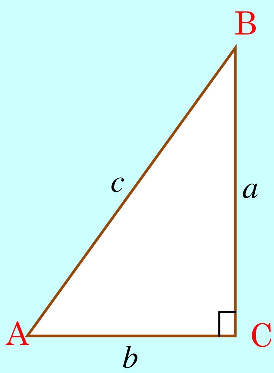


Figure 5.78

2 **a** $(\sin(\angle A))^2$

b $(\cos(\angle A))^2$

c Write the value of $\sin^2(\angle A) + \cos^2(\angle A)$.

Notation: We abbreviate $(\sin(\angle A))^2$ as $\sin^2(\angle A)$. Similarly, we write $\cos^2(\angle A)$ and $\tan^2(\angle A)$ instead of $(\cos(\angle A))^2$ and $(\tan(\angle A))^2$, respectively.

Do you notice any interesting results from the above Activity? State them.

You might have discovered that

1 If $m(\angle A) + m(\angle B) = 90^o$, i.e., A and B are complementary angles, then

i $\sin(\angle A) = \cos(\angle B)$ **ii** $\cos(\angle A) = \sin(\angle B)$

2 $\tan(\angle A) = \dfrac{\sin(\angle A)}{\cos(\angle A)}$

3 $\sin^2(\angle A) + \cos^2(\angle A) = 1$

How can you use the trigonometric table to find the sine, cosine and tangent of obtuse angles such as 95^o, 129^o, and 175^o?

Such angles are not listed in the table.

Before we consider how to find the trigonometric ratio of obtuse angles, we first redefine the trigonometric ratios by using directed distance. To do this, we consider the right angle triangle POA as drawn in Figure 5.79. Angle POA is the anticlockwise angle from the positive x-axis.

Note that the lengths of the sides can be expressed in terms of the coordinates of point P.

i.e., $OA = x$, $AP = y$, and using Pythagoras theorem, we have,

$$OP = \sqrt{x^2 + y^2}$$

As a result, the trigonometric ratios of $\angle POA$ can be expressed in terms of x, y and $\sqrt{x^2 + y^2}$, as follows:

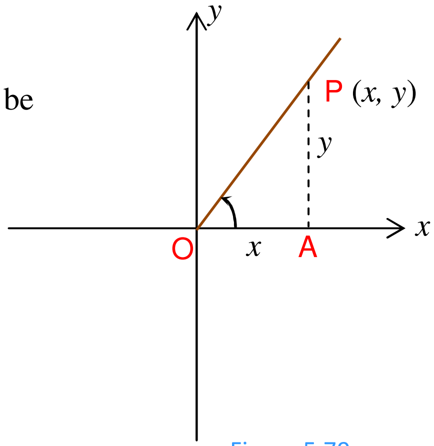


Figure 5.79

$$\sin(\angle POA) = \frac{\text{opp}}{\text{hyp}} = \frac{AP}{OP} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos(\angle POA) = \frac{\text{adj}}{\text{hyp}} = \frac{OA}{OP} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\tan(\angle POA) = \frac{\text{opp}}{\text{adj}} = \frac{AP}{OA} = \frac{y}{x}$$

i.e., $\sin(\angle POA) = \dfrac{y}{\sqrt{x^2 + y^2}}$; $\cos(\angle POA) = \dfrac{x}{\sqrt{x^2 + y^2}}$; $\tan(\angle POA) = \dfrac{y}{x}$

From the above discussion, it is possible to compute the values of trigonometric ratios using any point on the terminal side of the angle.

Let us now find the sine and cosine of 129^o using the table. To do this, we first put 129^o on the xy-plane, so that its vertex is at the origin and its initial side on the positive x-axis.

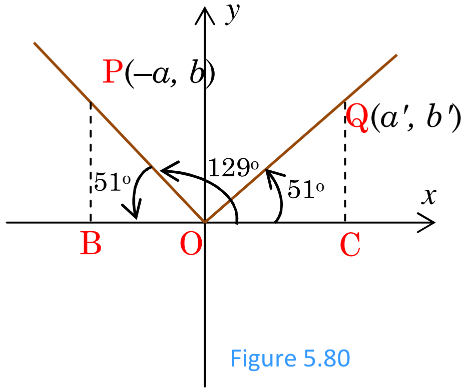


Figure 5.80

i To find $\sin 129^o$, we first express $\sin 129^o$ in terms of the coordinates of the point P($-a$, b). So, we have,

$$\sin 129^o = \frac{b}{\sqrt{a^2+b^2}}.$$

What acute angle put in the xy-plane has the same y value (that is b)?

If we draw the 51^o $\angle COQ$ so that OP = OQ, then we see that

$\Delta BOP \cong \Delta COQ$. So we have

BP = CQ and OB = OC

It follows that $\sin 129^o = \sin 51^o$. From the table $\sin 51^o = 0.777$.

Hence, $\sin 129^o = 0.777$

Notice that $\sin 129^o = \sin (180^o - 129^o)$

This can be generalized as follows.

If θ is an obtuse angle, i.e.,$90^o < \theta < 180^o$, then

$$\sin \theta = \sin (180 - \theta)$$

ii To find $\cos 129^o$.

Here also we first express $\cos 129^o$ in terms of the coordinates of P ($-a$, b). So,

$$\cos 129^o = \frac{-a}{\sqrt{a^2+b^2}}.$$

By taking $180^o - 129^o$, we find the acute angle 51^o.

Since $\Delta BOP \cong \Delta COQ$, we see that OC = OB, but in the opposite direction. So, the x value of P is the opposite of the x value of Q. That is $a = -a'$

Therefore, $\cos 129^o = \dfrac{a'}{\sqrt{a^2+b^2}} = -\cos 51^o$

From the trigonometric table, you have $\cos 51^o = 0.629$

Therefore, $\cos 129^o = -0.629$.

This discussion leads you to the following generalization.

If θ is an obtuse angle, then

$$\cos \theta = -\cos (180^o - \theta) \qquad \text{II}$$

Example 3 With the help of the trigonometric table, find the approximate values of:

a $\cos 100^o$ **b** $\sin 163^o$ **c** $\tan 160^o$

Solution: a Using the rule $\cos \theta = -\cos (180^o - \theta)$, we obtain;

$\cos 100^o = -\cos (180^o - 100^o) = -\cos 80^o$

From the trigonometric table, we have $\cos 80^o = 0.174$.

Therefore, $\cos 100^o = -0.174$.

b From the relation $\sin \theta = \sin (180 - \theta)$, we have

$\sin 163^o = \sin (180^o - 163^o) = \sin 17^o$

From the table $\sin 17^o = 0.292$

Therefore, $\sin 163^o = 0.292$

c To find $\tan 160^o$,

$$\tan 160^o = \frac{\sin 160^o}{\cos 160^o} = \frac{\sin 20^o}{-\cos 20^o} = -\left(\frac{\sin 20^o}{\cos 20^o}\right) = -\tan 20^o$$

From the table, we have $\tan 20^o = 0.364$.

Therefore, $\tan 160^o = -0.364$.

To summarize, for a positive obtuse angle θ,

$\sin \theta = \sin (180^o - \theta)$

$\cos \theta = -\cos (180^o - \theta)$

$\tan \theta = -\tan (180^o - \theta)$

Exercise 5.9

1 **i** Express each of the following radian measures in degrees:

a $\frac{\pi}{6}$ **b** π **c** $\frac{\pi}{3}$, **d** 2 **e** $\frac{3}{4}\pi$ **f** 5

ii Express each of the following in radian measure:

a 270^o **b** 150^o **c** 225^o **d** 15^o

2 Without using a table, find the value of each of the following. (Answers may be left in radical form.)

a $\sin \frac{\pi}{6}$ **b** $\tan \frac{3}{4}\pi$ **c** $\cos 150^o$ **d** $\tan \frac{2}{3}\pi$

3 In ΔABC, if m $(\angle A) = 53^o$, AC = 8.3 cm and m $(\angle C) = 90^o$, find BC, correct to the nearest whole number.

4 A ladder 20 ft long leans against a building, making an angle of 65^o with the ground. Determine, correct to the nearest ft, how far up the building the ladder reaches.

5 Express each of the following in terms of the sine or cosine of an acute angle:

a $\cos 165^o$ **b** $\sin 126^o$ **c** $\cos \frac{3}{5}\pi$ **d** $\sin 139^o$

6 In each of the following, find the length of the hypotenuse (a):

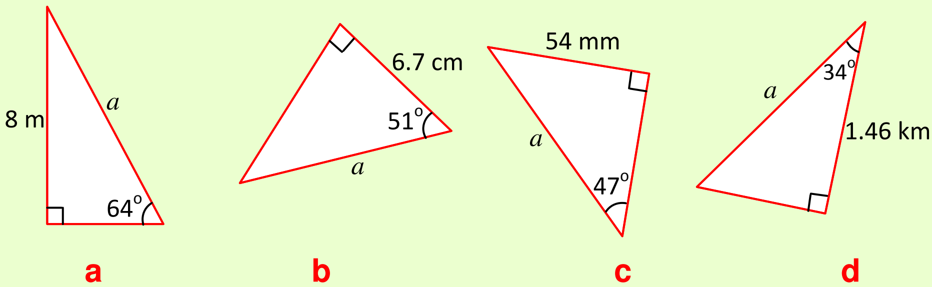


Figure 5.81

7 Find the sine, cosine and tangent of each of the following angles from the table.

a 25^o **b** 63^o **c** 89^o

d 135^o **e** 142^o **f** 173^o

8 Use the trigonometric table included at the end of the book to find the degree measure of $\angle P$ if:

a $\sin P = 0.83$ **b** $\cos P = 0.462$ **c** $\tan P = 0.945$

d $\sin P = \frac{1}{4}$ **e** $\cos P = 0.824$

5.4 CIRCLES

5.4.1 Symmetrical Properties of Circles

ACTIVITY 5.13

1. What is a circle?
2. What is a line of symmetry?
3. Which of the following figures have a line of symmetry?

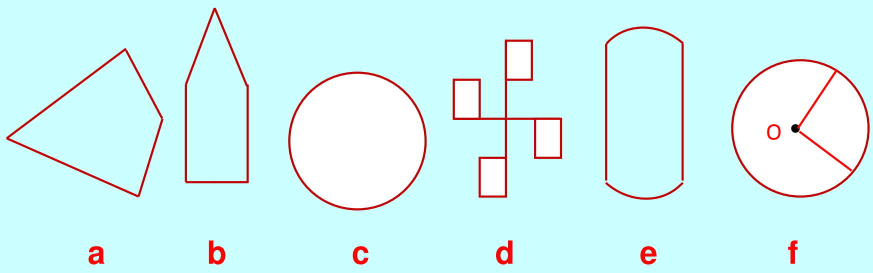


Figure 5.82

Recall that a circle is defined as the set of points in a given plane, each of which is at the same distance from a fixed point of the plane. The fixed point is called the centre, and the distance is the radius of the circle.

A line segment through the centre of a circle with end points on the circle is called a diameter. A chord of a circle is a line segment whose end points lie on the circle.

In Section 5.1.2, you learned that if one part of a figure can be made to coincide with the rest of the figure by folding it about a straight line, $\overleftrightarrow{AB}$, the figure is said to be symmetrical about $\overleftrightarrow{AB}$, and the straight line $\overleftrightarrow{AB}$ is called the line of symmetry. For example, each of the following figures is symmetrical about the line $\overleftrightarrow{AB}$.

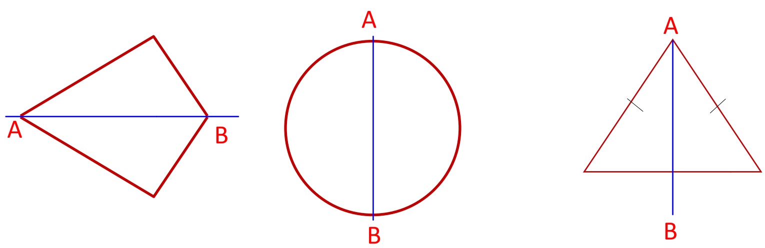


Figure 5.83

Observe that in a symmetrical figure the length of any line segment or the size of any angle in one half of the figure is equal to the length of the corresponding line segment or the size of the corresponding angle in the other half of the figure.

If in the figure on the right, P coincides with Q when the figure is folded about $\overleftrightarrow{AB}$ and if $\overline{PQ}$ intersects $\overleftrightarrow{AB}$ at N then, ∠PNA coincides with ∠QNA and therefore each is a right angle and PN = QN.

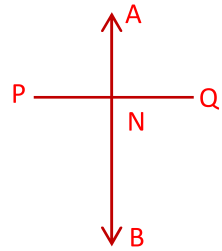


Figure 5.84

Therefore,

If P and Q are corresponding points for a line of symmetry $\overleftrightarrow{AB}$, the perpendicular bisector of $\overline{PQ}$ is $\overleftrightarrow{AB}$. Conversely, if $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{PQ}$, then P and Q are corresponding points for the line of symmetry $\overleftrightarrow{AB}$ and we say that Q is the image of P in $\overleftrightarrow{AB}$ and P is the image of Q in $\overleftrightarrow{AB}$.

In the adjacent figure, O is the centre and $\overline{AB}$ is a diameter of the circle. Note that a circle is symmetrical about its diameter. Therefore, a circle has an infinite number of lines of symmetry.

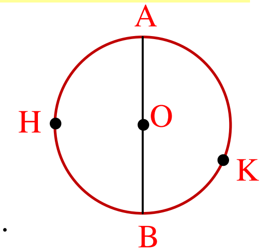


Figure 5.85

We now discuss some properties of a circle, stating them as theorems.

Theorem 5.9

The line segment joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

Proof:-

Given: A circle with centre O and a chord $\overline{PQ}$ whose midpoint is M.

We want to prove that ∠OMP is a right angle.

Construction: Draw the diameter ST through M. Then the circle is symmetrical about the line ST. But PM = QM.

So, ST is the perpendicular bisector of PQ.

This completes the proof.

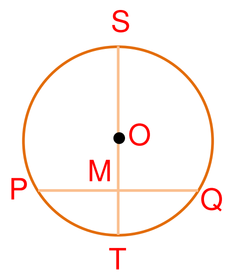


Figure 5.86

Theorem 5.10

The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord.

Proof:-

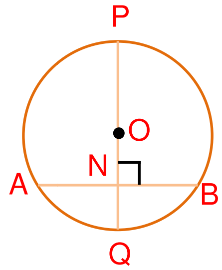


Figure 5.87

Given: A circle with centre O, and the line segment ON drawn from O perpendicular to the chord AB as shown in the adjacent figure.

We want to prove that AN = NB

Construction: Draw the diameter PQ through N.

Then the circle is symmetrical about PQ. But PQ ⊥ AB and A and B are on the circle.

Therefore, PQ is the perpendicular bisector of AB.

ACTIVITY 5.14

1. Prove Theorem 5.10 and 5.11 using congruency of triangles.
2. A chord of length 10 cm is at a distance of 12 cm from the centre of a circle. Find the radius of the circle.
3. A chord of a circle of radius 6 cm is 8 cm long. Find the distance of the chord from the centre.
4. $\overline{AB}$ and $\overline{CD}$ are equal chords in a circle of radius 5 cm. If each chord is 3 cm, find their distance from the centre of the circle.
5. Define what you mean by '*a line tangent to a circle*'.
6. How many tangents are there from an external point to a circle? Discuss how to compare their lengths.

Some other properties of a circle can also be proved by using the fact that a circle is symmetrical about any diameter.

Theorem 5.11

i. If two chords of a circle are equal, then they are equidistant from the centre.

ii. If two chords of a circle are equidistant from the centre, then their lengths are equal.

Theorem 5.12

If two tangent segments are drawn to a circle from an external point, then,

i the tangents are equal in length, and

ii the line segment joining the centre to the external point bisects the angle between the tangents.

Restatement: If TP is a tangent to a circle at P whose centre is O and TQ is another tangent to this circle at Q, then,

i $TP = TQ$ ii $m(\angle OTP) = m(\angle OTQ)$

Proof:-

i ΔOTP and ΔOTQ are right angled triangles with right angles at P and Q

(A radius is perpendicular to a tangent at the point of tangency).

ii Obviously $OT = OT$

and $OP = OQ$ (*why?*)

iii $\therefore\ \Delta OTP \cong \Delta OTQ$ (*why?*)

So, $TP = TQ$ and $m(\angle OTP) = (\angle OTQ)$, as required.

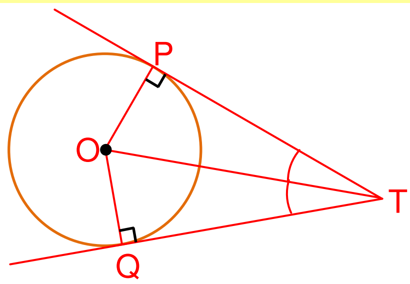


Figure 5.88

5.4.2 Angle Properties of Circles

We start this subsection by a review and discussion of some important terms. Referring to the diagrams in Figure 5.89 will help you to understand some of these terminologies. (In each circle, O is the centre.)

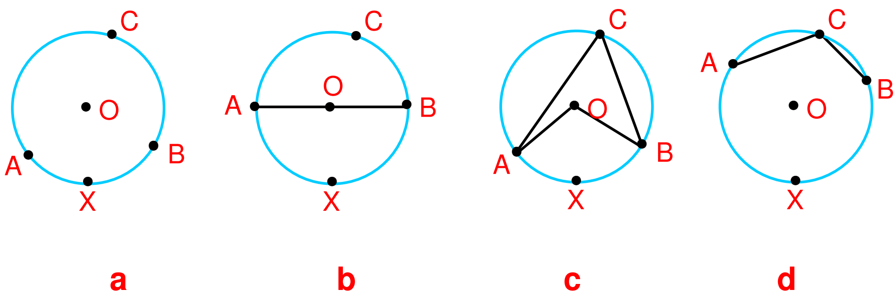


Figure 5.89

- ✓ A part of a circle (part of its circumference) between any two points on the circle, say between A and B, is called an arc and is denoted by $\overset{\frown}{AB}$. However, this notation can be ambiguous since there are two arcs of the circle with A and B as end points. Therefore, we either use the terms minor arc and major arc or we pick another point, say X, on the desired arc and then use the notation $\overset{\frown}{AXB}$. For example, in Figure 5.89a, $\overset{\frown}{AXB}$ is the part of the circle with A and B as its end points and containing the point X. The remaining part of the circle, i.e., the part whose end points are A and B but containing C is the arc $\overset{\frown}{ACB}$.

- ✓ If AB is a diameter of a circle (see Figure 5.89b), then the arc $\overset{\frown}{ACB}$ $\left(\text{or } \overset{\frown}{AXB}\right)$ is called a semicircle. Notice that a semicircle is half of the circumference of the circle. An arc is said to be a minor arc, if it is less than a semicircle and a major arc, if it is greater than a semicircle. For example, in Figure 5.89c, $\overset{\frown}{AXB}$ is minor arc whilst $\overset{\frown}{ACB}$ is a major arc.

A central angle of a circle is an angle whose vertex is at the centre of the circle and whose sides are radii of the circle. For example, in Figure 5.89c, the angle ∠AOB is a central angle. In this case, we say that ∠AOB is subtended by the arc $\overset{\frown}{AXB}$ (or by the chord AB). Here, we may also say that the angle ∠AOB intercepts the arc $\overset{\frown}{AXB}$.

Recall that the measure of a central angle equals the angle measure of the arc it intercepts. Thus, in Figure 5.89c,

$$m(\angle AOB) = m\left(\overset{\frown}{AXB}\right).$$

An inscribed angle in a circle is an angle whose vertex is on the circle and whose sides are chords of the circle. For example, in Figure 5.89c, ∠ACB is an inscribed angle. Here also, the inscribed angle ∠ACB is said to be subtended by the arc $\overset{\frown}{AXB}$ (or by the chord $\overline{AB}$).

- ✓ Observe that the vertex of an inscribed angle ∠ACB is on the arc $\overset{\frown}{ACB}$. This arc, $\overset{\frown}{ACB}$, can be a semicircle, a major arc or a minor arc. In such cases, we may say that the angle ∠ACB is inscribed in a semicircle, major arc or minor arc, respectively. For example, in Figure 5.89b, ∠ACB is inscribed in a semicircle, in Figure 5.89c ∠ACB is inscribed in the major arc, and in Figure 5.89d ∠ACB is inscribed in the minor arc.

Theorem 5.13

The measure of a central angle subtended by an arc is twice the measure of an inscribed angle in the circle subtended by the same arc.

Both drawings in Figure 5.90 illustrate this theorem.

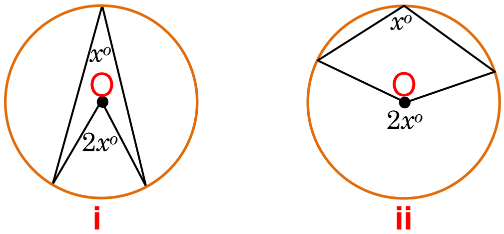


Figure 5.90

Exercise 5.10

In each of the following figures, O is the centre of the circle. Calculate the measure of the angles marked x.

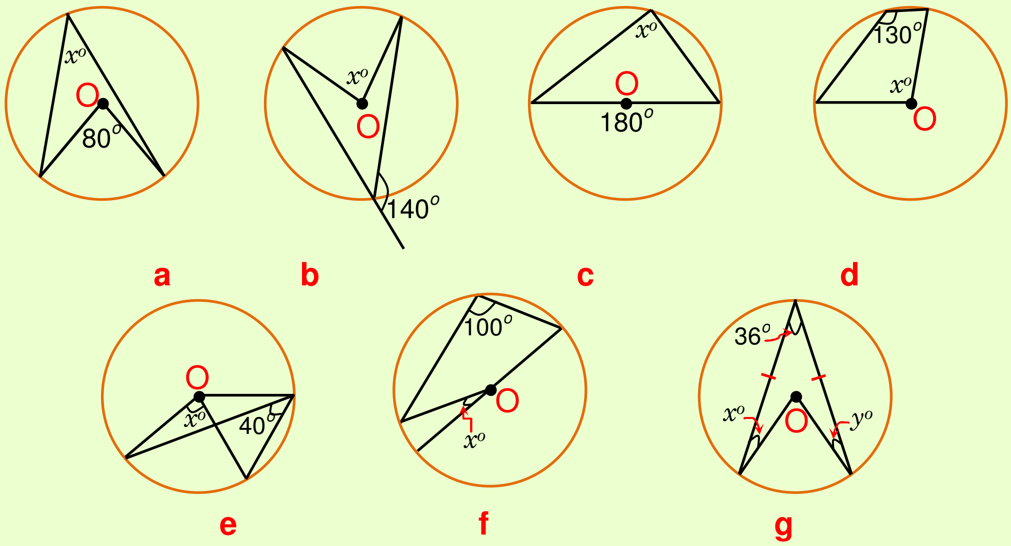


Figure 5.91

Corollary 5.13.1

Angles inscribed in the same arc of a circle (*i.e., subtended by the same arc*) are equal.

Proof:-

By the above theorem, each of the angles on the circle subtended by the arc is equal to half of the central angle subtended by the arc. Hence, they are equal to each other.

Corollary 5.13.2 Angle in a semicircle

The angle inscribed in a semi-circle is a right angle.

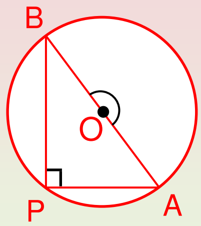


Figure 5.92

Proof:-

The given angle, $\angle APB$, is subtended by a semicircle. The corresponding central angle subtended by $\overline{AB}$ is straight angle. i.e., the central angle is 180^o. Hence, by Theorem 5.14 $m(\angle APB) = \frac{1}{2}m(\angle AOB) = \frac{1}{2}\times 180^o = 90^o$.

This completes the proof.

