ASOSSA GENERAL PREPARATORY SCHOOL
Tradition of Excellence
Est. 1975

Edited BY

Ahmed Mohammed
(AsossaSchool.com)

2020
ETHIOPIA

Chapter-02

Inverse Trigonometric Functions

- The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	**Domain**	**Range (Principal Value Branches)**
$y = sin^{-1}x$	[-1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	[-1, 1]	$[0, \pi]$
$y = \operatorname{cosec}^{-1} x$	R- [-1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ -{0}
$y = \sec^{-1} x$	R-[-1, 1]	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \tan^{-1} x$	R	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cot^{-1} x$	R	$[0, \pi]$

$sin^{-1}x$ should not be condused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ And similarly for other trigonometric functions

- The value of an inverse trigonometric functions which lies in its principal value branch is called the principal value of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = sin^{-1} x \Rightarrow x = sin\, y$
- $x = sin\; y \Rightarrow y = sin^{-1}\, x$
- $\sin\; (\sin^{-1} x) = x$
- $\sin^{-1}\; (\sin x) = x$

Key Notes

- $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x$
- $\cos^{-1}(-x) = \pi - \cos^{-1}x$
- $\cos^{-1}\frac{1}{x} = \sec^{-1}x$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x$
- $\tan^{-1}\frac{1}{x} = \cot^{-1}x$
- $\sec^{-1}(-x) = \pi - \sec^{-1}x$
- $\sin^{-1}(-x) = -\sin^{-1}x$
- $\tan^{-1}(-x) = -\tan^{-1}x$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
- $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$
- $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$
- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$
- $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$
- $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$
- $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \frac{1-x^2}{1+x^2}$

Chapter-3

Matrices

- A matrix is an ordered rectangular array of numbers or functions.
- A matrix having m rows and n columns is called a matrix of order m × n.
- $[a_{ij}]_{m\times 1}$ is a column matrix.
- $[a_{ij}]_{1\times n}$ is a row matrix.
- An m × n matrix is a square matrix if m = n.
- A = $A=[a_{ij}]_{m\times n}$ is a diagonal matrix if $a_{ij}=0$, when $i \neq j$
- $A = \left[a_{ji}\right]_{n\times n}$ is a scalar matrix if $a_{ij}=0$ when $i \neq j$, $a_{ij}=k$ (k is *some constant), when I=j.*
- $A = \left[a_{ij}\right]_{n\times n}$ is an identity matrix, if . $a_{ij} = 1$, *when* $i = j$, $a_{ij} = 0$, *when* $i \neq j$.
- A zero matrix has all its elements as zero.
- $A=\left[a_{ij}\right]=\left[b_{ij}\right]=B$ if (i) A and B are of same order, (ii) for all possible values of i *and* j.
- $kA = k\left[a_{ij}\right]_{m\times n} = \left[k\left(a_{ij}\right)\right]_{m\times n}$
- – A = (–1)A
- A – B = A + (–1) B
- A + B = B + A
- (A + B) + C = A + (B + C), where A, B and C are of same order.
- k(A + B) = kA + kB, where A and B are of same order, k is constant.
- (k + l) A = kA + lA, where k and l are constant.
- *If* $A = [a_{ij}]_{m\times n}$ *and* $B = [b_{jk}]_{n\times p}$, *then* $AB = C = [c_{ik}]_{m\times p}$, where $C_{tl} = \sum_{j=i}^{n} a_{ij}b_{jk}$

  (i) A(BC) = (AB)C,

(ii) A(B + C) = AB + AC,

(iii) (A + B)C = AC + BC

- $If\ A = \left[a_{ij}\right]_{m\times n},\ then\ A'\ or\ A^{T} = \left[a_{ji}\right]_{n\times m}$
- (i) (A′)′ = A,
- (ii) (kA)′ = kA′,
- (iii) (A + B)′ = A′ + B′,
- (iv) (AB)′ = B′A′
- A is a symmetric matrix if A′ = A.
- A is a skew symmetric matrix if A′ = –A.
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- Elementary operations of a matrix are as follows:
- (i) $R_1 \leftrightarrow R_j$ or $C_l \leftrightarrow C_j$

  (i) $R_1 \rightarrow kR_i$ or $C_l \leftrightarrow kC_l$

  (i) $R_1 \leftrightarrow R_j + kR_j$ or $C_l + kC_j$
- If A and B are two square matrices such that AB = BA = I, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B.
- Inverse of a square matrix, if it exists, is unique.

