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ETHIOPIA

MATHEMATICS FOR GRADE 12

HANDOUT FOR SECOND SEMESTER LESSONS

OROMIA EDUCATION BUREAU

In Collaboration With

EXCEL ACADEMY

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UNIT 4 APPLICATIONS OF DIFFERENTIAL CALCULUS

INTRODUCTION

The main task of this unit is to examine the use of calculus in general; and differential calculus in particular in finding extreme values, curve sketching and other applications.

Unit outcomes

After completing this unit, you will be able to:

- find local maximum or local minimum of a function in a given interval.
- find absolute maximum or absolute minimum of a function.
- apply the mean value theorem.
- solve simple problems in which the studied theorems, formulae, and procedures of differential calculus are applied.
- solve application problems.

4.1 EXTREME VALUES OF FUNCTIONS

4.1.1 Revision on Zeros of Functions

✓ Recall that a **zero** of a function f on its domain D is a number $x \in D$ such that $f(x) = 0$.

Examples 1. Find the zero(s) of the following functions.

a) $f(x) = x^2 - 2|x - 4|$ b) $f(x) = \sin 2x + \cos x$, for $0 < x < 2\pi$

Solution

a) $f(x) = x^2 - 2|x - 4|$. First redefine f using the definition of absolute value.

$\Rightarrow f(x) = x^2 - 2|x - 4| = \begin{cases} x^2 - 2x + 8\ , & if\ \ x \geq 4 \\ x^2 + 2x - 8\ , & if\ \ x < 4 \end{cases}$

$\Rightarrow\ f(x) = x^2 - 2x + 8\ \neq 0$ for all $x \geq 4$

For $x < 4,\ x^2 + 2x - 8 = (x + 4)(x - 2) = 0\ \ iff\ x = -4 < 4\ \ or\ \ x = 2 < 4$

Therefore, -4 and 2 are the zeros of $f(x)$.

b) $f(x) = \sin 2x + \cos x = 0\ \ \Rightarrow\ \ \sin 2x + \cos x = 2 \sin x \cos x + \cos x = 0$

$\Rightarrow\ \cos x\,(2 \sin x + 1) = 0\ \ \Rightarrow\ \ \cos x = 0\ \ or\ \ \sin x = -\frac{1}{2}$

On $(0\ ,\ 2\pi)$, $\cos x = 0\ iff\ x = \pi$ and

On $(0\ ,\ 2\pi)$, $\sin x = -\frac{1}{2}\ \ iff\ \ x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}\ or\ \ x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

Therefore, π , $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$ are the zeros of $f(x)$.

4.1.2 Critical Numbers and Extreme Values of Functions

Critical Numbers

Definition 1. (Critical Number)

Let f be a function and c be in the domain of f. Then if $f'(c) = 0$ or f has no derivative at c, then $x = c$ is said to be a **critical number** of f.

Note : A function may have one, more than one critical number or may not have any critical number.

Examples 2. Find the critical number(s) of each of the following functions.

a) $f(x) = 3x - x^3$ b) $f(x) = \frac{3}{8}x^{\frac{8}{3}} - 6x^{\frac{2}{3}}$ c) $f(x) = \begin{cases} 3x^2 + 12\,, & if\ x \geq 2 \\ 12x - 13\,, & if\ x < 2 \end{cases}$

Solution

a) Domain of $f = \mathbb{R}$ and $f'(x) = 3 - 3x^2 = 3(1 - x^2) = 3(1 - x)(1 + x) = 0$

$\Rightarrow\ x = -1$ or $x = 1$ are both in the domain.

Thus, − 1 and 1 are the critical numbers of f.

b) Domain of $f = \mathbb{R}$ and $f'(x) = x^{\frac{5}{3}} - 4x^{-\frac{1}{3}} = 0 \Rightarrow \frac{x^{\left(\frac{5}{3}\right)}x^{\left(\frac{1}{3}\right)} - 4}{\sqrt[3]{x}} = \frac{x^{\left(\frac{5}{3} + \frac{1}{3}\right)} - 4}{\sqrt[3]{x}} = \frac{x^2 - 4}{\sqrt[3]{x}} = 0$

$\Rightarrow x^2 - 4 = 0$, for $x \neq 0$ $\Rightarrow x = -2\ or\ x = 2$. $f'(0) = \nexists$, but 0 is in the domain.

Thus, 0 and 2 are the critical numbers of f.

c) $f(x) = \begin{cases} 3x^2 + 12\,, & if\ x \geq 2 \\ 12x - 13\,, & if\ x < 2 \end{cases} \Rightarrow f'(x) = \begin{cases} 6x\,, & if\ x \geq 2 \\ 12\,, & if\ x < 2 \end{cases} = 0$

Now, $6x = 0$ iff $x = 0$, but $0 \ngeq 2$. And $12 \neq 0$ for all $x < 2$.

Thus, f has no any critical number.

Extreme Values of Functions

There are two types of extreme value : **Absolute extreme value** and **Relative (or Local) extreme values.**

Definition 2. (Absolute or Global Extreme Values)

Let f be a function defined on set S. If there is some $c \in S$ such that

i. $f(c) \geq f(x)$ for every x in S, then $f(c)$ is called an **ABSOLUTE MAXIMUM** of f on S.

ii. $f(c) \leq f(x)$ for every x in S, then $f(c)$ is called an **ABSOLUTE MINIMUM** of f on S.

iii. The absolute maximum and absolute minimum of f on S are called **EXTREME VALUES** or the **ABSOLUTE or GLOBAL EXTREME VALUES** of f on S.

In fact, if a function f is continuous on a closed bounded interval, it can be shown that both the absolute maximum and absolute minimum must occur. This result is based on a crucial theorem, known as **Extreme Value Theorem**, which plays an important role in the application of differentiation.

Theorem 1. (Extreme Value Theorem)

Let f be a function continuous on a closed and bounded interval $[a, b]$. Then f has both the absolute maximum and absolute minimum values on $[a, b]$.

Note that the (absolute) extreme value of a continuous function f on an interval $[a, b]$ are either at the end points or at the critical point(s) of f on $[a, b]$.

Guideline for finding extrema on closed interval

To find the extrema of a continuous function f on $[a, b]$, follow the following procedure.

Step 1. Find the critical number of f in (a, b), if any.

Step 2. Evaluate f at each critical number of f in (a, b), if any, and at the end points $x = a$ and $x = b$.

Step 3. The least of these values is the absolute minimum value and the greatest is the absolute maximum value of f on $[a, b]$.

Examples 3. Find the extreme values of each of the following functions on the given intervals.

a) $f(x) = 4x^3 - 5x^2 - 8x + 20$ on $[-1\,,\ 1]$

b) $g(x) = \frac{e^{2x}}{x^2 - 2}$ on $[0\,,\ 2]$

c) $k(x) = 2 + \sqrt[3]{x-1}$ on $[0\,,\ 3]$

Solution

a) $f(x) = 4x^3 - 5x^2 - 8x + 20 \quad \Rightarrow \quad f'(x) = 12x^2 - 10x - 8 = 2(3x-4)(2x+1) = 0$

$\Rightarrow\ x = \frac{4}{3}\ or\ x = -\frac{1}{2}\ \Rightarrow\ -\frac{1}{2} \in (-1\,,\ 1)$ but $\frac{4}{3} \notin (-1\,,\ 1)$.

- $-\frac{1}{2}$ is the critical point of f.

Now, $f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 - 5\left(-\frac{1}{2}\right)^2 - 8\left(-\frac{1}{2}\right) + 20 = \frac{89}{4} = 22\frac{1}{4}$

$f(-1) = 4(-1)^3 - 5(-1)^2 - 8(-1) + 20 = 19$

$f(1) = 4(1)^3 - 5(1)^2 - 8(1) + 20 = 11$

Here, $11 < 19 < 22\frac{1}{4}$

Therefore, the absolute minimum value is 11 and the absolute maximum value is $22\frac{1}{4}$. Moreover, f attain its absolute minimum at the point $(1\,,\ f(1)) = (1\,,\ 11)$ and its absolute maximum at the point $\left(-\frac{1}{2}\,,\ f(-\frac{1}{2})\right) = (-\frac{1}{2}\,, -\frac{89}{4})$.

b) $g(x) = \frac{e^{2x}}{x^2 - 2}$ on $[0\,,\ 2] \Rightarrow g'(x) = \frac{2e^{2x}(x^2-2) - 2xe^{2x}}{(x^2-2)^2} = \frac{2e^{2x}(x^2 - x - 2)}{(x^2-2)^2} = 0$

$\Rightarrow\ 2e^{2x}(x^2 - x - 2) = 0\ \Rightarrow\ x^2 - x - 2 = 0$, since $2e^{2x} \neq 0\ \forall x \in \mathbb{R}$.

$\Rightarrow\ x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}\ \Rightarrow \mathrm{x} = 2 \notin (0\,,\ 2)$ or $\mathrm{x} = -1 \notin (0\,,\ 2)$

$\Rightarrow\ g$ has no critical point on $(0\,,\ 2)$. Hence, the extreme values of g occur at the end points.

$g(0) = \frac{e^{2(0)}}{0^2 - 2} = -\frac{1}{2}$ and $g(2) = \frac{e^{2(2)}}{2^2 - 2} = \frac{e^4}{2} > -\frac{1}{2}$

Therefore, $-\frac{1}{2}$ and $\frac{e^4}{2}$ are the minimum and the maximum values of g on [0, 2], respectively.

c) $k(x) = 2 + \sqrt[3]{x-1}$ on $[0\,,\ 3]\ \Rightarrow\ k'(x) = \frac{d}{dx}\left\{2 + (x-1)^{\frac{1}{3}}\right\} = \frac{1}{3}(x-1)^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{(x-1)^2}} = 0$, which gives 1 = 0, which is impossible for all real number x.

Here, k is continuous on $1 \in [0, 3]$, but $k'(1) = \nexists$. Thus, 1 is the critical number of k.

$k(0) = 2 + \sqrt[3]{0-1} = 1. \qquad k(1) = 2 + \sqrt[3]{1-1} = 2 \qquad k(3) = 2 + \sqrt[3]{3-1} = 2 + \sqrt[3]{2}$

Therefore, 1 and $2 + \sqrt[3]{2}$ are the minimum and the maximum values of g on [0, 2], respectively.

Note: The continuity of the function f on $[a, b]$ is the sufficient condition for the existence of extreme value on $[a, b]$. That is if f is not continuous on $[a, b]$, then f may not have extreme value on $[a, b]$.

Exercise 4.1.

1. Find the set of critical points of each of the given functions.

a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

b) $f(x) = \frac{x^2}{x^2 - 9}$

c) $(x) = \sin x + sin^2 x$, $-\pi$, $\pi)$

d) $f(x) = x - 3\sqrt[3]{x}$

e) $f(x) = \begin{cases} 4 - x^2 & if\ x \leq 1 \\ -2x & if\ x > 1 \end{cases}$

2. Find the absolute extreme values of each of the given functions on the given intervals.

a) $f(x) = 2x^5 - 5x^4 - 10x^3$ on $[-2, 1]$

b) $f(x) = |x^2 - x|$ on $[0, 2]$

c) $f(x) = x^2 e^{-x}$ on $[-1, 3]$

d) $f(x) = 2 + \sqrt{x - 1}$ on $[2,\ 5]$

Increasing and Decreasing (Monotonic) Functions

⌘ **Read and discuss the definition of monotonic (increasing or decreasing) functions and strictly monotonic (strictly increasing or decreasing) functions on page 175, on your text book.**

⌘ To understand the definition of monotonicity of a function, consider a function f whose graph is given below.

From the graph, we can see that

a) f is increasing on the intervals $[a,\ p]$ and on $[q,\ b]$, but f is not increasing on the union $[a,\ p] \cup [q,\ b]$.

b) f is strictly increasing on $[a\ ,\ p]$ and on $[r\ ,\ b]$.

c) f is decreasing on the intervals $[p\ ,\ r]$.

d) f is strictly decreasing on $[p\ ,\ q]$.

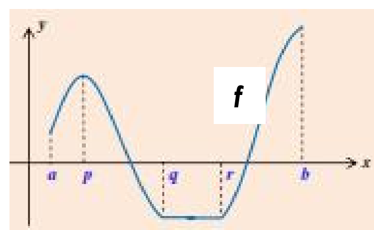


Figure 4.1. Graph of f on [a, b]

Note: i) f is strictly increasing $\Rightarrow$ f is increasing

ii) f is strictly decreasing $\Rightarrow$ f is decreasing

But the converses are **not** true.

Theorem 2. (Increasing and Decreasing Test)

Suppose that f is continuous on an interval I and differentiable in the interior of I.

i. If $f'(x) = 0$ for all x in the interior of I, then f is **increasing** on I.

ii. If $f'(x) = 0$ for all x in the interior of I, then f is **decreasing** on I.

iii. If $f'(x) > 0$ and $f'(x) = 0$ only for finite number of points on I, then f is **strictly increasing** on I.

iv. If $f'(x) < 0$ and $f'(x) = 0$ only for finite number of points on I, then f is **strictly decreasing** on I.

Note: To determine the interval on which a function f is increasing or decreasing, it will be easier to use sign chart for its derivative f' based on the critical point of f.

Examples 5. Find the interval(s) on which each of the following functions is increasing and on which it is decreasing.

a) $f(x) = x^3 + 3x^2 - 9x + 7$

b) $f(x) = 2 + 3^{(1-2x)}$

c) $f(x) = x\sqrt{x^2 + 1}$

Solution

a) $f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1) = 0$ iff $x = -3$ or $x = 1$

- $x = -3$ and $x = 1$ are the critical numbers of f.

You are going to find intervals in which $f'(x)$ is positive or negative. Use sign charts for this purpose, as follows:

	$-\infty$ to -3	-3 to 1	1 to ∞
$x+3$	$-$ (0 at -3)	$+$	$+$
$x-1$	$-$	$-$ (0 at 1)	$+$
$f'(x)$	$+$ (0 at -3)	$-$ (0 at 1)	$+$
Conclusion	**Increasing**	**Decreasing**	**Increasing**

Therefore, f is strictly increasing on the intervals $(-\infty, -3]$ and $[1, \infty)$ (But f is not increasing $(-\infty, -3] \cup [1, \infty)$), and f is decreasing on $[-3, 1]$.

Note: f is monotonic on intervals I_1 and I_2 has no guarantee for f to be monotonic on $I_1 \cup I_2$.

b) $f(x) = 2 + 3^{(1-2x)} \Rightarrow f'(x) = (-2)(\ln 3)(3^{(1-2x)}) \neq 0,\ \forall x \in \mathbb{R}$. **WHY?**

Now, since $-2 < 0$, $\ln 3 > 0$ *and* $3^{(1-2x)} > 0, \forall x \in \mathbb{R}$,

we have $f'(x) = (-2)(\ln 3)\left(3^{(1-2x)}\right) < 0, \forall x \in \mathbb{R}$.

Therefore, f (strictly) decreasing function (on $\mathbb{R}$).

c) $f(x) = x\sqrt{x^2+1} \Rightarrow f'(x) = \sqrt{x^2+1} + \frac{x(2x)}{2\sqrt{x^2+1}} = \frac{2(x^2+1)+2(x^2)}{2\sqrt{x^2+1}} = \frac{2x^2+1}{2\sqrt{x^2+1}} = 0$

$\Rightarrow 2x^2 + 1 = 0$. But $2x^2 + 1 > 0$ for all real number x.

Therefore, f strictly increasing function on its entire domain $\mathbb{R}$.

Exercise 4.2. Find intervals in which f is strictly increasing or strictly decreasing.

1. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
2. $f(x) = 2\sin x + \cos 2x$ *on* $[0, \pi]$
3. $f(x) = \sqrt{16 - x^2}$
4. $f(x) = x - 3\sqrt[3]{x}$
5. $f(x) = \frac{x^2+1}{x-1}$
6. $f(x) = \ln(2 - 3x)$
7. $f(x) = xe^x - 4$
8. $f(x) = x - 2|x - 3|$

Local (or Relative) Extreme Values of a Function on its Entire Domain

Recall that the local extreme values of a function f correspond to the **peak (local maximum)** and **valley (local minimum)** on the graph of f. **(See definition 4.2 and figure 4.2 on your text book : page 166)**

Theorem 3. (First Derivative Test for Local Extreme Values of a Function)

Suppose that f be a continuous on some interval I and let x = c $\in$ I be the critical number of f. Then

i. If f' changes its sign from **positive** to **negative** at c, then f has **local maximum** value at c. **(Here,** point (c , f(c)) is the **local max. point** and y = f(c) is the **local maximum value** of f).

ii. If f' changes its sign from **negative** to **positive** at c, then f has **local minimum** value at c.

(Here, point (c , f(c)) is the **local min. point** and y = f(c) is the **local minimum value** of f).

iii. If f' doesn't change its sign at c, then has no any local extreme value at c.

- We can relate monotonicity and local extreme value as follow:
  Let f be decreasing on $[a\ ,\ c]$ and increasing on $[c\ ,\ b]$. Then f has local minimum at $x = c$.
  Let f be increasing on $[a\ ,\ c]$ and decreasing on $[c\ ,\ b]$. Then f has local maximum at $x = c$.

Examples 6. Find the local maximum and minimum values of each of the following functions.

a) $f(x) = x^3 + 3x^2 + 3x + 4$

c) $f(x) = \begin{cases} x^2 - 4x + 3 \ for\ x \geq 1 \\ x^3 - 3x^2 - 9x + 11 \ \ for\ \ x < 1 \end{cases}$

b) $f(x) = (x^2 - 4)^{\frac{2}{3}}$

Solution

a) $f\,'(x) = 3x^2 + 6x + 3 = 3(x+1)^2 = 0$ iff $x = -1$

- $x = -1$ is the critical numbers of f.

		- 1	
$x + 1$	$-$	**0**	$+$
$f\,'(x) = 3(x+1)^2$	$+$		$+$
Conclusion	**Increasing**	**0**	**Increasing**

From the sign chart we can see that, there is no change of sign of $f\,'$ throughout the entire domain. That is f is strictly increasing function.
Therefore, by First Derivative Test, f has **no any local extreme value**.

b) $f(x) = (x^2 - 4)^{\frac{2}{3}} = \sqrt[3]{(x^2-4)^2}$. Notice that f is continuous on the entire domain of ℝ.

$\Rightarrow \quad f\,'(x) = \frac{2}{3}(x^2 - 4)^{-\frac{1}{3}} = \frac{2(2x)}{3\sqrt[3]{x^2-4}} = \frac{4x}{3\sqrt[3]{x^2-4}} = \frac{4x}{3\sqrt[3]{(x-2)(x+2)}}$

$\Rightarrow \quad f\,'(x) = 0$ when $x = 0 \in \mathbb{R}$, and $f\,'(x)$ doesn't exist when $x = -2$ or $2 \in \mathbb{R}$.

$\Rightarrow \quad -2\ , 0\ \ and\ \ 2$ are the critical numbers of f.

		- 2		**0**		**2**	
$\mathbf{4x}$	$-$		$-$	**0**	$+$		$+$
$\mathbf{3\sqrt[3]{x^2-4}}$	$+$	**0**	$-$		$-$	**0**	$+$
$\boldsymbol{f\,'(x)}$	$-$	$\nexists$	$+$	**0**	$-$	$\nexists$	$+$

By First Derivative Test,

- f has **relative maximum** at $x = 0$, and hence

  $f(0) = (0^2 - 4)^{\frac{2}{3}} = 2\sqrt[3]{2}$ is the **relative maximum value** of f.
- f has **relative minimum** at $x = -2$ and $x = 2$, and hence

  $f(-2) = f(2) = ((\pm 2)^2 - 4)^{\frac{2}{3}} = 0$ is the relative minimum value of f.

  In this example, we can see that f attains its minimum value twice at $x = -2$ and $x = 2$.

c) $f(x) = \begin{cases} x^2 - 4x + 3 \; for \; x \geq 1 \\ x^3 - 3x^2 - 9x + 11 \;\; for \;\; x < 1 \end{cases} \Rightarrow \;\; f\,'(x) = \begin{cases} 2x - 4 \quad for \quad x > 1 \\ \nexists \qquad\quad for \qquad x = 1 \\ 3x^2 - 6x - 9 \;\; for \;\; x < 1 \end{cases}$

Here we can observe that f is continuous at $x = 1$, but f is not differentiable at $x = 1$.
On $(1\,,\ \infty)$, $f'(x) = 2x - 4 = 2(x - 2) = 0$ iff $x = 2 \in (1\,,\ \infty)$.
On $(-\infty,\ 1)$, $f\,'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1) = 0$ iff $x = 3 \;\; or \;\; x = -1$.
Here, $x = -1 \in (-\infty,\ 1)$ but $x = 3 \notin (-\infty,\ 1)$.

- -1 , 1 and 2 are the critical points of f.