Exercise 5.11

1 Calculate the marked angles in each of the following figures:

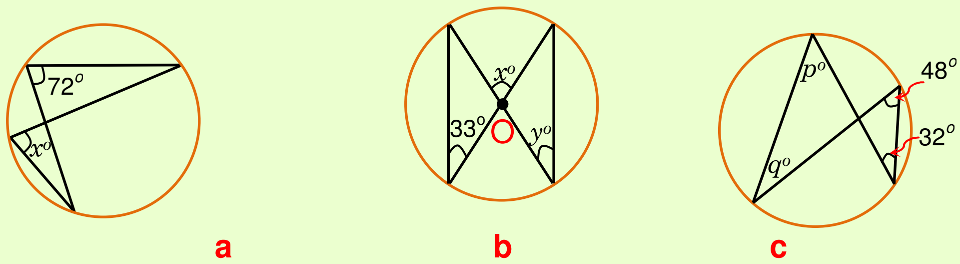


Figure 5.93

2 In each of the following figures, O is the centre and $\overline{AB}$ is the diameter of the circle. Calculate the value of x in each case.

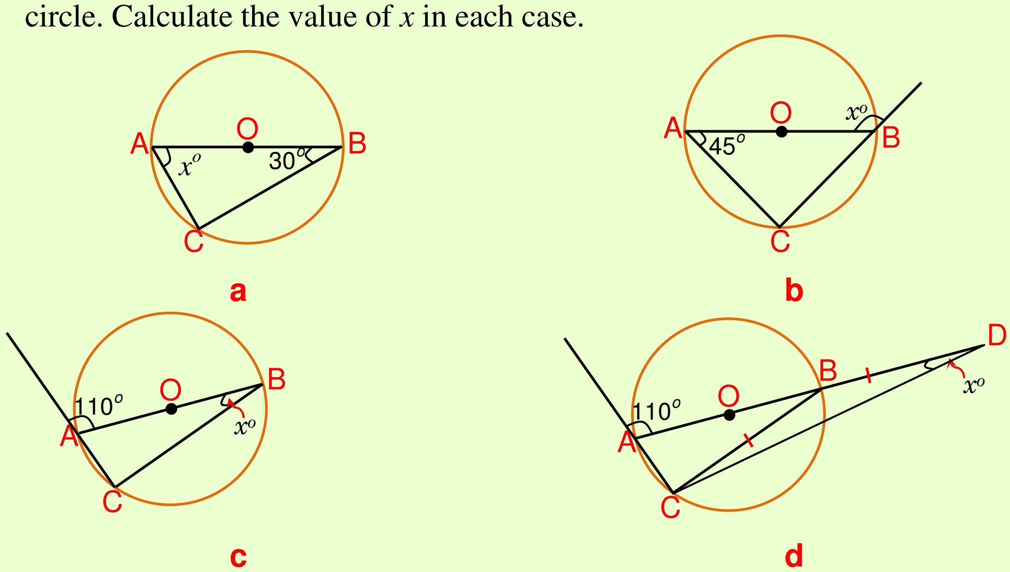


Figure 5.94

Corollary 5.13.3

Points P, Q, R and S all lie on a circle. They are called concyclic points.

Joining the points P, Q, R and S produces a cyclic quadrilateral.

The opposite angles of a cyclic quadrilateral are supplementary. i.e.,

$$m(\angle P) + m(\angle R) = 180° \text{ and } m(\angle S) + m(\angle Q) = 180°.$$

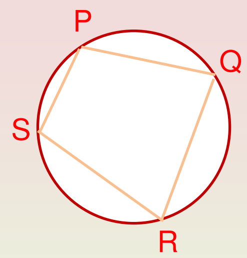


Figure 5.95

ACTIVITY 5.15

Calculate the lettered angles in each of the following:

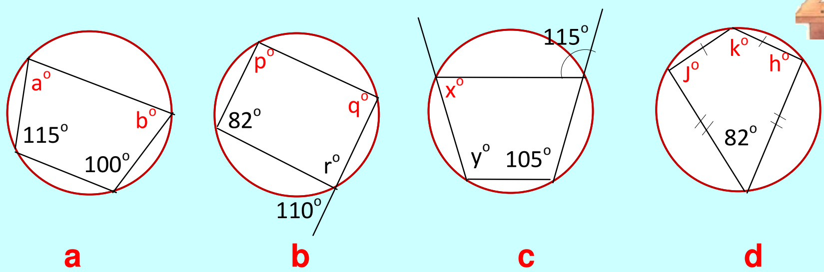


Figure 5.96

5.4.3 Arc Lengths, Perimeters and Areas of Segments and Sectors

Remember that:

- ✓ Circumference of a circle = $2\pi r$ or πd.
- ✓ Area of a circle = πr^2.
- ✓ Part of the circumference of a circle is called an arc.
- ✓ A segment of a circle is a region bounded by a chord and an arc.
- ✓ A sector of a circle is bounded by two radii and an arc.

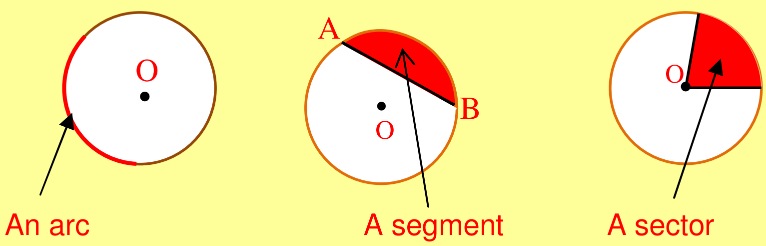


Figure 5.97

Group Work 5.5

- What fraction of a complete circle is the shaded region in Figure 5.98?
- What is the area of the shaded region in this figure?

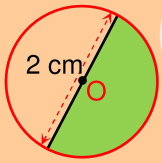


Figure 5.98

- What is the area if the shaded region is a quadrant (Figure 5.99)?

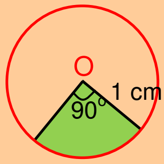


Figure 5.99

- What is the area if the shaded region is part of a quadrant bounded by a chord and an arc as shown in Figure 5.100?

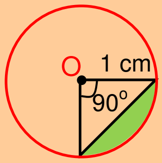


Figure 5.100

Discuss how to find the area of each of the shaded sectors shown below. Is the area of each sector proportional to the angle between the radii bounding the sector?

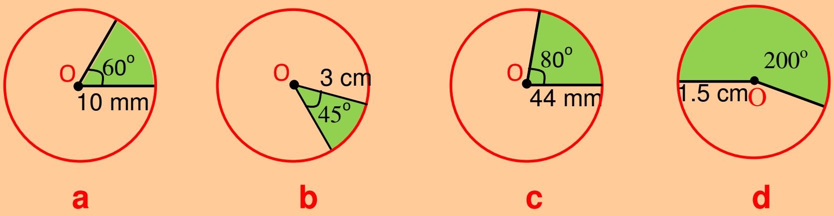


Figure 5.101

Discuss how to find the area of each of the shaded segments shown below:

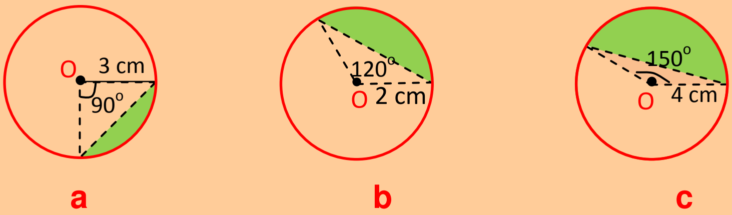


Figure 5.102

✓ Arc length

The length ℓ of an arc of a circle of radius r that subtends an angle of θ at the centre is given by

$$\ell = \frac{\theta}{360^o} \times 2\pi r = \frac{\pi r \theta}{180^o}$$

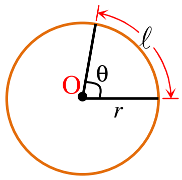


Figure 5.103

✓ The area and perimeter of a sector

The area A of a sector of radius r and central angle θ is given by

$$A = \frac{\theta}{360^o} \times \pi r^2 = \frac{\pi r^2 \theta}{360^o}$$

The perimeter P of the sector is the sum of the radii and the arc that bound it.

$$P = 2r + \frac{\pi r\theta}{180^o}$$

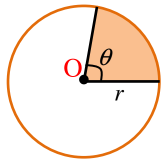


Figure 5.104

✓ The area and perimeter of a segment

The area A and perimeter P of a segment of a circle of radius r, cut off by a chord subtending an angle θ at the centre of a circle are given by

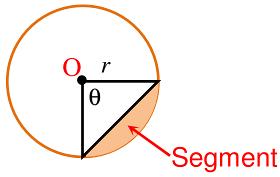


Figure 5.105

$$A = \frac{\pi\theta r^2}{360^o} - \frac{1}{2}r^2\sin\theta \quad (sector\ area\ - triangle\ area)$$

Note: The area formula for a triangle:

$A = \frac{1}{2}ab\sin\theta$ where a and b are the lengths of any two sides of the triangle and θ is the measure of the angle included between the given sides is discussed in the next section (Section 5.5 of this textbook).

$$P = 2r\sin\frac{\theta}{2} + \frac{\pi r\theta}{180^o} \quad (chord\ length + arc\ length)$$

Example 1 A segment of a circle of radius 12 cm is cut off by a chord subtending an angle 60^o at the centre of the circle. Find:

a the area of the segment. **b** the perimeter of the sector.

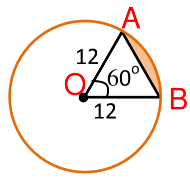


Figure 5.106

Solution:

a From the figure, area of the segment (the shaded part)

= area of the sector OAB – area of triangle OAB;

Area of the sector OAB = $\frac{\pi \times 12^2 \times 60}{360}$ = 24π cm^2.

Area of triangle OAB = $\frac{1}{2} \times 12^2 \times \sin\ 60^o = 36\sqrt{3}$ cm^2.

Therefore, segment area = $(24\pi - 36\sqrt{3})$ cm^2.

b Perimeter of the sector = 2 × radius + length of arc AB

$= 2 \times 12 + \frac{\pi \times 12 \times 60}{180} = (24 + 4\pi)$ cm

Exercise 5.12

1 Calculate the perimeter and area of each of the following figures. All curves are semicircles or quadrants.

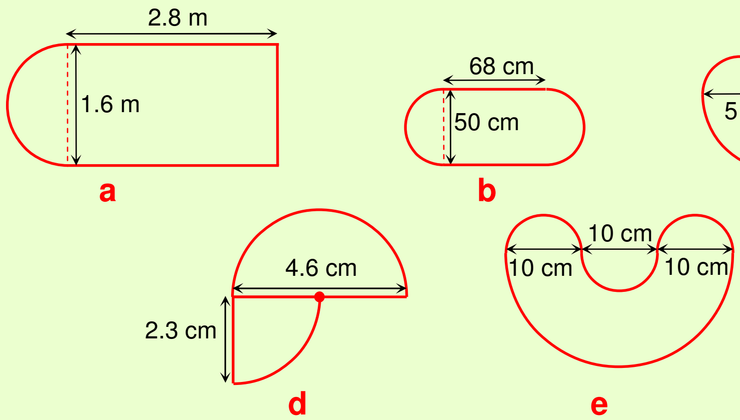


Figure 5.107

2 In each of the following sectors OPQ find:

i the length of arc PQ.

ii area of the sector OPQ.

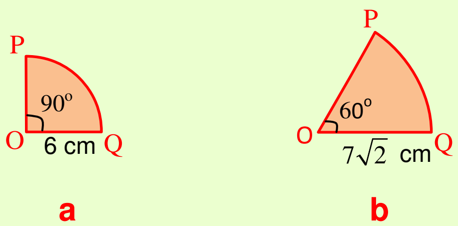


Figure 5.108

3 In Figure 5.109, O is the centre of the circle. If the radius of the circle is 4 cm and m(∠AKB) = 30^o, find the area of the segment bounded by the chord AB and arc AKB.

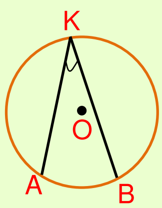
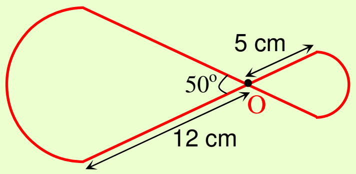


Figure 5.109

4 A square ABCD is inscribed in a circle of radius 4 cm. Find the area of the minor segment cut off by the chord $\overline{AB}$.

5 Calculate the perimeter and area of each of the following figures, where the curves are arcs of a circle with common centre at O.



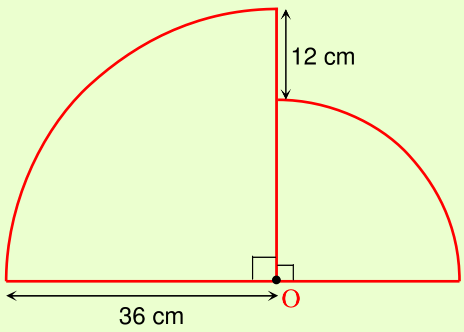


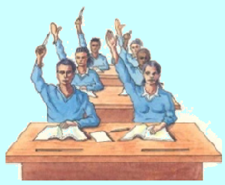
Figure 5.110

5.5 MEASUREMENT

5.5.1 Areas of Triangles and Parallelograms

A Areas of triangles

ACTIVITY 5.16



Given the right angle triangle shown below, verify that each of the following expressions give the area of ΔABC. In each case, discuss and state the formula used.

i Area of $\Delta ABC = \frac{1}{2}ac$

ii Area of $\Delta ABC = \frac{1}{2}bh$

iii Area of $\Delta ABC = \frac{1}{2}bc \sin(\angle A)$

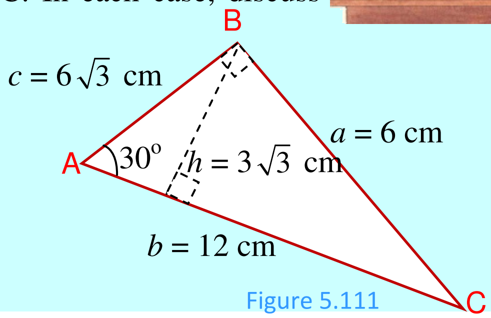


Figure 5.111

The above Activity should have reminded you what you studied in your lower grades except for case iii, which you have used in the preceding section and are going to learn about now.

Case i uses the following fact.

The area A of a right angle triangle with perpendicular sides of length a and b is given by

$$A = \frac{1}{2}ab$$

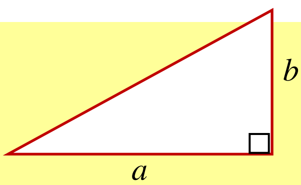


Figure 5.112

Case ii uses the following formula.

The area A of any triangle with base b and the corresponding height h is given by

$$A = \frac{1}{2} bh$$

The base and corresponding height of a triangle may appear in any one of the following forms.

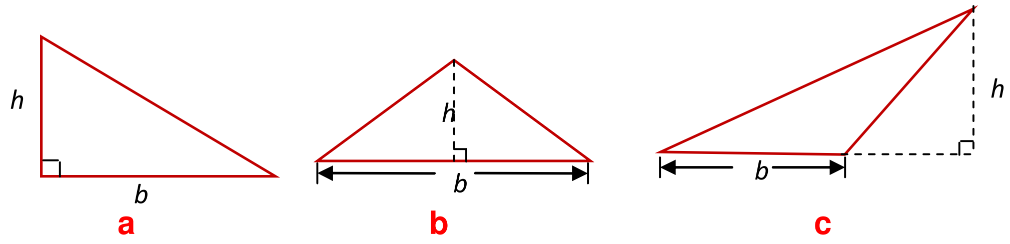


Figure 5.113

From the verification of Case iii, we come to the following formula.

The area A of any triangle with sides a and b units long and angle C ($\angle C$) included between these sides is

$$A = \frac{1}{2} ab \sin(\angle C)$$

Proof:-

Let Δ ABC be given such that BC = a and AC = b.

Case i Let $\angle C$ be an acute angle.

Consider the height h drawn from B to AC. It meets AC at D (see Figure 5.114).

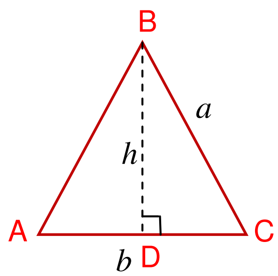


Figure 5.114

Now, area of $\Delta ABC = \frac{1}{2}\ bh \qquad (1)$

Since Δ BCD is right-angled with hypotenuse a,

$\sin(\angle C) = \frac{h}{a}$

$\therefore\ h = a \sin(\angle C)$

Replacing h by $a \sin(\angle C)$ in 1 we obtain

$$\text{Area of } \Delta ABC = \frac{1}{2}\ ab \sin(\angle C) \text{ as required.}$$

Case ii Let $\angle$C be an obtuse angle.

Draw the height from B to the extended base AC. It meets the extended base AC at D. Now,

area of ΔABC = Area of ΔABD – Area of ΔBDC

$$= \frac{1}{2}\text{AD}.h - \frac{1}{2}\text{CD}.h = \frac{1}{2}\ h(\text{AD} - \text{CD})$$

$$= \frac{1}{2}\ h.\text{AC} = \frac{1}{2}\ hb \qquad (2)$$

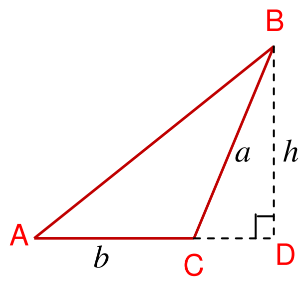


Figure 5.115

In the right-angled triangle BCD, $\sin(180° - \text{C}) = \dfrac{h}{a}$

$\therefore \quad h = a \sin(180° - \text{C})$

Since $\sin(180° - \text{C}) = \sin \text{C}$, we have $h = a \sin(\angle \text{C})$

For any two angles A and B if $m(\angle A) + m(\angle B) = 180^o$, then, $\sin A = \sin B$.

$\therefore$ Replacing h by $a \sin(\angle\text{C})$ in 2 we obtain;

Area of ΔABC = $\dfrac{1}{2}\ ab \sin(\angle\text{C})$ as required.

Case iii Let $\angle$C be a right angle.

$$A = \frac{1}{2}ab = \frac{1}{2}ab(\sin 90^o) \qquad (\sin 90^o = 1)$$

$$= \frac{1}{2}ab\sin(\angle C) \qquad (\textit{as required})$$

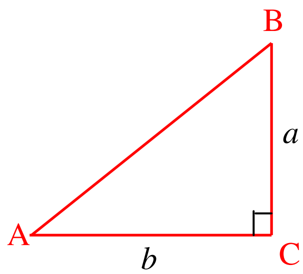


Figure 5.116

This completes the proof.

Group Work 5.6

1 Using this formula, show that the area A of a regular n-sided polygon with radius r is given by

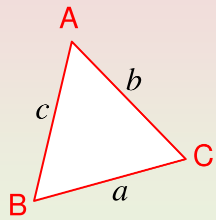
$$A = \frac{1}{2}nr^2 \sin\frac{360^o}{n}$$

2 Show that the area A of an equilateral triangle inscribed in a circle of radius r is

$$A = \frac{3\sqrt{3}}{4}r^2$$

Now we state another formula called Heron's formula, which is often used to find the area of a triangle when its three sides are given.

Theorem 5.14 **Heron's formula**



The area A of a triangle with sides a, b and c units long and semi- perimeter $s=\frac{1}{2}(a+b+c)$ is given by

$$A=\sqrt{s(s-a)(s-b)(s-c)}\,.$$

Figure 5.117

Example 1 Given ΔABC. If AB = 15 units, BC = 14 units and AC = 13 units, find

a the area of ΔABC.

b the length of the altitude from the vertex A.

c the measure of ∠B.

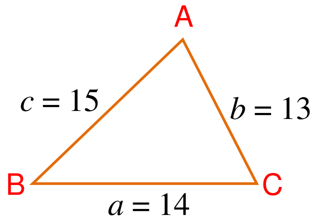


Figure 5.118

Solution:

a $a = 14$ $\quad s - a = 7$

$b = 13$ $\quad s - b = 8$

$c = 15$ $\quad s - c = 6$

$a + b + c = 42$ $\quad (s - a) + (s - b) + (s - c) = 21$

$$\therefore\ s = \frac{a+b+c}{2} = \frac{42}{2} = 21$$

Why is the sum of s – a, s – b, s – c equal to s? This provides a useful check.

$$\therefore \text{ area of } \Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{21(7)(8)(6)} = 84 \text{ unit}^2.$$

b Let the altitude from the vertex A to the corresponding base BC be h, meeting BC at D as shown.

Then, area of ΔABC = $\frac{1}{2}$ BC × h

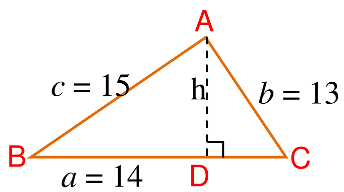


Figure 5.119

$$\therefore\ 84 = \frac{1}{2}\times 14\times h = 7h$$

$$\therefore\quad h = \frac{84}{7} = 12 \text{ units.}$$

Therefore, the altitude of ΔABC from the vertex A is 12 units long.

c In the right angle triangle ABD shown above in Figure 5.119, we see that

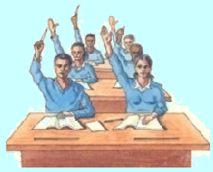
$$\sin(\angle B) = \frac{AD}{AB} = \frac{h}{c} = \frac{12}{15} = 0.8$$

Then, from trigonometric tables, we find that the corresponding angle is 53^o.

i.e., $m(\angle B) = 53^o$.

B Area of parallelograms

ACTIVITY 5.17



1 What is a parallelogram?

2 Show that a diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Theorem 5.15

The area A of a parallelogram with base b and perpendicular height h is

$$A = bh$$

Proof:-

Let ABCD be a parallelogram with base BC = b. Draw diagonal AC. You know that AC divides the parallelogram into two congruent triangles. Moreover, note that any two congruent triangles have equal areas. Now, the area of $\Delta ABC = \frac{1}{2}bh$.

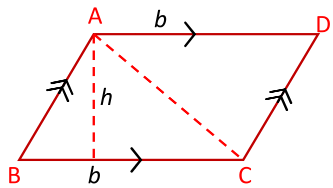


Figure 5.120

Therefore, the area of parallelogram ABCD $= 2\left(\frac{1}{2}bh\right) = bh$.

Example 2 If one pair of opposite sides of a parallelogram have length 40 cm and the distance between them is 15 cm, find the area of the parallelogram.

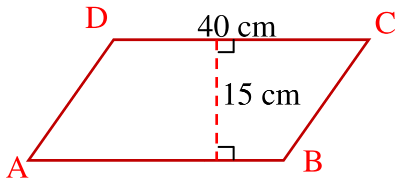


Figure 5.121

Solution: Area = 40 cm × 15 cm = 600 cm^2.

Exercise 5.13

1 In Δ ABC, $\overline{BE}$ and $\overline{CF}$ are altitudes of the triangle. If AB = 6 units, AC = 5 units and CF = 4 units find the length of BE.

2 In ΔDEF, if DE = 20 units, EF = 21 units and DF = 13 units find:

a the area of Δ DEF

b the length of the altitude from the vertex D

c sin (∠D)

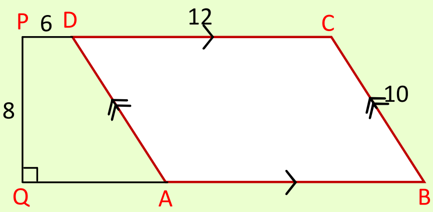


Figure 5.122

3 In the given figure, PD = 6 units, DC = 12 units, PQ = 8 units and BC = 10 units. Find:

a the area of the parallelogram ABCD.

b the height of the parallelogram that corresponds to the base AD.

4 PQRS is a parallelogram of area 18 cm^2. If PQ = 5 cm and QR = 4 cm, calculate the lengths of the corresponding heights.

5 In Δ MNO if MN = 5 cm, NO = 6 cm and MO = 7 cm, find:

a the area of Δ MNO. **b** the length of the shortest altitude.