Chapter-4

Determinant

- Determinant of a matrix $A = [a_{11}]_{1\times 1}$ is given by $|a_{11}| = a_{11}$

- Determinant of a matrix A $\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$ is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22}\ - a_{12}\ \ a_{21}$$

- Determinant of a matrix A $\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$ is given by (expanding along (R_1)

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

- **For any square matrix A, the |A| satisfy following properties.**
- $|A'| = |A|$, where A' = transpose of A.
- If we interchange any two rows (or columns), then sign of determinant changes.
- If any two rows or any two columns are identical or proportional, then value of determinant is zero.
- If we multiply each element of a row or a column of a determinant by constant k, then value of determinant is multiplied by k.
- Multiplying a determinant by k means multiply elements of only one row (or one column) by k.
- $If\ \ A = \left[a_{ij}\ \right]_{3\times 3}\ ,then\ |k.\ A| = k^3\ |A|$
- If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.

- If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added, then value of determinant remains same.
- Area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Minor of an element aij of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and denoted by M_{ij}
- Cofactor of aij of given by Aij = (– 1)i+ j Mij
- Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example, $|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13}A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example, $a_{11} A_{21} + a_{12}A_{22} + a_{13} A_{23} = 0$
- $A\ (adj\ A) = (adj\ A)\ A = |A|\ I,$ where A is square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A|| = 0$ *or* $|A| \neq 0.$
- If AB = BA = I, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B$ *or* $B^{-1} = A$ *and hence* $\left(A^{-1}\right)^{-1} = A.$
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1} = \frac{1}{|A|}(\text{adj } A)$
- If $a_1x + b_1y + c_1z = d_1$
- $a_2x + b_2y + c_2z = d_2$
- $a_3x + b_3y + c_3z = d_3$
- then these equations can be written as A X = B, where

Key Notes

- $$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = X \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$
- Unique solution of equation AX = B is given by $X = A^{-1}B,\ where\ |A|\ 0\ \neq 0$.
- A system of equation is consistent or inconsistent according as its solution exists or not.
- For a square matrix A in matrix equation AX = B
- |$|A| \neq 0,$ there exists unique solution
- |A| = 0 and (adj A) B ≠ 0, then there exists no solution
- $|A| = 0\ and\ (adj\ A)\ B = 0,$ then system may or may not be consistent.

Chapter-5

Continuity and Differentiability

- A real valued function is continuous at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.
- Sum, difference, product and quotient of continuous functions are continuous. i.e., if f and g are continuous functions, then

  $(f \pm g)\ (x) = f(x) \pm g(x)$ is continuous.

  (f . g) (x) = f (x) . g(x) is continuous.

  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (wherever g (x) ≠ 0) is continuous.

- Every differentiable function is continuous, but the converse is not true.
- Chain rule is rule to differentiate composites of functions. If f = v o u, t = u (x) and if both

  and if both $\frac{dt}{dx}$ *and* $\frac{dv}{dt}$ exist then $\frac{df}{dx} = \frac{dv}{dt} = \frac{dt}{dx}$

- Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=\frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(e^x\right)=e^x$$

$$\frac{d}{dx}(\log x)=\frac{1}{x}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$ Here both f (x) and u (x) need to be positive for this technique to make sense.
- **Rolle's Theorem:** If f: [a, b] → R is continuous on [a, b] and differentiable on (a, b) such that f (a) = f (b), then there exists some c in (a, b) such that f ′(c) = 0.
- Mean Value Theorem: If f: [a, b] → R is continuous on [a, b] and differentiable on (a, b). Then there exists some c in (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Chapter-6

Application of Derivatives

- If a quantity y varies with another quantity x, satisfying some rule ( ) y f x = ,then $\frac{dx}{dy}$ (or f '(x) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or f '(x_0) represents the rate of change of y with respect to x at0 x x = .
- If two variables x and y are varying with respect to another variable t, i.e., if $x=f(t)$ *and* $y=g(t)$ then by Chain Rule
- $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$, if $\frac{dx}{dt}\neq 0$

(a) A function f is said to be increasing on an interval (a, b) if

$x_1 < x_2 \text{ in } (a, b) \Rightarrow f(x_1) \leq f(x_2) \text{ for all } x_1, x_2 \in (a, b).$