		-1		1		2	
$2x-4$ = 2(x – 2)					$-$	0	$+$
$x-3$	$-$		$-$				
$x+1$	$-$	0	$+$				
$3x^2-6x-9$ $= (x-3)(x+1)$	$+$		$-$				
$f'(x)$	$+$	0	$-$	$\nexists$	$-$	0	$+$

From the sign chart, we can see that

- f is **increasing** on the intervals $(-\infty, 1]$ and on $[2\,,\ \infty)$.
- f is **decreasing** on $[-1\,,\ 1] \cup [1\,,\ 2] = [-1\,,\ 2]$. **(WHY we take the union?)**

By First Derivative Test,

- f has **relative maximum** at $x = -1$, and hence

  $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 11 = 16$ is the **relative maximum value** of f.
- f has **relative minimum** at $x = 2$, and hence

  $f(0) = (2)^2 - 4(2) + 3 = -1$ is the **relative minimum** value of f.

Exercise 4.3.

1. Find the local maximum and local minimum values of each of the following functions.

   a) $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$ d) $f(x) = \sqrt{2}x - \cos 2x \;\; on \;\; [0\,,\ 2\pi]$

b) $f(x) = \begin{cases} 3x+1 \; for \; x \leq 1 \\ 5-x^2 \; for \; x > 1 \end{cases}$

e) $f(x) = \frac{-x^2+4x-7}{x-1}$

c) $f(x) = |ln\, x|$

2. If $f(x) = 2x^3 + ax^2 - 24x + 1$ has local maximum at $x = -4$, then find the value of a.

The Second Derivative Test

Another important application of derivative is the determination of extreme values of a function using the derivative test known as the **Second Derivative Test**.

Theorem 3. (Second Derivative Test for Local Extreme Values of a Function)

Suppose f is twice differentiable and f'' is continuous at c

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
c) If $f'(c) = 0$ and $f''(c) = 0$, then we can not set any conclusion about the local extreme value of using Second Derivative Test.

Note : If $f''(c) = 0$ or $f''(c) = \nexists$, then we can't apply Second Derivative Test. In that case we turn back to the First Derivative Test to obtain local extreme value of f.

Examples 7. Using Second Derivative Test, find the local maximum and minimum values of each of the following functions.

a) $f(x) = xe^x + 1$ b) $f(x) = \frac{9}{10}x^{\frac{5}{3}}$

Solution

a) $f(x) = xe^x + 1 \Rightarrow f'^{(x)} = e^x + xe^x = e^x(1+x) = 0 \Rightarrow x = -1$

$\Rightarrow f''(x) = 2e^x + xe^x = (2+x)e^x$

- $f''(-1) = (2-1)e^{-1} = \frac{1}{e} > 0.$

Therefore, by Second Derivative Test, f has local minimum at $x = -1$, and $f(-1) = (-1)e^{-1} + 1 = 1 - \frac{1}{e}$ is the local minimum value of f .

b) $f(x) = \frac{9}{10}x^{\frac{5}{3}} \Rightarrow f'(x) = \left(\frac{9}{10}\right)\left(\frac{5}{3}\right)x^{\frac{2}{3}} = -\frac{3}{2}x^{\frac{2}{3}} = 0 \Rightarrow x = 0$

$\Rightarrow f''(x) = -\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)x^{-\frac{1}{3}} = -\frac{1}{\sqrt[3]{x}}$. But $f''(0) = \nexists$

Therefore, the Second Derivative Test fails to apply.

Concavity and inflection points

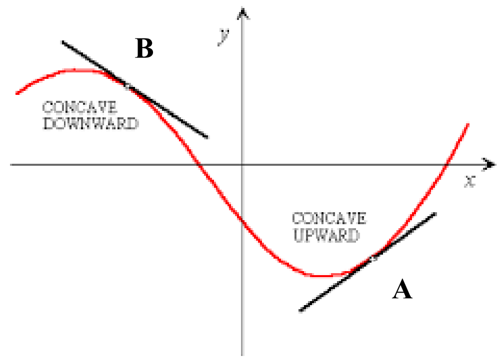
Definition 4.3

Let f be a function differentiable at c and let L be the line tangent to the graph of f at. $(c\,,\; f(c))$

i. The graph of f is said to be **CONCAVE UPWARD** at $(c\,,\; f(c))$ if there is an open interval I about c such that $\forall x \in I$ and $x \neq c$, the point $(x\,,\; f(x))$ lies above L.

ii. The graph of f is said to be **CONCAVE DOWNWARD** at $(c\,,\; f(c))$ if there is an open interval I about c such that $\forall x \in I$ and $x \neq c$, the point $(x\,,\; f(x))$ lies below L.

In the figure, the graph is concave upward at point A, and concave downward at B



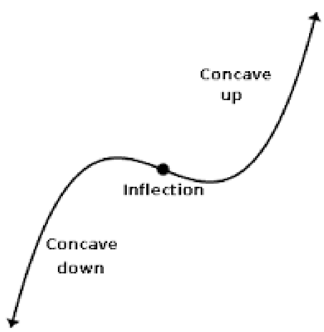
Definition 4.4

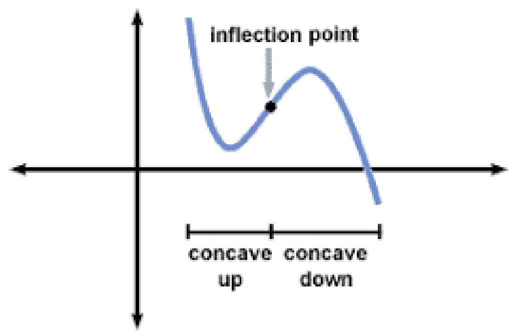
i. If the graph of a function f lies above all of its tangents on an interval I, then f is said to be **concave upward** on I.

ii. If the graph of a function f lies below all of its tangents on an interval I, then f is said to be **concave downward** on I.

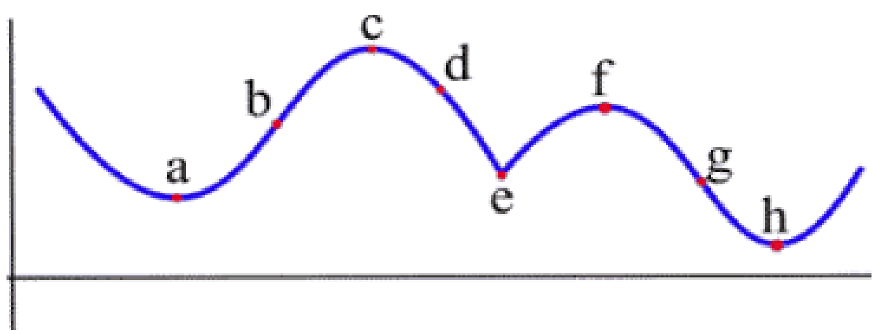
Definition 4.5

A point on a curve is called an **INFLECTION POINT**, if the curve changes either from concave up to concave down or from concave down to concave up.





Activity : In the following figure, identify which points indicated on the graph are inflection points.



➢ Another important application of the second derivative test is it helps to obtain the interval on which the graph is concave upward and on which the graph is concave upward, and also to find the inflection point of the function.

Theorem 4. (Concavity Test)

Let f be a function twice differentiable on an interval I.

i. If $f''(x) > 0$ for all x in I, then the graph of f is **concave upward** on I.

ii. If $f''(x) < 0$ for all x in I, then the graph of f is **concave downward** on I.

Examples 8. Discus the behaviour of the curves of the following functions w.r.t local maximum and local minimum, concavity and points of inflection.

a) $f(x) = 3x^4 - 4x^3$ b) $f(x) = \frac{x^2 + 1}{x}$

Solution

a) **Local extreme values :** $f'(x) = 12x^3 - 12x^2 = 0$

$\Rightarrow f'(x) = 12x^2(x-1) = 0 \Rightarrow x = 0$ or $x = 1$

$\Rightarrow$ 0 and 1 are the critical point of f.

Then sketch the sign chart for f'.

		0		1	
$12x^2$	$+$	0	$+$		$+$
$x-1$	$-$		$-$	0	$+$
$f'(x)$	$-$	0	$-$	0	$+$

Hence, from the sign chart, by 1st Derivative Test, $f(1) = 3(1)^4 - 4(1)^3 = -1$ is the local minimum value of f.

But f has no local maximum value.

Concavity and points of inflection : $f''(x) = (12x^3 - 12x^2)' = 36x^2 - 24x = 12x(3x-2) = 0$

$\Rightarrow$ 0 and $\frac{2}{3}$ are the critical point of f'.

Then sketch the sign chart for f''.

		0		$\frac{2}{3}$	
$12x$	$-$	0	$+$		$+$
$3x-2$	$-$		$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$

Hence, from the sign chart, by 2nd Derivative Test,

The graph of f is **concave upward** on $(-\infty, 0)$ and on $\left(\frac{2}{3}, \infty\right)$; and f is **concave downward** on $(0, \frac{2}{3})$.

Concavity of f changes at $x = 0$ and $x = \frac{2}{3}$.

Therefore, the inflection points of f are $(0, f(0)) = (0, 0)$ and $\left(\frac{2}{3}, f\left(\frac{2}{3}\right)\right) = (\frac{2}{3}, -\frac{16}{27})$

b) $f'(x) = \frac{x^2 - 1}{x^2} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = -1$ or $x = 1$ $\Rightarrow$ The critical numbers are -1 & 1.

$f''(x) = \frac{2x(x^2) - 2x(x^2 - 1)}{(x^2)^2} = \frac{2}{x^3} \neq 0$ for all real number $x \neq 0$. $\Rightarrow$ f' has no critical number.

Then, sketch the sign charts for f' and f''.

		-1		0		1	
x^2-1	$+$	0	$-$		$-$	0	$+$
x^2	$+$		$+$	0	$+$		$+$
$f'(x)$	$+$	0	$-$	$\nexists$	$-$	0	$+$

		0	
2	$+$		$+$
x^3	$-$	0	$+$
$f''(x)$	$-$	$\nexists$	$+$

- ✓ Hence, from the first sign chart, by 1st Derivative Test, f has local maximum value at the point $(-1\,,\ f(-1)) = (-1\,,-2)$ and local minimum value at the point $(1\,,\ f(1)) = (1\,,\ 2)$
- ✓ The graph of f is **concave upward** on the intervals $(-\infty\,,\ 0)$ and on $\left(\frac{2}{3}\ ,\ \infty\right)$; and f is **concave downward** on the interval on $(0\,,\ \frac{2}{3})$.
- ✓ Concavity of f changes at $= 0$, but $0 \notin$ domain of f. Therefore, the graph of f has no inflection point.

Exercise 4.4. Find the local extreme values, the intervals on which the graphs concave upward and concave downward, and the inflection point of each of the following functions.

1. $f(x) = x^4 - 6x^2 + 8x + 10$
2. $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$
3. $f(x) = \frac{x}{x^2+1}$
4. $f(x) = 2x - \tan x\,,\ on\ \left(-\frac{\pi}{2}\,,\ \frac{\pi}{2}\right)$

Comprehensive Curve Sketching

Now, you are ready to develop a procedure for curve sketching. To sketch the graph of a given function, say f, you should follow the following procedure.

Step 1. Determine the domain of the function f.
Step 2. Check whether the function is odd or even. This will help to understand the shape of the graph.
Step 3. Determine the x- and y-intercepts of f.
Step 4. Determine the asymptotes of f, if any.
Step 5. Determine the intervals of monotonicity and the local extreme values f.
Step 6. Determine the intervals of concavity and the inflection point of f.
Step 7. Lastly, using the information obtained in the above six steps sketch the graph of f.

Examples 9: Discuss the behaviors and sketch the graph of the function $f(x) = x^4 - 2x^2$.
of the following functions.

Solutions $f(x) = x^4 - 2x^2$

- Domain = $\mathbb{R}$
- $f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$. Hence, f is even function.
  $\Rightarrow$ The graph of f is symmetric w.r.t the y – axis.
- $f(x) = x^4 - 2x^2 = x^2(x^2-2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$
  $\Rightarrow -\sqrt{2}\,,$ 0 and $\sqrt{2}$ are the x-intercepts of f.
  $y = f(0) = 0$ is the y – intercept of f.
- Since f is polynomial, its graph has no any asymptote.
- $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1) = 0 \Rightarrow x = -1,\ 0\ or\ 1$
  $\Rightarrow x = -1,$ 0 and 1 are the critical numbers of f.
- $f''(x) = 12x^2 - 4 = 4(3x^2-1) = 4(\sqrt{3}x-1)(\sqrt{3}x+1) = 0 \Rightarrow x = -\frac{\sqrt{3}}{3}\ or\ \frac{\sqrt{3}}{3}$
  $\Rightarrow -\frac{\sqrt{3}}{3}$ and $\frac{\sqrt{3}}{3}$ are the critical numbers of f.

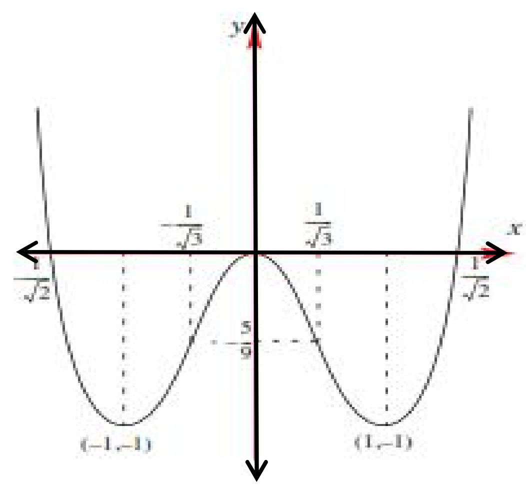
Sign chart for $f'(x)$

		-1		0		1	
$4x$	$-$		$-$	0	$+$		$+$
x^2-1	$+$	0	$-$		$-$	0	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$

Sign chart for $f''(x)$

		$-\frac{\sqrt{3}}{3}$		$\frac{\sqrt{3}}{3}$	
4	$+$		$+$		$+$
$\sqrt{3}x-1$	$-$		$-$	0	$+$
$\sqrt{3}x+1$	$-$	0	$+$		$+$
$f''(x)$	$+$	0	$-$	0	$+$

- Hence, from the **first sign chart**, f is **monotonically decreasing** on the intervals $(-\infty, -1)$ and on $(0, 1)$. f is **monotonically increasing** on the intervals $(-1, 0)$ and on $(1, \infty)$.
- Again, from the **first sign chart**, by **1st Derivative Test**, f has local maximum value at the point $\big(-1, f(-1)\big) = (-1, -1)$ and local minimum value at the point $\big(1, f(1)\big) = (1, -1)$
- From the **second sign chart**, the graph of f is **concave upward** on the intervals $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and on $\left(\frac{\sqrt{3}}{3}, \infty\right)$; and f is **concave downward** on the interval on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.
- This implies that $\left(-\frac{\sqrt{3}}{3}, f\left(-\frac{\sqrt{3}}{3}\right)\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ and $\left(\frac{\sqrt{3}}{3}, f\left(\frac{\sqrt{3}}{3}\right)\right) = \left(\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ are the inflection points of the graphs of f.
- Therefore, the graph of f can be sketched as follow.



Exercise 4.5: Discuss the behaviors and sketch the graph of the following function.

a) $f(x) = x^3 - 12x$

b) $f(x) = 2 + \frac{x}{x^2-1}$

c) $f(x) = \frac{1}{2^x - 1}$

d) $f(x) = \begin{cases} x^2 - 6x + 8 & if\ \ 1 \le x < 4 \\ x + 2 & if\ -5 \le x < 1 \end{cases}$

e) $f(x) = e^{-x}\cos x \ \ on\ \left[0, \frac{5\pi}{2}\right]$

4.2 MINIMIZATION AND MAXIMIZATION PROBLEMS (OPTIMIZATION PROBLEMS)

The techniques and theories that have been developed in previous sections and chapter have important application to real life problems in which the maximum or minimum value of quantity is required. Problems that require the use of this theory can be found in many real life situations such as manufacturers wanting to minimize their costs, designers wanting to maximize the available space to be used A businessperson wants to minimize costs and maximize profits, etc. Such types of problem sometimes known as **OPTIMIZATION (MINIMIZATION AND MAXIMIZATION) PROBLEMS**

Guideline to solve optimization problems

Step 1. Understand the problem

Step 2. Introduce notation : Assign a symbol to the quantity that is to be maximized or minimized and select symbols for the unknowns.

Step 3. Draw a diagram (if necessary), and label the given and the required quantities on the diagram.

Step 4. Write a mathematical equation that describes the problem. Determine the domain.

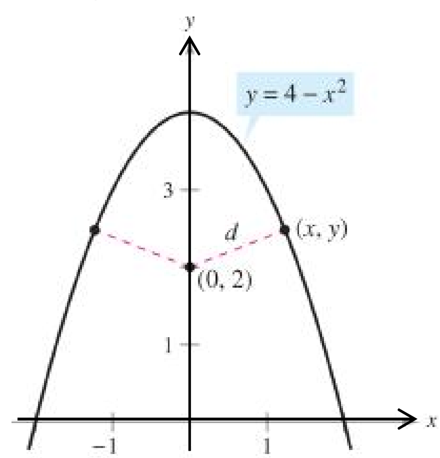
Step 5. Express the quantity which is going to be optimized in terms of the unknowns.

Step 6. Use the methods of solving maximization and minimization problems to get the quantity optimized.

Note: The maximum or minimum value does not always occur when the first derivative is zero. It is essential to also examine the values of the function at the endpoint(s) of the domain for global maxima and minima.

Examples 10. Find a point on the graph of $f(x) = 4 - x^2$ which is closest to the point (0, 2).

Solution: The graph of $f(x) = 4 - x^2$ shows that there are two points on the graph which have equal distance from (0, 2).



To optimize distance

Let (x, y) be a point on the graph of f which has least distance from the point (0, 2). Then by distance formula

$$d = \sqrt{(x-0)^2 + (y-2)^2} = d(x,y)$$

$\Rightarrow\ d(x,y) = \sqrt{x^2 + (y-2)^2}$ and domain of

Since $y = f(x) = 4 - x^2$, we have

$$\Rightarrow\ d(x) = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$$

The domain of $d(x) = \mathbb{R}$ and there is no end point on the domain.

Because of d is smallest when $x^4 - 3x^2 + 4$ is smallest, you need only to find the critical numbers of $g(x) = x^4 - 3x^2 + 4$.

$\Rightarrow\quad g'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0 \ \Rightarrow\ x = 0\,,\ -\sqrt{\frac{3}{2}}\,,\ \sqrt{\frac{3}{2}} \in \mathbb{R}$ are critical numbers of f.

$\Rightarrow\quad g(0) = (0)^4 - 3(0)^2 + 4 = 4$, and

$$g\left(-\sqrt{3/2}\right) = \left(-\sqrt{3/2}\right)^4 - 3\left(-\sqrt{3/2}\right)^2 + 4 = \tfrac{7}{4} = g\left(\sqrt{3/2}\right) < g(0)$$

The relative minimum g occurs when $x = -\sqrt{3/2}\ \ or\ \ x = \sqrt{3/2}$

Therefore, points $\left(-\sqrt{3/2}\ ,\ f\left(-\sqrt{3/2}\right)\right) = \left(-\sqrt{3/2}\ ,\ 5/2\right)$ and

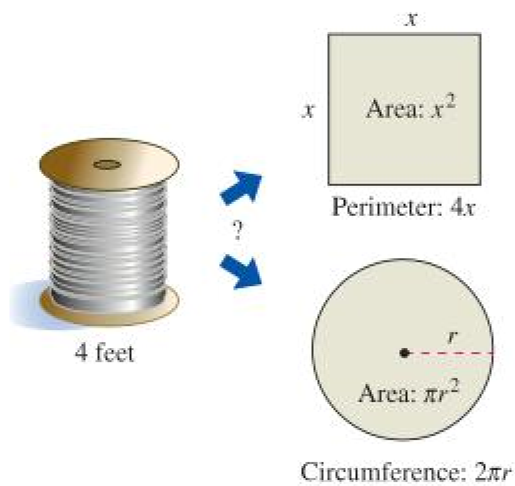
$\left(\sqrt{3/2}\ ,\ f\left(\sqrt{3/2}\right)\right) = \left(\sqrt{3/2}\ ,\ 5/2\right)$ are the closest points on the graph of f fro (0, 2).

Example 11. Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Solution Let L = length of the wire = 4 ft.

r = radius of the circle and

x = the length of a side of the square.



To optimize total area A

The total area A is given by

$$A = a(square) + a(circle)$$
$$A = x^2 + \pi r^2 = A(x, r)$$

This is the primary equation.

$$L = P(square) + C(circle)$$
$$4 = 4x + 2\pi r$$

Solving for r, we obtain

$$r = 2(1 - x)/\pi$$

Substitute this in A gives

$$A = x^2 + \pi[2(1 - x)/\pi]^2 = \tfrac{1}{\pi}[(\pi + 4)x^2 - 8x + 4] = A(x)$$

Based on the perimeter of the square, the feasible domain is $0 \le x \le 1$.