(*leave your answers in radical form.*)

6 In the parallelogram ABCD (shown in Figure 5.123 below), AB = 2 cm, AD = 3 cm and m (∠B) = 60^o. Find the length of the altitude from A to $\overline{DC}$.

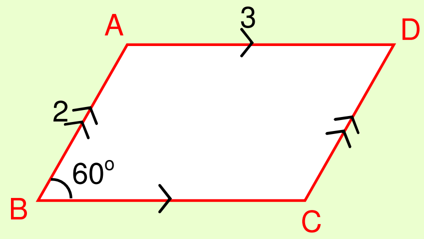


Figure 5.123

7 The lengths of three sides of a triangle are 6x, 4x and 3x inches and the perimeter of the triangle is 26 inches. Find:

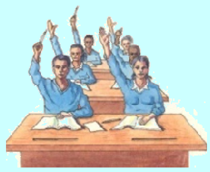
a the lengths of the sides of the triangle.

b the area of the triangle

8 Find the area of a rhombus whose diagonals are 5 inches and 6 inches long.

5.5.2 Further on Surface Areas and Volumes of Cylinders and Prisms

ACTIVITY 5.18



1 What is a solid figure?

2 Which of the following solids are prisms and which are cylinders? Which of them are neither prisms nor cylinders?

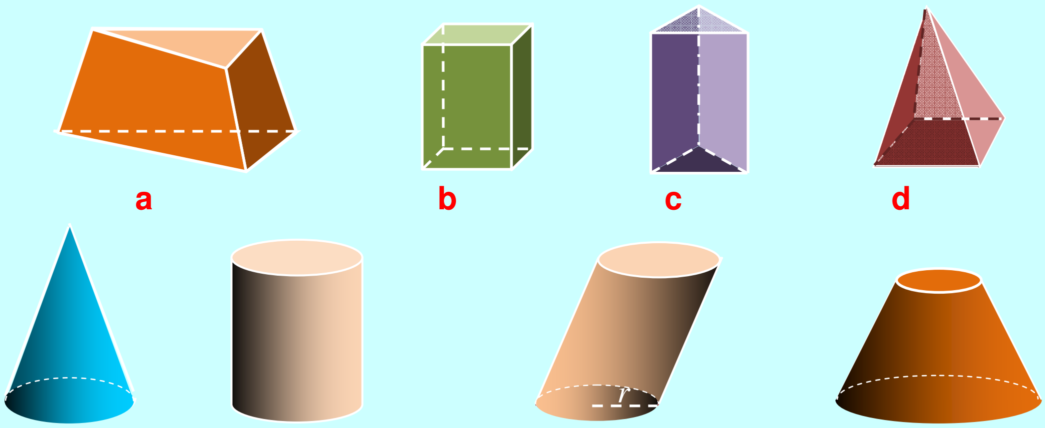


Figure 5.124

3 The radius of the base of a right circular cylinder is 2 cm and its altitude is 3 cm. Find its:

a curved surface area **b** total surface area **c** volume

4 Find a formula for the surface area of a right prism by constructing a model from simple materials.

5 Roll a rectangular piece in to a cylinder. Discuss how to obtain the surface area of a right circular cylinder.

A Prism

- A **prism** is a solid figure formed by two congruent polygonal regions in parallel planes, along with three or more parallelograms, joining the two polygons. The polygons in parallel planes are called **bases**.
- A prism is named by its base. Thus, a prism is called triangular, rectangular, pentagonal, etc., if its base is a triangle, a rectangle, a pentagon, etc., respectively.
- In a prism,
  - ✓ the lateral edges are equal and parallel.
  - ✓ the lateral faces are parallelograms.
- A **right prism** is a prism in which the base is perpendicular to a lateral edge. Otherwise it is an **oblique prism**.
- In a right prism
  - ✓ All the lateral edges are perpendicular to both bases.
  - ✓ The lateral faces are rectangles.
  - ✓ The altitude is equal to the length of each lateral edge.
- A regular prism is a right prism whose base is a regular polygon.

Surface area and volume of prisms

- The lateral surface area of a prism is the sum of the areas of its lateral faces.
- The total surface area of a prism is the sum of the lateral areas and the area of the bases.
- The volume of any prism is equal to the product of its base area and its altitude.

✓ If we denote the lateral surface area of a prism by A_L, the total surface area by A_T, the area of the base by A_B and its volume by V, then

i $A_L = Ph$

where P is the perimeter of the base and h the altitude or height of the prism.

ii $A_T = 2A_B + A_L$ **iii** $V = A_B\, h$.

Example 1 The altitude of a rectangular prism is 4 units and the width and length of its base are 3 and 2 units respectively. Find:

a the total surface area of the prism. **b** the volume of the prism.

Solution:

a To find A_T, first we have to find the base area and the lateral surface area.

$A_B = 2 \times 3 = 6$ unit^2.

and $A_L = Ph = (3 + 2 + 3 + 2) \times 4 = 40$ unit^2.

$\therefore\ A_T = 2A_B + A_L = 2 \times 6 + 40 = 52$ unit^2.

So, the total surface area is 52 unit^2.

b $V = A_B\, h = 6 \times 4 = 24$ unit^3

Example 2 Through the centre of a regular hexagonal prism whose base edge is 6 cm and height 8 cm, a hole whose form is a regular triangular prism with base edge 3 cm is drilled as shown in Figure 5.125. Find:

a the total surface area of the remaining solid.

b the volume of the remaining solid.

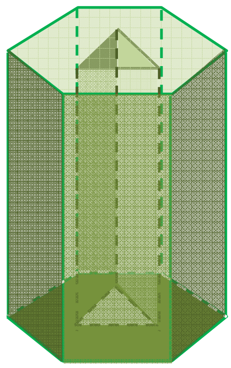


Figure 5.125

Solution: Recall that the area A of a regular n-sided polygon with radius r is

$$A = \frac{1}{2} nr^2 \sin \frac{360^o}{n}.$$

Also, the radius and the length of a side of a regular hexagon are equal.

So, Area of the given regular hexagon $= \frac{1}{2} \times 6 \times 6^2 \times \sin 60^o = 54\sqrt{3}$ cm^2.

Area of the equilateral triangle $= \frac{1}{2} ab \sin C = \frac{1}{2} \times 3 \times 3 \times \sin 60^o = \frac{9\sqrt{3}}{4}$ cm^2

a **i** Area of the bases of the remaining solid = $2 \times$ (area of hexagon – area of Δ)

$$= 2 \times \left(54\sqrt{3} - \frac{9\sqrt{3}}{4}\right) = 108\sqrt{3} - \frac{9\sqrt{3}}{2} = \frac{207}{2}\sqrt{3}\ \text{cm}^2$$

ii Lateral surface area of the remaining solid = lateral area of hexagonal prism + lateral area of triangular prism (inner)

= perimeter of hexagon $\times$ 8 + perimeter of triangle $\times$ 8

= $36 \times 8 + 9 \times 8 = 360\ \text{cm}^2$.

$\therefore$ total surface area of the remaining solid = $\left(\frac{207}{2}\sqrt{3} + 360\right)\ \text{cm}^2$.

b Volume of the remaining solid

= volume of hexagonal prism – volume of triangular prism

$$= \left(54\sqrt{3} \times 8 - \frac{9\sqrt{3}}{4} \times 8\right)\ \text{cm}^3 = 414\sqrt{3}\ \text{cm}^3$$

B Cylinder

Recall from your lower grades that:

➢ A circular cylinder is a simple closed surface bounded on two ends by circular bases. (See Figure 5.126). A more general definition of a cylinder replaces the circle with any simple closed curve. For example, the cylinder shown in Figure 5.127 is not a circular cylinder.

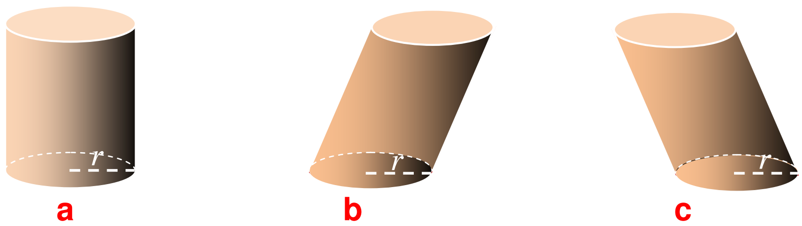


Figure 5.126 (circular cylinders)

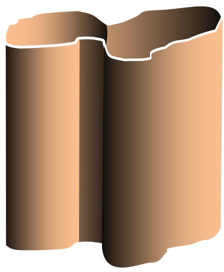


Figure 5.127

In our present discussion, we shall consider only cylinders whose bases are circles (i.e., circular cylinders).

A circular cylinder resembles a prism except that its bases are circular regions. In Figure 5.126a the cylinder is called a right circular cylinder. In such a cylinder the line segment joining the centres of the bases is perpendicular to the bases. The cylinders in Figures 5.126b and c above are not right circular cylinders; they are oblique cylinders.

Surface area and volume of circular cylinders

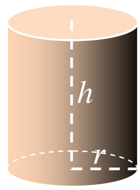


Figure 5.128

1 The lateral surface area (i.e., area of the curved surface) of a right circular cylinder denoted by A_L is the product of its height h and the circumference C of its base.

i.e. $A_L = hC$ OR $A_L = 2\pi rh$

2 The total surface area (or simply surface area) of a right circular cylinder denoted by A_T is two times the area of the circular base plus the area of the curved surface (lateral surface area). So, if the height of the cylinder is h and the radius of the base circle is r, we have

$$A_T = 2\pi rh + 2\pi r^2 = 2\pi r\,(h + r)$$

3 The volume V of the right circular cylinder is equal to the product of its base area and height.

So, if the height of the cylinder is h and its base radius is r then

$$V = \pi r^2\, h$$

Example 3 If the height of a right circular cylinder is 8 cm and the radius of its base is 5 cm find the following giving your answers in terms of π.

a its lateral surface area **b** its total surface area **c** its volume

Solution:

a The lateral surface area of the right circular cylinder is given by

$A_L = 2\pi rh$

$= 2\pi \times 5 \times 8 = 80\pi \text{ cm}^2$

b $A_T = 2\pi rh + 2\pi r^2$

$= 2\pi \times 5 \times 8 + 2\pi \times 5^2 = 80\pi + 50\pi = 130\pi \text{ cm}^2$

c The volume of the cylinder is

$V = \pi r^2 h$

$= \pi \times 5^2 \times 8 = 200\pi \text{ cm}^3$

Example 4 A circular hole of radius 2 units is drilled through the centre of a right circular cylinder whose base has radius 3 units and whose altitude is 4 units. Find the total surface area of the resulting figure.

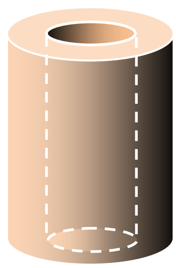


Figure 5.129

Solution: Let R be the radius of the bigger cylinder and r be the radius of the smaller cylinder then

i Area of the resulting base $= 2\,(\pi R^2 - \pi r^2)$

$$= 2\,(\pi \times 3^2 - \pi \times 2^2)\text{ unit}^2 = 10\pi\text{ unit}^2$$

ii Lateral surface area of the resulting figure

= lateral surface area of the bigger cylinder

+ lateral surface area of inner (smaller) cylinder

$= (2\pi Rh + 2\pi rh)\text{ unit}^2 = [\,2\pi\,(3)\,4 + 2\pi\,(2)\,4]\text{ unit}^2$

$= 40\,\pi\text{ unit}^2$

Therefore, total surface area of the resulting figure $= (10\pi + 40\pi) = 50\pi\text{ unit}^2$.

Exercise 5.14

1 Using the measurements indicated in each of the following figures, find:

a the total surface area of each figure. **b** the volume of each figure.

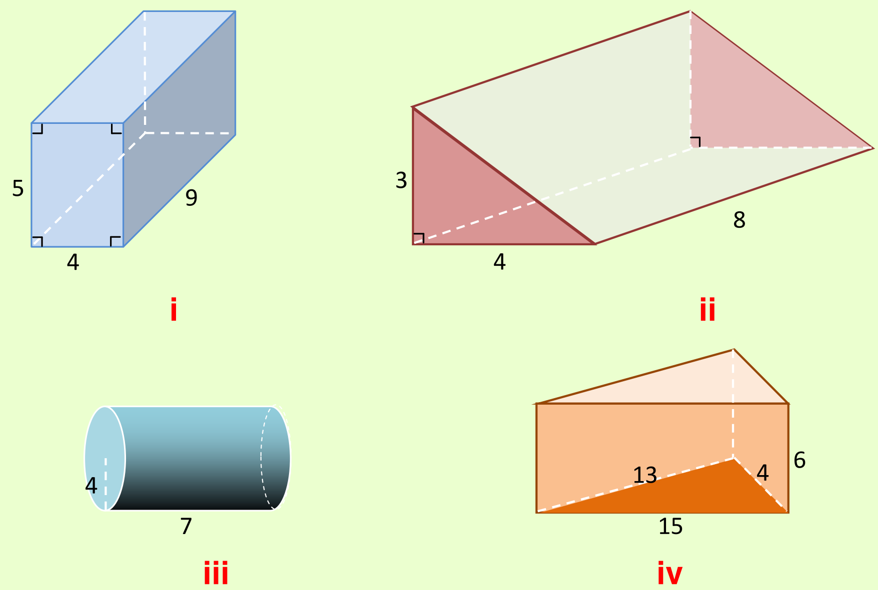


Figure 5.130

2 The base of a right prism is an isosceles triangle with equal sides 5 inches each, and third side 4 inches. The altitude of the prism is 6 inches. Find:

a the total surface area of the prism. **b** the volume of the prism.

3 Find the lateral surface area and total surface area of a right circular cylinder in which:

a $r = 4$ ft, $h = 12$ ft **b** $r = 6.5$ cm, $h = 10$ cm

4 Through a regular hexagonal prism whose base edge is 8 cm and whose height is 12 cm, a hole in the shape of a right prism, with its end being a rhombus with diagonals 6 cm and 8 cm is drilled (see Figure 5.131). Find:

a the total surface area of the remaining solid.

b the volume of the remaining solid.

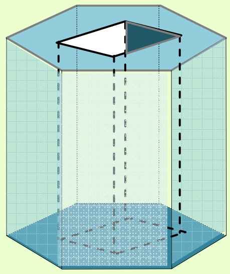


Figure 5.131

5 A manufacturer makes a closed right cylindrical container whose base has radius 7 inches and whose height measures 14 inches. He also makes another cylindrical container whose base has radius 14 inches and whose height measures 7 inches.

a Which container requires more metal?

b How much more metal does it require? Give your answer in terms of π.

Key Terms

apothem	lateral surface area	sector
arc	parallelogram	segment
area	polygon	similarity
circle	prism	total surface area
congruency	regular polygon	triangle
cylinder	rhombus	volume

Summary

1 A polygon is a simple closed curve formed by the union of three or more line segments no two of which in succession are collinear. The line segments are called the sides of the polygon and the end points are called its vertices.

2 **a** A polygon is said to be convex if each interior angle is less than 180^o.

b A polygon is said to be concave (non convex), if at least one of its interior angles is greater than 180^o.

3 A diagonal of a polygon is a line segment that joins any two of its non-consecutive vertices.

4 **a** The sum S of all the interior angles of an n-sided polygon is given by the formula

$$S = (n-2) \times 180^o$$

b The sum of all the exterior angles of an n-sided polygon is given by

$$S = 360^o$$

5 A regular polygon is a convex polygon with all sides equal and all angles equal.

6 **a** Each interior angle of a regular n-sided polygon is

$$\frac{(n-2)\times 180^o}{n}$$

b Each exterior angle of a regular n-sided polygon is $\frac{360^o}{n}$

c Each central angle of a regular n-sided polygon is $\frac{360^o}{n}$

7 A figure has a line of symmetry, if it can be folded so that one half of the figure coincides with the other half.

A figure that has at least one line of symmetry is called a symmetrical figure.

8 An n-sided regular polygon has n lines of symmetry.

9 A circle can be always inscribed in or circumscribed about any given regular polygon.

10 The apothem is the distance from the centre of regular polygon to a side of the polygon.

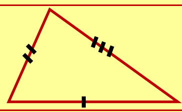
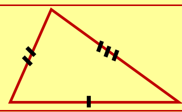
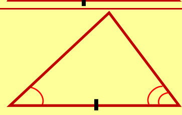
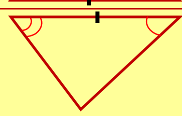
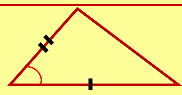
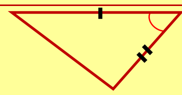
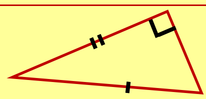
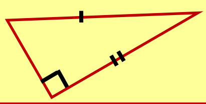
11 Formulae for the length of a side s, apothem a, perimeter P and area A of a regular polygon with n sides and radius r are given by

i $s = 2r\sin\frac{180^o}{n}$ **ii** $a = r\cos\frac{180^o}{n}$

iii $P = 2nr\sin\frac{180^o}{n}$ **iv** $A = \frac{1}{2}aP$ or $A = \frac{1}{2}nr^2\sin\frac{360^o}{n}$

12 **Congruency**

Two triangles are congruent, if the following corresponding parts of the triangles are congruent.

i	Three sides (SSS)		
ii	Two angles and the included side (ASA)		
iii	Two sides and the included angle (SAS)		
iv	A right angle, hypotenuse and a side (RHS)		

13 Similarity

i Two polygons of the same number of sides are similar, if their corresponding angles are congruent and their corresponding sides have the same ratio.

ii Similarity of triangles

a SSS-similarity theorem: If three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are similar.

b SAS-similarity theorem: Two triangles are similar, if two pairs of corresponding sides of the triangles are proportional and the included angles between the sides are congruent.

c AA-similarity theorem: If two angles of one triangle are correspondingly congruent to two angles of another triangle, then the two triangles are similar.

14 If the ratio of the lengths of any two corresponding sides of two similar polygons is k then

i the ratio of their perimeters is k. **ii** the ratio of their areas is k^2.

15 i Heron's formula

The area A of a triangle with sides a, b and c units long and semi-perimeter

$s = \frac{1}{2}(a + b + c)$ is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

ii If h is the height of the triangle perpendicular to base b, then the area A of the triangle is $A = \frac{1}{2}bh$

iii If the angle between the sides a and b is θ then the area A of the triangle is

$$A = \frac{1}{2}ab\sin\theta$$

16 Radians measure angles in terms of the lengths of the arc swept out by the angle. A radian (rad) is defined as the measure of the central angle subtended by an arc of a circle equal to the radius of the circle.

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3^{\circ}$$

$$1^{\circ} = \frac{\pi}{180} \text{ radian} \approx 0.0175 \text{ radian.}$$

✓ To convert radians to degree, multiply by $\frac{180^{\circ}}{\pi}$

✓ To convert degrees to radians, multiply by $\frac{\pi}{180^{\circ}}$

17 **i** for any acute angle θ

$\sin\theta = \cos(90^o - \theta)$

$\cos\theta = \sin(90^o - \theta)$

ii for any angle θ between 90^o and 180^o

$\sin\theta = \sin(180^o - \theta)$

$\cos\theta = -\cos(180^o - \theta)$

$\tan\theta = -\tan(180^o - \theta)$

18 **a** A circle is symmetrical about every diameter.

b A diameter perpendicular to a chord bisects the chord.

c The perpendicular bisector of a chord passes through the centre of the circle.

d In the same circle, equal chords are equidistant from the centre.

e A tangent is perpendicular to the radius drawn at the point of contact.

f Line segments that are tangents to a circle from an outside point are equal.

19 **Angle properties of a circle**

a The measure of an angle at the centre of a circle is twice the measure of an angle at the circumference subtended by the same arc.

b Every angle at the circumference subtended by the diameter of a circle is a right angle.

c Inscribed angles in the same segment of a circle are equal.

20 **a** The length l of an arc that subtends an angle θ at the centre of a circle with radius r is

$$\ell = \frac{\pi r\theta}{180^o}$$

b The area A of a sector with central angle θ and radius r is given by

$$A = \frac{\pi r^2\theta}{360^o}$$

c The area A of a segment associated with a central angle θ and radius r is given by

$$A = \frac{\pi r^2\theta}{360^0} - \frac{1}{2}r^2\sin\theta$$

21 If A_L is the lateral surface area of a prism, A_T is the total surface area of the prism, A_B is base area of the prism and V is the volume of the prism, then

i $A_L = Ph$, where P is the perimeter of the base and h the altitude or height of the prism.

ii $A_T = 2A_B + A_L$

iii $V = A_B\,h$

Review Exercises on Unit 5

1 ABCDE is a pentagon. If m(∠A) = m(∠B) = m(∠C) = m(∠D) = 115°, find m(∠E).

2 Given a regular convex polygon with 20 sides, find the measure of:

i each interior angle. ii each exterior angle. iii each central angle.

3 The measure of each interior angle of a regular convex polygon is 150°. How many sides does it have?

4 The angles of a quadrilateral, taken in order, are y^o, $3y^o$ $5y^o$, $7y^o$. Verify that two of its sides are parallel.

5 Find the area of a regular hexagon if each side is 8 cm long. (*leave the answer in radical form*).

6 The area of a regular hexagon is given as $384\sqrt{3}$ cm^2

a How long is each side of the hexagon?

b Find the radius of the hexagon. c Find the apothem of the hexagon.

7 Find the value of x in the following pair of congruent triangles:

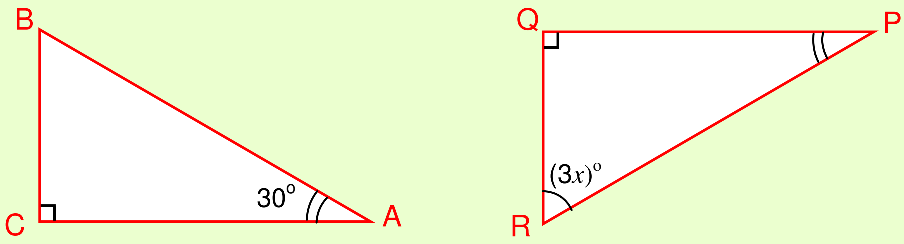


Figure 5.132

8 In Figure 5.133 below, BA = BC and KA = KC. Show that m(∠BAK) = m(∠BCK).

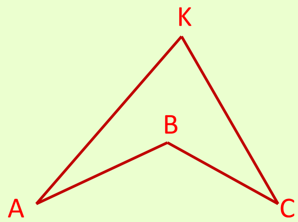


Figure 5.133

9 Two triangles are similar. The sides of one are 4, 6 and 7 cm. The shortest side of the other is 10 cm. Calculate the lengths of the other two sides of this triangle.

10 In the figure below, ∠ABC and ∠BDC are right angles; if AB = 5 cm, AD = 3 cm and BD = 4 cm, find BC and DC.

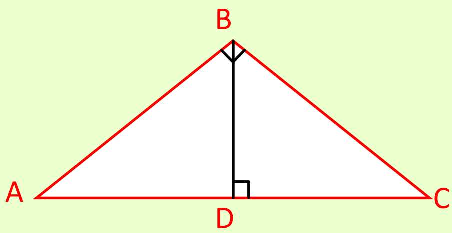


Figure 5.134

11 The areas of two similar triangles are 144 cm^2 and 81 cm^2. If one side of the first triangle is 6 cm, what is the length of the corresponding side of the second?

12 A chord of a circle of radius 6 cm is 8 cm long. Find the distance of the chord from the centre.

13 Two chords, AB and CD, of a circle intersect at right angles at a point inside the circle. If m(∠BAC) = 35^o, find m(∠ABD).

14 In each of the following figures, O is centre of the circle. In each figure, identify which angles are:

i supplementary angles. **ii** right angles. **iii** congruent angles.