Alternatively, if $f'(x) \geq 0$ *for each x in* (a, b)

(b) decreasing on (a,b) if

$x_1 < x_2 \text{ in } (a, b) \Rightarrow f(x_1) \geq f(x_2) \text{ for all } x_1, x_2 \in (a, b).$

Alternatively, if $f'(x) \leq 0$ for each x in (a, b)

- The equation of the tangent at (x_0, y_0) to the curve y = f (x) is given by
- $y-y_o=\left.\frac{dy}{dx}\right]_{(x_0,y_0)}(x-x_o)$
- If $\frac{dy}{dx}$ does not exist at the point (x_0,y_0) , then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$.

- If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x-axis, then $\left.\frac{dy}{dx}\right]_{x=x_0}$

- Equation of the normal to the curve y = f (x) at a point (x_o, y_o) is given by

$$y - y_o = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}}(x - x_0)$$

- If $\frac{dy}{dx}$ at the point (x_o, y_o) is zero, then equation of the normal is $x = x^o$.

- If $\frac{dy}{dx}$ at the point (x_o, y_o) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$.

- Let y = f (x), Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x, i.e., Δy = f (x + Δx) − f (x). Then dy given by $dy\, f'(x)\, dx$ or $dy = \left(\frac{dx}{dx}\right)$Δx is a good approximation of Δy when dx x = Δ is relatively small and we denote it by dy ≈ Δy.

- A point c in the domain of a function f at which either f ′(c) = 0 or f is not differentiable is called a critical point of f.

- **First Derivative** Test Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then,

  - If f ′(x) changes sign from positive to negative as x increases through c, i.e., if f ′(x) > 0 at every point sufficiently close to and to the left of c, and f ′(x) < 0 at every point sufficiently close to and to the right of c, then c is a point of local maxima.

  - If f ′(x) changes sign from negative to positive as x increases through c, i.e., if f ′(x) < 0 at every point sufficiently close to and to the left of c, and f ′(x) > 0 at every point sufficiently close to and to the right of c, then c is a point of local minima.

  - If f ′(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called point of inflexion.

- **Second Derivative Test** Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then, x = c is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

  The values f (c) is local maximum value of f.

  (i) x = c is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

  In this case, f (c) is local minimum value of f.

  (ii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

  In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.

- Working rule for finding absolute maxima and/or absolute minima

  Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

  Step 2: Take the end points of the interval.

  Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

  Step 4: Identify the maximum and minimum values of f out of the values calculated in Step

- This maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f .

Chapter-7

Integrals

- Integration is the inverse process of differentiation. In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation. Let $\frac{d}{dx}F(x)=f(x)$. Then we write $\int f(x)dx = F(x)+C$. These integrals are called indefinite integrals or general integrals, C is called constant of integration. All these integrals differ by a constant.
- From the geometric point of view, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along the y-axis.
- Some properties of indefinite integrals are as follows:

1. $\int[f(x)+g(x)]dx = \int f(x)dx + f(x)dx$

2. For any real number k, $kf(x)dx = k\int f(x)dx$

More generally, if $f_1, f_2, f_3, \ldots\ldots, f_n$ are functions and $k_1, k_2, \ldots\ldots, k_n$ are real numbers. Then

$$\int[k_1f_1(x)+k_2f_2(x)+\ldots\ldots+k_nf_n(x)]dx = k_1\int f_1(x)dx + k_2\int k_2(x)dx + \ldots. + k_n\int f_n(x)dx$$

- **Some standard integrals:**

(i) $\int x^n dx = \frac{x^{n+1}}{n+1}+C, n \neq -1.$ Particularly, $\int dx = x+C$

(ii) $\int \cos x\, dx = \sin x + C$

(iii) $\int \sin x\, dx = -\cos x + C$

(iv) $\int \sec^2 x\, dx = \tan x + C$

(v) $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

(vi) $\int \sec x \tan x \, dx = \sec x + C$

(vii) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-i} x + C$

(ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-i} x + C$

(x) $\int \frac{dx}{\sqrt{1+x^2}} = \tan^{-i} x + C$

(xi) $\int \frac{dx}{\sqrt{1+x^2}} = -\cot^{-i} x + C$

(xii) $\int e^x dx = e^x + C$

(xiii) $\int a^x dx = \frac{a^x}{\log a} + C$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} + C$

(xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} + C$

(xvi) $\int \frac{1}{x} dx = \log |x| + C$

Chapter-8

Application of Integrals

- The area of the region bounded by the curve y = f (x), x-axis and the lines x = a and x = b (b > a) is given by the formula: $Area=\int_a^b ydx=\int_a^b f(x)dx.$
- The area of the region bounded by the curve $x = \varphi(y)$), y-axis and the lines y = c, y = d is given by the formula: $Area=\int_c^b xdy=\int_c^d \theta(y)dy$
- The area of the region enclosed between two curves y = f (x), y = g (x) and the lines x = a, x = b is given by the formula, $Area=\int_a^b [f(x)-g(x)]dx, where,\ f(x)\geq g(x)\ in\ [a, b]$
- $If\ f(x) \geq g(x)\ in\ [a,c]\ and\ f(x)\leq g(x)\ in\ [c,b], a<c<b$, then

  $Area=\int_a^c [f(x)-g(x)]dx+\int_c^b [g(x)]dx.$

Chapter-9

Differential Equations

An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.

- Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
- Degree of a differential equation is defined if it is a polynomial equation in its derivatives.
- Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.
- To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.
- Variable separable method is used to solve such an equation in which variables can be separated completely i.e. terms containing y should remain with dy and terms containing x should remain with dx.
- A differential equation which can be expressed in the form $\frac{dy}{dx}$ $f\ (x, y)$ or $\frac{dx}{dy} g(x, y)$ where, $f\left(x,\ y\right)$ *and* $g\left(x,\ y\right)$ are homogenous functions of degree zero is called a homogeneous differential equation.
- A differential equation of the form $\frac{dy}{dx} + Py$, where P and Q are constants or functions of x only is called a first order linear differential equation.

Chapter-10

Vector Algebra

- Position vector of a point P (x, y) is given as $\overline{OP}(=\bar{r}) = x\hat{i} + y\hat{j} + z\hat{k}$ and its magnitude by $\sqrt{x^2 + y^2 + z^2}$
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude (r), direction ratios (a, b, c) and direction cosines (l, m, n) of any vector are related as: $l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$
- The vector sum of the three sides of a triangle taken in order is $\overline{O}$
- The vector sum of two conidial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- The multiplication of a given vector by a scalar λ, changes the magnitude of the vector by the multiple |λ|, and keeps the direction same (or makes it opposite) according as the value of λ is positive (or negative).
- For a given vector $\bar{a}$, the vector $\hat{a} = \frac{\bar{a}}{|\bar{a}|}$ gives the unit vector in the direction of $\bar{a}$
- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\bar{a}$ *and* $\bar{b}$ respectively, in the ratio $m:n$

  (i) internally, is given by $\frac{n\bar{a} + m\bar{b}}{m+n}$

  (ii) externally, is given by $\frac{m\bar{b} - n\bar{a}}{m-n}$
- The scalar product of two given vectors $\bar{a}$ and $\bar{b}$ having angle θ between them is defined as $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|\cos\theta$

Also, when $\overline{a}\,.\,\overline{b}$ is given, the angle 'θ' between the vectors $\overline{a}$ *and* $\overline{b}$ may be determined by

$$cos\theta = \frac{\overline{a}\,.\,\overline{b}}{|\overline{a}||\overline{b}|}$$

- If θ is the angle between two vector $\overline{a}$ and $\overline{b}$, then their cross product is given as

  $\overline{a}\times\overline{b} = |\overline{a}||\overline{b}|\sin\theta\hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the plane containing $\overline{a}$ *and* $\overline{b}$. Such that $\overline{a}$, $\overline{b}$, $\hat{n}$ form right handed system of coordinate axes.

- If we have two vectors $\overline{a}$ *and* $\overline{b}$ given in component form as $\overline{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\overline{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and λ any scalar, then, $a+b = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$

  $\lambda\overline{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k};$

  $\overline{a}.\overline{b} = a_1b_1 + a_2b_2 + a_3b_3$

  and $\overline{a}\times\overline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

Chapter-11

Three Dimensional Geometry

Direction cosines of a line are the cosines of the angles made by the line with the positive direct ions of the coordinate axes.