Then $\frac{dA}{dx} = \frac{2(\pi + 4)x - 8}{\pi} = 0$ iff $x = \frac{4}{\pi+4} \in (0\ ,\ 1)$

Hence, $x = \frac{4}{\pi+4}$ is the critical point of A(x).

Since there us a closed interval domain, we use Extreme Value Theorem.

$$A(0) = \tfrac{1}{\pi}[(\pi + 4)(0)^2 - 8(0) + 4] = \tfrac{4}{\pi}\ \text{ft}^2.$$

$$A\left(\tfrac{4}{\pi+4}\right) = \tfrac{1}{\pi}\left[(\pi + 4)(\tfrac{4}{\pi+4})^2 - 8(\tfrac{4}{\pi+4}) + 4\right] = \tfrac{4}{\pi+4}\ \text{ft}^2.$$

$$A(1) = \tfrac{1}{\pi}[(\pi + 4)(1)^2 - 8(1) + 4] = 1\ \text{ft}^2.$$

Hence, the maximum area occurs when $x = 0$, that is the whole wire is used to make the circle.

The maximum area so obtained is $A_{max} = A(0) = \frac{4}{\pi}$ ft^2

This type of maximum sometimes known as end point maximum.

Exercise 4.6:

1. The height 's' of an object in free fall on the moon is approximately given by $s = s_0 + u_0 t - \frac{27}{10}t^2$, where 's' is measured in feet and 't' in seconds.
   a) What is the acceleration due to "gravity" on the moon?
   b) If an object is thrown upward from the surface of the moon with an initial velocity of 54 feet per second, what is the maximum height it will reach, and when will it reach that height?

2. An open box is to be constructed with a square base and is required to have a volume of 48 cubic inches. The bottom of the box costs 3 cents per square inch, whereas the sides cost **2** cents per square inch. Find the dimensions that will minimize the cost of the box.
3. A farmer must fence in a rectangular field with one side along a stream; no fence is needed on that side. If the area must be 1800 square meters and the fencing cost Br. 2 per meter, what dimensions will minimize the cost?
4. A thin-walled cone-shaped cup is to hold 36 π cu. in. of water when full. What dimensions will minimize the amount of material needed for the cup?
5. Find the dimensions of the closed cylindrical can that will have a capacity of **k** volume units and used the minimum amount of material.

4.3 RATE OF CHANGE

There are many real-life applications of rates of change such as velocity, acceleration, population growth rates, unemployment rates, production rates, etc.. You can investigate the rate of change of one variable with respect to any other related variable.
When determining the rate of change of one variable with respect to another, you must be careful to distinguish between average and instantaneous rates of change.

- The **average rate of change** of a function f over the interval $[a\ ,\ b]$ is the slope of a secant line is determined by two points $(a\ ,\ f(a))$ and $(b\ ,\ f(b))$ which is given by

$$\textbf{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- The **instantaneous rate of change** of a function f at the point $x = c$ is the slope of a tangent line to the graph of f at the point $(c\ ,\ f(c))$ which is given by

$$\textbf{instantaneous rate of change} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

Related Rates

- In this section, you will study problems involving variables that are changing with respect to time. If two or more such variables are related to each other, then their rates of change with respect to time are also related.
- If two or more quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other quantities must be changing.
- Let x and y be variables which are both changing with time t and let $y = f(x)$.

  $\Rightarrow$ The rate of change of y is $\frac{dy}{dt}$ and the rate of change of x is $\frac{dx}{dt}$.

  Then by Chain Rule, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ related as

$$y = f(x) \quad \Rightarrow \quad y(t) = f(x(t))$$

  Differentiating both sides w.r.t x gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For instance, suppose variables x and y be related by $y = 2x^3$

$\Rightarrow \quad y(t) = 2(x(t))^3$

$\Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} = 6x^2\frac{dx}{dt}$

Examples 11. The volume of a cube increases at a constant rate of 10 cm^3/sec. Find the rate of change in its total surface area at the instant when its sides are 20 cm. long.

Solution : Let x be the lengths of the sides of the cube. Then the total surface area A and the volume V of the cube are

$A = 6x^2$ and $V = x^3$

$\Rightarrow \quad \frac{dA}{dt} = 12x\frac{dx}{dt}$ and $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

At the instant when $x = 20\ cm.$, $\frac{dV}{dt} = 3(20)^2\frac{dx}{dt} = 1200\frac{dx}{dt}$

$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{1200}\frac{dV}{dt} = \frac{1}{1200}(10) = \frac{1}{120}cm/s$

Thus, $\frac{dA}{dt} = 12x\frac{dx}{dt} = 12\left(\frac{1}{120}\right)(20) = 2\ cm^2/s$

Therefore, the surface area is increasing at the rate of $2cm^2s^{-1}$

Examples 12. The radius r of a right circular cone is increasing at a rate of 2 cm / min. The height h of the cone is related to the radius by $h = 3r$. Find the rate of change of the volume when $r = 3$cm.

Solution : The volume V of the cone is

$V = \frac{1}{3}\pi r^2 h$ and it is given that $h = 3r$

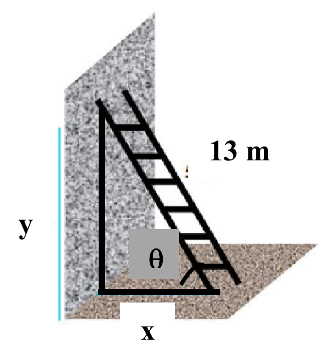
Hence, V can be written as

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 3r^2\frac{dr}{dt} = 3r^2 2cm/min = 6r^2 cm/min$

Then, at the instant r = 3 cm., the rate of change of the volume becomes

$\frac{dV}{dt} = 6(3cm)^2 cm/min = 54cm^3/min$

Examples 13. A ladder 13m long is leaning against a vertical wall. If the foot of the ladder slides away from the wall at a constant rate of 2 m / s, how fast is the angle formed between the ladder and the flat ground changing (rad. / s) at the instant when the top of the ladder is 12 m. above the ground?



Solution Let θ be the angle between the ladder and the flat ground.

Rate of change of x = $\frac{dx}{dt} = 2m/s$

To find $\frac{d\theta}{dt}$

From trigonometric identity $\cos\theta = \frac{x}{r} = \frac{x}{13}$

➢ $\frac{d}{dt}(\cos\theta) = \frac{d}{dt}\left(\frac{x}{13}\right)$

➢ $(-\sin\theta)\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$

We know that $\sin\theta = \frac{y}{r} = \frac{y}{13}$. At the instant y = 12 m., we have

$\sin\theta = \frac{y}{13} = \frac{12}{13}$

- $\frac{d\theta}{dt} = -\frac{1}{13}.\frac{13}{12}.2 = -\frac{1}{6}$ rad/s
- Therefore, the angle between the ladder and the flat ground decreases at the rate of $\frac{1}{6}$ rad/s.

Implicit Differentiation

Examples 12. If $x^3 + y^3 = 10$, then find $\frac{dy}{dx}$ and $\frac{dx}{dy}$, assuming that y is differentiable w.r.t x and x is differentiable w.r.t y

Solution $x^3 + y^3 = 10$

i) Differentiate both sides w.r.t x, and the solve for $\frac{dy}{dx}$

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(10)$$

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 0 \Rightarrow 3x^2\frac{dx}{dx} + 3y^2\frac{dy}{dx} = 0 \quad \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^2$$

ii) Differentiate both sides w.r.t y, and the solve for $\frac{dx}{dy}$

$$\frac{d}{dy}(x^3 + y^3) = \frac{d}{dy}(10)$$

$$\Rightarrow \frac{d}{dy}(x^3) + \frac{d}{dy}(y^3) = 0 \Rightarrow 3x^2\frac{dx}{dy} + 3y^2\frac{dy}{dy} = 0 \quad \Rightarrow 3x^2\frac{dx}{dy} + 3y^2 = 0$$

$$\Rightarrow \frac{dx}{dy} = -\left(\frac{y}{x}\right)^2$$

Exercise 4.7

1. Let $s = t^3 - 9t^2 + 24t$ describe the position **s** at time ***t*** of an object moving on a straight line.
   a) Find the velocity and acceleration.
   b) Determine when the velocity is positive and when it is negative.
   c) Determine when the acceleration is positive and when it is negative.
2. An open box is to be constructed with a square base and is required to have a volume of 48 cubic inches. The bottom of the box costs 3 cents per square inch, whereas the sides cost **2** cents per square inch. Find the dimensions that will minimize the cost of the box.
3. 4fts. of wire is to be used to form a square and a circle. How much of the wire should be used for the square and should be used for the circle to enclose the maximum total area?
4. The length of a rectangle having a constant area of 800 sq. mm is increasing at the rate of 4 mm/sec.
   a) What is the width of the rectangle at the moment when the width is decreasing at the rate of 0.5mm/sec?
   b) How fast is the diagonal of the rectangle changing when the width is 20 mm?
5. A cube of ice is melting. The side s of the cube is decreasing at the constant rate of 2 inches per minute. How fast is the volume *V* decreasing?

6. A conical tank (with vertex down) is 10ft. deep. If water is flowing into the tank at a rate of $10\ ft^3/min$, find the rate of change of the depth of the water when the water is 8ft. deep.
7. If the radius of a sphere is increasing at the constant rate of 3 $mm./sec.$, then how fast is the volume changing when the surface area is 10sq. mm.?

UNIT 5 INTRODUCTION TO INTEGRAL CALCULUS

INTRODUCTION

Integral Calculus is the process of finding the function itself when its derivative is known.

Unit Outcomes

After completing this unit, you will be able to:

- *understand the concept of definite integrals.*
- *integrate different polynomial functions, simple trigonometric functions, exponential and logarithmic functions.*
- *use the various techniques of integration to evaluate a given integral.*
- *use the fundamental theorem of calculus for computing definite integrals.*
- *apply the knowledge of integral calculus to solve real life mathematical problems*

5.1 INTEGRATION AS REVERSE PROCESS OF DIFFERENTIATION

5.1.1 The Concept of Indefinite Integral

Definition 5.1 (Anti derivative)

The process of finding $f(x)$ from its derivative $f'(x)$ is said to be **anti-differentiation** or **integration**. $f(x)$ is said to be the **anti-derivative** of $f'(x)$.

➢ **Integration is the reverse operation of differentiation.**

Definition 5.2

The set of all anti derivatives of a function $f(x)$ is called the **INDEFINITE INTEGRAL** of $f(x)$. The indefinite integral of $f(x)$ is denoted by $\int f(x)\,dx$, read as **"the integral of $f(x)$ with respect to x"**.

- The symbol $\int$ is said to be the **integral sign**.
- The function $f(x)$ is said to be the **integrand** of the integral.
- dx denotes that the **variable of integration** is x.
- If a function has an integral, then it is said to be **integrable**.
- If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is constant, and known as the **constant of integration**.

For instance, if $(x) = x$, then a function whose derivative is $f(x)$ is $F(x) = \frac{x^2}{2}$. Hence, F is an antiderivative of $f(x)$ and we say that $F(x) = \frac{x^2}{2}$ is the integral of $f(x)$. Moreover, $F_1(x) = \frac{x^2}{2} + 1$,

$F_2(x) = \frac{x^2}{2} - 3$, $F_3(x) = \frac{x^2}{2} + \sqrt{2}$, etc., are also the antiderivatives of $f(x)$. In general, for any constant c, $F(x) = \frac{x^2}{2} + c$ is the set of all antiderivatives of $f(x)$. This set of antiderivatives of f if denoted by

$$F(x) = \int x\, dx = \frac{x^2}{2} + c$$

which is called the **indefinite integral** of $f(x) = x$.

Example 1

a) $G(x) = \frac{x^3}{3}$ is an anti-derivative of $f(x) = x^2$, since $G'(x) = \frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 = f(x)$.
   Thus, the indefinite integral of f is given by
   $$F(x) = \int x^2\, dx = \frac{x^3}{3} + c \text{ , for any constant c.}$$

b) $G(x) = x^4$ is an anti-derivative of $f(x) = 4x^3$, since $G'(x) = \frac{d}{dx}(x^4) = 4x^3 = f(x)$.
   Thus, the indefinite integral of f is given by
   $$F(x) = \int 4x^3\, dx = x^4 + c \text{ , for any constant c.}$$

c) $G(x) = \sin x$ is an anti-derivative of $f(x) = \cos x$, since $G'(x) = (\sin x)' = \cos x$.
   Thus, the indefinite integral of f is given by
   $$F(x) = \int \cos x\, dx = \sin x + c \text{ , for any constant c.}$$

d) $G(x) = e^x$ is an anti-derivative of $f(x) = e^x$, since $G'(x) = \frac{d}{dx}(e^x) = e^x = f(x)$.
   Thus, the indefinite integral of f is given by
   $$F(x) = \int e^x\, dx = e^x + c \text{ , for any constant c.}$$

e) $G(x) = \ln x$ is an anti-derivative of $f(x) = \frac{1}{x}$, since $G'(x) = (\ln x)' = \frac{1}{x} = f(x)$.
   Thus, the indefinite integral of f is given by
   $$F(x) = \int \frac{1}{x}\, dx = \ln x + c \text{ , for any constant c.}$$

Note : If $F(x)$ and $G(x)$ are anti-derivatives of f, then they differ only by a constant.

$F(x) = G(x) + c$, where c is an arbitrary constant.

Remark: From the definition of indefinite integral and anti-differentiation, we have the following:

i) $\int \frac{d}{dx} f(x)\, dx = \int f'(x)\, dx = f(x) + c$, where c is arbitrary constant.

   For instance, $\int \frac{d}{dx}(\sin x)\, dx = \int \cos x\, dx = \sin x + c$, where c is arbitrary constant.

   Also $\int \frac{d}{dx}(x^4)\, dx = x^4 + c$, where c is arbitrary constant.

ii) $\frac{d}{dx}\int f(x)\, dx = f(x)$

   For instance, $\frac{d}{dx}\int \sin x\, dx = \frac{d}{dx}(-\cos x + c) = \frac{d}{dx}(-\cos x) + \frac{d}{dx}(c) = \sin x + 0 = \sin x$

   Also $\frac{d}{dx}\int x^4\, dx = x^4$

Integration (Indefinite Integral) of some simple functions

For an arbitrary constant c,

a) $\int 0\, dx = c$
b) $\int dx = \int 1\, dx = x + c$
c) $\int k\, dx = kx + c$, for any constant k.
d) $\int x\, dx = \frac{x^2}{2} + c$
e) $\int x^2\, dx = \frac{x^3}{3} + c$
f) $\int \frac{1}{x}\, dx = \int x^{-1}\, dx = \ln x + c$

Power Rule

Thus, for any real number n, $\int \boldsymbol{x^n\, dx} = \begin{cases} \frac{x^{n+1}}{n+1} + c & if \quad n \neq -1 \\ \ln|x| + c & if \quad n = -1 \end{cases}$

g) $\int \sqrt{x}\, dx = \int (x)^{\frac{1}{2}}\, dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{2}{3} x^{\frac{3}{2}} + c$
h) $\int \sin x\, dx = -\cos x + c$
i) $\int \cos x\, dx = \sin x + c$
j) $\int sec^2 x\, dx = \tan x + c$
k) $\int csc^{\,2} x\, dx = -\cot x + c$
l) $\int \sec x \tan x\, dx = \sec x + c$
m) $\int \csc x \cot x\, dx = -\csc x + c$

The above indefinite integrals are obtained directly from the differentiations of basic functions.

Examples 2. Evaluate the following indefinite integrals.

a) $\int x^5\, dx$
b) $\int x^{-0.3}\, dx$
c) $\int \sqrt[3]{x^5}\, dx$
d) $\int (x^{-7})^{-\frac{2}{3}}\, dx$
e) $\int \frac{x^2}{\sqrt[5]{x^3}}\, dx$
f) $\int \frac{\sqrt[5]{x^3}}{x^{\frac{8}{5}}}\, dx$

Solution : For an arbitrary constant c;

a) $\int x^5\, dx = \frac{x^6}{6} + c$
b) $\int x^{-0.3}\, dx = \frac{x^{-0.3+1}}{-0.3+1} + c = \frac{x^{0.7}}{0.7} = \frac{10}{7} x^{0.7}$
c) $\int \sqrt[3]{x^5}\, dx = \int x^{\frac{5}{3}}\, dx = \frac{3}{8} x^{\frac{8}{3}} + c$
d) $\int (x^{-7})^{-\frac{2}{3}}\, dx = \int x^{\frac{14}{3}} dx = \frac{3}{17} x^{\frac{17}{3}} + c$
e) $\int \frac{x^2}{\sqrt[5]{x^3}}\, dx = \int x^{\frac{7}{5}}\, dx = \frac{5}{12} x^{\frac{12}{5}} + c$
f) $\int \frac{\sqrt[5]{x^3}}{x^{\frac{8}{5}}}\, dx = \int x^{\frac{3}{5}-\frac{8}{5}}\, dx = \int x^{-1} dx = \ln x + c$

5.1.2 Properties of the Indefinite Integral

Properties of the Indefinite Integral

For two integrable functions f and g,

1. $\int kf(x)dx = k\int f(x)dx$, for any constant k
2. $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
3. $\int[f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

Remark : For a polynomial function of degree n given by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots \ldots \ldots . + a_2x^2 + a_1x + a_0 \text{ , } a_n \neq 0,$$

$$\int f(x)dx = \int[a_nx^n + a_{n-1}x^{n-1} + \cdots \ldots \ldots . + a_2x^2 + a_1x + a_0]dx$$

$$= \int a_nx^ndx + \int a_{n-1}x^{n-1}dx + \ \ldots \ldots \ldots . + \int a_1xdx + \int a_0dx$$

$$= a_n\frac{x^{n+1}}{n+1} + a_{n-1}\frac{x^n}{n} + \cdots \ldots \ldots . + a_1\frac{x^2}{2} + a_0x$$

which is also polynomial function of degree $n+1$.

Examples 2. Evaluate the following indefinite integrals.

a) $\int[\pi x^3 - 5x^2 + x - 3]dx$

b) $\int(2t + 3sec^2t\,)dt$

c) $\int\left(\frac{1}{2\sqrt{x}} - \frac{1}{sin^2x}\right)dx$

d) $\int\frac{(1-2y^2\,)^2}{\sqrt{y^3}}dy$

e) $\int\left(1 - 3z^2\sqrt[3]{z^2}\,\right)dz$

Solution

a) $\int[\pi x^3 - 5x^2 + x - 3]dx = \int \pi x^3dx - \int 5x^2dx + \int x\,dx - \int 3dx$

$= \pi\int x^3dx - 5\int x^2dx + \int x\,dx - 3\int dx = \pi\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - 3x + c,$ c is a constant.

b) $\int(2t + 3sec^2t\,)dt = 2\int tdt + 3\int sec^2t\,dt = 2\left(\frac{x^2}{2}\right) + 3\tan t + c,$ c is a constant.

c) $\int\left(\frac{1}{2\sqrt{x}} - \frac{1}{sin^2x}\right)dx = \int\left(\frac{1}{2}x^{-\frac{1}{2}} - csc^2x\right)dx = \sqrt{x} + \cot x + c$, c is a constant.

d) $\int\frac{(1-2y^2\,)^2}{\sqrt{y^3}}dy = \int(1 - 4y^2 + 4y^{\,4})y^{-\frac{3}{2}}dy = \int\left(y^{-\frac{3}{2}} - 4y^{\frac{1}{2}} + 4y^{\frac{5}{2}}\right)dy$

$= -2y^{-\frac{1}{2}} - 4.\frac{2}{3}y^{\frac{3}{2}} + 4.\frac{2}{7}y^{\frac{5}{2}} + c = -2y^{-\frac{1}{2}} - \frac{8}{3}y^{\frac{3}{2}} + \frac{8}{7}y^{\frac{5}{2}} + c,\ \ c \in \mathbb{R}$

e) $\int\left(1 - 3z^2\sqrt[3]{z^2}\,\right)dz = \int\left(1 - 3z^{\frac{8}{3}}\right)dz = z - 3.\frac{3}{11}z^{\frac{11}{3}} + c = z - \frac{9}{11}z^{\frac{11}{3}} + c\ ,\ \ c \in \mathbb{R}$

Examples 3. For $a > 0\ ,\ a \neq 1$ *and any real number* x, show that

$$\int a^xdx = \frac{a^x}{\ln a} + c \text{ , where c is arbitrary constant.}$$

Solution We know that $(a^x)' = a^x\ln a \ \Rightarrow\ a^x = \frac{1}{\ln a}(a^x)' = \left(\frac{a^x}{\ln a}\right)'$, since $\frac{1}{\ln a}$ is constant.

