# FRACTAL GEOMETRY OF THE PARABOLIC ANDERSON MODEL IN 2D AND 3D WITH WHITE NOISE POTENTIAL

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ABSTRACT. We study the parabolic Anderson model (PAM)

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2} \Delta u(t,x) + u(t,x) \xi(x), & t > 0, x \in \mathbb{R}^d, \\ u(0,x) \equiv 1, & x \in \mathbb{R}^d, \end{cases}$$

where  $\xi$  is spatial white noise on  $\mathbb{R}^d$  with  $d \in \{2,3\}$ . We show that the peaks of the PAM are macroscopically multifractal. More precisely, we prove that the spatial peaks of the PAM have infinitely many distinct values and we compute the macroscopic Hausdorff dimension (introduced by Barlow and Taylor [BT89, BT92]) of those peaks. As a byproduct, we obtain the exact spatial asymptotics of the solution of the PAM. We also study the spatio-temporal peaks of the PAM and show their macroscopic multifractality. Some of the major tools used in our proof techniques include paracontrolled calculus and tail probabilities of the largest point in the spectrum of the Anderson Hamiltonian.

Keywords: Parabolic Anderson model, Anderson Hamiltonian, macroscopic Hausdorff dimension, Paracontrolled Calculus.

AMS 2020 subject classification: Primary. 60H15; Secondary. 35R60, 60K37.

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#### 1. Introduction

We consider the parabolic Anderson model on  $\mathbb{R}^d$  with  $d \in \{2,3\}$ 

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\xi(x), & t > 0, x \in \mathbb{R}^d, \\ u(0,x) \equiv 1, & x \in \mathbb{R}^d. \end{cases}$$
(1.1)

where the random potetial  $\xi$  is the spatial white noise on  $\mathbb{R}^d$  which is a mean zero Gaussian field with delta correlation between any two spatial points. PAM is one of the prototypical framework for modelling conduction of electron in crystals filled with defects. There is a competition between the two terms appearing in the operator: While the eigenfunctions of the Laplacian which depicts the behavior of the electron waves being spread out over the whole space, the multiplication-by- $\xi$  operator which models the random defects, tends to concentrate the mass of the eigenfunctions in very small regions. A discrete version of the above Hamiltonian was introduced in a seminal paper of Anderson [And58] where he showed that the bottom part of the spectrum consists of localized eigenfunctions. This phenomenon is often termed as the Anderson localization which triggered an enormous amount of research activities in last several decades (see [DL20] for detailed references).

The solution theory of (1.1) is obtained by using a mollified version of noise  $\xi_{\epsilon}$  minus a correction  $c_{\epsilon} = \frac{1}{2\pi} \log \epsilon$  and it is proved that the solution  $u_{\epsilon}$  of the PAM with potential  $\xi_{\epsilon} - c_{\epsilon}$  has a limit as  $\epsilon \to 0$ . It was first constructed on torus  $\mathbb{T}^2$  by Hairer [Hai14] using the regularity structure and by Gubinelli, Imkeller and Perkowski [GIP15a] by using the framework of para-controlled calculus. Later Hairer and Labbé extended the solution theory for the whole  $\mathbb{R}^2$  in [HL15] and furthermore for whole  $\mathbb{R}^3$  in [HL18a].

With some particular choices of random potential, PAM admits an intriguing concentration property for its tall peaks on large space-time scales which is often referred as *intermittency*. A vast amount of previous works of the PAM on  $\mathbb{Z}^d$  with i.i.d. potential, and on  $\mathbb{R}^d$  with regular potential, have revealed that the solution of PAM is highly concentrated on few small islands that are far from each other and carry most of the total mass of the solution. This phenomenon can be attributed to the following spectral representation in terms of the eigenvalues  $\lambda_1 \geq \lambda_2 \geq \cdots$  and corresponding  $L_2$ -orthonormal basis of eigenfunctions  $e_1, e_2, e_3, \ldots$  of  $\frac{1}{2}\Delta + \xi$ ,

$$u(t,x) = \sum_{n} e^{tn} e_n(x) e_n(0).$$
 (1.2)

From this representation, the intermittency of the system comes as consequence of Anderson localization which dictates the leading eigenfunctions  $e_1, e_2, \cdots$  to be concentrated in small islands. This phenomenon has been proved inside large centered boxes for few instances including the case where  $\xi$  an i.i.d. potential on  $\mathbb{Z}^d$  with double exponential tails [BKdS18]. See [KÏ6] for the past developments on PAM.

In the case of white noise potential, the phenomenon of intermittency or Anderson localization makes sense for dimensions d=1,2,3. The one dimensional case is well understood due to three beautiful works by Laure Dumaź and Cyril Labbé [DL20, DL, DL21]. There is no known solution theory for  $d \geq 4$  since the PAM with white noise potential is scaling-critical/supercritical in those cases. As we have mentioned earlier, the cases d=2,3 are dealt with regularity structure as developed by Hairer or paracontrolled calculus by Gubinelli and Perkowski.