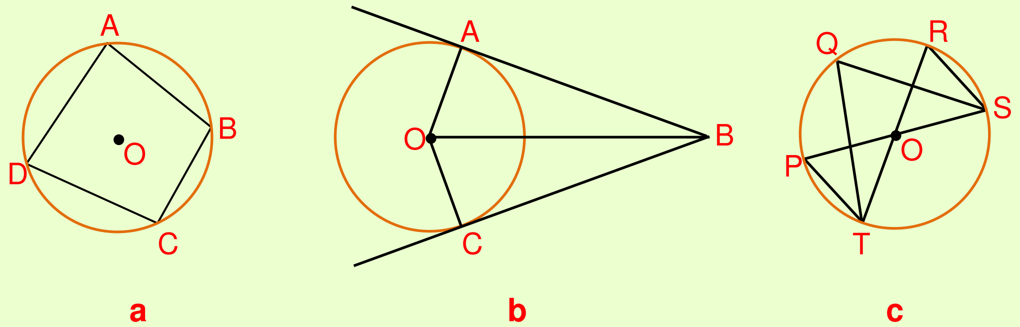


Figure 5.135

15 Find the perimeter and area of a segment of a circle of radius 8 cm, cut off by a chord that subtends a central angle of:

a 120^o **b** $\frac{3}{4}\pi$ radians.

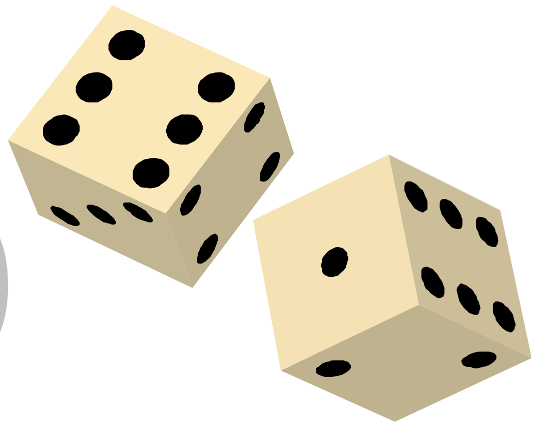
16 Calculate the volume and total surface area of a right circular cylinder of height 1 m and radius 70 cm.

17 A 40 m deep well with radius $3\frac{1}{2}$ m is dug and the earth taken out is evenly spread to form a platform of dimensions 28 m by 22 m. Find the height of the platform.

18 A glass cylinder with a radius of 7 cm has water up to a height of 9 cm. A metal cube of $5\frac{1}{2}$ cm edge is immersed in it completely. Calculate the height by which the water rises in the cylinder.

19 An agriculture field is rectangular, with dimensions 100 m by 42 m. A 20 m deep well of diameter 14 m is dug in a corner of the field and the earth taken out is spread evenly over the remaining part of the field. Find the increase in the level of the field.

Unit 6



STATISTICS AND PROBABILITY

Unit Outcomes:

After completing this unit, you should be able to:

- *know methods and procedures in collecting and presenting simple statistical data.*
- *know basic concepts about statistical measures.*
- *understand facts and basic principles about probability.*
- *solve simple mathematical problems on statistics and probability.*

Main Contents

INTRODUCTION

You have some knowledge about statistics and its basics, such as collection of data, presentation of data, etc., from your primary grade mathematics. In this unit, we formally define 'statistics' as a branch of applied mathematics and learn more about collection, presentation and analysis of numerical data. The unit also reviews the concept of probability, which was introduced in Grade 8 and teaches you more about experimental and theoretical approaches to probability and helps you solve simple problems based on these approaches.

6.1 STATISTICAL DATA

6.1.1 Collection and Tabulation of Statistical Data

Group Work 6.1

1 Split the class into three groups. Let Group A find the last year's mathematics result of the school in EGSECE from the school office's records.

Let group B collect information about the diseases treated in your nearest health centre, hospital, or health post. Let group C measure the height of each student in your class and consider its distribution by age and sex.

Answer the following questions using the information gathered by each group.

a How many students appeared for the exam?

b How many students scored A in the national exam?

c What was the score obtained by most of the students?

d Which diseases are treated most frequently?

e What is the average height of the class?

f Are males or females taller?

2 Discuss more about the importance and purpose of statistics.

3 What is the annual birth rate and death rate in Ethiopia? Which governmental agency is responsible for the preparation of such records?

4 Why does the government of Ethiopia carry out a census every ten years? Discuss.

There are many definitions of the term statistics given by different scholars. However, for the purpose of this unit, we will confine ourselves to the following:

Definition 6.1

Statistics is the science of *collecting*, *organizing*, *presenting*, *analyzing* and *interpreting data* (quantitative information) in order to draw conclusions.

Basically, statistics is a procedural process performing five logical steps on numerical data. These are:

collection of data → organization of data → presentation of data → analysis of data → interpretation of data

1 Collection of data

The first step of statistics is collection of data. This is the process of obtaining measurements or counts. For example, measuring the heights of students in your class, or counting the number of persons admitted to a certain hospital are examples of data collection.

2 Organization of data

The second step of statistics refers to the organization of data. Collected data has to be organized in a suitable form to understand the information gathered. The collected data must be edited, classified and tabulated.

3 Presentation of data

The main purpose of data presentation is to facilitate statistical analysis. This can be done by illustrating the data using graphs and diagrams like bar graph, histograms, pie-charts, pictograms, frequency polygons, etc.

4 Analysis of data

In order to meet the desired purpose of investigation, data has to be analyzed. The purpose of analyzing data is to highlight information useful for decision making.

5 Interpretation of data

Based on analyzed data, conclusions have to be drawn. This step usually involves decision making about a large collection of objects (the population) based on information gathered from a small collection of similar objects (the sample).

The decision making processes used by the managers of modern businesses and industry is governed by statistical application. Statistical methods can be applied to any situation where numerical information is gathered with the objective of making rational decisions in the face of uncertainty.

The following examples show us how statistics plays a major role in decision making in different sectors.

Example 1 Information gathered about the incidence or prevalence of diseases in a community provides useful information on changing trends in health status, mortality, nutritional status or environmental hazards.

Example 2 Statistics is used to study existing conditions and the prevalence rate of HIV/AIDS in order to design new programs or to study the merits of different methods adopted to control HIV/AIDS. It assists in determining the effectiveness of new medication and the importance of counselling.

Example 3 Demographic data about population size, its distribution by age and sex and the rate of population growth, etc., all help policy makers in determining future needs such as food, clothing, housing, education, health facilities, water, electricity and transportation systems.

Example 4 Recording annual temperatures in a country provides the community with timely warning of environmental hazards.

Example 5 Statistical data collected on customer services provides feedback that can help to reform policies and systems.

In the absence of accurate and timely data, it is impossible to form suitable policies. Statistics also plays a vital role in monitoring the proper implementation of programs and policies.

In its ordinary usage, population refers to the number of people living in an area or country. In statistics, however, **population** refers to the complete collection of individuals, objects or measurements that have a common characteristic.

Gaining access to an entire group (or population) is often difficult, expensive and sometimes destructive. Therefore, instead of examining the entire group, a researcher examines a small part of the group, called a **sample**.

Data can be classified as either **qualitative** or **quantitative**. However, statistics deals mainly with quantitative data.

Example 6 Data collected from the population of students in Ethiopia could be;

i **Qualitative** if the data is based on some characteristic whose values are not numbers, such as their eye colour, sex, religion or nationality.

ii **Quantitative** if the data is numerical such as height, weight, age or scores in tests.

A rule which gives a corresponding value to each member of a population is called a **population function**.

Example 7 Here is a table showing the approximate sizes of major lakes in Ethiopia.

TABLE 6.1: Size of major lakes in Ethiopia

Name of Lake	Length (km)	Width (km)	Area (km^2)
Abaya	60	20	1160
Abayata	17	15	205
Ashenge	5	4	20
Hawassa	16	9	229
Chamo	26	22	551
Hayk	7	5	35
Koka	20	15	205
Langano	18	16	230
Shalla	28	12	409
Tana	70	60	3600
Ziway	25	20	434

We can think of the set of the eleven lakes as the population and the functions L: Length, W: width, A: area, etc as functions on this population.

Example 8 The following table shows the age of 10 students in a certain class.

TABLE 6.2: Age of students

Name of student x	Age (in years) A(x)
Abebe	18
Abdu	17
Bayissa	16
Fatuma	17
Hiwot	15
Kidane	14
Lemlem	18
Meseret	17
Omod	15
Zehara	16

The students are members of the population and their age, A is the population function. Statistics can be classified into two types: Descriptive statistics and inferential statistics.

Definition 6.2

Descriptive statistics is a branch of statistics concerned with summarizing and describing a large amount of data without drawing any conclusion about a particular bit of data.

Descriptive statistics describes information collected through numerical measurement, charts, graphs and tables. The main purpose of descriptive statistics is to provide an overview of the information collected.

Definition 6.3

Inferential statistics is a branch of statistics concerned with interpreting data and drawing conclusions.

We can classify data as primary data and secondary data.

1 Primary data

Data is said to be primary, if it is obtained first hand for the particular purpose on which one is currently working. Primary data is original data, obtained personally from primary sources by observation, interview or direct measurement.

Example 9 If you measure the heights of students in your class, this is primary data.

Example 10 The data gathered by the Ministry of Education about the number of students enrolled in different universities of Ethiopia is primary data for the Ministry itself. (*If you were to use this data, it would be secondary data for you.*)

2 Secondary data

Data which has been collected previously (for similar or different purpose) is known as secondary data. Secondary data refers to that data which is not originated by the researcher himself/herself, but which he/she obtains from someone else's records. Some sources of secondary data are official publications, journals, newspapers, different studies, national statistical abstracts, etc.

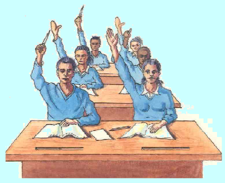
Example 11 Reports on the number of victims of HIV/AIDS by the Ministry of Health is secondary data for anyone other than the Ministry.

Example 12 The 2007 census of population size of regions by sex reported by the Central Statistical Agency (CSA) is secondary data for the government.

Information expressed in quantitative form can result in such a large amount of data that unless these figures are presented in some organized manner, their significance is easily lost. One of the basic methods of presenting statistical data is putting it into tables. To do this, often the data needs to be classified.

Classification is the process of arranging things into groups or classes.

ACTIVITY 6.1



1. Classify the employees in your school by household income.
2. Group the number of HIV/AIDS victims recorded in your nearest health centre according to their age group.
3. Collect data on age, height and mathematics exam score of the students in your class. Classify or tabulate the data collected.

Different people or organizations collect data for different reasons and the basis of classification they use is also different accordingly. To see this, consider the following examples.

Example 13 An economist in the Labour Department of a Regional Social Affairs Bureau may classify the households in a certain locality by household income as shown in the table below.

TABLE 6.3: Monthly income of 300 households

Income (in Birr)	Number of households
Under 350	85
Between 350 and 650	72
Between 651 and 950	64
Between 951 and 1250	48
Between 1251 and 1550	21
above 1550	10
Total	300

Example 14 According to the 2007 Ethiopian Census, the Ethiopian Central Statistical Agency (CSA) has classified the population by sex as follows.

TABLE 6.4: Population by sex (2007 Ethiopian census)

Region	Male (in 1000)	Female (in 1000)	Both sexes (in 1000)
Tigray	2124.8	2189.6	4314.4
Affar	786.3	624.7	1411.0
Amhara	8636.9	8577.2	17214.1
Oromiya	13676.2	13482.3	27158.5
Somale	2468.8	1970.4	4439.2
Benshangul	340.4	330.5	670.9
SNNP	7482.0	7560.5	15042.5
Gambela	159.7	147.2	306.9
Harari	92.3	91.1	183.4
Addis Ababa	1304.5	1433.7	2738.2
Dire Dawa	171.9	170.9	342.8
Total	37243.8	36577.4	73821.2

A statistical table is a systematic presentation or organization of numerical data in columns and rows. Columns are vertical arrangements and rows are horizontal. The main purpose of a statistical table is to allow the reader to quickly access relevant information. A title and row and column headers are important.

Exercise 6.1

1. What are the steps used in doing a statistical study?
2. What do we mean by organizing or presenting data?
3. Explain each of the following statistical terms by giving examples.
   - a qualitative data
   - b quantitative data
   - c population
   - d population function
   - e sample
4. Mention four uses of statistics.
5. What is descriptive statistics?
6. Describe in your own words the difference between a population and a sample.
7. Determine whether the following data is qualitative or quantitative.
   - a Gender
   - b Temperature
   - c Zip code
   - d Number of days
   - e Religions
   - f Occupations
   - g Ages
   - h Colours
   - i Nationality

8 Mention some advantages of tabular presentation of data.

9 Why is it necessary to organize data in a systematic manner after it has been collected?

10 Draft a table to show the following data, collected from employees in a company.

   a sex

   b three ranks: supervisors, assistants and clerks

   c years: 2000 and 2001

   d age group: 18 years and under, over 18 but less than 50 years, over 50 years

6.1.2 Distributions and Histograms

Information (data) is obtained from a census, existing data sources, surveys or designed experiments. After data is collected, it must be organized into a manageable form. Data that is not organized is referred to as raw data.

Definition 6.3

A quantity that we measure from observation is called a variate or variable denoted by V. The distribution of a population function is the function which associates with each variate of the population function a corresponding frequency denoted by f.

Methods for organizing raw data include the drawing of tables or graphs, which allow quick overview of the information collected.

Example 1 Suppose there are 10 students in a group whose scores in a mathematics quiz were as follows:

13, 12, 14, 13, 12, 12, 13, 14, 15, 12

Organize the data in tabular form. What are the variates? Give the frequency of each variate.

Solution: The data given above is raw data.

We may now tabulate the given data in the form given below.

Score (V)	12	13	14	15
Number of students (f)	4	3	2	1

The table given above is called the frequency distribution table. The scores are the variate and the number of students getting a particular score is the frequency of the variate.

Definition 6.4

A frequency distribution is a tabular or graphical representation of a data showing the frequency associated with each data value.

Example 2 Organize the data below into a frequency distribution table.

8,	9,	8,	7,	10,	9,	6,	4,	9,	8,
7,	8,	10,	9,	8,	6,	9,	7,	8,	8

Solution: (Write the values in ascending order.)

Value(V)	4	5	6	7	8	9	10	Total
Frequency(f)	1	0	2	3	7	5	2	20

Quantitative data can also be represented graphically, through a histogram.

Definition 6.5

A histogram is a graphical representation of a frequency distribution in which the variate (V) is plotted on the x-axis (horizontal axis) and the frequency (f) is plotted on the y-axis (vertical axis).

When drawing a histogram:

i Construct a frequency distribution table of the given data.

ii The x-axis

- **a** Determine a suitable scale for the horizontal axis and determine the number of rectangles needed to represent each variate or group of variates as desired.
- **b** Try not to break the x-axis.

iii The y-axis

- **a** Display information about frequency on the vertical (y) axis. .
- **b** Determine the length of the y-axis.

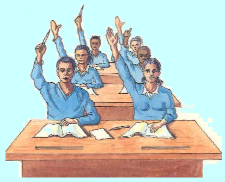
iv Draw bars for each variate (V)

v Label the histogram with a title, and label the axes.

Note: **i** The height of each rectangle is the frequency.

ii The width of each rectangle should be the same.

ACTIVITY 6.2



Consider the following data that shows the number of days 25 individuals participated in soil and water conservation tasks:

3	8	7	4	8
5	9	8	5	9
7	8	3	7	5
8	5	6	8	8
10	7	4	4	7

Construct a frequency distribution table and a histogram for the above data.

Example 3 The temperature in ^{o}C for the first 14 days of September in a certain town were recorded as

22,	27,	19,	23,	19,	18,	27,
27,	25,	23,	26,	27,	28,	23

Construct a frequency distribution table and a histogram for the given data.

Solution: Now construct the frequency distribution table from the raw data.

Temperature (in ^{o}C) (V)	18	19	20	21	22	23	24	25	26	27	28
frequency (f)	1	2	0	0	1	3	0	1	1	4	1

Using the above table, we draw a histogram as shown below.

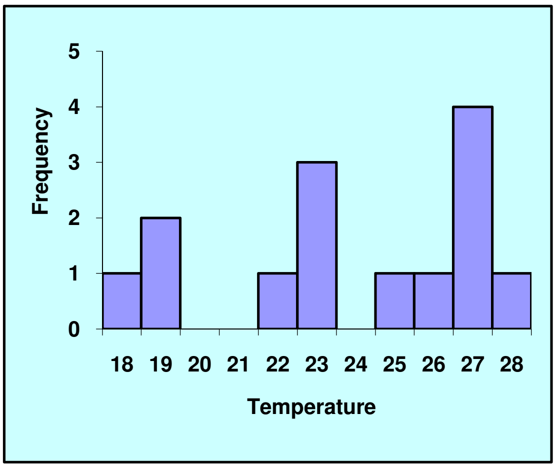


Figure 6.1

Example 4 The following histogram shows the daily income (in Birr) of 30 employees in a factory.

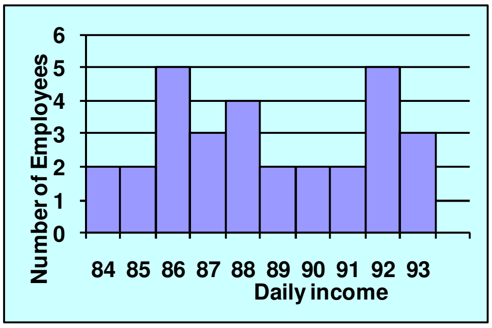


Figure 6.2

From the histogram, answer the following questions:

a How many employees have a daily income of Birr 92?

b How many employees collect a daily income of more than Birr 90?

c What is the highest frequency?

d What percent of the employees earn a daily income of more than Birr 89?

Solution:

a 5 employees have a daily income of Birr 92.

b 10 employees earn a daily income of more than Birr 90.

c The highest frequency is 5.

d Percentage

$$= \frac{\text{Sum of the frequencies of employees earning more than 89}}{\text{Total Frequency}} \times 100\%$$

$$= \frac{2+2+5+3}{30} \times 100\% = \frac{12}{30} \times 100\% = 40\%$$

i.e., 40% of the employees earn a daily income of more than Birr 89.

Exercise 6.2

1 Give two reasons why raw data should be summarized into a frequency distribution.

2 What is the difference between a frequency distribution table and a histogram?

3 The ages (to the nearest year) of 40 children in a certain village are as follows:

10	7	4	5	1	9	3	6	5	4
2	7	5	3	2	5	6	2	8	9
5	8	9	9	5	2	1	3	9	4
3	5	7	9	6	3	6	8	1	2

Prepare a frequency distribution table and a histogram for the given data.

4 Collect the score the students in your class obtained in their mid–semester mathematics exam and

a Prepare a frequency distribution table.

b Draw a histogram.

c What score is most frequent?

d What is the least score obtained?

5 A sample of 50 couples married for 10 years were asked how many children they had. The result of the survey is as follows:

0	4	2	2	1	3	0	3	2	4
3	3	1	3	3	3	3	3	2	2
1	3	3	2	4	3	1	5	2	2
2	0	0	2	1	2	2	2	3	2
3	3	3	4	3	1	3	0	3	2

a Construct a frequency distribution.

b Construct a histogram.

c What percentage of couples have two children?

d What percentage of couples have at least two children?

6 Here are quintals of fertilizer distributed to 50 farmers.

20	24	22	19	20	10	18	24	10	15
21	20	20	19	20	10	14	22	10	18
18	15	14	18	20	15	14	22	14	20
15	14	15	20	21	10	20	20	15	24
10	10	15	22	14	21	20	14	15	10

a Construct a frequency distribution.

b Construct a histogram.

7 Suppose the following data represents the number of persons who took counselling on HIV/AIDS on 40 consecutive days:

10	5	10	3	4	5	12	9	11	13
10	9	6	10	8	7	3	7	9	10
4	6	8	6	7	6	4	4	11	8
10	9	5	8	8	7	8	8	6	12

a Construct a frequency distribution table from the data.

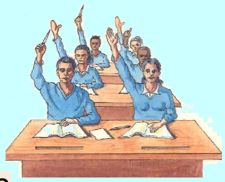
b Construct a histogram.

c On what percent of days did more than 10 people take counselling?

6.1.3 Measures of Location (Mean, Median and Mode(s))

Quantitative variables contained in raw data or in frequency tables can also be summarized by means of a few numerical values. A key element of this summary is called the measure of average or measure of location. The three commonly used measures of location are the arithmetic mean (or the mean), the median and the mode(s).

ACTIVITY 6.3



1 After completing a unit, a mathematics teacher gave a test marked out of 10, and the scores of 22 students were as follows:

6, 5, 8, 10, 6, 7, 3, 9, 3, 2, 9, 6, 7, 2, 6, 5, 4, 8, 6, 4,8, 3

- a Did the group do well in the test?
- b Prepare a frequency distribution table from the given data.
- c What is the average score of the group?
- d How many students score above average?
- e From the average obtained, can we say something about the performance of the group?
- f What relation can we see between the single value obtained in c and the marks of the students? Can the single value summarize the data?

2 Record the height and age of each student in your class.

- a What is the average height and age of the students?
- b What is the middle value of height and age of the students?
- c What value of height and age is most frequent (or has the highest frequency)?

3 Suppose a student scored the following marks in five subjects:

85, 89, 78, 92, 91

- a What is the average score of the student?
- b What is the middle value of the score?

4 Considering the following data

20, 21, 21, 22, 23, 23, 25, 27, 27, 27, 29, 98, 98

- a Find the mean, median and mode.
- b Which measure of location does not give a clear indication of the centre of the distribution?

5 Could you find the arithmetic mean of qualitative data? What about median and mode?

1 The arithmetic mean

When used in everyday language the word "average" often stands for the arithmetic mean.

Definition 6.6

The **arithmetic mean** (or the **mean**) of a variable is the sum of all the data values, divided by the total frequency (number of observations).

If x_1, x_2, x_3,....., x_n are the n observations of a variable, then the mean, $\bar{x}$, is given by

$$\text{Mean}: \ \bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\text{sum of values}}{\text{total number of values}}$$

Example 1 Find the mean of the following data:

7, 21, 2, 17, 3, 13, 7, 4, 9, 7, 9

Solution:

$$\bar{x} = \frac{7+21+2+17+3+13+7+4+9+7+9}{11} = \frac{99}{11} = 9$$

Note: The mean of a population function can also be calculated from its frequency distribution. So, if the values x_1, x_2, x_3, …, x_n occur f_1, f_2, f_3, …, f_n times, respectively, then the mean ($\bar{x}$) is given by

$$\text{Mean: } \bar{x} = \frac{x_1 f_1 + x_2 f_2 + \ldots + x_n f_n}{f_1 + f_2 + .. + f_n}$$

Example 2 The following table shows the age of 14 students in a certain class:

Age in years (V)	12	13	16	18
Number of students (f)	3	4	2	5

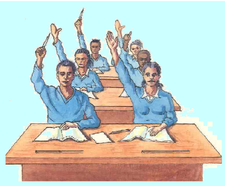
Compute the mean age of the students.

Solution:

$$\bar{x} = \frac{12\times3+13\times4+16\times2+18\times5}{3+4+2+5} = \frac{36+52+32+90}{14} = \frac{210}{14} = 15 \text{ years}$$

Properties of the mean

ACTIVITY 6.4



There are five students in a group. Lemlem wants to know how much money each student has and asked all the members of the group. She found the following amounts:

Birr 6, Birr 9, Birr 8, Birr 4 and Birr 3.

a What is the mean of the amount of money within the group?

b If Lemlem gives Birr 2 to each member of the group, what will be the new mean?

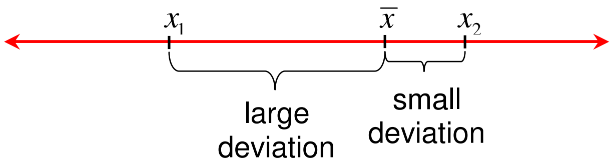
c If the amount of money in the pocket of each member is multiplied by 3, what will be the new mean?

d If you subtract the mean of the data obtained from each value, what will be the sum of the differences obtained?

e Discuss, what you observed from your answers to **a**, **b**, **c** and **d**.

The above Activity should help you to observe different properties of the mean.