Then by definition of indefinite integral , we have

$$\int a^xdx = \frac{a^x}{\ln a} + c \text{ , where c is arbitrary constant.}$$

Thus, the indefinite integral of expnonential function is given as follow:

The Indefinite Integral of Expnonential Function

For $a > 0$, $a \neq 1$ *and any real number* x, show that

$$\int a^x dx = \frac{a^x}{\ln a} + c\,, \quad \text{where c is arbitrary constant.}$$

- **If $a = e$, then**

$$\int e^x dx = \frac{e^x}{\ln e} + c = e^x + c\,, \quad \text{where c is arbitrary constant.}$$

Examples 4. Evaluate the following indefinite integrals.

a) $\int 2^x\, dx$ b) $\int \frac{3(2^x) - 2}{\sqrt{5^x}} dx$

Solution

a) $\int 2^x\, dx = \frac{2^x}{\ln 2} + c$, where c is arbitrary constant.

b) $\int \frac{3(2^x) - 2}{\sqrt{5^x}} dx = \int \frac{3(2^x) - 2}{\sqrt{5}^x} dx = \int \left[3\left(\frac{2}{\sqrt{5}}\right)^x - 2\left(\frac{1}{\sqrt{5}}\right)^x\right] dx = 3\frac{\left(\frac{2}{\sqrt{5}}\right)^x}{\ln\left(\frac{2}{\sqrt{5}}\right)} - 2\frac{\left(\frac{1}{\sqrt{5}}\right)^x}{\ln\left(\frac{1}{\sqrt{5}}\right)} + c$,

$= 3\left(\frac{2}{\sqrt{5}}\right)^x\left[\ln\left(\frac{2}{\sqrt{5}}\right)\right]^{-1} - 2\left(\frac{1}{\sqrt{5}}\right)^x\left[\ln\left(\frac{1}{\sqrt{5}}\right)\right]^{-1} + c$, for arb. constant.

Exercise 5.1 Evaluate the following indefinite integrals.

a) $\int 3^{-x}\, dx$ d) $\int \frac{2^x - 3^x}{6^x} dx$ g) $\int \frac{y^2 - 4y+10}{y^2\sqrt[3]{\sqrt{y}}} dy$

b) $\int (1 - s^2)^2(3s + s^3)\, ds$ e) $\int (2e^{2x} - e^{-3x})\, dx$ h) $\int \left(2t + \frac{1}{t}\right)^2 dt$

c) $\int (x^2 - 5^{2x})\, dx$ f) $\int \frac{\cos x}{1 - \cos^2 x} dx$ i) $\int e^{7-2x} dx$

5.2 TECHNIQUES OF INTEGRATION

There are different techniques of integrations. The most commonly used methods are : **substitution, partial fractions,** and **integration by parts**.

5.2.1 Integration by Substitution

Integration by substitution is a counter part to the chain rule of differentiation. It is a method of finding integrals by changing variables. In integration by substitution, we may

- Use **pattern recognition** to evaluate indefinite integral.
- Use **change of variable** to evaluate indefinite integral.
- Use **general power rule of integration** to evaluate indefinite integral.

Rule of Integration by Substitution

Let f and g be functions such that fog and g' are continuous on an interval I. If F is an anti-derivative of f on I, then

$$\int \boldsymbol{f(g(x))g'(x)dx = F(g(x)) + c}, \textbf{ for an arbitrary c.}$$

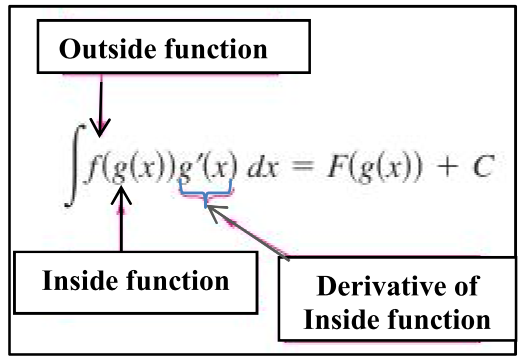
Change of variable

In the above rule, letting $u = g(x)$ gives $du = g'(x)dx$. Then the rule can be written as

$$\int \boldsymbol{f(g(x))g'(x)dx = \int f(u)du = F(u) + c}$$

$$\Rightarrow \qquad \int \boldsymbol{f(g(x))g'(x)dx = (g(x)) + c}, \quad \textbf{for an arbitrary c}$$

- To apply Rule of Integration by Substitution given above, you must recognize the presence of $f(g(x))$, and $g'(x)$ in the integrand.
- The composite function $f(g(x))$ in the integrand has **the outside function** f **the inside function** g.



Examples 5. Evaluate the following indefinite integrals.

(pattern recognition and change of variable)

a) $\int 2x(x^2+1)\,dx$

b) $\int xe^{x^2}dx$

c) $\int \sqrt[3]{(1-x)^4}\,dx$

d) $\int sin^4x\ \cos x\,dx$

Solution

a) Let $u = x^2+1 \Rightarrow du = 2xdx$. Then by Substitution Rule

$$\int 2x(x^2+1)\,dx = \int (x^2+1)\,2xdx = \int u\,du = \frac{u^2}{2} + c = \frac{(x^2+1)^2}{2} + c$$

b) Let $u = x^2 \Rightarrow du = 2xdx \Rightarrow xdx = \frac{1}{2}du$. Then by Substitution Rule

$$\int xe^{x^2}dx = \int e^{x^2}xdx = \int \frac{1}{2}e^u du = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c$$

c) Let $u = 1-x \Rightarrow du = -dx \Rightarrow dx = -\,du$. Then by Substitution Rule

$$\int \sqrt[3]{(1-x)^4}\,dx = \int -1.\sqrt[3]{u^4}\,du = -\int u^{\frac{4}{3}}\,du = \frac{3}{7}u^{\frac{7}{3}} + c = \frac{3}{7}(1-x)^{\frac{7}{3}} + c$$

d) Let $u = \sin x \Rightarrow du = \cos x\,dx$. Then by Substitution Rule

$$\int sin^4x\ \cos x\,dx = \int u^4 du = \frac{u^5}{5} + c = \frac{sin^5x}{5} + c\ ,$$

Note : If F is an antiderivative of f and $a \neq 0$ is constant, then by change of variable

$$\int \boldsymbol{f(ax)\,dx = \frac{1}{a}F(ax) + c}$$

For instance,

i) $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

ii) $\int e^{\sqrt{3}\,x}\,dx = -\frac{1}{\sqrt{3}}e^{\sqrt{3}\,x} + c$

iii) $\int \cos(5x)\,dx = \frac{1}{5}\cos\,(5x) + c$

Other version of change of variable

Examples 6. Evaluate the following indefinite integrals.

a) $\int x\sqrt{x+1}\,dx$ b) $\int \frac{x^2}{2x-1}\,dx$

Solution From the nature of the integrands, the above rule of integration may not helpful to evaluate these integrals. The reason is that the integrands have no the form $f\big(g(x)\big)g'(x)$. But still you can use change of variable to evaluate them.

a) Let $u = x + 1 \Rightarrow du = dx$ and $x = u - 1$. Then

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du = \int (u\sqrt{u} - \sqrt{u}\ )du$$

$$= \int (u\sqrt{u} - \sqrt{u}\ )du = = \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}$$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

b) Let $u = 2x - 1 \Rightarrow du = 2dx$ and $x = \frac{u+1}{2} \Rightarrow x^2 = \left(\frac{u+1}{2}\right)^2 = \frac{u^2+2u+1}{4}$. Then

$$\int \frac{x^2}{2x-1}\,dx = \int \frac{u^2+2u+1}{4u}du = \int \left(\frac{u}{4} + \frac{1}{2} + \frac{1}{4}u^{-1}\right)du = \frac{u^2}{8} + \frac{u}{2} + \frac{\ln|u|}{4}$$

$$= \frac{(2x-1)^2}{8} + \frac{2x-1}{2} + \frac{\ln|2x-1|}{4} + c$$

Other version of Integration by Substitution

i) $\int \frac{f\prime(x)}{f(x)}\,dx$; letting $u = f(x) \Rightarrow du = f'(x)dx$. Then the given integral can be written as

$\int \frac{f\prime(x)}{f(x)}\,dx = \int \frac{du}{u}$, then evaluate the resulting integral.

ii) $\int f(x)f'(x)\,dx$; letting $u = f(x) \Rightarrow du = f'(x)dx$. Then the given integral can be written as

$\int f(x)f'(x)\,dx = \int u\,du$, then evaluate the resulting integral.

Examples 7. Evaluate the following indefinite integrals.

a) $\int \frac{1}{x\ln x}\,dx$ b) $\int \sin x\cos x\ dx$

Solution

a) Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$. Then

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}du = \ln|u| = \ln|\ln x| + c$$

b) Let $u = \sin x \Rightarrow du = \cos x\,dx$. Then

$$\int sinx\ cosx\ dx = \int u\,du = \frac{u^2}{2} = \frac{1}{2}sin^2x + c$$

Indefinite Integral of $f(x) = tan\ x$ and $f(x) = cot\ x$

i) $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$; letting $u = cos\ x$ then $du = -\sin x\, dx$

$$= \int -\frac{du}{u} = -\ln|\, u\,| + c = -\ln|\cos x| + c$$

ii) $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$; letting $u = \sin x$. Then $du = \cos x\, dx$

$$= \int \frac{du}{u} = \ln|\, u\,| + c = \ln|\sin x| + c$$

The Indefinite Integral of $f(x) = tan\ x$ and $f(x) = cot\ x$

1) $\int \tan x\, dx = -\ln|\cos x| + c$

2) $\int \cot x\, dx = \ln|\sin x\,| + c$

For an arbitrary constant c.

Note that also we can use Integration by substitution to show :

The Indefinite Integral of $f(x) = sec\ x$ and $f(x) = csc\ x$

1. $\int \sec x\, dx = \ln|sec\ x + tan\ x \cos x| + c$
2. $\int \csc x\, dx = -\ln|csc\ x + cot\ x\,| + c$

For an arbitrary constant c.

Exercise 5.2 Evaluate the following indefinite integrals.

1. $\int 5x^2 \cos(1 - 2x^3)\, dx$
2. $\int \sqrt{x}\,\sqrt{1 + x\sqrt{x}}\, dx$
3. $\int \frac{4x - 5}{\sqrt{2x^2 - 5x + 1}}\, dx$
4. $\int 5^{(2-3x)} dx$
5. $\int x\sqrt{x + 3}\, dx$
6. $\int \tan x \sec^2 x\, dx$

7) $\int \sin^2 x\, dx$

8) $\int \cos^2 x\, dx$

5.2.2 Integration by Partial Fractions

In this sub-unit, we shall use decomposition of a rational expression into partial fractions along with the method of substitution in order to evaluate the indefinite integrals of some rational expressions.

Example 8: Decompose each of the following rational expressions into partial fractions

a) $f(x) = \frac{1}{x^2 - x}$

b) $f(x) = \frac{x + 2}{x^3 - 3x^2}$

Solution :

a) $x^2 - x = x(x-1)$

Let $\frac{1}{x^2 - x} = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$, where A and B are constants to be determined.

$\Rightarrow \frac{1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$

$\Rightarrow (A+B)x - A = 1 \quad \Rightarrow \quad A + B = 0$ and $-A = 1 \quad \Rightarrow \quad A = -1$ and $B = 1$

$\Rightarrow -\frac{1}{x}$ *and* $\frac{1}{x-1}$ are the partial fractions of f.

The partial fraction decomposition of $f(x)$ is

$$f(x) = \frac{1}{x^2 - x} = -\frac{1}{x} + \frac{1}{x-1} = \frac{1}{x-1} - \frac{1}{x}$$

b) $f(x) = \frac{x+2}{x^3 - 3x^2} = \frac{x+2}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{(A+C)x^2 + (-3A+B)x - 3B}{x^2(x-3)}$

$\Rightarrow (A+C)x^2 + (-3A+B)x - 3B = x + 2 \quad \Rightarrow \quad A + C = 0,\ -3A + B = 1, -3B = 2$

$\Rightarrow A = -\frac{5}{9}$, $B = -\frac{2}{3}$, $C = \frac{5}{9}$

Hence, $f(x) = \frac{x+2}{x^3 - 3x^2} = -\frac{5}{9x} - \frac{2B}{3x^2} + \frac{5}{9(x-1)}$ is the partial decomposition of f.

Note : Indefinite integral of some rational functions need not partial decomposition.

Example 8: Devaluate the following indefinite integrals of rational functions.

a) $\int \frac{1}{3-2x}\,dx$ b) $\int \frac{2x-1}{x+1}\,dx$ c) $\int \frac{x^3}{x-2}\,dx$

Solution: The above indefinite integrals need not partial decomposition **.**

a) Let $u = 3 - 2x \Rightarrow du = -2dx$. Then

$$\int \frac{1}{3-2x}dx = \int -\frac{1}{2u}du = -\frac{1}{2}\int \frac{1}{u}du = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|3-2x| + c$$

b) Let $u = x + 1 \Rightarrow du = dx$ and $x = u - 1 \Rightarrow \frac{2x-1}{x+1} = \frac{2(u-1)-1}{u} = \frac{2u-3}{u} = 2 - \frac{3}{u}$. Then

$$\int \frac{2x-1}{x+1}\,dx = \int \left(2 - \frac{3}{u}\right) du = 2u - 3\ln|u| = 2(x+1) - 3\ln|x+1| + c$$

c) The integrand $\frac{x^3}{x-2}$ is improper rational function. Write $\frac{x^3}{x-2}$ in the form of

$\Rightarrow \frac{x^3}{x-2} = (x^2 + 4x + 4) + \frac{8}{x-2}$

$\Rightarrow \int \frac{x^3}{x-2}dx = \int \left(x^2 + 4x + 4 + \frac{8}{x-2}\right)dx = \frac{x^3}{3} + 2x^2 + 4x + 8\ln|x-2| + c$

Example 9: Devaluate the following indefinite integrals of rational functions.

a) $\int \frac{x+1}{x^2-x}\,dx$ b) $\int \frac{x^3}{x^2-9}\,dx$

Solution : The above indefinite integrals need not partial decomposition **.**

a) $x^2 - x = x(x-1)$

Let $\frac{x+1}{x^2 - x} = \frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$, where A and B are constants to be determined.

$\Rightarrow \frac{x+1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$

$\Rightarrow (A+B)x - A = x + 1 \Rightarrow A + B = 1$ and $-A = 1 \quad \Rightarrow \quad A = -1$ and $B = 2$

$$\int \frac{x+1}{x^2-x}dx = \int\left(\frac{-1}{x}+\frac{2}{x-1}\right)dx = -\int\frac{1}{x}dx + 2\int\frac{1}{x-1}dx = -\ln|x| + 2\ln|x-1| + c$$

b) $\frac{x^3}{x^2-9} = x + \frac{9x}{x^2-9}$. Let $\frac{9x}{x^2-9} = \frac{9x}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow (A+B)x + 3(A-B) = 9x$

$$\Rightarrow \quad A = B = \frac{9}{2} \quad \Rightarrow \quad \frac{x^3}{x^2-9} = x + \frac{9x}{(x-3)(x+3)} = x + \frac{9/2}{x-3} + \frac{9/2}{x+3}$$

$$\int \frac{x^3}{x^2-9}dx = \int\left(x + \frac{9/2}{x-3} + \frac{9/2}{x+3}\right)dx = \frac{x^2}{2} + \frac{9}{2}(\ln|x-3| + \ln|x+3|) + c = \frac{x^2}{2} + \frac{9}{2}(\ln|x^2-3|)$$

Exercise 5.3 Evaluate the following indefinite integrals.

1. $\int \frac{2x-1}{x+1}dx$
2. $\int \frac{x+2}{x^2(x-3)}dx$
3. $\int \frac{x^2+1}{x^2-1}dx$
4. $\int \frac{\sqrt{x}}{1+\sqrt{x}}dx$
5. $\int \frac{dx}{4x^2+4x+1}$
6. $\int \frac{1}{e^x-e^{-x}}dx$
7. $\int \frac{x^3-1}{x^3+x}dx$

5.2.3 Integration by Parts

Integration by parts is a method which is a counter part of the **product rule of differentiation**. This technique can be applied to a wide variety of functions and is particularly useful for integrands involving products of **algebraic** and **transcendental functions**. For instance, Integration by parts well with integrals such as

$\int x\ln x\,dx$, $\int x^2e^x dx$, $\int e^x \sin x\,dx$, etc

Rule of Integration by Parts

Let f and g be functions that have continuous derivatives. Then

$$\int \boldsymbol{f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx}$$

For instance, consider the indefinite integral $\int x\sin x\,dx$.

Let $f(x) = x$ and $g'(x) = \sin x\,dx$

$\Rightarrow$ $f'(x) = dx$ and $g(x) = \int \sin x\,dx = -\cos x$

Then by the rule of integration by parts, we have

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\Rightarrow \quad \int x\sin x\,dx = x(-\cos x) - \int(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c,$$

Note : For computational purpose, integration by parts can be written in more convenient and simpler form as follow:

Let $u = f(x)$ and $dv = g'(x)dx$

$\Rightarrow$ $du = f'(x)dx$ and $v = g(x)$

Then rule of integration by parts can be written as

$$\int \boldsymbol{u\,dv = uv - \int v\,du}$$

Remark: In integration by part, we choose u for a function which is easily differentiable and dv for a function which is easily integrable.

Example 10: Evaluate the following indefinite integrals.

a) $\int xe^{2-3x}\,dx$ b) $\int e^{3x}\cos 2x\,dx$ c) $\int \ln x\ dx$

<u>Solution</u> : The above indefinite integrals need not partial decomposition **.**

a) $\int xe^{2-3x}\,dx$; Let $u = x$ and $dv = e^{2-3x}\,dx$

$\Rightarrow du = dx$ and $v = \int e^{2-3x}\,dx = e^2\int e^{-3x}\,dx = e^2\left(-\frac{1}{3}\right)e^{-3x} = -\frac{1}{3}e^{2-3x}$

$\Rightarrow \int xe^{2-3x}\,dx = uv - \int v\,du = -\frac{1}{3}xe^{2-3x} - \int -\frac{1}{3}e^{2-3x}\,dx = \frac{1}{3}xe^{2-3x} + \frac{1}{3}\int e^{2-3x}\,dx$

$= -\frac{1}{3}xe^{2-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{2-3x}\right) = -\frac{1}{3}xe^{2-3x} - \frac{1}{9}e^{2-3x} + c$

b) $\int e^{3x}\cos 2x\,dx$; Let $u = e^{3x}$ and $dv = \cos 2x\,dx$

$\Rightarrow \quad du = 3e^{3x}dx$ and $v = \int \cos 2x\,dx = \frac{1}{2}\sin 2x$

$\Rightarrow \quad \int e^{3x}\cos 2x\,dx = uv - \int v\,du = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x\ dx$ --------- **(*)**

Repeat integration by part for $\int e^{3x}\sin 2x\ dx$:

Let $u = e^{3x}$ and $dv = \sin 2x\,dx$

$\Rightarrow du = 3e^{3x}dx$ and $v = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

$\Rightarrow \int e^{3x}\sin 2x\,dx = uv - \int v\,du = -\frac{1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2x\ dx$ --------- **(**)**

Now substitute (**) in (*) gives

$\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x\ dx$

$= \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\left[-\frac{1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2x\ dx\right]$

$= \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x - \frac{9}{4}\int e^{3x}\cos 2x\ dx$

Then, collect like terms gives

$\int e^{3x}\cos 2x\,dx + \frac{9}{4}\int e^{3x}\cos 2x\ dx = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x$

$\Rightarrow \quad \frac{13}{4}\int e^{3x}\cos 2x\ dx = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x$

Therefore, $\int e^{3x}\cos 2x\ dx = \frac{4}{13}\left(\frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x\right) + c$

$= \frac{2}{13}e^{3x}\left(\sin 2x + \frac{3}{2}\cos 2x\right) + c$

c) $\int \ln x\ dx = \int 1.\ln x\ dx$

Let $u = \ln x$ and $dv = 1dx$

$\Rightarrow du = \frac{1}{x}\,dx$ and $v = \int 1\,dx = x$

$\Rightarrow \quad \int \ln x\,dx = uv - \int v\,du = x\,ln\,x - \int x.\frac{1}{x}\,dx = x\,ln\,x - \int\ dx = x\,ln\,x - x + c$

Therefore,

$$\boldsymbol{\int \ln x\,dx = x\,ln\,x - x + c}$$

Indefinite Integral of Logarithm Functions

1. $\int \ln x\, dx = x \ln x - x + c$
2. For $a > 0$ and $a \neq 1$,

$$\int \log_a x\, dx = \int \frac{\ln x}{\ln a} dx = \frac{1}{\ln a} \int \ln x\, dx = \frac{1}{\ln a}(x \ln x - x) = x \log_a x - \frac{x}{\ln a} + c$$

Example 11: Evaluate the following indefinite integrals.

a) $\int \log(1-3x)\, dx$ c) $\int \ln^2 x\, dx$

b) $\int (2x+1) \ln 5x\, dx$ d) $\int x^{-3} \ln x\, dx$

Solution : The above indefinite integrals need not partial decomposition **.**

a) Let $u = 1-3x \Rightarrow du = -3dx$

$$\int \log(1-3x)\, dx = -\frac{1}{3}\int \log u\ du = -\frac{1}{3}\int \log u\ du = -\frac{u}{3\ln 10}(\ln u - 1)$$
$$= -\frac{1-3x}{3\ln 10}[\ln(1-3x) - 1] + c$$

b) Let $u = 2-3x$ and $dv = e^{2-3x}\, dx$

$$\Rightarrow du = dx \text{ and } v = \int e^{2-3x}\, dx = e^2 \int e^{-3x}\, dx = e^2\left(-\frac{1}{3}\right)e^{-3x} = -\frac{1}{3}e^{2-3x}$$

$$\Rightarrow \int xe^{2-3x}\, dx = uv - \int v\, du = -\frac{1}{3}xe^{2-3x} - \int -\frac{1}{3}e^{2-3x}\, dx = \frac{1}{3}xe^{2-3x} + \frac{1}{3}\int e^{2-3x}\, dx$$
$$= -\frac{1}{3}xe^{2-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{2-3x}\right) = -\frac{1}{3}xe^{2-3x} - \frac{1}{9}e^{2-3x} + c$$

c) Let $u = \ln^2 x$ and $dv = 1\, dx \Rightarrow du = 2\frac{\ln x}{x}dx$ and $v = \int 1\, dx = x$

$$\Rightarrow \int \ln^2 x\ dx = uv - \int v\, du = x\ln^2 x - \int 2x\frac{\ln x}{x}dx = x\ln^2 x - 2\int \ln x\, dx$$
$$\Rightarrow \int \ln^2 x\ dx = x\ln^2 x - 2(x\ln x - x) + c$$

d) Let $u = \ln x$ and $dv = x^{-3}dx$

$$\Rightarrow du = \frac{1}{x}dx \text{ and } v = \int x^{-3}\, dx = \frac{x^{-2}}{-2}$$

$$\Rightarrow \int x^{-3}\ln x\, dx = uv - \int v\, du = \frac{x^{-2}}{-2}\ln x - \int \frac{1}{x}\frac{x^{-2}}{-2}dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}\int x^{-3}dx$$

$$\Rightarrow \int x^{-3}\ln x\, dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}\frac{x^{-2}}{-2} = -\frac{1}{4x^2}(2\ln x + 1) + c$$

Exercise 5.3 Evaluate the following indefinite integrals.