Intermittency is intimately tied with macroscopic fractality which was studied in [KKX17, KKX18] for a large collection of parabolic stochastic PDEs including the (1+1)-d stochastic heat equation with multiplicative space-time white noise. They had shown that when the intermittency holds, the peaks of those stochastic PDEs form complex macroscopic multifractal structures. More precisely, their results show that the macroscopic Hausdorff dimension (introduced by Barlow and Taylor [BT89, BT92], see Definition B.1) of the tall peaks take distinct and nontrivial values as the level of the peaks vary, a property which symbolizes the multifractality. The same phenomenon does not hold in the case of Brownian motion where the tall peaks demonstrate a constant Hausdorff dimension (see [KKX17, Theorem 1.4]) along a different length scale.

In a recent work, [KPvZ20] showed that the sizes of the tall peaks in boxes of width  $t^{\alpha}$  for  $\alpha \in (0,1)$  and deep valleys in parabolic Anderson model in 2 dimension are asymptotically same. They have also commented that similar result is expected for PAM in 3 dimension. This property is in apparent contradiction with the intermittency property for the PAM. In this paper, we seek to study the fractality of the PAM. Our main theorems which are stated below shows that the spatial (Theorem 1.1) and spatio-temporal peaks of the PAM (Theorem 1.2) are macroscopically multifractal (see Section B for definition) for d = 2, 3.

Before proceeding to the main statement of those results, we introduce few notations. For  $\alpha, \beta, v, t > 0$ , define the set of peaks

$$\mathcal{P}_t^d(\alpha) := \left\{ x \in \mathbb{R}^d : u(t, x) \ge e^{\alpha t (\log|x|)^{\frac{2}{4-d}}} \right\},\,$$

and

$$\mathcal{P}^d(\beta, v) := \left\{ (e^{t/v}, x) \in (e, \infty) \times \mathbb{R}^d : u(t, x) \ge e^{\beta t^{\frac{6-d}{4-d}}} \right\}.$$

We also introduce

$$\mathfrak{c}_d := \frac{8}{d^{\frac{d}{2}} (4 - d)^{2 - \frac{d}{2}} \kappa_d^4},\tag{1.3}$$

where

$$\kappa_d := \sup_{f \in H^1(\mathbb{R}^d)} \frac{\|f\|_{L^4(\mathbb{R}^d)}}{\|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/4} \|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}}.$$

Our first result which is stated below finds the macroscopic Hausdorff dimension (denoted as  $Dim_{\mathbb{H}}[\cdot]$ ) of the peaks of PAM for d=2,3 in the spatial direction for all large time t. Furthermore, it also finds the asymptotic shape of the peaks in the spatial direction for any fixed large t.

Theorem 1.1 (Spatial Multifractality and Asymptotics of the PAM). For  $\alpha > 0$ , there exists  $t_0 = t_0(\alpha, d) > 0$  such that for all  $t \geq t_0$ , we have

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_t^d(\alpha)] = (d - \alpha^{\frac{4-d}{2}} \mathfrak{c}_d) \vee 0, \quad a.s.$$
 (1.4)

In addition, there exists  $t_1 = t_1(d) > 0$  such that for all  $t > t_1$ ,

$$\limsup_{|x| \to \infty} \frac{\log_+ u(t, x)}{(\log |x|)^{\frac{2}{4-d}}} \stackrel{a.s.}{=} \left(\frac{d}{\mathfrak{c}_d}\right)^{\frac{1}{2-d/2}} t. \tag{1.5}$$

For d=2, the long time asymptotics of the solution of PAM had been found in [KPvZ20]. They have show that  $\sup_{x\in \log u(t,x)}$  is approximately equal to  $\frac{2t}{\mathfrak{c}_2}$  as t gets larger when the initial data is Dirac delta. Our result shows that the tall peaks of  $\log u(t,x)$  in the spatial direction take the shape of  $\frac{2t}{\mathfrak{c}_2}$  even for finite value of t. For one dimensional PAM, similar results were proven by Xia Chen [Che15] using the moment asymptotics. However, the proof techniques for t=1 breaks down in the case of t=1, 3 since the moments of PAM in dimension larger than 1 blow up in finite time. This poses a serious technical difficulty which we are able to circumvent in this paper by

introducing new techniques. Our next result shows the macroscopic Hausdorff dimension of the spatio-temporal peaks of PAM for d = 2, 3.