The difference between a single data value x and the mean is called the deviation from the mean (or simply the deviation) and is given by $(x-\bar{x})$. A data point that is close to the mean will have a small deviation, whereas data points far from the mean will have large deviations as shown in the figure below.



1 The sum of the deviations of individual observations from mean ($\bar{x}$) is zero. That is, let x_1, x_2, x_3,…, x_n be n observations with mean $\bar{x}$. Then the sum of the deviations of the observations from the mean is given by

$$(x_1-\bar{x})+(x_2-\bar{x})+(x_3-\bar{x})+\ldots+(x_n-\bar{x})=0$$

Proof:-

Since the mean of n observations x_1, x_2, x_3, …, x_n is given by $\bar{x}$,

$$\bar{x}=\frac{x_1+x_2+x_3+\ldots+x_n}{n} \text{ which shows } x_1+x_2+x_3+\ldots+x_n=n\bar{x}$$

Now, $(x_1-\bar{x})+(x_2-\bar{x})+(x_3-\bar{x})+\ldots+(x_n-\bar{x})$

$$=(x_1+x_2+x_3+\ldots+x_n)-\underbrace{(\bar{x}+\bar{x}+\bar{x}\ldots+\bar{x})}_{n \text{ times}}$$

$$=(x_1+x_2+x_3+\ldots+x_n)-n\bar{x}$$

$$=n\bar{x}-n\bar{x}=0 \text{ as required.}$$

Example 3 Let the ages of 5 children be 2, 3, 6, 9, 10. Then, the mean age

$$\bar{x}=\frac{2+3+6+9+10}{5}=\frac{30}{5}=6$$

The sum of the deviations from the mean is:

$$(2-6)+(3-6)+(6-6)+(9-6)+(10-6)=-4-3+0+3+4=0$$

2 If a constant k is added to (or subtracted from) each data value, then the new mean is the sum (or the difference) of the old mean and the constant k.

Proof:- Let $\bar{x}$ be the mean of the data values x and k be the constant.

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \bar{x}$$

Adding k to each data value, the new mean is then

$$\frac{(x_1+k)+(x_2+k)+(x_3+k)+\ldots+(x_n+k)}{n} =$$

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n + k + k + k + \ldots + k}{n}$$

$$= \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} + \frac{nk}{n}$$

$= \bar{x} + k$ (the old mean plus k).

A similar proof can be done for the case when k is subtracted from each data value.

3 The mean of the sum or difference of two population functions (of equal numbers of observations) is equal to the sum or difference of the means of the two population functions.

Proof:-

Let $\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \bar{x}$ and $\frac{y_1 + y_2 + y_3 + \ldots + y_n}{n} = \bar{y}$

Then the mean of their sum,

$$\text{Mean } (x + y) = \frac{(x_1 + y_1) + (x_2 + y_2) + \ldots + (x_n + y_n)}{n}$$

$$= \frac{(x_1 + x_2 + x_3 + \ldots + x_n) + (y_1 + y_2 + y_3 + \ldots + y_n)}{n}$$

$$= \frac{(x_1 + x_2 + x_3 + \ldots + x_n)}{n} + \frac{(y_1 + y_2 + y_3 + \ldots + y_n)}{n}$$

$= \bar{x} + \bar{y}$ (the sum of the means)

Example 4 The mean of 2, 4, 6, 8 is 5 and the mean of 5, 7, 9, 7 is 7. Then, the mean of the sum 7, 11, 15, 15 is 5+7 = 12.

4 The mean of a constant times a population function is equal to the constant times the mean of the population function. That is,
if $\bar{x}$ is the mean of the population function $x_1, x_2, x_3, \ldots, x_n$ and if k is a constant, then the mean of $kx_1, kx_2, kx_3, \ldots, kx_n$ is equal to $k\bar{x}$.

Proof:-

$$\frac{kx_1 + kx_2 + kx_3 + \ldots + kx_n}{n} = \frac{k(x_1 + x_2 + x_3 + \ldots + x_n)}{n} = k\overline{x}$$

Example 5 The mean of 8, 9, 6, 8, 4, is 7. If you multiply each of value by 5, you will obtain 40, 45, 30, 40, 20. Then the new mean is $5 \times 7 = 35$

Note:

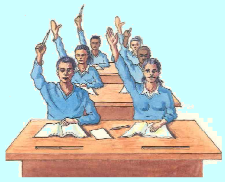
1 The mean is unique.

2 The mean is affected by extreme values.

2 The median

The following Activity will help you to revise what you learned in previous grades.

ACTIVITY 6.5



1 Find the median for each of the following sets of data.

a 5, 2, 9, 7, 3 b 12 , 8, 10, 14, 13, 9

2 What did you observe about the middle term when the number of observations is odd or even?

A second measure of location of quantitative data is the median.

Definition 6.7

The median is the value that lies in the middle of the data when it is arranged in ascending or descending order. So, half the data is below the median and half the data is above the median.

Example 6 Find the median of each of the following:

a 6, 7, 9, 7, 11, 13, 15 b 27, 23, 36, 38, 27, 40, 45, 39

Solution:

a First arrange the data in ascending order as 6, 7, 7, 9, 11, 13, 15

There are seven values (an odd number of values) and the middle value is the 4th element of the list which is 9.

Therefore 9 is the median of the data.

b First, arrange the data in ascending order as 23, 27, 27, 36, 38, 39, 40, 45

There are eight values (an even number). The two middle values are the 4th and 5th elements of the list which are 36 and 38. The median is half the sum of 36 and 38. So, the median is $\frac{36+38}{2} = 37$.

Example 7 Find the median of the following distribution

V	1	2	3	4	5
f	2	3	2	4	2

Solution: There are 13 data values. So, the median is the 7^{th} piece of data, which is 3.

Note that the median of a set of data with values arranged in ascending or descending order is:

i the middle value of the list if there is an odd number of values.

ii half of the sum of the two middle values if there is an even number of values.

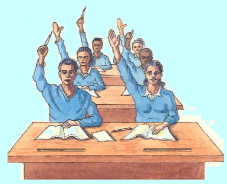
Properties of the median

1. The median can be obtained even when some of the data values are not known.
2. It is not affected by extreme values.
3. It is unique for a given data set.

3 The mode

The following activity should help you to recall what you have learnt about mode previously.

ACTIVITY 6.6



1. Find the mode(s) of the following data

   a 5, 7, 8, 7, 9, 11 **b** M, F, M, F, F

2. Can you find the mean and median for the above data?
3. Discuss your observation.

A third measure of location is the mode. The mode can be found for both quantitative and qualitative data.

Definition 6.8

The value of the variable which occurs most frequently in a data set is called the mode.

Example 8 Find the mode of each of the following data sets:

a 4, 6, 12, 10, 7 **b** 12, 10, 11, 13, 10, 14, 12, 18, 17

c 9, 8, 7, 10, 6, 8

Solution:

a It has no mode because each value occurs only once.

b The values 10 and 12 both occur twice, while the others occur only once. It has two modes and the data is a bimodal.

c 8 is the mode because it occurred twice (most frequently).

Example 9 Find the mean, median and mode of the following distribution of temperatures in a certain town for one month.

Temperature in °C(V)	20	21	23	24	26	28
Number of days(f)	2	4	5	9	3	7

Solution: Mean: $\bar{x} = \dfrac{(20\times 2)+(21\times 4)+(23\times 5)+(24\times 9)+(26\times 3)+(28\times 7)}{2+4+5+9+3+7}$

$$= \frac{40+84+115+216+78+196}{30} = \frac{729}{30} = 24.3$$

Therefore, the mean is 24.3°C.

The number of observations is an even number which is 30. So, the median is half the sum of the 15^{th} and 16^{th} values.

i.e., median $= \dfrac{15^{\text{th}}\ \text{value} + 16^{\text{th}}\ \text{value}}{2} = \dfrac{24+24}{2} = 24$

Therefore, the median is 24°C.

The value with highest frequency is the number 24. Therefore, the mode is 24°C.

Note that a set of data can have no mode, one mode (unimodal), two modes (bimodal) or more than two modes (multimodal). If there is no observation that occurs with the highest frequency, we say the data has no mode.

Properties of the Mode

1. The mode is not always unique.
2. It is not affected by extreme values.
3. The mode can also be used for qualitative data.

Exercise 6.3

1 **a** Find the mean, mode and median of the following data.

11, 9,14, 3, 11, 4, 10, 21, 8, 15, 350

b Which measure of location is preferable for this data?

2 Given below is a frequency distribution of values V.

a Find the mean, mode and median of the following distribution.

b How many of the values are non-negative?

V	–2	–3	0	1	2	3
f	3	2	3	6	5	1

3 Given the numbers 5, 6, 7, 10, 12, which number must be removed in order to make the mean of the resulting values 7.5?

4 Given the numbers 10, 12, 9, 15, 8, what number could be included so that the median is 11? (Explain)

5 Given 3, 4, x, 5, y,12. Find the values of x and y, if the mode of the data is 3 and the mean is 6.

6 If the mean of a, b, c, d is k, then what is the mean of

a $a + b$, $2b$, $c + b$, $d + b$? **b** ab, b^2, cb, db?

7 Calculate the mean, median and mode of the following data;

Value	10	15	20	25	30	35	40
Frequency	15	10	50	4	10	8	3

8 In a survey of the number of occupants of cars, the following data resulted.

Number of occupants	1	2	3	4
Number of cars	7	11	7	x

a If the mean number of occupants is $2\frac{1}{3}$, find x.

b If the mode is 2, find the largest possible value of x.

c If the median is 2, find the largest possible value of x.

9 A researcher tabulated the number of cases heard by 8 judges on a given day in a court and found the following data:

Judges	1	2	3	4	5	6	7	8
Count of cases	6	3	1	2	0	5	5	4

a Find the mean, median and mode.

b The researcher reported that over half of the judges heard above "average". What does the researcher mean by the "average"?

10 The following raw data represents the number of HIV/AIDS patients waiting for counselling at 8:00 am on 40 consecutive Saturdays at a certain hospital.

11	6	5	8	11	6	3	7	4	6
5	4	13	14	9	11	13	8	10	9
10	9	6	5	10	7	8	7	8	3
8	7	8	9	6	10	11	8	8	4

a Draw a frequency distribution table.

b Calculate the mean, median and modal number of HIV/AIDS patients.

c Draw a histogram.

11 In a mathematics test the scores for boys were 6, 7, 8, 7, 5 and the scores for girls were 6, 3, 9, 8, 2, 2, 5, 7,3

a Find the mean score for the boys.

b Find the mean score for the girls.

c Find the mean score for both the boys and girls.

d What do you conclude?

12 The mode of some data is 20. If each value in the data is increased by 2, what will be the mode of the new data?

13 Find the mean, median and mode of the data represented by the histogram below.

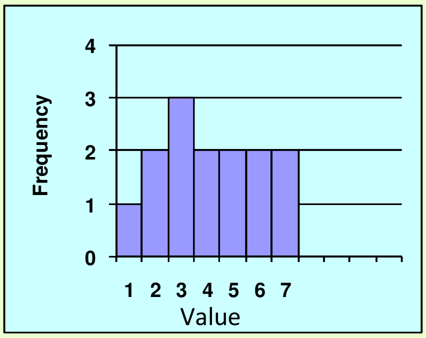


Figure 6.3

14 An Agricultural Development Station sells seedlings of plant through the post. It claims that the average height of the plants after one year's growth will be 85 cm. A sample of 24 of the plants were measured after one year with the following results (in cm).

6	7	7	9	34	56	85	89	89	90	90	91
91	92	93	93	94	95	95	96	97	97	99	93

a Find the mean and the median height of the sample.

b Is the station's claim about average height justified?

15 In order to receive a grade of A in her mathematics exam, Abeba needs a mean score of 90 and above on 4 tests. So far Abeba had scored 80, 91 and 93 on 3 tests. What is the lowest score that she must get in her last test in order to receive a grade of A?

6.1.4 Measures of Dispersion for Ungrouped Data

When comparing sets of data, it is useful to have a way of measuring the scatter or spread of the data.

Group Work 6.2

Consider the following three sets of data.

Group	Values							Total	Mean	Mode	Median
A	7	7	7	7	7	7	7				
B	4	5	6	7	7	9	1				
C	1	7	12	7	2	19	1				

a Complete the table by finding the sum of each group and the mean, median and mode.

b Are the means equal? Are the modes equal? Are the medians the same?

c Compare the variation of each group?

- **i** Which group shows most variation?
- **ii** Which group shows no variation?
- **iii** Which group shows slight variation?

d Compare the difference between the mean and each observed value in Group A, B and C.

- **i** In which group is the mean closest to each value?
- **ii** In which group is the difference between the mean and each data value the largest?

e Calculate the range for each group.

Dispersion or Variation is the scatter (or spread) of data values from a measure of central tendency.

There are several measures of dispersion that can be calculated for a set of data. In this section, we will consider only three of them, namely, the range, variance and the standard deviation.

1 Range

The simplest and the most crude measure of dispersion of quantitative data is the range.

Definition 6.9

The range R of a set of numerical data is the difference between the highest and the lowest values. i.e.,

Range = Highest value – Lowest value

Example 1 The ages of six students are 24, 20, 18, 13, 16, 15 years, respectively. What is the range?

Solution: Range = highest value – lowest value = 24 – 13 = 11 years.

Example 2 Find the range of the distribution given in the table below.

V	2	8	9	13	15	18
f	3	4	2	1	5	4

Solution: The maximum value is 18 and the minimum value is 2.

Range = maximum value – minimum value = 18 – 2 = 16

2 Variance (σ^2)

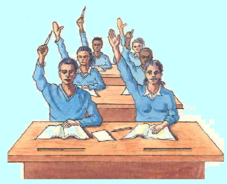
Definition 6.10

Variance, denoted by (σ^2), is defined as the mean of the squared deviations of each value from the arithmetic mean.

3 Standard deviation (σ)

The following Activity will help you to learn the steps used to find variance and standard deviation.

ACTIVITY 6.7



Consider the following data set:

2, 3, 10, 6, 9

a Find the mean $\bar{x}$.

b Find the deviation of each data value from the mean $(x - \bar{x})$.

c Square each of the deviations $(x - \bar{x})^2$.

d Find the mean of these squared deviations and its principal square root.

The standard deviation is the most valuable and widely used measure of dispersion.

Definition 6.11

Standard deviation, denoted by σ, is defined as the positive square root of the mean of the squared deviations of each value from the arithmetic mean.

The actual method of calculating σ can be summarized in the following steps:

Step 1 Find the arithmetic mean $\bar{x}$ of the distribution.

Step 2 Find the deviation of each data value from the mean $(x-\bar{x})$.

Step 3 Square each of these deviations, $(x-\bar{x})^2$.

Step 4 Find the mean of these squared deviations. This value is called the variance and is denoted by σ^2.

Step 5 Take the principal square root of σ^2, i.e.

Standard deviation = $\sqrt{\text{variance}}$.

Example 3 Find the variance σ^2 and the standard deviation σ for the following data:

3, 5, 8, 11, 13

Solution:

x	$(x-\bar{x})$	$(x-\bar{x})^2$.
3	–5	25
5	–3	9
8	0	0
11	3	9
13	5	25
Total 40		68

Variance $(\sigma^2) = \frac{68}{5} = 13.6$

Standard deviation $(\sigma) = \sqrt{\sigma^2} = \sqrt{13.6} \approx 3.7$

Example 4 Find the variance and standard deviation of the population function whose distribution is given in the following table.

V	2	3	5	6	8
f	3	4	4	5	4

Solution: First, the mean has to be calculated.

$$\bar{x} = \frac{3\times2+4\times3+4\times5+5\times6+4\times8}{3+4+4+5+4} = \frac{100}{20} = 5$$

x	f	xf	$x-\overline{x}$	$(x-\overline{x})^2$	$f(x-\overline{x})^2$
2	3	6	-3	9	27
3	4	12	-2	4	16
5	4	20	0	0	0
6	5	30	1	1	5
8	4	32	3	9	36
Total	20	100	0		84

Variance $(\sigma^2) = \dfrac{84}{20} = 4.2$

Standard deviation $(\sigma) = \sqrt{4.2} \approx 2.05$

Therefore, the population variance and standard deviation are 4.2 and 2.05 respectively.

Properties of variance and standard deviation

Group Work 6.3

Consider the following data which shows the amount of sugar in kilograms sold by a small shop for five days.

6, 4, 8, 9, 3

- **i** Find the mean.
- **ii** Find the variance and standard deviation.
- **iii** In the next five days, if the daily sales each increase by two kg.
  - **a** Find the mean of sales for the next five days.
  - **b** Find the variance and standard deviation of the sales for the next five days.
  - **c** Compare your answers above with those obtained in a and b above.
  - **d** Discuss the comparison you did above.
- **iv** If the daily sales given for the first five days were doubled i.e. if the daily sales were 12, 8, 16, 18 and 6,
  - **a** find the mean, variance and standard deviation.
  - **b** compare the above result with those one obtained in a and b and discuss the results.

The above Group Work will help you to observe the following properties.

1 If a constant c is added to each value of a population function, then the new variance is the same as the old variance. The new standard deviation is also the same as the old standard deviation.

Proof:-

Let $x_1, x_2, x_3,..., x_n$ be n observations with mean $\bar{x}$ and variance σ^2.

Adding c: $x_1+c,\ x_2+c,\ x_3+c,\ ...,\ x_n+c$. Then the new mean is $\bar{x}+c$.

$$\text{New variance} = \frac{\left[(x_1+c)-(\bar{x}+c)\right]^2+\left[(x_2+c)-(\bar{x}+c)\right]^2+...+\left[(x_n+c)-(\bar{x}+c)\right]^2}{n}$$

$$= \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+...+(x_n-\bar{x})^2}{n} = \sigma^2. \text{ (}\textit{The original variance}\text{)}$$

and the new standard deviation is $\sqrt{\text{variance}} = \sqrt{\sigma^2} = \sigma$ (*the original standard deviation*)

Example 5 Given 1, 2, 6, 3

a Find the variance **b** Find the standard deviation.

c Add 2 to each value and find the variance and standard deviation of the resulting numbers

Solution: $\bar{x} = \frac{1+2+6+3}{4} = 3$

$x-\bar{x}: -2, -1, 3, 0$ and $(x-\bar{x})^2: 4, 1, 9, 0$

a $\sigma^2 = \frac{4+1+9+0}{4} = \frac{14}{4} = 3.5$

b $\sigma = \sqrt{3.5} \approx 1.87$

c Adding 2: 3, 4, 8, 5

New mean: $\bar{x} = \frac{3+4+8+5}{4} = \frac{20}{4} = 5 = 3+2$

$x-\bar{x}: -2, -1, 3, 0$ and $(x-\bar{x})^2: 4, 1, 9, 0$

New: $\sigma^2 = \frac{4+1+9+0}{4} = \frac{14}{4} = 3.5$

New $\sigma = \sqrt{3.5} \approx 1.87$

Therefore, the old variance = the new variance

The old standard deviation = the new standard deviation.

2 If each value of a population function is multiplied by a constant c, then

i The new variance is c^2 times the old variance

ii The new standard deviation is $|c|$ times the old standard deviation.

Proof:-

Consider $x_1, x_2, \ldots, x_n$ whose mean is $\bar{x}$ and variance is σ^2.

Multiplying each data value by c gives us a new mean of $c\bar{x}$.

$$\text{Then, new variance} = \frac{(cx_1 - c\bar{x})^2 + (cx_2 - c\bar{x})^2 + (cx_3 - c\bar{x})^2 + \ldots + (cx_n - c\bar{x})^2}{n}$$

$$= \frac{c^2[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2]}{n}$$

$$= c^2 \times \text{the old variance} = c^2\sigma^2$$

Therefore, new standard deviation $= \sqrt{c^2\sigma^2} = |c|\sigma$

Exercise 6.4

1 Find the range, variance and standard deviation of the following data.

4, 2, 3, 3, 2, 1, 4,3, 2, 6

2 Find the range, variance and standard deviation of the distribution in the table below.

V	–1	–2	0	1	2
f	2	1	3	3	1

3 Find the range, variance and standard deviation from the histogram in the figure below.

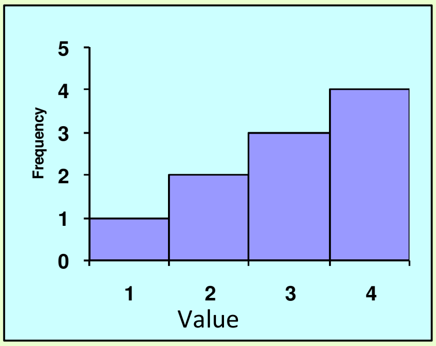


Figure 6.4

4 What is the value of y, if the standard deviation of 8, 8, 8, 8, y, 8 is 0?

5 If the variance of a, b, c, d is k, then what is

a the variance of $a + c, b + c, 2c, d + c$?

b the standard deviation of $a + c, b + c, 2c, d + c$?

c the variance of ac, bc, c^2, dc?

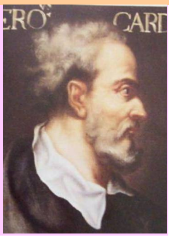
d the standard deviation of ac, bc, c^2, dc?

6 If a population function x has mean $M(x) = 2$ and $M(x^2) = 8$, find its standard deviation.

6.2 PROBABILITY

"The true logic of this world is the calculus of probabilities". James Clerk Maxwell

HISTORICAL NOTE:



The first inquiry into the science of Probability was made by Girolamo Cardano (1501-1576), an Italian physician and mathematician. Cardano predicted the date of his own death. Since he was healthy at the end of the day, he poisoned himself to make his prediction come true!

In your Grade 8 lessons, you have discussed the word probability as you often use it. "*The probability of winning a game is low*", or "*there is a high probability that it will rain today*", etc. In these two sentences, the word probability describes estimates of the possibilities.

Probability is a numerical value that describes the likelihood of the occurrence of an event in an experiment.

The following group work will help you recall what you have learned on this topic in Grade 8.

Group Work 6.4

Abel throws a fair die once. Based on this experiment, discuss the following:

1. Is it possible to predict the number that shows on the upper face of the die? Why?
2. List the set of all possible outcomes.
3. Write an example of an event from the experiment.
4. What can you say about the following events?
   - i The number on the upper face of the die is seven.
   - ii The number on the upper face of the die is an integer.
     - a Which of the above events i or ii is certain?
     - b Which of the above events i or ii is impossible?
5. Determine the probabilities of the following events.
   - a The number on the upper face of the die is 2.
   - b The number on the upper face of the die is 7.
   - c The number on the upper face of the die is less than 7.
6. Discuss the following terms.
   - a experiment
   - b possibility set
   - c event
   - d impossible event
   - e certain event

Definition 6.12

An experiment is a trial by which an observation is obtained but whose outcome cannot be predicted in advance.

Experimental probability

Probability determined using data collected from repeated experiments is called experimental probability.

Example 1 The numbers 1 to 20 are each written on one of 20 identical cards. One card is chosen at random.

a List the set of all possible outcomes.

b List the elements of the following events:

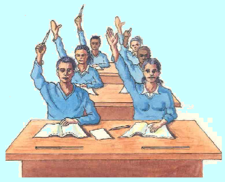
- **i** The number is less than 5.
- **ii** The number is greater than 15.
- **iii** The number is greater than 21.
- **iv** The number is divisible by 5.
- **v** The number is a prime.

Solution:

a S = {1, 2, 3, …, 19, 20}

b
- **i** {1, 2, 3, 4}
- **ii** {16, 17, 18, 19, 20}
- **iii** { } or ∅, since no card has a number greater than 20.
- **iv** {5, 10, 15, 20}
- **v** {2, 3, 5, 7, 11, 13, 17, 19}

ACTIVITY 6.8



Arrange yourselves into groups of 5. Let each group perform the following activities.

1 Take a coin, toss it 5 times, 10 times and 15 times, and record your observations in the following table.

	Number of tosses			Total
Number of times a coin is tossed	5	10	15	
Number of times the coin shows up Heads				
Number of times the coin shows Tails				

What proportion of the number of tosses shows Heads? a Tails? What is the probability that the outcome is Head?