- *If* l, m, n are the direct ion cosines of a line, then $l^2 + m^2 + n^2 = 1$
- Direct ion cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_2}{PQ}$
- Where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Direction ratios of a line are the numbers which are proportional to the direct ion cosines of a line.
- If l, m, n are the direct ion cosines and a, b, c are the direct ion ratios of a line

Then, $l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

- Skew lines are lines in space which are neither parallel nor intersecting. They lie in different planes.
- Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then,

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\bar{a}$ and parallel to a given vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda\bar{b}$

Key Notes

- Equation of a line through a point (x_1, y_1, z_1) and having direct ion cosines $l,\ m,\ n$ is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- The vector equation of a line which passes through two points whose posit ion vectors are $\bar{a}$ and $\bar{b}$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

- If θ is the acute angle between $\bar{r} = \bar{a}_1 + \lambda \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda \bar{b}_2$ then, $\cos\theta = \left| \frac{\bar{b}.\bar{b}_2}{|\bar{b}_1||\bar{b}_2|} \right|$

- If $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

- Shortest distance between two skew lines is the line segment perpendicular to both the lines.

- Shortest distance between $\bar{r} = \bar{a}_1 + \wedge \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \wedge \bar{b}_2$ $\left| \frac{(\bar{b}_1 \times \bar{b}_2).(\bar{a}_2 - \bar{a}_1)}{|\bar{b}_1 \times \bar{b}_2|} \right|$

- Shortest distance between the lines: $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_1} = \frac{y - y_2}{b_1} = \frac{z - z_2}{c_1}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

- Distance between parallel lines $\bar{r} = \bar{a}_1 + \wedge \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \wedge \bar{b}_2$ $\left| \frac{(\bar{b}) \times (\bar{a}_2 - \bar{a}_1)}{|\bar{b}_1|} \right|$

- In the vector form, equation of a plane which is at a distance d from the origin, and n ˆ is the unit vector normal to the plane through the origin is $\bar{r}.\hat{n} = d$
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose posit ion vector is a and perpendicular to the vector $\bar{N}$ $is(\bar{r} - \bar{a})$. N=0
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is
- $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
- Equation of a plane passing through three non collinear points (x_1, y_1, z_1)
- $(x_2, y_2, z_2)$and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
- Vector equation of a plane that contains three non collinear points having position vectors
- $\bar{a}, \bar{b}$ and $\bar{c}$ is $(\bar{r} - \bar{a}).\left[(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})\right] = 0$
- Equation of a plane that cuts the coordinates axes at $(a,\ 0,\ 0),\ (0,\ b,\ 0)\ and\ (0,\ 0,\ c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
- Vector equation of a plane that p asses thro ugh the in the section of planes $\bar{r}.\bar{n}_1 = d_1$ and $\bar{r}.\bar{n}_2 = d_2$ is $\bar{r}.(\bar{n}_1 + \lambda\bar{n}_2) = d_1 + \lambda d_2$ where λ is any nonzero constant.
- Cartesian equation of a plane that passes that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2 = 0$
- Two lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$and $\bar{r} = \bar{a}_2 + \mu\bar{b}_2$ are coplanar if $(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times -\bar{b}_2) = 0$

- In the Cartesian form above lines passing through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2 z_2)$
- $= \frac{y-y_2}{b_2} = \frac{z-z_2}{C_2}$ are coplanar if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
- In the vector form, if θ is the angle between the two planes, $\bar{r}.\bar{n}_1 = d_1$ and $\bar{r}.\bar{n}_2 = d_2$, then $\theta = \cos^{-1}\frac{|\bar{n}_1.\bar{n}_2|}{|\bar{n}_1||\bar{n}_2|}$
- The angle ϕ between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r}.\hat{n} = d$ $\sin\phi = \left|\frac{\bar{b}.\hat{n}}{|\bar{b}|\,|\hat{n}|}\right|$
- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by $\cos\theta = \left|\frac{A_1A_2 + +B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\,\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$
- The distance of a point whose position vector is $\bar{a}$ from the plane $\bar{r}.\hat{n} = d$ is $|d - \bar{a}.\hat{n}|$
- The distance from a point (x_1, y_1, z_1) to the plane Ax + By + Cz + D = 0 is $\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$

Chapter-12

Linear Programming

- A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum) of a linear function of several variables (called **objective function**) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear **constraints**). Variables are sometimes called decision variables and are non-negative.
- A few important linear programming problems are:

  (i) Diet problems

  (ii) Manufacturing problems

  (iii) Transportation problems
- The common region determined by all the constraints including the non-negative constraints $x \geq 0, y \geq 0$ of a linear programming problem is called the **feasible region** (or **solution region**) for the problem.
- Points within and on the boundary of the feasible region represent **feasible solutions** of the constraints. Any point outside the feasible region is an **infeasible solution.**
- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**
- The following Theorems are fundamental in solving linear programming problems:

  Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

  Theorem 2 Let R be the feasible region for a linear programming problem, and let be the objective function. If R is **bounded,** then the objective function Z has both a **maximum a**nd a **minimum v**alue on R and each of these occurs at a corner point (vertex) of R.

- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.
- **Corner point method** for solving a linear programming problem. The method comprises of the following steps:

  (i) Find the feasible region of the linear programming problem and determine its corner points (vertices).

  (ii) Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.

  (iii) If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective function.

  If the feasible region is unbounded, then,

  (i) M is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.

  (ii) m is the minimum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.
- If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

Chapter-13

Probability

The salient features of the chapter are –

- The conditional probability of an event E, given the occurrence of the event F is given by

  $P(E|F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$

  $0 \leq P(E|F) \leq 1,$
  $P(E'|F) = 1 - P(E|F)$
  $P((E \cup F)|G) = P(E|G) + P(F|G) - P((E \cap F)|G)$

- $P(E \cap F) = P(E)P(E|F), P(E) \neq 0$
  $P(E \cap F) = P(F)P(E|F), P(F) \neq 0$

- $P(E \cap F) = P(E) P(F)$
  $P(E|F) = P(E), P(F) \neq 0$
  $P(F|E) = P(F), P(E) \neq 0$

- **Theorem of total probability:**

  Let $\{E_1, E_2, ..., E_n)$ be a partition of a sample space and suppose that each of $E_1, E_2, ..., E_n$ has non zero probability. Let A be any event associated with S, then

  $P(A) = P(E_1)P(A|E_1) + P(E_2) + P(A|E_2) + ... + P(E_n)P(A|E_n)$

- **Bayes' theorem:** If $E_1, E_2, ..., E_n$ are events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1 4, E_2 4, ..., 4E_n = S$ *and* A be any event with non-zero probability, then, $P(E_i|A) = \frac{P(E_i) P(A|E_i)}{\sum_{j 1}^{n} P(E_i) P(A|E_i)}$

- A random variable is a real valued function whose domain is the sample space of a random experiment.

- The probability distribution of a random variable X is the system of numbers

X : x_1 x_2 ..... x_n

P(X) : p_1 p_2 ...... p_n

Where, $p_i > o, \sum_{i=1}^{n} p_i = 1, \ i = 1, 2, \ldots\ldots, n$

- Let X be a random variable whose possible values $x_1, x_2, x_3 \ldots.\ \ldots, x_n$ occur with probabilities $p_1, p_2, p_3 \ldots.\ \ldots, p_n$ respectively. The mean of X, denoted by μ is the number $\sum_{i=1}^{n} x_i p_i$. The mean of a random variable X is also called the expectation of X, denoted by E (X).
- Let X be a random variable whose possible values $x_1, x_2, x_3 \ldots.\ \ldots, x_n$ occur with probabilities $p(x_1), p(x_2), \ldots\ldots p(x_n)$ respectively. Let $\mu = E(X)$ be the mean of X. The variance of X, denoted by Var (X) or σ_x^2 is defined as $x^2\ Var\ (X) = \sum_{i=1}^{n} (x_i\ \mu)^2\ \ p(x_i)$ or equivalently $\sigma_x^2 = E\ (X - \mu)^2$. The non-negative number, $_x\sqrt{Var\ (X)} = \sqrt{\sum_{1=i}^{n}\ (x_i\ \mu)^2\ p(x_i)}$ is called the standard deviation of the random variable X.

$Var\ (X) = E\left(X^2\right) - \left[E(X)\right]^2$

- Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions:

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes: success or failure.

(iv) The probability of success remains the same in each trial.

For Binomial distribution $B(n, p)$, $P(X{=}x) = {}^n C_x q^{n-x} P^x, x = 0, 1, \ldots\ldots, n (q = 1 - p)$