1. $\int e^{3x}\sin(2x-1)\, dx$ 3. $\int x^{-1}e^{3x}dx$ 5) $\int sec^3 x\, dx$

2. $\int x^3\ln^2 x\ dx$ 4. $\int (x^2+3x)\sin 2x\ dx$

5.3 DEFINITE INTEGRALS, AREA AND THE FUNDAMENTAL THEOREM OF CALCULUS

5.3.1 Definite Integrals

- Unlike the previous sub-units where the indefinite integral of a function resulted in a new function, when finding the **definite integral** we produce a **numerical value**.

- Definite integrals are important because they can be used to find different types of measures such as areas, volumes, lengths and so on.

Definition 5.3 (Fundamental Theorem of Calculus)

Let f be f function continuous on the interval $[a,\ b]$ and F be the antiderivative of f. Then the **DEFINITE INTEGRAL** f on $[a,\ b]$ is the value of the integral $\int_a^b f(x)dx$ which is given by

$$\int_a^b \boldsymbol{f(x)dx} = (\boldsymbol{F(x)})|_a^b = \boldsymbol{F(b) - F(a)}$$

which is read as "the (definite) integral of $f(x)$ w.r.t x from a to b".

Note : The constant of integration c is omitted, because it would be cancel itself out upon carrying out the subtraction :

$$\int_a^b \boldsymbol{f(x)dx} = (\boldsymbol{F(x) + c})|_a^b = [\boldsymbol{F(b) + c}] - [\boldsymbol{F(a) + c}] = \boldsymbol{F(b) - F(a)}$$

In the expression $\int_a^b \boldsymbol{f(x)dx}$,

- x is called the **variable of integration**, and
- a and b are called the **lower limit** and **upper limit**, respectively.

Properties of Definite Integrals

Let f and g be two integrable functions on $[a,\ b]$ and k be constant. Then

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
4. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
5. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for any c where $a < c < b$

Example 12: Evaluate the following definite integrals.

a) $\int_1^4 (\frac{1}{x} + 2\sqrt{x}\,)dx$ b) $\int_{-4}^0 \frac{x}{\sqrt{x^2+9}}\,dx$

Solution

a) $F(x) = \int\left(\frac{1}{x} + 2\sqrt{x}\,\right)dx = \ln x + \frac{4}{3}\sqrt{x^3} + c$. Then

$\int_1^4 (\frac{1}{x} + 2\sqrt{x}\,)dx = \left(\ln x + \frac{4}{3}\sqrt{x^3}\right)\Big|_1^4 = \left(\ln 4 + \frac{4}{3}\sqrt{4^3}\right) - \left(\ln 1 + \frac{4}{3}\sqrt{1^3}\right) = \ln 4 + \frac{28}{3}$

b) Let $u = x^2 + 9 \Rightarrow \quad du = 2xdx \Rightarrow xdx = \frac{du}{2}$

$\int\left(\frac{x}{\sqrt{x^2+9}}\right)dx = \int \frac{1}{2\sqrt{u}}du = \sqrt{u} = \sqrt{x^2+9}$

$\Rightarrow \int_{-4}^0 \frac{x}{\sqrt{x^2+9}}\,dx = \sqrt{x^2+9}\Big|_{-4}^0 = \sqrt{(0)^2 + 9} - \sqrt{(-4)^2 + 9} = 3 - 5 = -2$

Notice that in (b), we use change of variable. But we do not change the limit of integrations. This is because we express the antiderivative F interms of the original variable x.

But it is possible to change the limit of integration to the new variable when we use change of variable in evaluation of the integral.

Change of variable in Definite Integral

$$\int_a^b f(g(x))g'(x)dx = F(g(x)) + c, \text{ for an arbitrary c.}$$

letting $u = g(x)$ gives $du = g'(x)dx$.

$x = a \Rightarrow u = g(a)$ and $x = b \Rightarrow u = g(b)$

Then the rule can be written as

$$\int_a^b f(g(x))g'(x)dx = \int_a^b f(u)du = F(u)\Big|_{g(a)}^{g(b)}$$

Example 13: Evaluate the following definite integrals.

a) $\int_0^{\frac{\pi}{3}} \tan x \, sec^2 x \, dx$ b) $\int_{ln2}^{ln5} \frac{1}{e^x - e^{-x}} \, dx$

Solution

a) Let $u = \tan x \Rightarrow du = sec^2 x \, dx$

$x = 0 \Rightarrow u = g(0) = \tan(0) = 0$ and $x = \frac{\pi}{3} \Rightarrow u = g\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$$\Rightarrow \int_0^{\frac{\pi}{3}} \tan x \, sec^2 x \, dx \, dx = \int_0^{\sqrt{3}} u \, du = \frac{u^2}{2}\Big|_0^{\sqrt{3}} = \frac{\sqrt{3}^2}{2} - \frac{0^2}{2} = \frac{3}{2} - 0 = \frac{3}{2}$$

b) $\int_{ln2}^{ln5} \frac{1}{e^x - e^{-x}} \, dx = \int_{ln2}^{ln5} \frac{e^x}{e^{2x} - 1} \, dx.$ $u = e^x \Rightarrow du = e^x dx$

$x = \ln 2 \Rightarrow u = e^{\ln 2} = 2$ and $x = \ln 5 \Rightarrow u = e^{\ln 5} = 5$

$$\Rightarrow \int_{ln2}^{ln5} \frac{1}{e^x - e^{-x}} \, dx = \int_{ln2}^{ln5} \frac{e^x}{e^{2x} - 1} \, dx = \int_2^5 \frac{du}{u^2 - 1} = \frac{1}{2}\int_2^5 \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$$

$$\Rightarrow \int_{ln2}^{ln5} \frac{1}{e^x - e^{-x}} \, dx = \int_2^5 \frac{du}{u^2 - 1} = \frac{1}{2}(\ln|u-1| - \ln|u+1|)\Big|_2^5 = \frac{1}{2}[(\ln 4 - \ln 6) - (\ln 1 - \ln 3)]$$

$$= \frac{1}{2}[\ln 4 - \ln 6 + \ln 3] = \frac{1}{2}\ln\frac{4 \times 3}{6} = \frac{1}{2}\ln 2 = \ln\sqrt{2}$$

Exercise 5.4 Evaluate the following definite integrals.

a) $\int_1^3 \sqrt{x}(1 - \frac{1}{x}) \, dx$

b) $\int_{-1}^3 \frac{2x}{x+3} \, dx$

c) $\int_{-\frac{\pi}{2}}^{\frac{4\pi}{3}} |\sin x| dx$

d) $\int_{-1}^3 \frac{\cos x}{\sqrt{1 + \sin x}} \, dx$

e) $\int_1^3 \frac{x^3+1}{x^2(x-4)} \, dx$

f) $\int_e^{e^2} \left(\frac{\ln x}{x^2}\right) dx$

g) $\int_{-1}^3 (e^x + e^{-x})^2 \, dx$

h) $\int_0^1 \frac{2^{2x-1} - 5^{2x+1}}{10^x} \, dx$

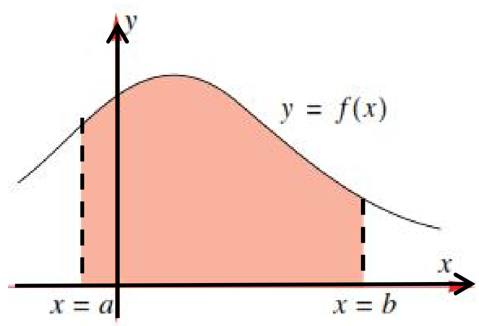
5.4 APPLICATIONS OF INTEGRAL CALCULUS

- **Mathematical application** : To calculate area of a region bounded by curves of continuous functions defined on a closed interval $[a , b]$ and the volume of a solid of revolution.
- **Physical applications** : To calculate work done by a variable force along a straight line, acceleration, velocity and displacement.

5.4.1 The Area Between Two Curves

1. Area between a curve and the x-axis

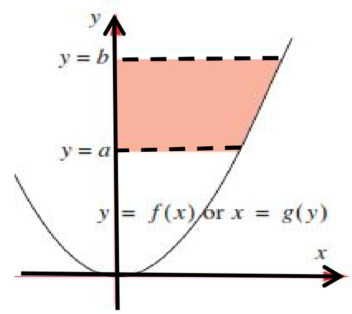
If $f(x)$ is a **positive** and continuous function on the interval $[a, b]$, then the **area A** bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by



$$A = \int_a^b f(x)\, dx$$

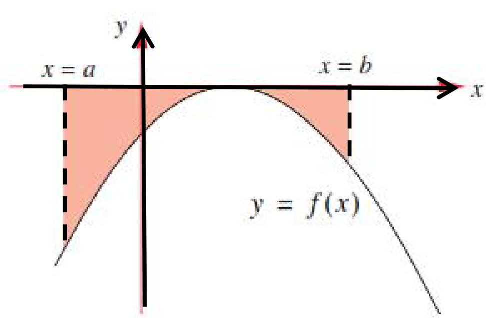
2. Area between a curve and the y-axis

If $f(y)$ is a **positive** and continuous function on the interval $[a, b]$, then the **area A** bounded by the graph of $x = f(y)$, the y-axis, and the horizontal lines y= a and $y = b$ is given by



$$A = \int_a^b f(y)\, dy$$

Remark : If $f(x)$ is a **negative** and continuous function over the interval $[a, b]$, then the **area A** bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines x $= a$ and x $= b$ is given by



$$A = \left| \int_a^b f(x)\, dx \right|$$

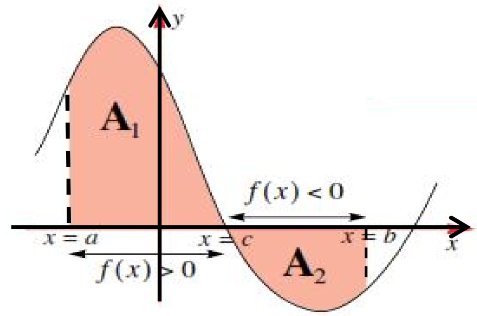
Note: It is possible for a continuous function $y = f(x)$ to alternate between negative and positive values over the interval $[a , b]$. In this case there is atleast a number $c \in (a , b)$ where the graph of $y = f(x)$ crosses the x-axis. For instance, as shown in the figure below,

$A = \int_a^b f(x)\,dx = A_1 + A_2$

$\Rightarrow \quad \boldsymbol{A = \int_a^c f(x)\,dx + \left(-\int_c^b f(x)\,dx\right)}$

But instead, we can write A, in general, as

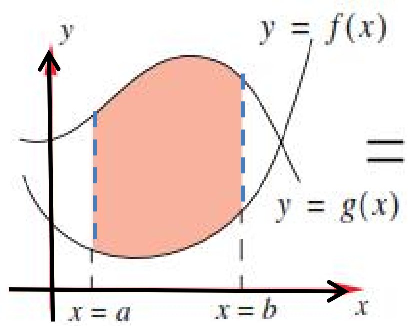
$\Rightarrow \quad \boldsymbol{A = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|}$



3. The Area Between Two Curves

Suppose f and g are two continuous functions on $[a, b]$ with $f(x) \geq g(x)$ on $[a, b]$. Then the area A bounded by the curves of $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$ is

$$\boldsymbol{A = \int_a^b [f(x) - g(x)]\,dx}$$



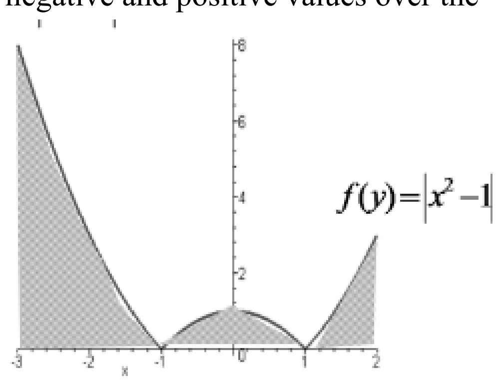
Examples 14. Find the area of the region bounded by the graph of the function $f(x) = |x^2 - 1|$ and the x-axis between the lines $x = -3 \; and \; x = 2$

Solution

Notice that the values of $f(x) = |x^2 - 1|$ to alternate between negative and positive values over the interval $[-3, 2]$.

$x^2 - 1 = 0 \; iff \; x = -1 \; or \; x = 1$

$$f(x) = |x^2 - 1| = \begin{cases} \geq 0 \; on \; [-3, -1] \\ \leq 0 \; on \; [-1, 1] \\ \geq 0 \; on \; [1, 2] \end{cases}$$



Hence, $A = A_1 + A_2 + A_3$

$A = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$

$A = \int_{-3}^{-1} (x^2 - 1)\,dx + \int_{-1}^{1} (1 - x^2)\,dx + \int_{1}^{2} (x^2 - 1)\,dx$

$= \left(\frac{x^3}{3} - x\right)_{-3}^{-1} + \left(x - \frac{x^3}{3}\right)_{-1}^{1} + \left(\frac{x^3}{3} - x\right)_{1}^{2} = \frac{20}{3} + \frac{4}{3} + \frac{4}{3} = \frac{28}{3} \; sq.u$

Examples 15. Find the area of the region bounded by the graph of the function $y = x^2 + x - 2$ and $y = x + 2$

Solution

$x^2 + x - 2 = x + 2$

$\Rightarrow \quad x^2 - 4 = 0$

$\Rightarrow \quad x = -2$ or $x = 2$

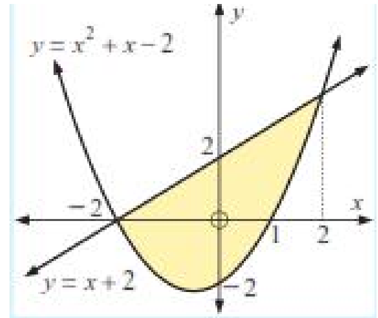
The two functions intersect at $x = -2$ and $x = 2$.

On $[-2, 2]$, $x + 2 \geq x^2 + x - 2$

$\Rightarrow$ Then the area of the region is

$$A = \int_{-2}^{2}[f(x) - g(x)]\,dx = \int_{-2}^{2}[(x-2) - (x^2 + x - 2)]\,dx$$

$$= \int_{-2}^{2}[4 - x^2]\,dx = \frac{32}{3} \text{ sq. u}$$



Examples 15. Find the area of the region bounded by the graph of the function $x = 4 - y^2$ and $x = y - 2$

Solution

$4 - y^2 = y - 2$

$\Rightarrow \quad y^2 + y - 6 = (y+3)(y-2) = 0$

$\Rightarrow \quad y = -3$ or $y = 2$

The two functions intersect at $y = -3$ and $y = 2$.

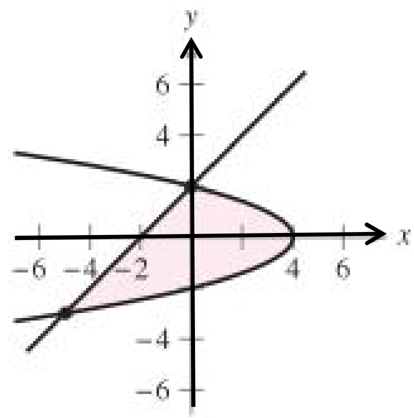
$y - 2 \leq 4 - y^2$ for $-3 \leq y \leq 2$

$\Rightarrow$ Then the area of the region is

$$A = \int_{-3}^{2}[(4 - y^2) - (y - 2)]dy$$

$$= \int_{-3}^{2}[(6 - y - y^2)]dy$$

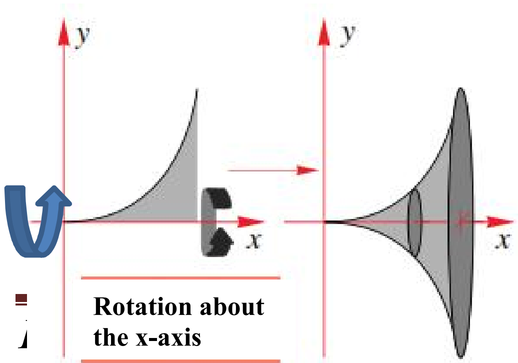
$$= \left(6y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-3}^{2} = \frac{32}{3} + \frac{27}{2} = \frac{145}{6} \ sq.u$$



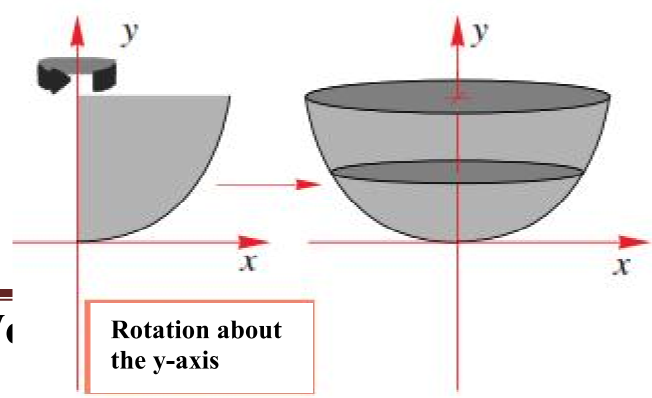
5.4.2 Volume of Revolution

The volume of a solid of revolution

- Another important application of definite integral is its use in finding the volume of a three dimensional solid generated by revolving a plane region about a line.
- If a region is revolving about a line, the resulting solid is called a **SOLID OF REVOLUTION** and the **AXIS OF ROTATION** is called axis of revolution.
- The volume of a solid of revolution is said to be a **VOLUME OF REVOLUTION.**
- In this section we consider only two axis of rotation : the **x-axis** and the **y-axis.**



Rotation about the x-axis



Rotation about the y-axis

Rule 1 : Disk Method

1. Let $f(x)$ be a function continuous on $[a\ ,\ b]$. The volume V of the solid of revolution obtained by revolving the plane region enclosed by the graph of $y = f(x)$, the x-axis and the lines $x = a\ \ and\ \ x = b$ is given by
$$V = \pi \int_a^b [f(x)]^2 dx$$
2. Let $f(y)$ be a function continuous on $[c\ ,\ d]$. The volume V of the solid of revolution obtained by revolving the plane region enclosed by the graph of $x = f(y)$, the y-axis and the lines $y = c\ \ and\ \ y = d$ is given by
$$V = \pi \int_c^d [f(y)]^2 dy$$

This method is sometimes known as **DISK METHOD**.

Examples 15. Find the volume of the solid generated by revolving the region bounded by the graph of $f(x) = \sqrt{e^x + 1}$ and the *x*-axis between $x = 0$ and $x = \ln 2$ about the *x*-axis.

Solution : Clearly f is continuous on [0 , ln x]

$$V = \pi \int_a^b [f(x)]^2 dx = \pi \int_0^{\ln 2} [\sqrt{e^x + 1}]^2 dx$$
$$V = \pi \int_0^{\ln 2} (e^x + 1) dx = (e^x + 1)_0^{\ln 2} = (e^{\ln 2} + 1) - (e^0 + 1) = 2\ cu.u$$

Examples 16. Find the volume of the solid generated by revolving the region bounded by the graph of $f(x) = \sqrt{e^x + 1}$ and the *x*-axis between $x = 0$ and $x = \ln 2$ abType equation here.out the *x*-axis.