2 Throw a die 20 times. Record the observation in each trial and complete the following table.

Number on the upper face of the die	1	2	3	4	5	6
Number of times it shows up						

- **a** Find the number of times 3 is on the upper face of the die.
- **b** Find the number of times 6 is on the upper face of the die.
- **c** Find the number of times 7 is on the upper face of the die.
- **d** Write the proportion of each number.
- **e** What is the probability that the number that shows up on the upper face of the die is 4?

Suppose we toss a coin 100 times and get a head 45 times, and a tail 55 times. Then we would say that in a single toss of a coin, the probability of getting a head is $\frac{45}{100} = \frac{9}{20}$. Again suppose we toss a coin 500 times and get a head 260 times, and a tail 240 times. Then we say that in a single toss of a coin, the probability of getting a head is $\frac{260}{500} = \frac{13}{25}$. So from various experiments, we might obtain different probabilities for the same event. However, if an experiment is repeated a sufficiently large number of times, the relative frequency of an outcome will tend to be close to the theoretical probability of that outcome.

Definition 6.13

The possibility set (or sample space) for an experiment is the set of all possible outcomes of the experiment.

Example 2

- **a** Give the sample space for tossing a coin.
- **b** What is the sample space for throwing a die?

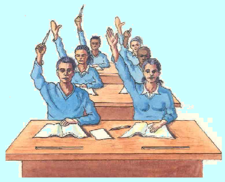
Solution:

- **a** When we toss a coin there are only two possible outcomes: Heads (H) or Tails (T). So S = {H,T}.
- **b** When we throw a die the score can be any of the six numbers 1, 2, 3, 4, 5, 6, so S = {1,2,3,4,5,6}.

Definition 6.14

An event is a subset of the possibility set (sample space).

ACTIVITY 6.9



Suppose we toss a coin 1000 times and obtain 495 heads.

a How many times was the experiment performed?

b If our event is Heads, how many times does this event occur?

c What is the probability of Heads based on the result of this experiment?

Definition 6.15

If an experiment has n equally likely outcomes and if m of these represent a particular event, then the probability of this event occurring is $\frac{m}{n}$.

Example 3 In an experiment of selecting students at random a researcher found the following result after 50 trials.

Student	Boy	Girl	Total
Number	20	30	50

What is the probability that a randomly selected student is a girl?

Solution The probability that a randomly selected student is a girl will be the ratio of the number of girls to the total number of trials.

$$P(\text{a girl will be selected}) = \frac{30}{50} = \frac{3}{5}$$

In decimal form the probability is 0.6.

A tree diagram is one way of showing the possible outcomes of a repeated experiment.

Example 4 In an experiment of tossing two coins,

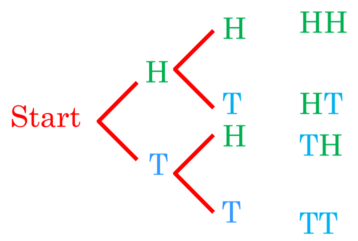
a What are the possible outcomes?

b How many different possible outcomes are there?

c What is the probability of the coins landing with

i two heads? **ii** two tails? **iii** one head?

Solution: Using a tree diagram, we get



a The set of possible outcomes is S = { HH,HT,TH,TT}.

b There are 4 possible outcomes.

c **i** The event two heads = {HH} has one member, so

$$\text{P (two heads)} = \frac{1}{4}$$

ii $\text{P (two tails)} = \frac{1}{4}$

iii The event one head = {HT,TH} has two members, so

$$\text{P (one head)} = \frac{2}{4} = \frac{1}{2}$$

In real situations, it might not always be possible to perform an experiment and calculate probability. In such situations, we need to develop another approach to find the probability of an event.

In the next section, you will discuss a theoretical approach of finding probabilities.

Theoretical probability of an event

Definition 6.16

The theoretical probability of an event E, written as $P(E)$ is defined as follows:

$$P(E) = \frac{\text{Number of outcomes favourable to the event } E}{\text{Total number of possible outcomes } (S)}$$

You can write the probability of an event as a fraction, a decimal, or a percentage.

Example 5 A fair coin is tossed once. What is the probability of getting a head?

Solution:

S = { H, T}

E = {H}

$$\text{P (head)} = \frac{n(E)}{n(S)} = \frac{1}{2} = 0.5$$

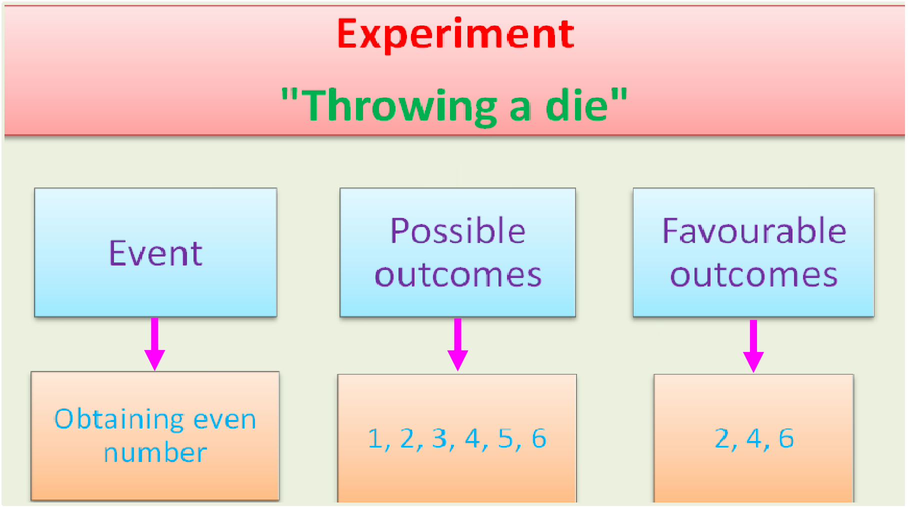
Example 6 If we throw a die once, what is the probability that an even number will show on the upper face of the die?

Solution:

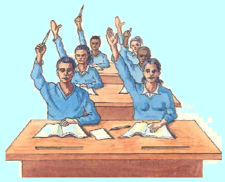
S = {1, 2, 3,4, 5, 6}

E = {2, 4, 6}

$$P(\text{even}) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$



ACTIVITY 6.10



We are going to investigate whether the theoretical probability of a coin landing on Heads is backed up by experimental results.

a Toss a coin 10 times, 20 times, 30 times . . . .

b Keep a record of your results,

Number of throws	Number of heads
10	
20	
30	

c For each row in the table, what proportion of the number of throws landed as heads?

How do your answers compare with P (head) = $\frac{1}{2}$? (the theoretical probability)

Definition 6.17

Let S be the possibility set of an experiment and each element of S be equally likely to occur. Then the probability of the event E occurring, denoted by $P(E)$, is defined as:

$$P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}$$

Example 7 A die is thrown once. What is the probability that the number appearing will be

a 3? **b** a number less than 5?

Solution: There are six possible outcomes: {1, 2, 3, 4, 5, 6}. Hence $n = 6$.

a Only one of these outcomes is 3. Hence the probability that 3 will be on the upper face of the die is $\frac{1}{6}$.

b {1, 2, 3, 4} is the required set, which has four elements. Hence the probability is $\frac{4}{6} = \frac{2}{3}$.

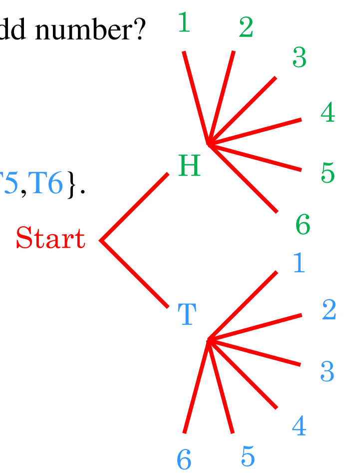
Example 8 A die and a coin are tossed together.

a Sketch a tree diagram showing the outcomes of this experiment.

b What is the probability of getting a head and an even number?

c What is the probability of getting a tail and an odd number?

Solution:



a The outcomes of this experiment are:

S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5,T6}.

So, $n(S) = 12$

b E_1 = {H2,H4,H6} . Hence $P(E_1) = \frac{3}{12} = \frac{1}{4}$.

c E_2 = {T1, T3, T5}. Hence $P(E_2) = \frac{3}{12} = \frac{1}{4}$.

Example 9 Use a tree diagram to list the sample space (possibility set) showing the possible arrangement of boys and girls in a family with exactly three children.

a What is the probability that all three children are boys?

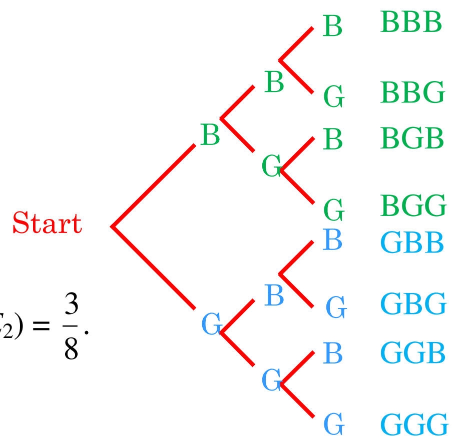
b What is the probability that two children are boys and one is a girl?

c What is the probability that none of the children is a boy?

d What is the probability that at least one of the children is a girl?

e What is the probability that all three children are of the same sex?

Solution:



S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

Thus, $n(S) = 8$.

a E_1= {BBB}. Hence $P(E_1) = \frac{1}{8}$.

b E_2= {BBG, BGB, GBB}. Hence $P(E_2) = \frac{3}{8}$.

c E_3 = {GGG}. Hence $P(E_2) = \frac{1}{8}$.

d E_4= {BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

Hence $P(E_4) = \frac{7}{8}$.

(Alternatively, having at least one girl is all outcomes except BBB.

i.e., $8 - 1 = 7$ outcomes, giving the same result, $P(E_4) = \frac{7}{8}$).

e E_5 = {BBB, GGG}. Hence, $P(E_5) = \frac{2}{8} = \frac{1}{4}$.

Note: For any event E,

✓ $0 \leq P(E) \leq 1$.

✓ If $P(E) = 1$ then the event is certain.

✓ If $P(E) = 0$, then the event is impossible.

For example, if a ball is taken from a box containing only red balls, then

P (ball is red) = 1 and P (ball is black) = 0.

Exercise 6.5

1 Two dice are simultaneously thrown once. List the ways in which the following events can occur.

a A= the same number is shown on each die.

b B = the sum of the numbers is 13.

c C = the product of the two numbers is 1.

d D = the quotient of the two numbers is 7.

2 Three coins are tossed at the same time. Sketch a tree diagram for the outcomes of this experiment. What is the possibility set?

3 A bag contains four red balls and three black balls. What is the possibility set for colour, if 2 balls are chosen at random?

4 Toss a coin and keep a record of whether it lands on Heads or Tails. Do this at least 20 times for each experiment and perform at least five experiments. Enter your results in a table like the following.

Experiment	Number of coin tosses	Number of heads obtained
1		
2		
3		
4		
5		
Total		

a Do you feel that the two outcomes “head” and “tail” are equally likely?

b Do your experimental results support this feeling?

c What is the ratio of the number of heads to the number of tosses in each experiment?

d What ratio do you have for the total number of heads to the total number of tosses?

5 A fair die is rolled once. Calculate the probability of getting:

a an odd number **b** a score of 5

c a prime number **d** a score of 0

6 A number is selected at random from the set of whole numbers 1 to 20, both inclusive. Find the probability that the number selected is:

a even **c** a multiple of 3 **e** the square of 2

b a multiple of 2 and 3 **d** even or odd **f** the square of 6

7 A bag contains five red balls, three black balls and four white balls. A ball is drawn out of the bag at random. What is the probability that the ball drawn is:

a white? **b** red? **c** black?

8 A bag contains 100 identical cards on which the numbers 1 to 100 are marked. A card is drawn randomly. Find the probability that the number on the card is:

a an even number **b** an odd number **c** a multiple of 7

d a multiple of 5 **e** a multiple of 3 **f** less than 76

g greater than 32 **h** a factor of 24

Key Terms

analysis	measure of central tendency	range
arithmetic mean	measure of dispersion	raw data
average	measure of location	sample
classification	median	sample space
collection	mode	secondary data
descriptive statistics	outcomes	standard deviation
equally likely	presentation	statistical data
event	population	statistics
frequency	population function	tabulation
frequency distribution	possibility set	variable (or variate)
histogram	primary data	variance
interpretation	probability	

Summary

1 Statistics is the science of collecting, organizing, presenting, analysing and interpreting data in order to draw conclusions.

2 A population is the complete collection of individuals, objects or measurements that have a characteristic in common.

3 A small part (or a subset) of a population is called a sample.

4 If the categories of a classification are based on some attribute or characteristics whose values are not numbers, then it is called qualitative classification.

5 If the characteristic of interest is numerical, then it is called quantitative classification.

6 Descriptive statistics is a branch of statistics concerned with summarizing and describing a large amount of data.

7 Data is said to be primary, if it is obtained first-hand for the particular purpose on which one is currently working.

8 Data that has been previously collected for a similar or different purpose is called secondary data.

9 A statistical table is a systematic presentation of data in columns and rows.

10 The quantity that we measure from observation is called a variate (or variable).

11 The distribution of a population function is the function that associates with each variate of the population function the corresponding frequency.

12 A frequency distribution is a distribution showing the number of observations associated with each data value.

13 A histogram is a pictorial representation of a frequency distribution in which the variables (*V*) are plotted on the *x*-axis and the frequency of occurrence is plotted on the *y*-axis.

14 If $x_1, x_2, x_3, \ldots x_n$ are the n observations of a variable then the mean ($\bar{x}$) is given by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

15 The median of a variable is the value that lies in the middle of the data when arranged in ascending or descending order.

16 The mode of a variable is the most frequent observation of the variable that occurs in the data set.

17 The range R of a set of numerical data is the difference between the maximum and minimum values.

Range = maximum value − minimum value

18 Standard deviation is the square root of the mean of the squared deviation of each value from the arithmetic mean.

19 The outcomes of an experiment are said to be equally likely if, when the experiment is repeated a large number of times, each outcome occurs equally often.

20 The possibility set for an experiment is the set of all possible outcomes of the experiment. It is also known as the sample space.

21 An event is a subset of the possibility set.

22 If S is the possibility set of an experiment and each element of S is equally likely, then the probability of an event E occurring, denoted by $P(E)$, is defined as:

$$P(E) = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{n(E)}{n(S)}$$

Review Exercises on Unit 6

1 What is meant by summarizing and describing data?

2 The marks of 30 students in a mathematics test are given below:

3	5	4	6	8	12	14	5	6	5
8	5	9	10	9	10	12	10	12	10
12	13	10	15	14	15	14	15	14	14

a Construct a frequency distribution table.

b Draw a histogram to represent the data.

c What percent of the students have scored less than 15?

3 Refer to the following histogram to answer the questions that follow.

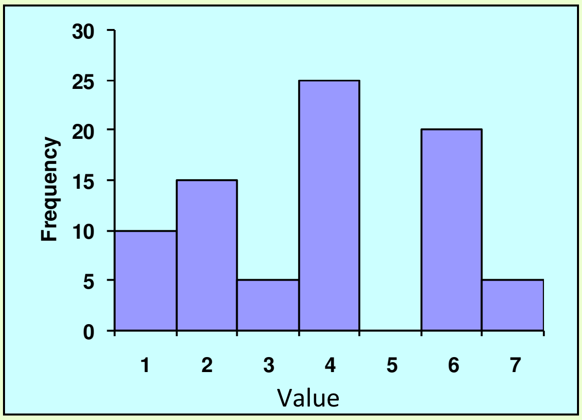


Figure 6.5

a Prepare a frequency distribution table.

b What is the highest variable?

c What is the highest frequency?

d How many variates occur 5 times?

e Which variates have the minimum frequency?

4 Find the mean, median, mode, range, variance and standard deviation of the population function whose distribution is given in the table below.

V	2	3	4	5	6
f	2	4	1	2	3

5 Find the mean, median, mode, range and variance from the histogram given below.

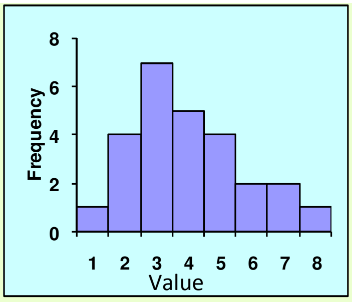


Figure 6.6

6 Why can the probability of an event not be $2\sqrt{2}-\pi$, $\frac{\pi}{2}$ or $\frac{3}{2}$?

7 An integer n, $1 \leq n \leq 144$, is picked at random. What is the probability that n is the square of an integer?

8 Given the following values of a population function:

5, 4, 7, 3, 6, 5, 3, 1, 5, 7, 5, 9.

Find the probability that a randomly chosen value from the data is

a a modal value;

b below the mean value;

c any of the numbers 1, 4, 6 or 9;

d an odd number greater than the mean value.

9 Two fair dice are rolled once. What is the probability that the difference of the two numbers shown is 1?

10 Given below is the frequency distribution of a population function V.

V	−10	−5	0	5	10
f	5	10	5	20	10

If an element is drawn randomly from the population find the probability that it is:

a non-negative; **b** non-zero;

c less than or equal to −5; **d** positive;

11 The median of $x - 4$, x, $2x$ and $2x + 12$ is 9, where x is a positive integer. Find the value of x.

12 The table below shows the number of students who scored marks 3, 4 or 5 in a maths test.

Mark	3	4	5
Number of Students	3	x	4

If the mean mark is 4.1, how many students scored 4?

13 In a class of boys and girls, the mean weight of 8 boys is 55 kg and the mean weight of a group of girls is 48 kg. The mean weight of all the children is 50.8 kg. How many girls are there?

14 There are 24 right-handed students in a class of 30. What is the probability that a student chosen at random will be left-handed?

15 Suppose you write the days of the week on identical pieces of paper. You mix them in a bowl and select one at a time. What is the probability that the day you select will have the letter r in it?

16 A pair of dice are rolled. Find the probability that the sum of the numbers on the upper faces is:

a 9; **b** greater than 9; **c** even; **d** not greater than 9;

e greater than 9 and even; **f** greater than 9 or even.

17 From the members of a farmers' association 50 farmers cultivated wheat. An agricultural expert wants to study the farmers' yield in terms of quintals they harvested per hectare and found the following

50	45	45	50	46	48	55	48	52	54
51	52	45	55	46	50	55	54	49	51
48	46	51	52	47	45	49	54	46	48
53	52	48	46	55	47	51	47	50	53
47	53	48	45	54	48	50	46	52	54

a Prepare a frequency distribution that represents the data.

b Draw a histogram.

c Find the mode of the data.

d If the wereda agriculture office wants to praise farmers who produced more than 52 quintals per hectare, how many farmers will they praise?

18 Which of the following is true?

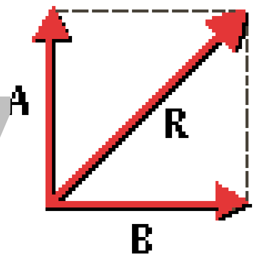
a The mean, mode and median of a population function cannot be equal.

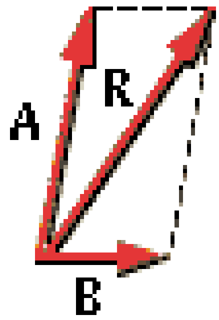
b The range and the standard deviation of a population function are inversely related.

c The range of a population function cannot be a non-positive number.

d The sum of the deviations of each value of a population from the mean will always be zero.

Unit 7





VECTORS IN TWO DIMENSIONS

Unit Outcomes:

After completing this unit, you should be able to;

- *know basic concepts and specific facts about vectors.*
- *perform operations on vectors.*

Main Contents

INTRODUCTION

From previous grades, you know about measurements of different kinds such as height, weight, temperature, distance, angle measure, area, etc. Such quantities assume real numbers as their measure (with some unit of measurement). For example, the height of a room is 3 m, the weight of a quintal is 100 kg, the distance between two walls in a classroom is 8 m, the temperature of a normal person is 36.5°C, the area of a triangle ABC is 6 cm^2, etc. Not all quantities, however, assume only a single real number as their measure. There are some quantities that assume measures involving directions.

Example Suppose we are in School A, and someone has told us that he studies at a nearby school B that is *d* m away. Do we have enough information to find B? Of course not, because B could be at any point on a circle of radius *d* m centred at A. In addition to the distance, we need to know the direction in order to find B.

There are many physical quantities whose measurements involve both magnitude and direction. These include velocity, force, acceleration, electric or magnetic fields, etc. Such quantities are called vectors. Today vectors have many applications. All branches of classical and modern physics are represented by using the language of vectors. Vectors are also used with increasing frequency in the social and biological sciences. In this unit, you will deal with vectors, in particular vectors in two dimensions.

HISTORICAL NOTE:

Sir William Rowan Hamilton (1805-1865)

The study of vectors started with Hamilton's invention of quaternions. Quaternions were developed as mathematical tools for the exploration of physical space. As quaternions contained both scalar and vector parts, difficulties arose when these two parts were treated simultaneously.

Scientists soon learned that many problems could be dealt with by considering vector parts separately, and the study of vector analysis began.

7.1 INTRODUCTION TO VECTORS AND SCALARS

Group Work 7.1

1. Discuss some quantities that can be expressed completely using a single measurement (with units) .
2. Discuss some quantities that require both size and direction.

In general, there are two types of physical measurements: those involving only magnitude and no direction, called scalars and others involving magnitude and a definite direction, called vectors. In many applications of mathematics to the physical and biological sciences and engineering, scientists are concerned with quantities that have both magnitude and direction. As mentioned above, examples include the notion of force, velocity, acceleration, and momentum. It is useful to be able to express these quantities (vectors) both geometrically and algebraically.

ACTIVITY 7.1

Consider the following quantities and identify whether each is a scalar quantity or a vector quantity.

a Amount of rainfall in mm

b Area of a plane figure

c Temperature in a room

d Force of water hitting a turbine

e Gravity

f Acceleration of a motorbike

g Volume of a solid figure

h Speed of an airplane

Scalar quantities

Definition 7.1

Scalar quantities are those quantities of measures that have only magnitude and no direction. (Simply represented by a real number and a specified unit).

Example 1 The length of a side of a triangle is 4 cm. Since 4 is a real number with no direction the length represents a scalar.

Example 2 The height of Mount Ras Dashen is 4550 metres. Here the height is represented by a single real number. Hence it represents a scalar.

Example 3 The daytime temperature of Mercury rises to 430°C. Since 430 is a real number, this temperature represents a scalar.

Vector quantities

Definition 7.2

Vector quantities are those quantities of measure that have both magnitude (length) and direction.

Example 4 The velocity of a car is 80 km/h in the direction of north. This is a vector.

Example 5 Suppose Helen moves, from A, 10 m to the East [E] and then 7 m to the North [N] to reach at B. Show, as a vector, Helen's final displacement.

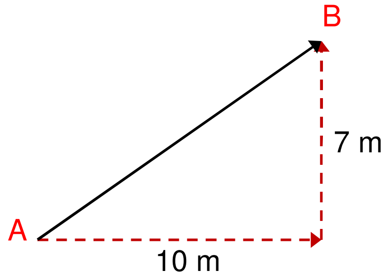


Figure 7.1

Solution: Taken together, the distance and direction of the line from A to B is called the displacement from A to B, and is represented by the arrow in Figure 7.1.

The arrow-head tells us that we are talking about the displacement of Helen from A to B. This is an example of a vector.

7.2 REPRESENTATION OF A VECTOR

ACTIVITY 7.2

1. Discuss algebraic and geometric representations of vectors.
2. Represent the vector $\overrightarrow{OP}$ geometrically, where O is the origin and P = (2, 3) in the *xy*-coordinate system.
3. Discuss the magnitude and direction of a vector.
4. Find the magnitude and the direction of the vector $\overrightarrow{OP}$.
5. When are two vectors equal?

A vector can be represented either algebraically or geometrically. Often, the most convenient way of representing vectors is geometrically, where a vector is represented by an arrow or directed line segment.