Solution : Clearly f is continuous on [0 , ln x]

$$V = \pi \int_a^b [f(x)]^2 dx = \pi \int_0^{\ln 2} [\sqrt{e^x + 1}]^2 dx$$
$$V = \pi \int_0^{\ln 2} (e^x + 1) dx = (e^x + 1)_0^{\ln 2} = (e^{\ln 2} + 1) - (e^0 + 1) = 2\ cu.u$$

Examples 17. Find the volume of the solid generated by revolving the region bounded by the graph of $f(x) = \ln x$, the lines $x = 1$ and $x = e$ about the y-axis.

Solution : Clearly f is continuous on [1, e].

First express x interms of y.

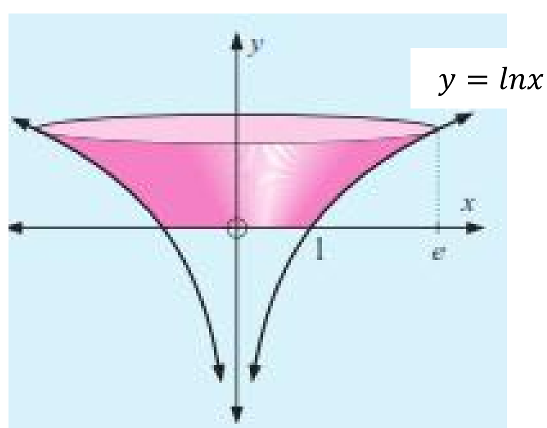
$y = f(x) = \ln x \ \Rightarrow\ x = e^y = g(y)$

$x = 1 \ \Rightarrow\ y = \ln 1 = 0 = c$ and

$x = e \ \Rightarrow\ y = \ln e = 1 = d$. Then apply the formula.

$$V = \pi \int_c^d [g(y)]^2 dy = \pi \int_0^1 [e^y]^2 dy = \pi \int_0^1 e^{2y} dy$$

$$\Rightarrow\ V = \left(\frac{1}{2}\pi e^{2y}\right)_0^1 = \frac{1}{2}\pi e^2 - \frac{1}{2}\pi e^0 = \frac{1}{2}\pi(e^2 - 1)$$



Examples 18. Find the volume of the solid generated by revolving the region bounded by the graph of $x = 4y - y^2$, the y-axis from $y = -2$ to y = 2 about the y-axis.

Solution : Clearly $x = g(y) = 4y - y^2$ is continuous on for $-2 \leq y \leq 2$.

$$V = \pi \int_c^d [g(y)]^2 dy = \pi \int_{-2}^{2} [4y - y^2]^2 dy = \pi \int_0^1 [16y^2 - 8y^3 - y^4] dy$$

$$\Rightarrow \; V = \left(16\frac{y^3}{3} - 8\frac{y^4}{4} - \frac{y^5}{5}\right)_{-2}^{2} = \frac{256}{3} - \frac{64}{5} = \frac{1088}{15}$$

Rule 2 : Washer Method

Let $f(x)$ and $g(x)$ be two functions continuous on $[a\;,\;b]$ with $(x) \geq g(x)\;\forall x \in [a, b]$. Then the volume V of the solid of revolution generated by revolving the plane region enclosed by the graphs of $y = f(x)$ and $y = g(x)$ over $a \leq x \leq b$ about the x-axis is given by

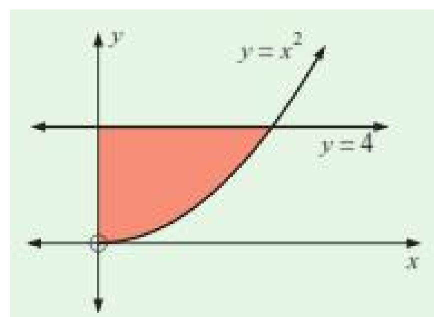
$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

This method is sometimes known as **WASHER METHOD**.

Examples 19. Find the volume of revolution generated by rotating the region bounded by the graphs of $y = 4 \; and \; y = x^2$, and the y-axis, in the first quadrant about the x-axis.

Solution : $4 = x^2 \;\Rightarrow\; x = -2 \;\; or \;\; x = 2$. But $x \geq 0$. Hence $0 \leq x \leq 2$.

On [0, 2], $4 \geq x^2$. Then the required volume is



$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

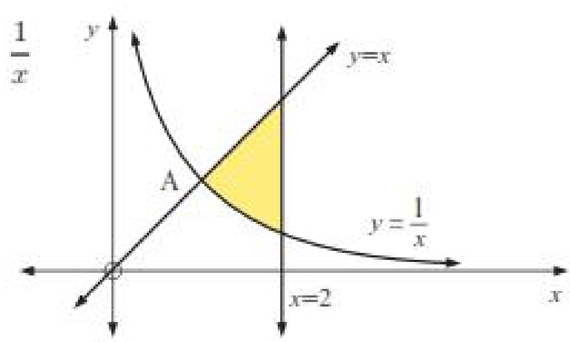
$$= \pi \int_0^2 ([4]^2 - [x^2]^2) dx$$

$$= \pi \int_0^2 (16 - x^4) dx$$

$$\Rightarrow \qquad V = \left(16x - \frac{x^5}{5}\right)_0^2 = 32 - \frac{32}{5} = \frac{118}{5} \text{ cubic u.}$$

Examples 20. In the figure, the shaded region between the graphs of $y = x,\; y = \frac{1}{x} \;\; and \;\; x = 2$.

Find the volume of revolution generated by rotating the region about the x-axis.



Solution : $x = \frac{1}{x} \;\Rightarrow\; x^2 = 1 \;\;\Rightarrow x = 1$, since $x > 0$.

The shaded region lies between x = 1 and x = 2, and on [1, 2], $x \geq \frac{1}{x}$. Hence

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

$$= \pi \int_1^2 \left([x]^2 - [\tfrac{1}{x}]^2\right) dx$$

$$= \pi \int_1^2 (x^2 - x^{-2}) dx = \pi \left(\frac{x^3}{3} + x^{-1}\right)_1^2 = \frac{19}{6} - \frac{4}{3} = \frac{11}{6} \text{ cubic u.}$$

UNIT 6 3-D GEOMETRY AND VECTORS IN SPACE

INTRODUCTION

This unit has two main tasks. The first task is to enable students setup a coordinate system in space and locate or determine the coordinates of a point using ordered triple of numbers. The second task is to enable students identify basic facts about coordinates and their use in determining geometric concepts in space and to solve some practical problems in real life.

After completing this unit, students will be able to:

- *know methods and procedures in setting up coordinate system in space.*
- *know basic facts about coordinates and their use in determining geometric concepts in space.*
- *apply facts and principles about coordinates in space to solve related problems.*
- *know specific facts about vectors in space.*

6.1 COORDINATE AXES AND COORDINATE PLANES IN SPACE

There is a one – to – one coorespondence between the set of points of the plane and the set of all ordered pairs of real numbers.

You can construct three dimensional coordinate system by passing a z-axis perpendicular to both the x- and y- axes at the origin.

Figure 6.1 shows the positive portion of each coordinate axis. Taken as a pair, the axes determine three **coordinate planes :** the **xy-plane**, the **xz-plane**, and the **yz-plane**. These three coordinate planes separate three-space into eight **octants**.

The first four octants : O-1, O-2, O-3, O-4 lies above the xy-plane, and The next four : O-5, O-6, O-7, O-8 lies below the xy-plane.

1^{st} Octant $= \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ 5^{th} Octant $= \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^-$

2^{nd} Octant $= \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}^+$ 6^{th} Octant $= \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}^-$

3^{rd} Octant $= \mathbb{R}^- \times \mathbb{R}^- \times \mathbb{R}^+$ 7^{th} Octant $= \mathbb{R}^- \times \mathbb{R}^- \times \mathbb{R}^-$

4^{th} Octant $= \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^+$ 8^{th} Octant $= \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^-$

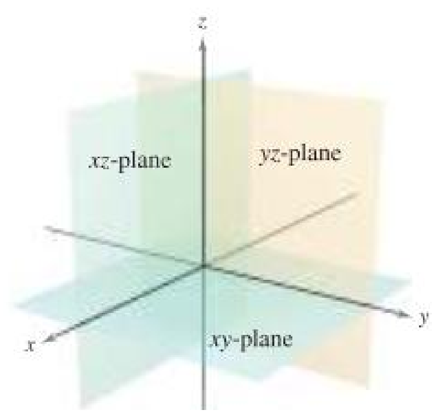


Figure 6.1. The 3-Dimensional coordinate system

6.2 COORDINATES OF A POINT IN SPACE

Figure 6.2 shows alongside the coordinate planes divide space into eight regions, each pair of planes intersecting on the axes.

In figure 6.3, the positive direction of each axis is a solid line whereas the negative direction is represented by dashed line. A point P in space can be specified by an **ordered triple** of numbers (x, y, z) where x, y and z are the steps in the X, Y and Z directions from the origin O, to P.

Figure 6.4 shows a rectangular prism or box with origin O as one vertex, the axes as sides adjacent to it, and P being the vertex opposite O. This helps you to visualize the 3-D position of a point on your 2-D paper.

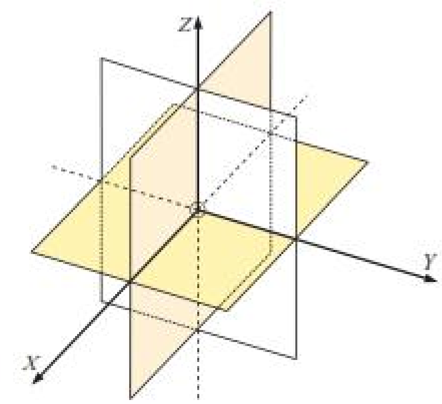


Figure 6.2 Coordinate planes and octants in the space

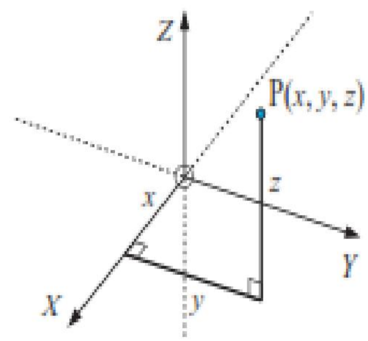


Figure 6.3 Points in the space

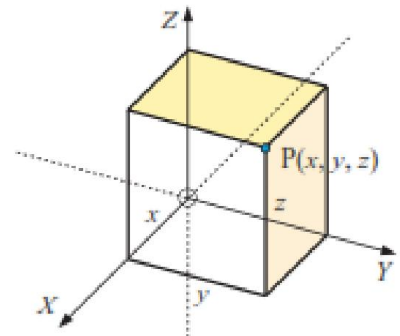


Figure 6.4 3-D point

Example 1. Locate each of the following points in space using reference axes x, y and z, using the same or different coordinate systems in each case.

a) P(2, –5, 3) b) Q(–2, 5, 4) c) R(1, 6, 0) d) T(3, 3, –2)

Solution

The process of locating the point **P(2, –5, 3)** : Start from the origin O and move 3 units in the direction of the positive x-axis. Then move 3 units in the direction of the negative y-axis and finally move 3 units up in the direction of the positive z-axis to get point B. (See figure 6.5)

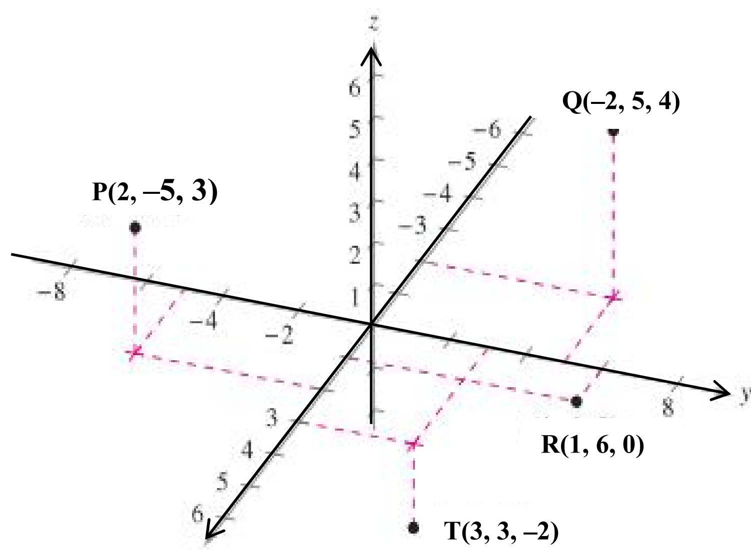


Figure 6.5 Locating points in 3-D

Example 2. Give the equation of

a) the x-axis in a coordinate plane (2-D)
b) the 2^{nd} quadrant
c) the x-axis in the space (3-D)
d) the yz-plane

Solution

a) x-axis in 2-D = $\{(x, y) : x = 0 \text{ and } y \in \mathbb{R}\} = \{(0, y) : y \in \mathbb{R}\} = \{0\} \times \mathbb{R}$
b) Second quadrant = $\{(x, y) : x < 0 \text{ and } y > 0\} = \mathbb{R}^- \times \mathbb{R}^+$
c) x-axis in 3-D = $\{(x, y, z) : x \in \mathbb{R}, y = z = 0\} = \{(x, 0, 0) : x \in \mathbb{R}\} = \mathbb{R} \times \{0\} \times \{0\}$
d) yz-plane = $\{(x, y, z) : x = 0, y \text{ and } z \in \mathbb{R}\} = \{(0, y, z) : y \in \mathbb{R}\} = \{0\} \times \mathbb{R} \times \mathbb{R}$

6.3 DISTANCE BETWEEN TWO POINTS IN SPACE

Distance Formula in Space

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space. Then the distance d between P and Q is given by

$$\boldsymbol{d = PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \qquad \text{------------} \quad \textbf{(6.1)}$$

Example 3. Find the distance between points A(−3, 0, 1) and B(1, −2, −5).

Solution : Taking $(x_1, y_1, z_1) = (-3, 0, 1)$ and $(x_2, y_2, z_2) = (1, -2, -5)$,

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(-3-1)^2 + (0-(-2))^2 + (1-(-5))^2} = \sqrt{(-4)^2 + (2)^2 + (6)^2} = 2\sqrt{14}$$

Example 4. Find the coordinates of points on the y-axis which are 5 u. from the point $(-1, 3, -\sqrt{8})$.

Solution : Let (a, b, c) be a point on the y-axis whose distance from $(-1, 3, -\sqrt{8})$ is 5 units.

$\Rightarrow a = 0 = c$. By distance formula

$$d = \sqrt{(a+1)^2 + (b-3)^2 + (c-\sqrt{8})^2} \quad \Rightarrow \quad 5 = \sqrt{(0+1)^2 + (b-3)^2 + (0-\sqrt{8})^2}$$

$$\Rightarrow 25 = 1 + (b-3)^2 + 8 \quad \Rightarrow \quad (b-3)^2 = 16 \quad \Rightarrow \quad b - 3 = \pm 4 \quad \Rightarrow \quad b = 7 \text{ or } -1.$$

Therefore, the coordinates of the possible points are $(0, 7, 0)$ and $(0, -1, 0)$

6.4 MIDPOINT OF A LINE SEGMENT IN SPACE

Midpoint Formula in Space

Then the coordinates of the midpoint M of a line segment in space whose end points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) \qquad \text{------------} \qquad \textbf{(6.2)}$$

Example 5. Find the midpoint of the line segment with end points A(5, 3, −1) and B(−2, 1, 0).

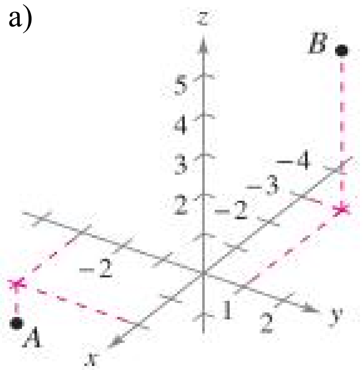
Solution : Taking $(x_1, y_1, z_1) = (5, 3, -1)$ and $(x_2, y_2, z_2) = (-2, 1, 0)$,

$$M\left(\frac{5+(-2)}{2}, \frac{3+1}{2}, \frac{-1+0}{2}\right) = M\left(\frac{3}{2}, 2, -\frac{1}{2}\right)$$

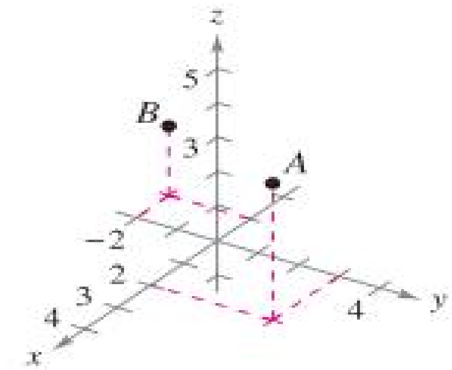
Exercise 6.1

1. Approximate the coordinates of points A and B in the following figures.

a)



b)



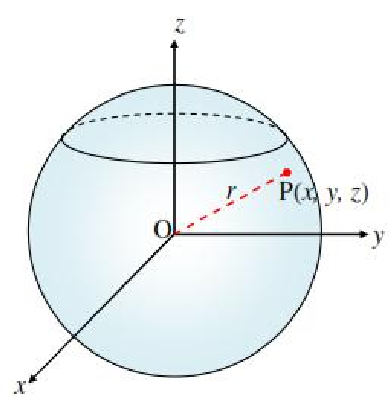
2. Find the coordinates of the point
   a) located 3 units behind the yz-plane, 4 units to the right of the xz-plane, and 5 units above the xy-plane.
   b) located on the x-axis, 5 units in front of the yz-plane.
3. Determine the location of a point (x, y, z) that satisfies the condition(s)
   a) $y = -3$ b) $x > 0$ c) $|x| \geq 1$ d) $xy < 0$, $z = 4$
4. If a point $(3, -5, 1)$ is translated 4 units downward along the z-axis. Then determine the coordinates of the translated point.
5. Find the coordinates of two points on the *x*-axis which are $\sqrt{12}$ the point P(–1, –1, 2).
6. The midpoint of a line segment with one endpoint $(2, -4, -2)$ is $(3, -1, 0)$. Find the coordinates of the other endpoint.

6.5 EQUATION OF SPHERE

A sphere is a three dimensional solid.

A **SPHERE** is a set of points in a space which have equal distance from a fixed given distance.

The fixed point is called the **CENTER** of the sphere, and the distance of the center from any point on the sphere is its **RADIUS**.



Equation of a Sphere : An equation of a sphere with center $A(a, b, c)$ and radius r is given by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \text{ ----- } \quad (6.3)$$

Equation (6.3) is called the **STANDARD FORM OF EQUATION** of the sphere.

- Thus, the equation of a sphere with center at the origin and radius r is $x^2 + y^2 + z^2 = r^2$

Remark : Given a point P and a sphere, one and only one of the following is always true :

i) P is the interior point of the sphere.
ii) P lies on the sphere.
iii) P is the exterior point of the sphere.

These conditions can be described as follow:

Consider a sphere with center $A(a,\ b,\ c)$ and radius r. Let P be a point on space.

i) P is the interior point of the sphere if and only if $AP < r$.
ii) P lies on the sphere if and only if $AP = r$.
iii) P is the exterior point of the sphere if and only if $AP > r$.

- Consider the standard form of equation of the sphere with center $A(a,\ b,\ c)$ and radius r given by

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Expanding the equation, gives

$$x^2-2ax+a^2+b^2-2by+b^2+z^2-2cz+c^2=0$$

This can be written in simplest form as

$$\boldsymbol{x^2+y^2+z^2+Bx+Cy+Dz+E=0} \qquad \text{------------} \quad \textbf{(6.5)}$$

where $B=-2a\,,\ C=-2b\,,\ D=-2c\,,\ D=\sqrt{a^2+b^2+c^2}$

Equation (6.5) is called the **GENERAL FORM OF EQUATION** of the sphere.

Note : The general form of equation of a sphere can be changed into its standard form using completing the square method.

QUESTION : What condition must be satisfied by the coefficients so that equation (6.5) is equation of a sphere?

Example 6: Find the equation of a sphere with center $A(-3,\ 2,-1)$ and passes through the point $P(0,\ 1,-2)$.

Solution : The sphere has center A and passes through P

$\Rightarrow\ r=AP=\sqrt{(-3-0)^2+(2-1)^2+(-1-(-2))^2}=\sqrt{9+1+1}=\sqrt{11}$

Then the equation of the sphere is

$(x-(-3))^2+(y-2)^2+(z-(-1))^2=\sqrt{11}^2$

$\Rightarrow\ (x+3)^2+(y-2)^2+(z+1)^2=11$ This is the standard form of the equation of the sphere.

The general form of the equation is obtained as follow :

$x^2+6x+9+y^2-4y+4+z^2+2z+1=11$

$x^2+y^2+z^2+6x-4y+2z+3=0$

Example 7: Find the center and radius $x^2+y^2+z^2-2x+6y+8z+1=0$

Solution : $x^2+y^2+z^2-2x+6y+8z+1=0$

$\Rightarrow\ x^2-2x+y^2+6y+z^2+8z+1=0$

$\Rightarrow\ x^2 - 2x + (1-1) + y^2 + 6y + (9-9) + z^2 + 8z + (16-16) + 1 = 0$

$\Rightarrow\ x^2 - 2x + 1 + y^2 + 6y + 9 + z^2 + 8z + 16 + 1 - 1 - 9 - 16 = 0$

$\Rightarrow (x-1)^2 + (y+3)^2 + (z+4)^2 = 25 = 5^2$

Hence, the center is $(1\,, -3,\ 4)$ and the radius is r = 5

Exercise 6.2

1. Find the center and radius of the sphere with equation

   $4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z = 23$

2. Determine the shortest and the longest distance of each of the following points from the sphere of equation $(x-2)^2 + (y+1)^2 + z^2 = 10$

3. Decide whether or not each of the following points is inside, outside or on the sphere whose equation is $x^2 + y^2 + z^2 - 2x + 6z + 2 = 0$

   $A(3, -1,\ 0)$ $B(2,\ 1, -1)$. $C(3, -1,\ 0)$.

4. Decide whether or not each of the following is an equation of a sphere. If it is an equation of a sphere, determine its centre and radius.

   a) $x^2 + y^2 + z^2 - 2y = 4$

   b) $x^2 + y^2 + z^2 - x + 2y - 3z + 4 = 0$

   c) $x^2 + y^2 + z^2 - 2x + 4y - 6z + 13 = 0$

6.6 VECTORS IN SPACE

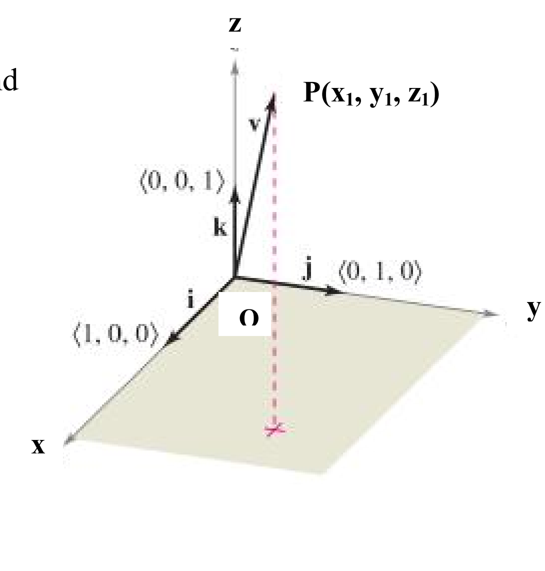
The notion of vectors in space (In component form)

Consider a point $P(x_1,\ y_1,\ z_1)$ in space. The x, y and z-steps from the origin to P are x_1 , y_1, and z_1, respectively. Hence, the vector which emanates from the origin O is denoted by ordered triple

$$\vec{v} = \overrightarrow{OP} = (x_1\,,\ y_1\,,\ z_1)$$

Using the **unit vectors :** $\vec{\imath} = (\mathbf{1}\,,\ \mathbf{0},\ \mathbf{0})$, $\vec{\jmath} = (\mathbf{0}\,,\ \mathbf{1},\ \mathbf{0})$ and $\vec{k} = (\mathbf{0}\,,\ \mathbf{0},\ \mathbf{1})$ along the direction of positive x-, y- and z-axis, the **standard unit vector notation** for $\vec{v}$ is given by

$$\vec{v} = \overrightarrow{OP} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$$



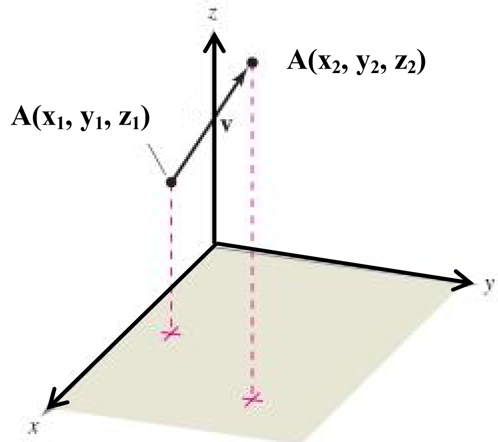
In general, if $A(x_1\ ,\ y_1\ ,\ z_1)$ and $B(x_2\ ,\ y_2\ ,\ z_2)$ are two points in space, then a vector with **initial point** A and **terminal point** B, denoted by $\overrightarrow{AB}$,

is given by

$$\overrightarrow{AB} = \boldsymbol{v} = (x_2 - x_1\ ,\ y_2 - y_1,\ z_2 - z_1)$$

or using standard unit vectors

$$\overrightarrow{AB} = \boldsymbol{v} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}$$



Properties of vectors in space

Let $\boldsymbol{a} = (a_1\ ,\ a_2\ ,\ a_3) = a_1\boldsymbol{i} + a_2\,\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = (b_1\ ,\ b_2\ ,\ b_3) = b_1\boldsymbol{i} + b_2\,\boldsymbol{j} + b_3\boldsymbol{k}$ be two vectors in space and c be a scalar.

1. **Zero Vectors** : $\mathbf{0} = (0\ ,\ 0,\ 0)$
2. **Opposite Vector to $\boldsymbol{a}$** $= -\boldsymbol{a} = -(a_1\ ,\ a_2\ ,\ a_3) = (-a_1\ ,-a_2\ ,-a_3) = -a_1\boldsymbol{i} - a_2\,\boldsymbol{j} - a_3\boldsymbol{k}$
3. **Equality of Vectors** : $\boldsymbol{a} = \boldsymbol{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$
4. **Length or Norm or Magnitude :** $|\,\boldsymbol{a}\,| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$
5. **Parallel Vectors :** $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if there is a some real constant c such that $\boldsymbol{b} = c\boldsymbol{a}$**.** That is $\boldsymbol{a} \parallel \boldsymbol{b}$ if and only if $(a_1\ ,\ a_2\ ,\ a_3) = c(b_1\ ,\ b_2\ ,\ b_3)$ for some $c \in \mathbb{R}$.
6. **Unit Vector in the Direction of $\boldsymbol{a}$:**
$$\frac{\boldsymbol{a}}{|\,\boldsymbol{a}\,|} = \frac{1}{|\,\boldsymbol{a}\,|}(a_1\ ,\ a_2\ ,\ a_3) = \frac{a_1}{|\,\boldsymbol{a}\,|}\boldsymbol{i} + \frac{a_2}{|\,\boldsymbol{a}\,|}\,\boldsymbol{j} + \frac{a_3}{|\,\boldsymbol{a}\,|}\boldsymbol{k},\ \boldsymbol{a} \neq \mathbf{0}$$
7. **Vector Addition :** $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1\ ,\ a_2 + b_2\ ,\ a_3 + b_3)$
$$= (a_1 + b_1)\,\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$$
8. **Scalar Multiplication :** $c\,\boldsymbol{a} = c(a_1\ ,\ a_2\ ,\ a_3) = (ca_1\ ,\ ca_2\ ,\ ca_3) = c(a_1\boldsymbol{i} + a_2\,\boldsymbol{j} + a_3\boldsymbol{k})$

Exercise 6.3:

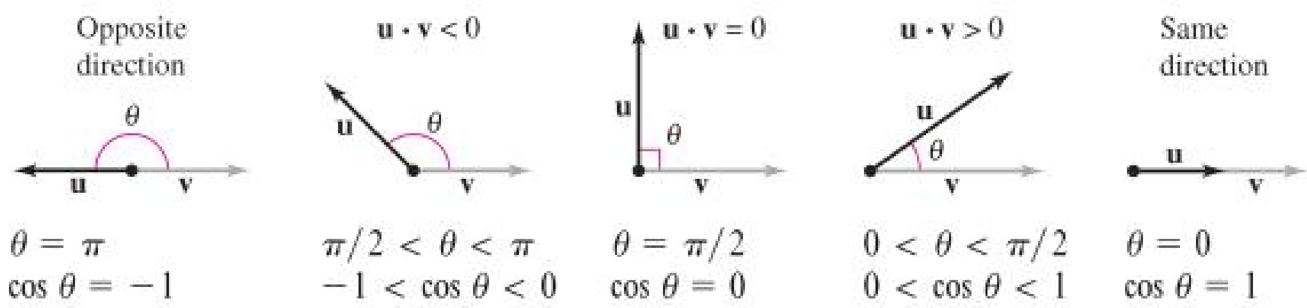
1. Find the position vector having initial point A and terminal point B, where
   a) $\mathrm{A}(2\ ,-1,\ 0)$, $\mathrm{B}(0\ ,\ 3,-4)$ b) $\mathrm{A}(-3\ ,\ 1,\ 4)$, B is the origin
2. Find the values of p and q that satisfy $p(-3\ ,\ 1,\ 4) - q(6\ ,-6,-5) = (-12\ ,24\ ,1)$ **.**
3. Find the values of p and q that satisfy $p(7\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k}) - 3p(3\boldsymbol{i} - \boldsymbol{j} + q\boldsymbol{k}) = -37\boldsymbol{i} - 25\boldsymbol{j} + 5\boldsymbol{k}$
4. Given three vectors $\boldsymbol{u} = (2\ ,\ 2,\ 3)$, $\boldsymbol{v} = (1\ ,-3,\ 4)$ and $\boldsymbol{w}\ (-3\ ,\ 6\ ,-4)$. Then find
   a) $3\boldsymbol{v} - 2\boldsymbol{w} + \boldsymbol{u}$ c) the unit vector along the direction of $\boldsymbol{w} - \frac{1}{2}\,\boldsymbol{u}$
   b) $|2\boldsymbol{u} + \boldsymbol{w}|$ d) the unit vector opposite the direction of $2\boldsymbol{v} - \boldsymbol{u}$
5. Find the value of λ so that $\vec{a} = (4,-2,\ 6)$ *and* $\vec{b} = (\ \lambda,-1,\ 3)$ are parallel.

Scalar or Dot Product of Vectors in Space

Definition (Dot Product of Vectors in Space)

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors in space, and let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, where $0 \le \theta \le \pi$. Then the **Dot Product** of $\boldsymbol{a}$ and , denoted by $\boldsymbol{a}\ .\ \boldsymbol{b}$, is a scalar value given by

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\, cos\,\theta \qquad \text{------------} \qquad \textbf{(6.6)}$$



Algebraic Properties of the Dot Product

For vectors , $\boldsymbol{b}$ *and* $\boldsymbol{c}$ **,** and $\mathrm{k} \in \mathcal{R}$,

1) $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$
2) $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$
3) $k(\boldsymbol{a}.\boldsymbol{b}) = (k\boldsymbol{a}).\boldsymbol{b}$
4) $k(\boldsymbol{b}+\boldsymbol{c}) = k\boldsymbol{b} + k\boldsymbol{c}$
5) $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|^{\boldsymbol{2}}$

Note: from the definition of dot product, we can see that for two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$**,**

1) $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel (i.e if $\theta = 0\ \ or\ \ \theta = \pi$) if and only if $\boldsymbol{a}.\boldsymbol{b} = \pm|\boldsymbol{a}||\boldsymbol{b}|$
2) $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular (i.e if $\theta = \frac{\pi}{2}$) if and only if $\boldsymbol{a}.\boldsymbol{b} = 0$

Example 8: If the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\theta = \frac{\pi}{3}$, $|\ \boldsymbol{u}\ | = 1$ and $|\ \boldsymbol{v}\ | = 5$ find $|\ 3\boldsymbol{u} - \boldsymbol{2v}|$.

Solution : From the property of vector, we have

$$|\ 3\boldsymbol{u} - \boldsymbol{2v}|^2 = (3\boldsymbol{u} - \boldsymbol{2v}).(3\boldsymbol{u} - \boldsymbol{2v}) = 9\boldsymbol{u}.\boldsymbol{u} - 12\boldsymbol{u}.\boldsymbol{v} + 4\boldsymbol{v}.\boldsymbol{v}$$

$$= 9|\boldsymbol{u}|^2 - 12|\boldsymbol{u}||\boldsymbol{v}| + 4|\boldsymbol{v}|^{\boldsymbol{2}} = 9(1)^2 - 12(1)(5) + 4(5)^2 = 49$$

$$|\ 3\boldsymbol{u} - \boldsymbol{2v}| = \sqrt{49} = 7$$

Scalar or Dot Product of Vectors in interms of Components

Let $\boldsymbol{a} = (a_1\ ,\ a_2\ ,\ a_3) = a_1\boldsymbol{i} +\ a_2\,\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = (b_1\ ,\ b_2\ ,\ b_3) = b_1\boldsymbol{i} +\ b_2\,\boldsymbol{j} + b_3\boldsymbol{k}$ be two vectors in space. Then the **Dot Product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a}.\boldsymbol{b} = (a_1\,,\ a_2\,,\ a_3).(b_1\,,\ b_2\,,\ b_3) = a_1b_1 + a_2b_2 + a_3b_3 \quad \text{------------} \quad (6.7)$$

Example 9: Let $\vec{a} = 2\vec{\imath} - 4\vec{\jmath} + \frac{1}{2}\vec{k}$ and $\vec{b} = \vec{\imath} + \frac{1}{2}\vec{\jmath} + \vec{k}$. Find $\boldsymbol{a}.\boldsymbol{b}$

Solution $\vec{a} = \left(2, -4, \frac{1}{2}\right)$, $\vec{b} = (1, \frac{1}{2}, 1)$

$$\boldsymbol{a}.\boldsymbol{b} = \left(2,\ -4,\ \frac{1}{2}\right).\left(1,\ \frac{1}{2},\ 1\right) = (2)(1) + (-4)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)(1) = 2 - 2 + \frac{1}{2} = \frac{1}{2}$$

Definition :

Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be **ORTHOGONAL** if $\boldsymbol{a}.\boldsymbol{b} = 0$.

Definition

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors in space, and let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Then $cos\,\theta$ is given by

$$cos\,\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} \quad \text{------------} \quad (6.8)$$

Example 10: Find the acute angle and the obtuse angle between $\boldsymbol{a} = (1, 0, 1)$and $\boldsymbol{b} = (1, 1, 0)$.

Solution : $\boldsymbol{a}.\boldsymbol{b} = (1\,,\ 0,\ 1)\ .\ (1\,,\ 1,\ 0)) = (1)(1) + (0)(1) + (1)(0) = 1$

$$|\,\boldsymbol{a}\,| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{\mathbf{2}} \text{ and } |\,\boldsymbol{b}\,| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$cos\,\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{1}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2}$$

Thus, the acute angle is $\theta = \frac{\pi}{3}$ and the acute angle is $\pi - \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Example 11: Show that the vectors $\boldsymbol{a} = 4\vec{\imath} - 5\vec{\jmath} - 7\vec{k}$ and $\boldsymbol{b} = -\vec{\imath} + 2\vec{\jmath} - 2\vec{k}$ are perpendicular.

Solution : $\boldsymbol{a}.\vec{b} = (4)(-1) + (-5)(2) + (-7)(-2) = -4 + -10 + 14 = 0$

Therefore $\vec{a}$ *and* $\boldsymbol{b}$ are perpendicular.

Example 12: Let $\boldsymbol{u} = m\vec{\imath} - \vec{\jmath} + 2\vec{k}$ and $\boldsymbol{v} = \vec{\imath} - \vec{\jmath}$. Find the value of *m* so that the angle between $\boldsymbol{u}$ *and* $\boldsymbol{v}$ is 45^0.

Solution : $\boldsymbol{u}\,.\boldsymbol{v} = \left(m\vec{\imath} - \vec{\jmath} + 2\vec{k}\right).(\vec{\imath} - \vec{\jmath}\,) = m(1) + (-1)(-1) + 2(0) = m + 1$

$$|\,\boldsymbol{u}\,| = \sqrt{m^2 + 5} \text{ and } |\,\boldsymbol{v}\,| = \sqrt{2}$$

$$\boldsymbol{u}\,.\boldsymbol{v} = |\,\boldsymbol{u}\,||\,\boldsymbol{v}\,|\cos 45^0 = \sqrt{m^2 + 5}.\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = \sqrt{m^2 + 5}$$

$$\Rightarrow \quad m + 1 = \sqrt{m^2 + 5} \quad \Rightarrow m^2 + 2m + 1 = m^2 + 5 \ \Rightarrow\ 2m = 4 \quad \Rightarrow m = 2$$

Exercises 6.4

1. Find a vector equation, parametric equation and symmetric equations of a line determined by
   a) point $P_0(1,\ 2,\ 3)$ and a vector $\vec{L} = -\vec{\imath} + \vec{\jmath} + 3\vec{k}$
   b) points $P_1(-3,\ 4,\ 5)$ and $P_2(-2,\ 1,\ 6)$.

2. Let $\vec{u} = 2\vec{\imath} + \vec{\jmath} + 3\vec{k}, \vec{v} = -2\vec{\imath} + \vec{\jmath} + \vec{k}$ and $\vec{w} = \vec{\jmath} + 2\vec{k}$. Show that $\vec{u}$ is perpendicular to $\vec{v}$.
3. Find the angle between the vectors $\vec{a} = (1,\ -1,\ 1)$ and $\vec{b} = (\sqrt{6},\ 1,\ 1)$.
4. Find a unit vector having the same direction as vector $\boldsymbol{t} = \sqrt{2}\,\vec{\imath} - \vec{\jmath} + \vec{k}$.

Review Exercises on Unit 6

1. Let $\boldsymbol{u} = \left(1, -\frac{1}{2}\ ,\ 0\right)$, $\boldsymbol{v} = (-1,\ 1,\ 3)$, $\boldsymbol{w} = 2\vec{\imath} + 3\vec{\jmath} - 4\vec{k}$. Find

   a) $(3\boldsymbol{w} - \boldsymbol{v}).\boldsymbol{u}$ b) $(2\boldsymbol{u} + \boldsymbol{v}).\left(\boldsymbol{w} - \frac{3}{5}\boldsymbol{v}\right)$. c) $\left|\frac{1}{2}\boldsymbol{u} + \boldsymbol{v}\right|$

3. Show that the line containing the points (1, 2, 1) and (2, 1, 3) is parallel to the line containing the points (2, 6, −7) and (6, 2, 1)
4. Find a unit vector having opposite direction as vector $\boldsymbol{n} = \pi\vec{\imath} - \sqrt{\pi}\,\vec{\jmath} + \sqrt{\pi}\vec{k}$.
5. If the angle between $\vec{a}$ and $\vec{b}$ is 60^{0}, and $\|\vec{a}\| = \|\vec{b}\| = 3$ find $\|\vec{a} - \vec{b}\|$.
6. For what value of λ, the vectors $\vec{a} = \vec{\imath} + \vec{\jmath} + \vec{k}$ and $\vec{b} = \lambda\vec{\imath} - 2\lambda\vec{\jmath} + \vec{k}$ are perpendicular ?
7. Given $\vec{a} = 2\vec{\imath} + \vec{\jmath} + \vec{k}$ *and* $\vec{b} = \vec{\imath} - 3\vec{\jmath} - 2\vec{k}$, find

   a) $\vec{a}.\vec{b}$ b) the cosine of the angle between $\vec{a}$ and $\vec{b}$
8. Show that the line containing points (1, 2, 4) and (3, 4, 2) is perpendicular to the line containing points (3, 1, 6) and (4, 2, 8).
9. For what value of x is $\vec{a} = (x,\ 1,\ -1)$ *and* $\vec{b} = (x,\ -1,\ 3)$ are orthogonal.
10. Find symmetric equations of the line l that contains the point $(6, -4,\ 3)$ and is parallel to the vector $3\vec{\jmath} - 4\vec{k}$.
11. Find a vector equation, parametric equation and symmetric equations of a line determined by

    a) point $P_0(1,\ 2,\ 3)$ and a vector $\vec{L} = -\vec{\imath} + \vec{\jmath} + 3\vec{k}$

    b) points $P_1(-3,\ 4,\ 5)$ and $P_2(-2,\ 1,\ 6)$.

UNIT 7 MATHEMATICAL PROOF

INTRODUCTION

Unit Outcomes

After completing this unit, you will be able to:

- *develop the knowledge of logic and logical statements.*
- *understand the use of quantifiers and use them properly.*
- *determine the validity of arguments.*
- *apply the principle of mathematical induction for a problem that needs to be proved inductively.*
- *realize the rule of inference.*

Revision of Statements and Logical Connectives

Recall that the five **logical connectives (or operators) : negation** ($\neg$), or ($\vee$), **and** ($\wedge$), **implication** ($\Rightarrow$) and **bi-implication** ($\Leftrightarrow$).

Let p and q be two simple propositions. The rules of the five logical connectives are shown below.

p	**q**	**$\neg$ p**	**p $\vee$ q**	**p $\wedge$ q**	**p $\Rightarrow$ q**	**p $\Leftrightarrow$ q**
T	**T**	F	T	T	T	T
T	**F**	F	T	F	F	F
F	**T**	T	T	F	T	F
F	**F**	T	F	F	T	T

Example 1. Write the **negations** of the following compound open proposition in verbal form.