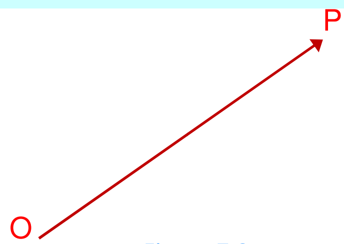


Figure 7.2

When a vector is represented by an arrow (*see* $\overrightarrow{OP}$ above), the point O is called the initial point and P is called the terminal point. Sometimes, vectors are represented using letters or a letter with a bar over it such as $\vec{u}$, $\vec{v}$, etc.

Example 1 What does the vector in the following figure represent?

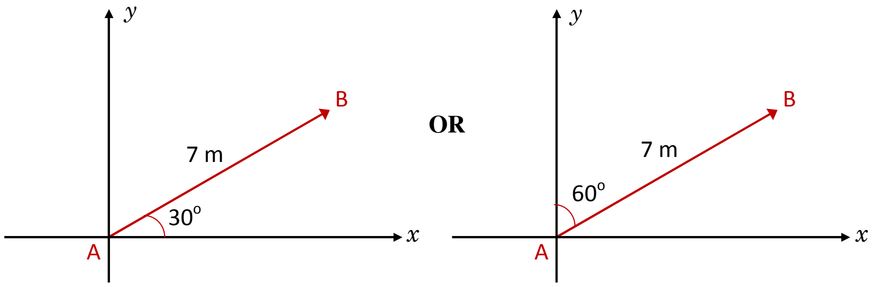


Figure 7.3

Solution: The vector $\overrightarrow{AB}$ has a length of 7 m and direction of East 30° North [E30°N] (or a direction of North 60° East [N60°E]. Its initial point is A and its terminal point is B.

What do you think is the magnitude (length) of a vector and the direction of a vector?

Example 2 The following are examples of vector representation. Can you determine their lengths and directions?

Hint: Use ruler and protractor.

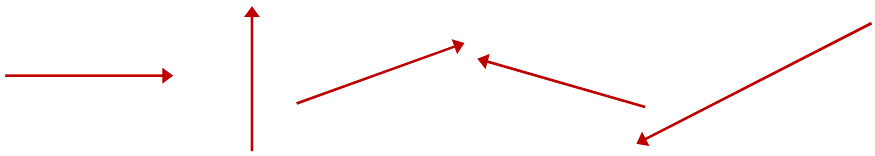


Figure 7.4

Magnitude (length) of vectors

The magnitude (length) of a vector $\overrightarrow{OP}$ or simply $\vec{u}$ is the length of the line segment from the initial point O to the terminal point P, (the length of the directed line segment).

Notation: Magnitude of vector $\overrightarrow{OP}$ is denoted as $\left|\overrightarrow{OP}\right|$.

Example 3 Determine the length of the vector $\overrightarrow{OP}$.

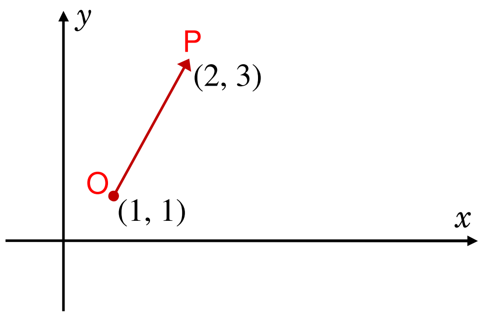


Figure 7.5

Solution: The magnitude of the vector $\overrightarrow{OP}$ is $\left|\overrightarrow{OP}\right| = \sqrt{5}$ (How?)

Example 4 Determine the length in centimetres of each of the vectors shown in the following Figure 7.6.

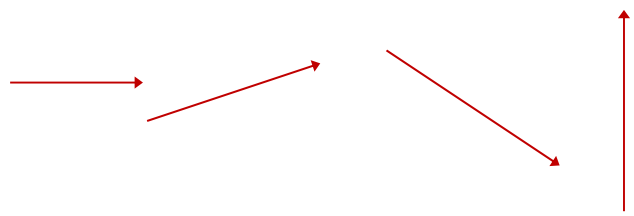


Figure 7.6

Example 5 A force of 10 pounds is exerted vertically down to the surface of the earth and a force of 20 pounds is exerted parallel to the surface of the earth from left to right. The geometric representation is

Figure 7.7

Here, notice that the arrows (directed line segments) are drawn with lengths proportional to the magnitudes. The arrow representing 10 pounds is half the length of the arrow representing 20 pounds.

From this, we realize that the magnitude of a vector is represented by the length of the arrow that represents the vector.

Direction of vectors

The direction of a vector is the angle that is formed by the arrow (that represents the vector) with the horizontal line at its initial point (or with the vertical line in the case of compass directions).

Example 6 The direction of the vector $\vec{u}$ from the horizontal line at its initial point, as represented below, is 45°. (or N45°E)

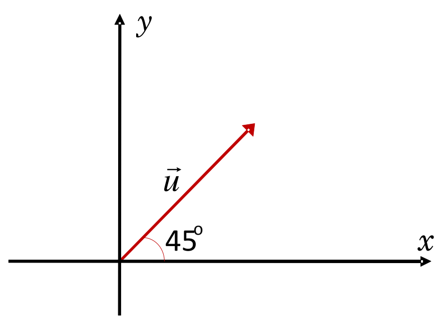


Figure 7.8

Consider the following pair of vectors $\vec{u}$ and $\vec{v}$.

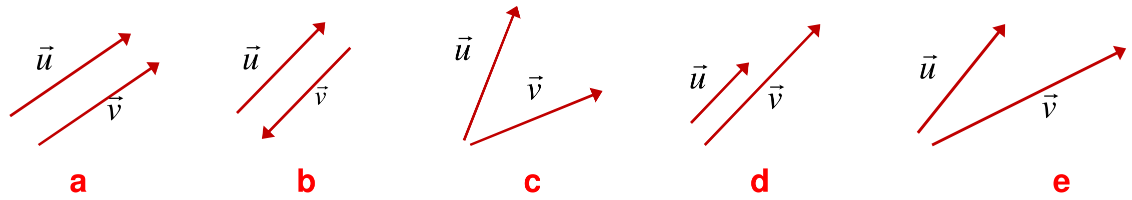


Figure 7.9

What do you observe? Do they have the same length? Do they have the same direction?

The two vectors in a have the same length and they have the same direction. The two vectors in b have the same length but they have opposite directions. The two vectors in c have the same length and different directions. The two vectors in d have different length, but they have same direction. And the two vectors in e differ in both length and direction.

Note:
1. If two vectors have opposite directions, they are called opposite vectors.
2. Vectors that have either the same or opposite directions are called parallel vectors.

Example 7 From the vectors given in Figure 7.9 above, a, b and d are parallel vectors.

When we represent vectors by using directed arrows as given above, we can observe similarities or differences in length or direction. What do you observe from the following vectors?

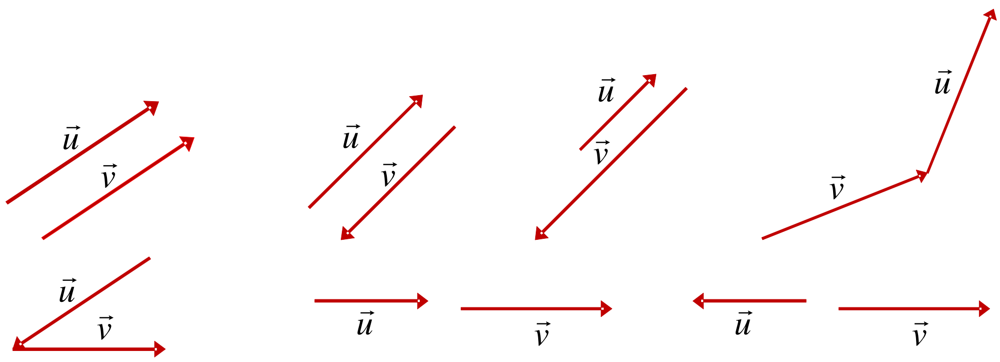


Figure 7.10

Equality of vectors

Two vectors are said to be equal, if they have the same length and the same direction.

Example 8 The following two vectors, $\vec{u}$ and $\vec{v}$, are equal since they have the same length and the same direction. The actual location of these vectors is not specified. We call such vectors free vectors.

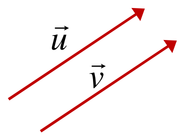


Figure 7.11

Example 9 In each of the diagrams below, all the vectors are equal.

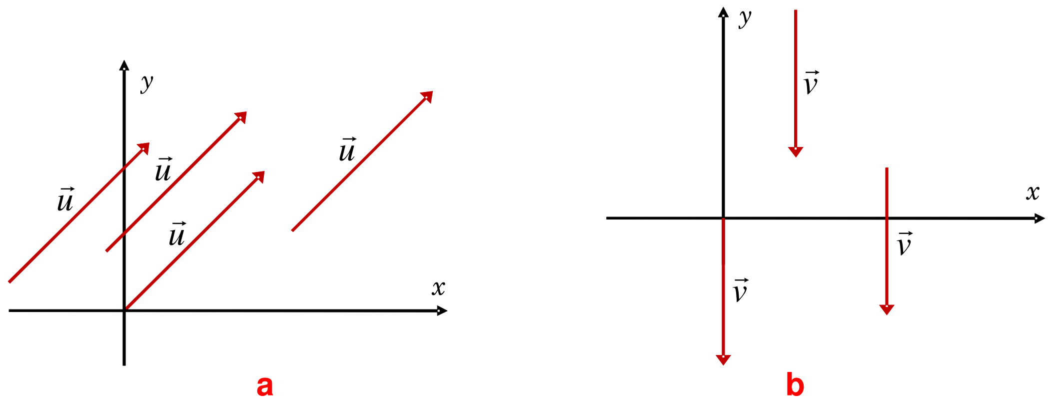


Figure 7.12

Group Work 7.2

1 Suppose vectors $\vec{u}$ and $\vec{v}$ are equal,

- a Can we conclude that they have the same initial point? Why?
- b Do they have the same length? Why?
- c Do they have the same direction? Why?

2 Suppose vectors $\vec{u}$ and $\vec{v}$ are opposite,

- a Can we conclude that they must start from the same initial point? Why?
- b Do they have the same length? Why?
- c Do they have the same direction? Why?

3 Summarize what you have concluded.

Exercise 7.1

1 Determine the magnitude and direction of each of the following vectors.

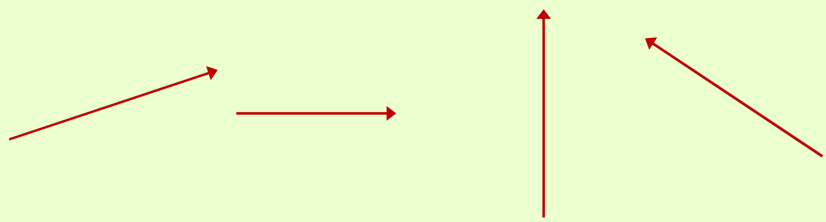


Figure 7.13

2 Locate each of the following vectors on a coordinate system.

- a $\overrightarrow{OP}$ whose length is 3 cm and direction is [N40°E].
- b $\overrightarrow{AB}$ whose length is 5 cm and direction is [S45°E].
- c $\overrightarrow{CD}$ whose initial point is (1, 2), length is 3 cm and direction is [N60°W].

3 From the following, identify the paired vectors that are equal, or opposite.

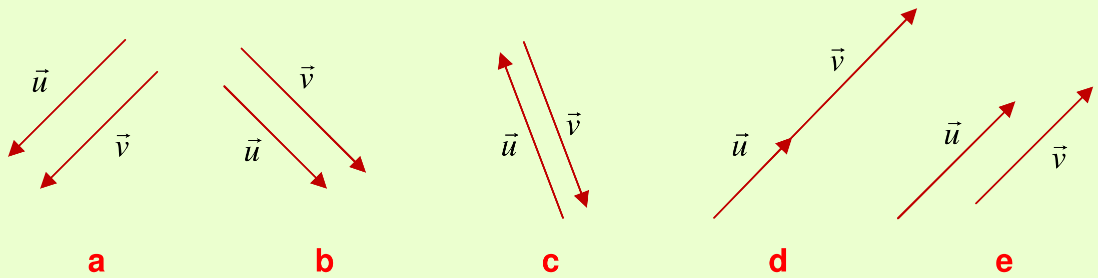


Figure 7.14

7.3 ADDITION AND SUBTRACTION OF VECTORS AND MULTIPLICATION OF A VECTOR BY A SCALAR

7.3.1 Addition of Vectors

ACTIVITY 7.3

Given below are pairs of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Translate $\overrightarrow{CD}$ so that its initial point is at the terminal point of $\overrightarrow{AB}$. Then, how do you express $\overrightarrow{AD}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{CD}$?

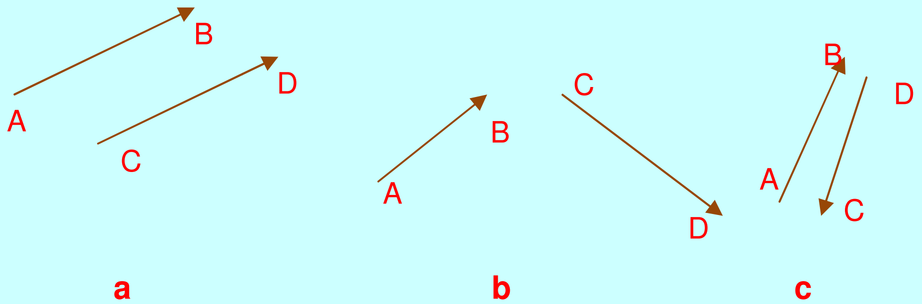


Figure 7.15

1 What is meant by addition of vectors?

2 How would you add vectors?

3 Is the length of the sum of two vectors always equal to the sum of the lengths of each vector? Why?

Suppose you have three cities A, B and C. Assuming you know the distance and direction from A to B and from B to C as shown in Figure 7.16.

If you want to go directly from A to C, what would be the distance and direction?

The first thing to notice is that if the three cities do not lie in a straight line, then the distance from A to C will not be equal to the sum of the distances from A to B and from B to C.

Also, the direction may not be related in a simple or obvious way to the two separate directions.

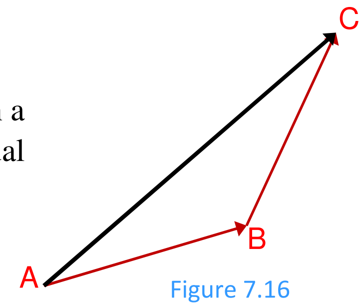


Figure 7.16

You will see, however, that the solution is easy if we work with the components of the displacement vectors. Let the components of the vector from A to B in the east and north directions be a and b, and from B to C in the east and north directions be a' and b', respectively. Then we can see that the component of the displacement vector from A to C in the east direction is $a + a'$, and in the north direction is $b + b'$.

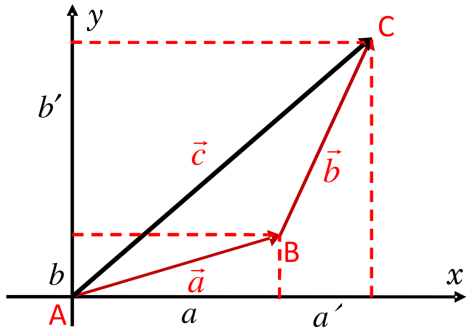


Figure 7.17

From this, we can conclude that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\vec{a} + \vec{b} = \vec{c}$.

We shall discuss addition of vectors using two approaches: the triangle law and the parallelogram law of addition of vectors.

Group Work 7.3

1 Discuss the Triangle law of vector addition.
2 Discuss the Parallelogram law of vector addition.
3 What relation and difference do both laws have?

Triangle law of addition of vectors

Consider the following (Figure 7.18).

Observe that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

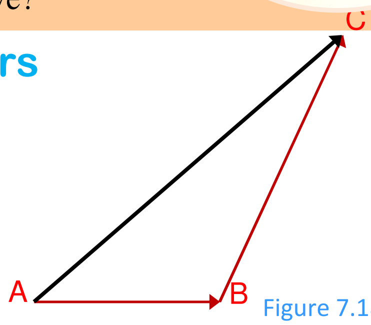


Figure 7.18

Definition 7.3 Triangle law of vector addition

Let $\vec{a}$ and $\vec{b}$ be two vectors in a coordinate system. If $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$, then their sum, $\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BC}$ is the vector represented by the directed line segment $\overrightarrow{AC}$. That is $\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

ACTIVITY 7.4

1 Consider the following figures. Determine the sum $\overrightarrow{AB} + \overrightarrow{BC}$ of each pair of vectors.

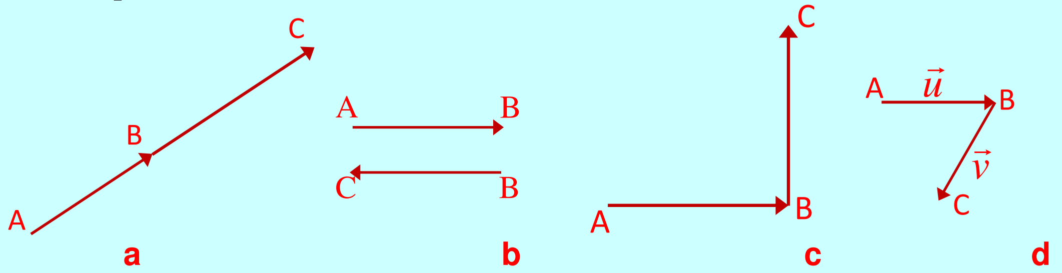


Figure 7.19

By writing the vector addition $\overrightarrow{AB} + \overrightarrow{BC}$, we are looking for that vector whose initial point is A and whose terminal point is C. This vector $\overrightarrow{AC}$ is sometimes called the resultant displacement.

Vector addition can be done either graphically or by separate addition of vector components. We shall discuss the addition of vector components later in this unit.

Example 1 A car travels 4 km to the North and then $4\sqrt{3}$ km to the East. What is the displacement of the car from A to the final position C?

Solution:

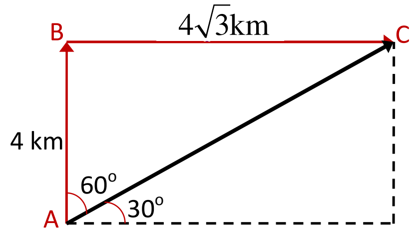


Figure 7.20

The magnitude AC = $\sqrt{4^2 + \left(4\sqrt{3}\right)^2} = \sqrt{16+48} = \sqrt{64} = 8$ and

$\tan(\angle BAC) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{4\sqrt{3}}{4} = \sqrt{3}$. Therefore, $\angle BAC = 60^o$

So the displacement is the vector $\overrightarrow{AC}$, which is 8 km in the direction of North 60^o East.

Example 2 A person moved 10 m to the East from A to B and then 10 m to the West from B to A. Find the resultant displacement.

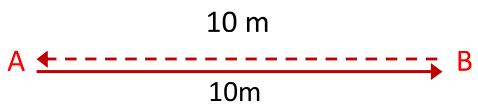


Figure 7.21

Solution: Here we see that the person ends up at A, hence his displacement is zero. From this we see that if we have $\overrightarrow{AB}$ and $\overrightarrow{BA}$, then the sum of these vectors $\overrightarrow{AB}+\overrightarrow{BA}$ vanishes in the sense that the initial point and the terminal point coincide. Such a vector is called a null vector and is denoted by $\vec{0}$ or simply 0. i.e., $\overrightarrow{AB}+\overrightarrow{BA}=0$.

Given $\overrightarrow{AC}$, if $\vec{u}$ is a vector parallel to $\overrightarrow{AC}$ but in opposite direction, then $\vec{u}$ is said to be an opposite vector to $\overrightarrow{AC}$. $-\overrightarrow{AC}$ represents the vector equal in magnitude but opposite in direction to $\overrightarrow{AC}$. That is, $-\overrightarrow{AC}=\overrightarrow{CA}$. Notice that $\overrightarrow{AC}+\overrightarrow{CA}=\overrightarrow{AC}-\overrightarrow{AC}=0$

Example 3 The following are all opposite to vector $\overrightarrow{AC}$.

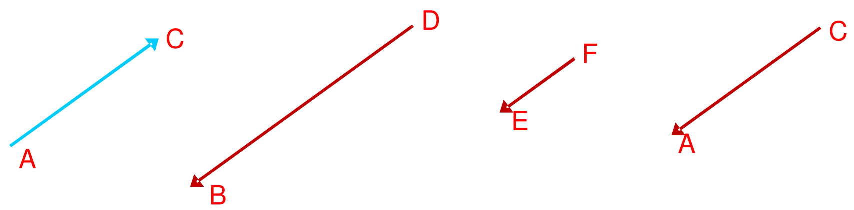


Figure 7.22

That is, vectors $\overrightarrow{CA}$, $\overrightarrow{DB}$ and $\overrightarrow{FE}$ are all opposite to $\overrightarrow{AC}$ (but not equal in magnitude)

Example 4 Consider the vectors $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{CB}$ and $\overrightarrow{AD}$. Determine the following vectors.

a $\overrightarrow{AC}+\overrightarrow{CB}$ **b** $\overrightarrow{AC}+\overrightarrow{CA}+\overrightarrow{AD}$

c $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}$ **d** $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}+\overrightarrow{DA}$

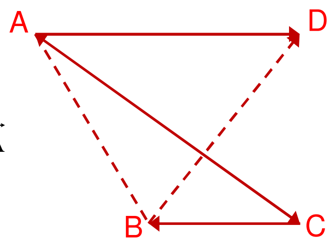


Figure 7.23

Solution:

a $\overrightarrow{AC}+\overrightarrow{CB}=\overrightarrow{AB}$ **b** $\overrightarrow{AC}+\overrightarrow{CA}+\overrightarrow{AD}=\overrightarrow{AD}$

c $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}=\overrightarrow{AD}$ **d** $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}+\overrightarrow{DA}=\vec{0}$

Parallelogram law of addition of vectors

In the above, we saw how the triangle law of addition of vectors is applicable where the initial point of one vector is the terminal point of the other. We may sometimes have vectors whose initial point is the same, yet we need to find their sum.

ACTIVITY 7.5

Consider the vectors $\overrightarrow{AC}$ and $\overrightarrow{AD}$ as given below.

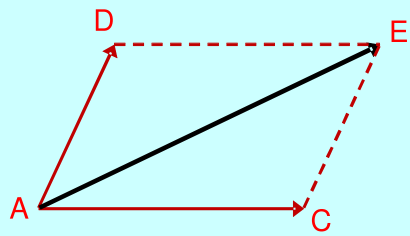


Figure 7.24

Discuss how to determine $\overrightarrow{AC}+\overrightarrow{AD}$.

From previous discussions you notice that if two vectors have the same length and direction, then they are equal.

In Figure 7.25, $\overrightarrow{AD}$ and $\overrightarrow{CE}$ are equal. So $\overrightarrow{AC}+\overrightarrow{AD}$ and $\overrightarrow{AC}+\overrightarrow{CE}$ represent the same vector. But, from triangle law, $\overrightarrow{AC}+\overrightarrow{CE}=\overrightarrow{AE}$

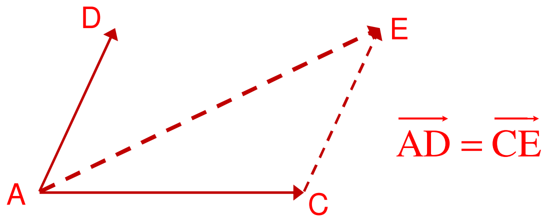


Figure 7.25

Therefore $\overrightarrow{AC}+\overrightarrow{AD}=\overrightarrow{AE}$.

Now let's see how we can construct a parallelogram and see the sum of two vectors (with the same initial point) as the diagonal of the constructed parallelogram.