" If a number is neither odd nor divisible by 3, then it is prime. "

Solution : One possible answer is

"It is not true that if a number is neither odd nor divisible by 3, then it is prime."

The other possibility is as follow **:** First change the statement into symbolic form.

Let p ≡ A number is odd. q ≡ A number is divisible by 3. r ≡ A number is prime.

Then the symbolic form the given statement is $(\neg p \wedge \neg q) \Rightarrow r$

Negations $\equiv \neg [(\neg p \wedge \neg q) \Rightarrow r] \equiv (\neg p \wedge \neg q) \wedge \neg r$

Therefore, the verbal form of the negation is

"A number is neither odd nor divisible by 3, but it is not prime."

Example 2. Find the truth value of

a) $(p \Rightarrow \neg q) \wedge (\neg p \vee q)$ if p is F. b) q , if $(\neg p \Rightarrow q) \wedge (\neg q \Leftrightarrow \neg p)$ and p are true.

Solution :

a) $p \equiv F$ implies $(p \Rightarrow \neg q) \wedge (\neg p \vee q) \equiv (F \Rightarrow \neg q) \wedge (\neg F \vee q) \equiv T \wedge T \equiv T$, because $F \Rightarrow s$ and $T \vee s$ are always TRUE for any statement s.

b) $p \equiv T$ and $(\neg p \Rightarrow q) \wedge (\neg q \Leftrightarrow \neg p) \equiv T$

$$(F \Rightarrow q) \wedge (\neg q \Leftrightarrow F) \equiv T$$
$$T \wedge (\neg q \Leftrightarrow F) \equiv T$$
$$(\neg q \Leftrightarrow F) \equiv T$$
$$\neg q \equiv F. \quad \text{therefore, } q \equiv T$$

Exercise 7.1 : Revision on Logical Connectives

1. Let $p \equiv$ 'Some even integers are prime.', $q \equiv$ '111 is prime.' And $s \equiv$ 'There is no smallest positive integer.' Then determine the truth value of the following compound statements.
   a) $\neg p \Rightarrow (q \wedge \neg s)$ b) $(s \vee \neg p) \Leftrightarrow \neg(q \Rightarrow p)$ c) $(q \Rightarrow p) \vee \neg(s \Leftrightarrow \neg q)$
2. Write the negations of the following compound open proposition in verbal form.
   a) A number is divisible by 3 if and only if it is both prime and even.
   b) The weather is cold if it rains.
3. If $\neg(\neg p \Rightarrow q)$ is False, then find the truth value of
   a) $\neg q \vee (p \Leftrightarrow \neg q)$ b) $(\neg q \Leftrightarrow p) \Rightarrow (r \vee \neg p)$
4. For three propositions p, q and r; if $(p \Rightarrow \neg q) \vee (\neg p \Leftrightarrow q)$ is False, then find the truth value of $\neg p \Leftrightarrow \neg(\neg q \Rightarrow r)$.
5. Assume $\neg p$, q and $\neg r$ are false. Then find the truth value of the following.
   a) $\neg(\neg q \Leftrightarrow p) \wedge [(\neg r \Rightarrow \neg p) \vee q]$ b) $[(r \wedge \neg q) \Rightarrow (p \wedge q)] \vee (r \Leftrightarrow \neg p)$
6. Write the symbolic form of the following.
   a) Ayantu likes either playing tennis or jogging whenever the weather is cold.
   b) It is false that Kenenisa likes neither playing tennis nor jogging when the weather is cold.
7. Find the truth value of
   a) $\neg p$, if $(\neg q \Rightarrow p) \Leftrightarrow \neg(q \wedge \neg p)$ and q are false.
   b) $\neg(\neg q \wedge p)$, if $(p \Rightarrow \neg q) \vee (\neg p \Leftrightarrow q)$ is false.
8. Construct truth table for the following compound propositions
   a) $p \Rightarrow (p \Rightarrow q)$ b) $[(r \wedge \neg p) \Rightarrow \neg q] \vee (\neg r \Leftrightarrow p)$
9. Check whether the following compound propositions are tautology, contradiction or neither.
   a) $(q \wedge \neg p) \Rightarrow q$ b)) $\neg p \Rightarrow (\neg p \vee q)$
10. Using the rules of connectives (without using truth table), simplify the following.
    a) $q \Rightarrow \neg[(p \wedge q) \wedge \neg p]$ b) $[p \wedge (\neg p \vee q)] \vee [q \wedge \neg(p \wedge q)]$

Open Statements and Quantifiers

An **open statement (or open proposition),** usually denoted by P(x), Q(x), S(x), etc, is a sentence involve variables or unknowns and become statements when the variables or the unknowns are replaced by specific numbers or individuals.
In P(x) **:** x stands for the **unknown** and P stands for some **property** that is to be satisfied by x.

Quantifiers

There are two types of **quantifiers** : **UNIVERSAL QUATIFIER** and **EXISTENTIAL QUATIFIER**

1. **Universal Quantifier**
   - It is denoted by ' ∀ ' **:** read as 'for all', 'for every', 'for each'
   - Let P(x) stands for an open proposition. '**If every individual x in the universe satisfy the property P',** then we write it as (∀x)P(x)

   For instance, let P(x) stands for $x^2 \geq 0,\ x \in \mathbb{R}$.
   (∀x ∈ ℝ)P(x) or (∀x)P(x), x ∈ ℝ read as
   For every real no. x, $x^2 \geq 0$ (**OR**) For all x ∈ ℝ , $x^2 \geq 0$ (**OR**) $x^2 \geq 0$ for all x ∈ ℝ.
2. **Existential Quantifier**
   - It is denoted by ' ∃ ' **:** read as 'there is', 'there exist', 'for some'
     Which is to mean 'there exist atleast one'
   - Let P(x) stands for an open proposition. If **'there is an individual x in the universe that satisfy the property P'**, then we write it as (∃x)P(x).

   For instance, let P(x) ≡ **x is an even integer**.
   (∃x ∈ ℤ)P(x) or (∃x)P(x), x ∈ ℤ read as
   There is an integer x such that x is even (OR) some integers are even.

Truth Values of Quantifiers

Let P(x) stands for an open proposition

1. (∀x)P(x) is **TRUE** if every individual x in the domain satisfy the property P
   (∀x)P(x) is **FALSE** if there exists atleast one individual x in the domain that do not satisfy the property P.
   To show (∀x)P(x) is **FALSE ,** it is enough to obtain one x in the domain that do not satisfy the property P.
2. (∃x)P(x) is **TRUE** if there is an individual in the given universe that satisfy the property P.
   (∃x)P(x) is **FALSE** if every individual x in the domain do not satisfy the property P.
3. To show (∃x)P(x) is **TRUE,** it is enough to obtain one x in the domain that satisfy the property P.

Negations of Quantifiers

Symbolically, the negations of quantifiers are given by

1. ¬ (∀x)P(x) ≡ (∃x) ¬ P(x)
2. ¬ (∃x)P(x) ≡ (∀x) ¬ P(x)

Example 1. Determine the truth values of the following.

a) (∀x)P(x) , where P(x) ≡ x is a white dog in the world

b) $(\exists x \in \mathbb{R})\ (x^2 + 1 = 0)$,

c) $(\forall x)(x + 2x^2 \geq x^2 - 2x - 4)$, $x \in \mathbb{R}$

d) $(\forall x)\ (\exists y)[(x - 1 < 0) \Rightarrow y^2 < x]$, x, $y \in \mathbb{R}$

Solution

a) FALSE, since there is some dog which is not white.

b) FALSE, since for all real number x, $x^2 \neq -1$

c) TRUE, $x + 2x^2 \geq x^2 - 2x - 4 \ \Rightarrow\ x^2 + 3x + 4 \geq 0$, which is always true, because $b^2 - 4ac < 0\ and\ a > 0$

d) It is equivalent to $\forall x < 1,\ \exists y \in \mathbb{R}$ such that $y^2 < x$. Now take x = 0. But $y^2 \nless 0$
Hence, it is FALSE.

Example 2. Determine the **negation** of the following.

a) $(\exists y)(\forall x)\ [(x - 1 < 0) \Rightarrow y^2 < x]$,

b) $(\forall x)\ (\forall y)(x < y \vee x \geq y)$

c) Every integer has reciprocal.

d) Somebody is a child of everybody.

Solution

a) **Negation :** $(\forall y)(\exists x)\ \neg[(x - 1 < 0) \Rightarrow y^2 < x] \equiv (\forall y)(\exists x)\ [(x - 1 < 0) \wedge (y^2 \geq x)]$

b) **Negation :** $(\exists x)\ (\exists y) \neg(x < y \vee x \geq y) \equiv (\exists x)\ (\exists y)\ (x \geq y \wedge x < y)$

c) **Negation :** Some integer has no reciprocal.

d) **Negation :** Everybody is a child of somebody.

Exercise 7.2

1. Write the following in symbolic form (using quantifiers). Write the negations in verbal form.
   a) No problem is wrong.
   b) Some student who likes Biology but not Chemistry.
   d) . Addition is associative on $\mathbb{R}$
2. Let the universal set be $\cup = \{1, 2, 3, 4\}$. Let
   P(x) : x is prime. H(x) : x is divisible by 2. R(x) : x is odd Q(x) : $x \leq 4$.
   Determine the truth value of the following statements.
   a) $(\exists x)\ P(x)$ b) $(\forall y)\ [Q(y) \wedge \neg H(y)]$ c) $(\forall x)\ \neg[H(x) \Rightarrow P(x)]$
3. Determine the truth value of the following statements.
   a) $(\exists x)(\forall y)(x^2 + 2y < 10)$; $\cup = \{1, 2, 3\}$
   b) $(\forall x)(\exists y)(x^2 + y^2 = 1)$; $\cup = \mathbb{R}$
   c) $(\exists x)(\forall y)(x + y = 0 \Leftrightarrow y = 4)$; $\cup = \mathbb{R}$
   d) $(\forall x)[\ (\exists y)(x + y = 0) \Rightarrow (\exists y)(x.y = 1)]$; $\cup = \mathbb{R}$
4. Give the negations in symbolic form.
   a) $(\exists x)\ P(x)$
   b) $(\forall x)\ [P(x) \wedge \neg H(x)]$
   c) $(\forall x)(\exists y)(x + y + 2xy = 0)$
   d) $(\exists x)(\forall y)(x + y = x \Rightarrow x.y > 0)$

Arguments and Validity of Arguments

Definition 7.1

A **LOGICAL DEDUCTION** or an **ARGUMENT FORM** is an assertion that a given set of statements P_1 , P_2 , P_3 , -- - -, Pn called the **PREMISE** (or **HYPOTHESIS**), yields another statement Q, called the **CONCLUSION** (or **CONSEQUENT**).

Such a logical deduction is denoted by

$$P_1 \ , P_2 \ , P_3 \ , \text{- - - - - -}, Pn \vdash Q$$

Definition 7.2

An argument is said to be **VALID**, if and only if the conjunction of all the premises always implies the conclusion. In other words, if we assume that the statements in the premises are all **true**, then (for a valid argument), the conclusion must be **true**. An argument which is not valid is called a **FALLACY**.

i.e To show the validity of an argument, you have to show that the conclusion is true whenever all the premises are true.

Thus, an argument is valid if the compound statement

$P_1 \wedge P_2 \wedge P_3 \wedge \text{- - - -} \wedge Pn \Rightarrow Q$ is always TRUE or Tautology.

Note : The validity of an argument can be checked in two ways **:** one **by constructing a truth table**, and the other **by Rule of Inference** .

i) By constructing a truth table

- First construct the truth table that contains all the premises and the conclusion.
-Choose rows that contain all true premises.
-If the conclusion is TRUE for all such rows, then the argument is valid. Otherwise, it is Fallacy.

Example 3. Check the validity of the following argument form using truth table.

a) $\neg p \Rightarrow q , q \vdash p$ b) $p \Rightarrow (q \vee r)$, $\neg r$, $p \vdash q$

Solution $\neg p \Rightarrow q , q \vdash p$

P	q	$\neg p$	$\neg p \Rightarrow q$	$(\neg p \Rightarrow q) \wedge q$	$[(\neg p \Rightarrow q) \wedge q] \Rightarrow p$
T	T	F	T	T	T
T	F	F	T	F	T
F	T	T	T	T	F
F	F	T	F	F	T

Therefore, the argument is invalid or fallacy.

b) $p \Rightarrow (q \vee r)$, $\neg r$, $p \vdash q$

p	q	r	$\neg r$	$q \vee r$	$P \Rightarrow (q \vee r)$	$P \Rightarrow (q \vee r) \wedge \neg r \wedge p$	$[P \Rightarrow (q \vee r) \wedge \neg r \wedge q] \Rightarrow q$
T	T	T	F	T	T	F	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	F	T
T	F	F	T	F	F	F	T
F	T	T	F	T	T	F	T
F	T	F	T	T	T	F	T
F	F	T	F	T	T	F	T
F	F	F	T	F	T	F	T

Therefore, the argument is valid.

By Rule of Inference

The construction of such a big truth table may be avoided by studying and correctly applying the following rules by which we check whether a given argument is valid or not. They are called **RULES OF INFERENCE** and are listed as follow

1	$\dfrac{p}{p \vee q}$	Principle of adjunction addition. It states that "If p is true, then $p \vee q$ is also true for any proposition q "
2	$\dfrac{p \wedge q}{p}$	Principle of detachment simplification. It states that "If $p \wedge q$ is true, then p is true".
3	$\dfrac{\begin{array}{c} p \\ q \end{array}}{p \wedge q}$	Principle of conjunction. It states that whenever p and q are true the statement $p \wedge q$ is also true.
4	$\dfrac{\begin{array}{c} p \Rightarrow q \\ p \end{array}}{q}$	Modus ponens. It states that whenever the implication $p \Rightarrow q$ is true and the hypothesis p is true, then the consequent q is also true. Recall the rule of implication.
5	$\dfrac{\begin{array}{c} p \Rightarrow q \\ \neg q \end{array}}{\neg p}$	Modus Tollens. It states that whenever, $p \Rightarrow q$ is true and q is false, then p is also false.
6	$\dfrac{\begin{array}{c} p \Rightarrow q \\ q \Rightarrow r \end{array}}{p \Rightarrow r}$	Principle of syllogism (Law of syllogism). It may be remembered as the transitive property of implication. This law was one of Aristotle's (384 – 322 B.C.) main contributions to logic.
7	$\dfrac{\begin{array}{c} p \vee q \\ \neg p \end{array}}{q}$	Modus Tollends Ponens. This rule is also called the Disjunctive syllogism.

Example 4. Check the validity of the following argument using Rules of Inference.

a) $p \Rightarrow (q \vee r)$, $\neg r$, $p \vdash q$ b) $(p \wedge q) \Rightarrow r$, $p \Rightarrow \neg r$, $p \vdash q$

Solution

a)

i) p is True -------- Premise

ii) $p \Rightarrow (q \vee r)$ is true ------- Premise

iii) $q \vee r$ is true ------------------ by i, ii and Modes Ponens
iv) $\neg r$ ------------ premise
v) q is true ------------ (iv) Modus Tollends Ponens
Therefore, the argument is valid

b)

i) $(p \wedge q) \Rightarrow r$ ----------- premise
ii) $p \Rightarrow \neg r$ ----------- premise
iii) $r \Rightarrow \neg p$ true ------------ by i, ii and contrapositive
iv) $(p \wedge q) \Rightarrow \neg p$ ------------ by i, iii and Principle of Syllogism
v) $\neg (p \wedge q)$ is true ------------ by iv, v, and Modes Tollens
vi) $\neg p \vee \neg q$ is true ------------- by v and De Morgan's Law
vii) p is true ------------ premise
viii) $\neg q$ is true ------------ vi, vii and Modus Tollends Ponens
ix) q is False ------------ viii and definition of negation
Therefore, the argument is invalid.

Example 5. Check the validity of the following argument using Rules of Inference.
If the moon is made up of bananas, then there are monkeys on the moon. The moon is not made up of banana. Therefore, there are no banana on the moon.

Solution First, change the argument into symbolic form.

Let p ≡ The moon is made up of banana.
q ≡ There are monkeys on the moon.
r ≡ There are banana on the moon.

Then the symbolic form of the argument is

$$p \Rightarrow q\ ,\ \neg p \vdash \neg q$$

i) $\neg p$ is true. ----------- premise
ii) $p \Rightarrow q$ is true ----------- premise
iii) $\neg q \Rightarrow \neg p$ true ----------- by ii and contrapositive
iv) $\neg q$ can be T or F ----------- iii and definition of implication

Therefore, the argument is fallacy.

Exercise 7.3

1. Test the validity of the following arguments using any using truth table.
   a) $\neg p \Rightarrow q\ ,\ q \vdash p$ e) $\neg p \wedge q\ ,\ (q \vee r) \Rightarrow p \vdash \neg r$
2. Test the validity of the following arguments using rule of inference.
   a) $p \Rightarrow q\ ,\ \neg r \Rightarrow \neg q \vdash \neg r \Rightarrow \neg p$ b) $p \Rightarrow \neg q\ ,\ p\ ,\ r \Rightarrow q \vdash \neg r$

REVIEW EXERCISE ON UNIT 7

1.Find the truth value of

a. $p \wedge \neg q$, if $\neg [(\neg p \Rightarrow q) \wedge (\neg q \Leftrightarrow \neg p)]$ is true.

b. $(q \vee p) \Leftrightarrow (\neg r \wedge s)$, if $\neg(r \Rightarrow \neg q)$ is true.

c. $(\neg p \Rightarrow q) \Leftrightarrow \neg(p \Leftrightarrow q)$, if $(q \wedge p) \Leftrightarrow (q \vee p)$ is false

2.Construct truth table for the following compound propositions

a) $[\neg q \wedge (p \Rightarrow \neg q)] \vee (\neg p \Leftrightarrow q)$

b) $[(r \wedge \neg p) \Rightarrow \neg q] \vee (\neg r \Leftrightarrow p)$

3. Check whether the following compound propositions are tautology, contradiction or neither.

a. $(p \Rightarrow \neg q) \vee (p \wedge q)$

b. $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

c) $\neg p \Rightarrow (\neg p \vee q)$

d) $(\neg p \vee q) \wedge p$

4. Using the rules of connectives (without using truth table), simplify the following.

a. $q \Rightarrow \neg[(p \wedge q) \wedge \neg p]$

b. $(q \Rightarrow \neg p) \wedge (p \Rightarrow q)$

c. $p \wedge [\neg q \wedge (\neg p \vee q)]$

d. $[p \wedge (\neg p \vee q)] \vee [q \wedge \neg(p \wedge q)]$

5.Verify the following equivalence **without using truth table**.

a. $(p \Rightarrow q) \Rightarrow q \equiv p \vee q$

b. $(p \wedge q) \Rightarrow r \equiv (p \Rightarrow r) \wedge (q \Rightarrow r)$

6. Determine the truth value of the following statements.

a. $(\forall x)(\exists y)(x^2 + y^2 = 1)$; $\cup = \mathbb{R}$

b) $(\exists y)(\forall x)(y \leq x^2 + 2x - 3)$; $\cup = \mathbb{R}$

c) $(\forall x)[(\exists y)(x + y = 0) \Rightarrow (\exists y)(x.y = 1)]$; $\cup = \mathbb{R}$

7. Give the negations in symbolic form.

a. $(\exists x) [\neg P(x) \Rightarrow (Q(x) \vee P(x))]$

b. $(\forall x) \neg[H(x) \Leftrightarrow P(x)]$

c. $(\exists x)(\exists y)[x > y \Leftrightarrow$ (x is prime and $y \neq 0)]$

8. Test the validity of the following arguments using any appropriate method.

a. $p \Rightarrow (q \vee r)$, $\neg r$, $p \vdash q$

b. $p \Rightarrow (s \Rightarrow \neg r)$, $p \Rightarrow r$, $p \vdash \neg s$

c. $(p \vee q) \Rightarrow (r \vee s)$, $\neg (r \vee s)$, $\neg t \Rightarrow (p \vee q) \vdash t$

d. $(p \vee q) \Rightarrow (r \vee s)$, $p \vee q$, $(r \vee s) \Rightarrow t \vdash t$

9. For each of the following argument forms

- Use appropriate symbols to represent the statements in the arguments and
- Write the argument forms using symbols and check its validity.

a) If I pass mathematics, then I will graduate. I didn't pass mathematics. Therefore, I didn't graduate.

b) Either the fridge is plugged in or I can see the milk. If the fridge door is open and the fridge is plugged in, then the fridge light is on. If the fridge light is off, then I cannot see the milk. Therefore, if the fridge door is open, then the fridge light is on.

c) If mathematics is good subject, then it is worth learning. Either the grading system is not fair or Mathematics is not worth learning. But the grading system is fair. Therefore, Mathematics is not good subject.