Example 5 Given the vectors $\overrightarrow{AC}$, $\overrightarrow{AE}$, $\overrightarrow{AD}$, $\overrightarrow{AF}$, $\overrightarrow{AK}$ and $\overrightarrow{AG}$ described in Figure 7.26, determine the following vectors.

a $\overrightarrow{AC}+\overrightarrow{AE}$ **b** $\overrightarrow{AC}+\overrightarrow{AD}$

c $\overrightarrow{AE}+\overrightarrow{AD}$

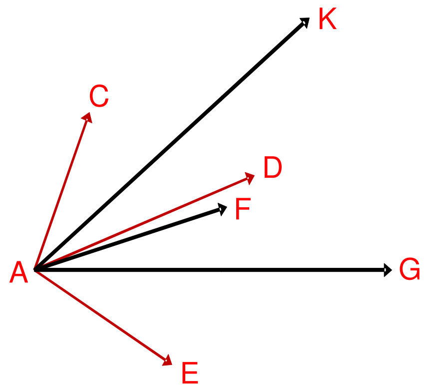


Figure 7.26

Solution: Construct a parallelogram and see that

a $\overrightarrow{AC}+\overrightarrow{AE}=\overrightarrow{AF}$

b $\overrightarrow{AC}+\overrightarrow{AD}=\overrightarrow{AK}$

c $\overrightarrow{AE}+\overrightarrow{AD}=\overrightarrow{AG}$

Subtraction of vectors

Group Work 7.4

If you have vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ such that $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

a How would you represent $-\overrightarrow{AB}$ geometrically?

b Can you show geometrically, that $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$?

c Discuss vector subtraction, and multiplication of a vector by a scalar.

d How do you represent vector subtraction and scalar multiplication of a vector geometrically?

You have discussed addition of vectors that are geometrically described by placing the initial point of one vector onto the terminal point of the other without changing the magnitude and direction. Now you shall consider the geometric subtraction of vectors.

ACTIVITY 7.6

Consider the following vectors, $\overrightarrow{AC}$ and $\overrightarrow{AD}$. What do you think the other vectors in diagrams a, b and c describe?

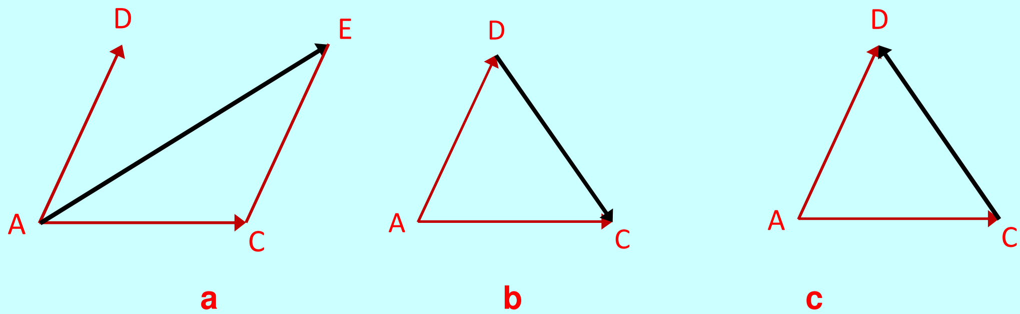


Figure 7.27

From addition of vectors, we recall that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, from which we can see that $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$.

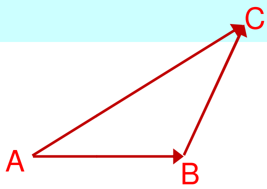


Figure 7.28

Example 6 Describe the vectors $\overrightarrow{BC}$ and $\overrightarrow{BD}$ and determine $\overrightarrow{CD}$ in terms of $\overrightarrow{BC}$ and $\overrightarrow{BD}$.

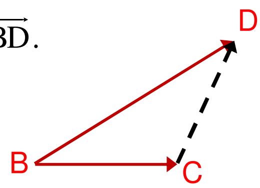


Figure 7.29

Solution: $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$ from Triangle Law.

Therefore, $\overrightarrow{CD} = \overrightarrow{BD} - \overrightarrow{BC}$.

Multiplication of vectors by scalars

ACTIVITY 7.7

Consider a vector $\overrightarrow{AC}$ and determine,

a $\overrightarrow{AC}+\overrightarrow{AC}$ **b** $\overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC}$ **c** $-\overrightarrow{AC}-\overrightarrow{AC}-\overrightarrow{AC}$

What do you observe? It seems very natural that $\overrightarrow{AC}+\overrightarrow{AC} = 2\overrightarrow{AC}$. Geometrically this means we are doubling the magnitude (length) of the vector $\overrightarrow{AC}$ without changing the direction.

In the same way if we can have $3\overrightarrow{AC}$, $\frac{1}{2}\overrightarrow{AC}$, $-\overrightarrow{AC}$ and $-2\overrightarrow{AC}$, then in $3\overrightarrow{AC}$ we are tripling the magnitude of $\overrightarrow{AC}$ and we are taking half of the magnitude of $\overrightarrow{AC}$ in $\frac{1}{2}\overrightarrow{AC}$.

What do you think $-\overrightarrow{AC}$ and $-2\overrightarrow{AC}$ mean?

If we have $k\overrightarrow{AC}$ where k is any real number, then depending on the value of k either we are enlarging the vector $\overrightarrow{AC}$ or we are shortening the vector $\overrightarrow{AC}$. When $k > 0$, the direction of $\overrightarrow{AC}$ and $k\overrightarrow{AC}$ are the same, but if $k < 0$, then the vectors $\overrightarrow{AC}$ and $k\overrightarrow{AC}$ are in opposite directions.

Scalar multiplication of a vector

Definition 7.4

Let $\overrightarrow{AC}$ be any given vector and k be any real number. The vector $k\overrightarrow{AC}$ is the vector whose magnitude is k times the magnitude of $\overrightarrow{AC}$ and,

a the direction of $k\overrightarrow{AC}$ is the same as the direction of $\overrightarrow{AC}$ if $k > 0$

b the direction of $k\overrightarrow{AC}$ is opposite to that of $\overrightarrow{AC}$ if $k < 0$.

Example 7 Figure 7.30 shows vector $\overrightarrow{AB}$ and the result of multiplying it by 2 and the result of multiplying it by -1.

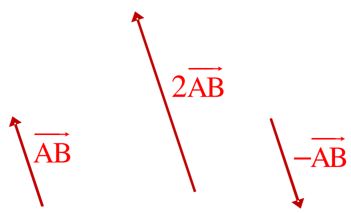


Figure 7.30

Example 8 Consider the following vectors.

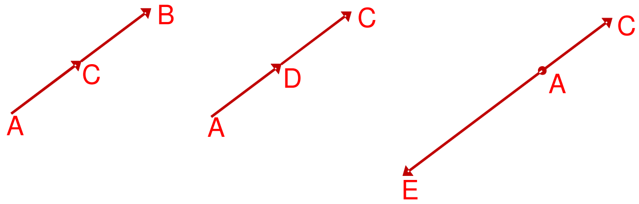


Figure 7.31

$\overrightarrow{AB} = 2\overrightarrow{AC}$, $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC}$ and $\overrightarrow{AE} = -2\overrightarrow{AC}$

Note: If $k \neq 0$, then any vector $\overrightarrow{AC}$ and $k\overrightarrow{AC}$ are parallel vectors.

Exercise 7.2

1. Almaz walks 3 m south and then 4.35 m west. What is her displacement from the initial point?
2. Simon moves 2 km West from A and 5 km north towards B. What is the displacement of Simon from A to B?
3. By drawing tip to tail add the following three vectors:

   $\vec{u} = 25.0$ m north, $\vec{v} = 35.0$ m at 45 degrees east of north and $\vec{w} = 12.0$ m east.
4. From Figure 7.32, give a single vector to represent the following

   a $\overrightarrow{AC} + \overrightarrow{CB}$ **b** $\overrightarrow{AD} - \overrightarrow{AC}$

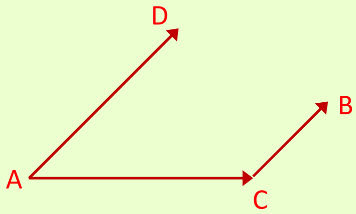


Figure 7.32

5. If $\vec{u} = 20$ m due North and $\vec{v} = 10$ m at 30 degrees E of N, find $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$.
6. From the vector $\overrightarrow{AC}$ given here, draw

   a $-3\overrightarrow{AC}$ **b** $\frac{4}{3}\overrightarrow{AC}$ **c** $4\overrightarrow{AC}$ **d** $-\overrightarrow{AC}$

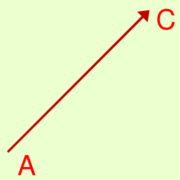


Figure 7.33

7.4 POSITION VECTOR OF A POINT

Up until now, you have used the geometric representation of vectors. Next, you will discuss components of vectors, and vector operations that include determining magnitude and direction by the use of components of a vector. You will also learn how to use vectors to describe the position of a point in a coordinate system.

Group Work 7.5

1 Consider the following vectors, which are on the *xy*-coordinate system. Move each vector so that their initial points are at the origin.

- **a** How do you differentiate one vector from another?
- **b** The initial point of any of those vectors is (0, 0). How do you express their terminal point?

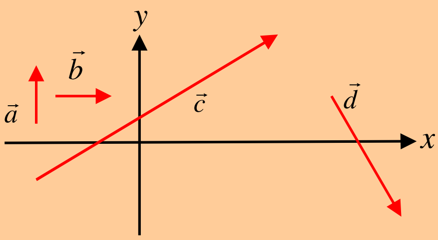


Figure 7.34

2 If $\overrightarrow{AB}$ is the vector with initial point A= (1, 2) and the terminal point (3, 4) what will its terminal point be if its initial point is moved to the origin?

3 If $\vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ represents a vector with initial point at the origin, then how do you express $\vec{v}$ in terms of the coordinates (2, 0) and (0, 5)?

From previous discussions, notice that the vectors represented in Figure 7.35 that have different initial points are equal. Of these vectors, the one whose initial point is the origin is called the standard form of the presentation of the vector (or simply, the position vector).

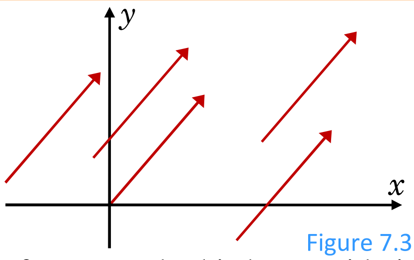


Figure 7.35

Analytically, we usually express vectors in component form. We do this by considering the vector with the origin as its initial point and write the coordinates of its terminal point as a "column vector". For example, in two dimensions, if $\vec{u} = \overrightarrow{OP}$, where O is (0, 0) and P is the point (x, y) then $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Note: Such column vectors are written vertically, to distinguish them from coordinates. Its geometric representation is as given below.

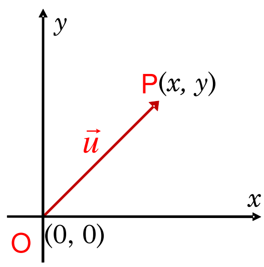


Figure 7.36

Since the vector $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ has O(0, 0) as its initial point and P(x, y) as its terminal point, its magnitude is $|\vec{u}| = \sqrt{x^2 + y^2}$ which is the length of the directed line from O (0, 0) to P (x, y).

ACTIVITY 7.8

Consider a vector $\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

1. Represent it geometrically.
2. Applying the Triangle law of vector addition to determine the components of $\vec{u}$.
3. Find the magnitude of $\vec{u}$.
4. Determine the direction of the vector $\vec{u}$.

Consider the following figure,

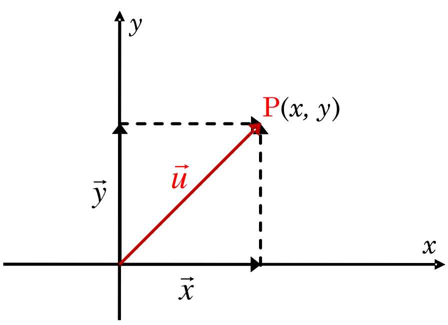


Figure 7.37

You can see that the vector $\vec{u}$ can be expressed as sum of the vectors $\vec{x}$ and $\vec{y}$ as $\vec{u} = \vec{x} + \vec{y}$ from the parallelogram law or the Triangle law where vectors $\vec{x} = x\vec{i}$ and $\vec{y} = y\vec{j}$ in which $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. From this, when $\vec{u} = x\vec{i} + y\vec{j}$, the x and y in $\vec{u} = x\vec{i} + y\vec{j}$ are called components of $\vec{u}$.

These components are useful in determining the direction of any vector.

Example 1 Represent the following vector as a position vector.

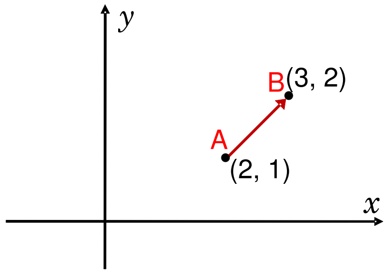


Figure 7.38

To represent a position vector of $\overrightarrow{AB}$ we need to construct a vector which has the same length and same direction as the vector $\overrightarrow{AB}$. That is, we need to construct a vector whose origin is **O** (0, 0) and whose terminal is $P(x_2 - x_1, y_2 - y_1)$ where (x_1, y_1) is the initial and (x_2, y_2) is the terminal point of the given vector.

Hence the position vector of the vector $\overrightarrow{AB}$ given above is

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } \vec{u} = \vec{i} + \vec{j} \text{ where } \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From this, we can determine the magnitude and the direction of the vector.

Example 2 For the vector given by $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, its geometric representation is given below. Find the magnitude and direction of the vector.

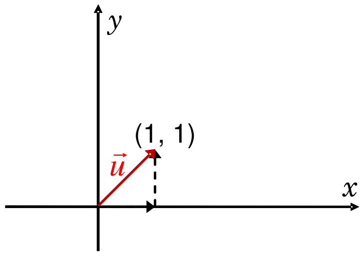


Figure 7.39

Solution: From this geometric representation and from the trigonometric identities that you discussed in chapter five, we can determine the direction of the vector.

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} \Rightarrow \tan\theta = \dfrac{1}{1} = 1.$ The acute angle whose tangent value is 1 is 45°.

Hence, the direction of the vector is 45°.

The magnitude of the vector is also $|\vec{u}| = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$

Example 3 Find the position vector of the following vectors whose initial and terminal points are as given below.

a Initial point (1, 2) and terminal point (2, 5)

b Initial point (–2, 3) and terminal point (1, 4)

Solution:

a The position vector $\vec{u}$ of the vector whose initial point is (1, 2) and terminal point is (2, 5) is (2–1, 5–2) = (1, 3).

That is, $\vec{u} = \vec{i} + 3\vec{j}$ which will be represented as $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

b The position vector $\vec{u}$ of the vector whose initial point is (–2, 3) and terminal point is (1, 4) is (1–(–2), 4–3) = (3, 1). That is, $\vec{u} = 3\vec{i} + \vec{j}$ which will be $\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

The geometric representation of these vectors is given below.

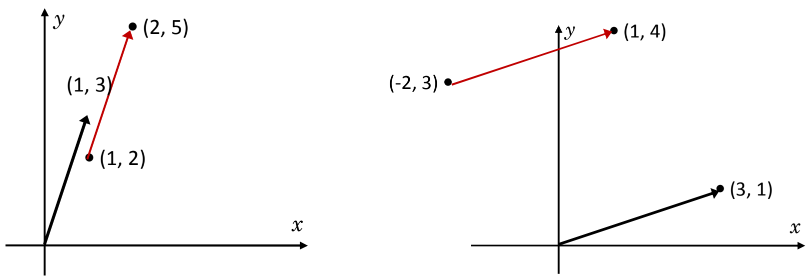


Figure 7.40

Exercise 7.3

1 Write each the following position vectors in Figure 7.41 as column vectors.

a $\overrightarrow{OA}$ **b** $\overrightarrow{OB}$

c $\overrightarrow{OC}$ **d** $\overrightarrow{OD}$

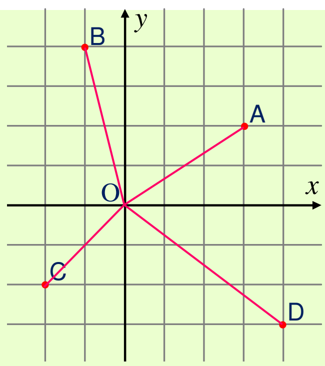


Figure 7.41

2 The coordinates of some points are as follows

P (2, 5), Q (–1, –2), R (0, –4) and S (3, –7)

What is the position vector of

a Q relative to P? **b** R relative to S? **c** S relative to Q?

3 The position vector of X relative to Y is $\overrightarrow{YX} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

a What is the position vector of Y relative to X?

b If X has coordinates (–1, 3), what are the coordinates of Y?

c If M is the midpoint of $\overrightarrow{XY}$ what is $\overrightarrow{XM}$?

d What is the position vector of $\overrightarrow{OM}$?

4 Represent the vectors, whose initial (I) and terminal points (T) are given below, geometrically on a coordinate system.

a I(1, 4) and T(3, 2) **b** I(−2, 2) and T(1, 6)

5 Determine the position vector of each of the vectors given in Question 4 above.

6 Determine the magnitude and the direction of each of the vectors given in Question 4 above.

Key Terms

addition of vectors	position vector
direction of a vector	scalar quantities
equality of vectors	subtraction of vectors
magnitude(length of) a vector	triangle law of vector addition
parallelogram law of vector addition	vector quantities

Summary

1 A scalar is a measure that involves only magnitude and no direction while a vector involves both magnitude and direction.

2 A vector is denoted by a directed arrow. Its length is called the magnitude. The direction it points is called the direction of the vector.

3 Vectors include velocity, force, acceleration, electric or magnetic fields, etc.

4 A vector is represented by an arrow ($\overrightarrow{OP}$); the point O is called the initial point and P is called the terminal point. Sometimes, vectors are represented by using letters or a letter with a bar over it such as $\vec{u}$, $\vec{v}$, etc.

5 The magnitude of a velocity is the speed; the magnitude of a displacement is distance. Thus, speed and distance are scalar quantities.

6 A magnitude is always a positive number.

7 Vectors can be described geometrically or algebraically: geometrically as a directed arrow and algebraically as a column vector.

8 Two vectors are said to be equal if they have the same magnitude and the same direction.

9 If two vectors have same or opposite directions then they are parallel.

10 For any two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ (the Triangle law)

11 A vector that has no magnitude and direction is called a zero vector or null vector.

12 The diagonal of a parallelogram is the sum of the side vectors. This is called the Parallelogram Law.

13 Subtraction of vectors $\overrightarrow{AE}$ and $\overrightarrow{AC}$, given as $\overrightarrow{AE} - \overrightarrow{AC} = \overrightarrow{CE}$ is the same as $\overrightarrow{AC} + \overrightarrow{CE} = \overrightarrow{AE}$

14 Multiplying a vector by a scalar k either enlarges or shortens the vector. If $|k| > 1$, it enlarges the vector and if $0 < |k| < 1$ it shortens the vector. If $k > 0$, the direction of the vector is unchanged; multiplying a vector by $k < 0$ changes the direction of the vector into the opposite direction.

15 If the initial and terminal points of a vector are (x_1, y_1) and (x_2, y_2) then its position vector can be calculated as P = $(x_2 - x_1, y_2 - y_1)$ and is denoted by $P = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$.

Review Exercises on Unit 7

1 Describe what is meant by scalar and vector quantities.

2 Write down some examples of scalar and vector quantities.

3 How do you represent a vector geometrically?

4 Sketch a vector whose magnitude is 3 cm in the direction of

a East **b** North 30° East

5 Sketch a vector of length 5 cm whose direction is

a North 45° East **b** West **c** South 20° East

6 When are two vectors parallel?

7 Show, by a diagram, that the sum of two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AC}$.

8 If the magnitude of $\overrightarrow{AC}$ is 4 cm, find the magnitude of

a $3\overrightarrow{AC}$ **b** $\frac{1}{4}\overrightarrow{AC}$ **c** $-\overrightarrow{AC}$

9 If A is the point (4, –2) and B is the point (–3, –6), what is the position vector of B relative to A?

10 The position vector of P (relative to the origin) is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$. If the magnitude of $\overrightarrow{OP}$ is 5 units, find the set of all possible values of $\begin{pmatrix} x \\ y \end{pmatrix}$, with $x, y \in \mathbb{Z}$.

Table of Trigonometric Functions

	sin	cos	tan	cot	sec	csc	
0°	0.0000	1.0000	0.0000	.....	1.000	.....	90°
1°	0.0175	0.9998	0.0175	57.29	1.000	57.30	89°
2°	0.0349	0.9994	0.0349	28.64	1.001	28.65	88°
3°	0.0523	0.9986	0.0524	19.08	1.001	19.11	87°
4°	0.0698	0.9976	0.0699	14.30	1.002	14.34	86°
5°	0.0872	0.9962	0.0875	11.43	1.004	11.47	85°
6°	0.1045	0.9945	0.1051	9.514	1.006	9.567	84°
7°	0.1219	0.9925	0.1228	8.144	1.008	8.206	83°
8°	0.1392	0.9903	0.1405	7.115	1.010	7.185	82°
9°	0.1564	0.9877	0.1584	6.314	1.012	6.392	81°
10°	0.1736	0.9848	0.1763	5.671	1.015	5.759	80°
11°	0.1908	0.9816	0.1944	5.145	1.019	5.241	79°
12°	0.2079	0.9781	0.2126	4.705	1.022	4.810	78°
13°	0.2250	0.9744	0.2309	4.331	1.026	4.445	77°
14°	0.2419	0.9703	0.2493	4.011	1.031	4.134	76°
15°	0.2588	0.9659	0.2679	3.732	1.035	3.864	75°
16°	0.2756	0.9613	0.2867	3.487	1.040	3.628	74°
17°	0.2924	0.9563	0.3057	3.271	1.046	3.420	73°
18°	0.3090	0.9511	0.3249	3.078	1.051	3.236	72°
19°	0.3256	0.9455	0.3443	2.904	1.058	3.072	71°
20°	0.3420	0.9397	0.3640	2.747	1.064	2.924	70°
21°	0.3584	0.9336	0.3839	2.605	1.071	2.790	69°
22°	0.3746	0.9272	0.4040	2.475	1.079	2.669	68°
23°	0.3907	0.9205	0.4245	2.356	1.086	2.559	67°
24°	0.4067	0.9135	0.4452	2.246	1.095	2.459	66°
25°	0.4226	0.9063	0.4663	2.145	1.103	2.366	65°
26°	0.4384	0.8988	0.4877	2.050	1.113	2.281	64°
27°	0.4540	0.8910	0.5095	1.963	1.122	2.203	63°
28°	0.4695	0.8829	0.5317	1.881	1.133	2.130	62°
29°	0.4848	0.8746	0.5543	1.804	1.143	2.063	61°
30°	0.5000	0.8660	0.5774	1.732	1.155	2.000	60°
31°	0.5150	0.8572	0.6009	1.664	1.167	1.942	59°
32°	0.5299	0.8480	0.6249	1.600	1.179	1.887	58°
33°	0.5446	0.8387	0.6494	1.540	1.192	1.836	57°
34°	0.5592	0.8290	0.6745	1.483	1.206	1.788	56°
35°	0.5736	0.8192	0.7002	1.428	1.221	1.743	55°
36°	0.5878	0.8090	0.7265	1.376	1.236	1.701	54°
37°	0.6018	0.7986	0.7536	1.327	1.252	1.662	53°
38°	0.6157	0.7880	0.7813	1.280	1.269	1.624	52°
39°	0.6293	0.7771	0.8098	1.235	1.287	1.589	51°
40°	0.6428	0.7660	0.8391	1.192	1.305	1.556	50°
41°	0.6561	0.7547	0.8693	1.150	1.325	1.524	49°
42°	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°
43°	0.6820	0.7314	0.9325	1.072	1.367	1.466	47°
44°	0.6947	0.7193	0.9667	1.036	1.390	1.440	46°
45°	0.7071	0.7071	1 .0000	1.000	1.414	1.414	45°
	cos	sin	cot	tan	csc	sec	

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