Theorem 1.2 (Spatio-Temporal Multifractality of the PAM). For every  $\beta > 0$  and v > 0, we have

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}^{d}(\beta, v)] = (d + 1 - \beta^{\frac{4-d}{2}} v \mathfrak{c}_{d}) \vee d, \quad a.s.$$
 (1.6)

Macroscopic fractal dimension of the spatio-temporal peaks of the parabolic stochastic PDEs with multiplicative white noise had been investigated by Khoshnevisan, Kim and Xiao [KKX18]. This class of stochastic PDEs contains the (1+1)-dimensional stochastic heat equation with multiplicative spatio-temporal white noise. Recently macroscopic fractal dimension of the the peaks and valleys of (1+1)-d Kardar-Parisi-Zhang (KPZ) equation has been found in [DG21, GY21]. The case of (2+1)- dimensional stochastic heat equation with spatio-temporal white noise remained completely unclear since the solution theory was only known for the sub-critical regime so far [CD20, CSZ20]. Although the solution theory of parabolic Anderson model in (2+1)-d and (3+1)-d are well studied by now, the depiction of the macroscopic fractal structures in those cases were missing. Theorem 1.2 filled this gap by showing the multifractality of spatio-temporal peaks for higher dimensional PAM.

Multifractality of the peaks of intermittent systems were discussed in many occasions in the previous literature including [GD05] in the context of turbulence and [ZTPSM00] for stochastic Allen-Cahn equation with multiplicative forcing. [KKX18, Theorem 1.1] showed that the spatiotemporal peaks of the (1+1)-dimensional stochastic heat equation (SHE) with space-time white noise form multifractals with peaks of height  $e^{\beta t}$  for every  $\beta > 0$ . This result leverages on the (moment) intermittency of the (1+1)-d, which means the exponential moment  $\mathbb{E}[u(t,x)^p]$  of the solution behaves as  $\exp(\gamma(p)t)$  where  $p \mapsto \gamma(p)$  is a strictly convex function (see [CM94]). Indeed, the proof of [KKX18, Theorem 1.1] utilized this exponential moment to obtain the tail estimates of the solution. However, the moments of parabolic Anderson model in (2+1)-d or (3+1)-d cases blow up. As a result, the previous approach based on moment intermittency breaks down in those two cases. We rather use the asymptotics of the spectrum of the Anderson Hamiltonian and the Feynman-Kac representation of the PAM built using the theory of para-controlled distributions [GIP15b]. Theorem 1.2 exposes that even though there is no moment intermittency, the spatiotemporal tall peaks of the solution to (2+1)-d (resp. (3+1)-d) PAM exhibit multifractality of order  $e^{\beta t^2}$  (resp.  $e^{\beta t^3}$ ), which displays a chaotic nature of the high dimensional multiplicative noise. In a recent work [GGL22], the first author of this paper and his collaborators have introduced the idea of finite time intermittency for the PAM in higher dimension with asymptotically singular noise. We believe that many of our proof techniques can be extended to study the macroscopic fractality of the peaks in those settings.

1.1. **Proof Ideas.** In this section, we discuss the proof ideas behind Theorem 1.1 and Theorem 1.2. Proving fractal dimension of any given set can be done in two steps: first showing a lower bound to the fractal dimension and finally, showing the appropriate upper bound. While showing an appropriate upper bound can pose serious challenges, proving a lower bound to the fractal dimension very often requires more precise insights about the geometry of the associated models. The case of PAM in higher dimension is not exception to this folklore. One of the major challenges in showing both upper and lower bounds to the fractal dimension is to control the tail probabilities of the maximum of PAM solution in compact sets. Since the moments of (d+1)-dimensional PAM blows up in finite time when d=2,3, previous approaches based on moment asymptotics [KKX18, DG21] fail to work in this situation. To get around this difficulty, we seek to use the connection between the solution of PAM and the spectrum of Anderson Hamiltonian as formally stated in (1.2).

Showing a lower bound to the fractal dimension of a set  $E \subset \mathbb{R}^d$  requires to show that the associated set is 'sufficiently thick'. See the illustration in Figure 1 for the definition of thickness

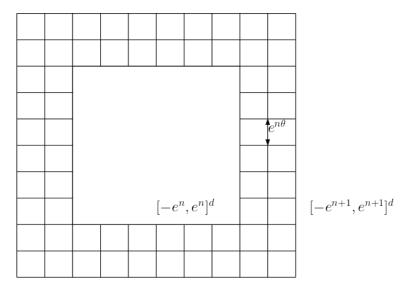


FIGURE 1. A set E is called  $\theta$ -thick for some  $\theta \in (0,1)$  if E contains points each cell of side-length  $e^{n\theta}$  in the outer shell of  $[-e^n, e^n]^d$  for all large n.

of a set E. In order to show enough thickness of E, we first embed E into a large d-dimensional box and then divide the large box into smaller boxes. It is then sufficient to show there are enough of such small boxes which carries the points from E or rather the probability of the set of points of E escaping most of those small boxes is close to 0. Controlling this probability will require two important ingredients which were lacking before: (a) near independence between the solutions of PAM restricted on any two such smaller boxes and (b) upper bound on the probability of the PAM to be bounded above by a large value.

On the other hand, following the definition of macroscopic Hausdorff dimension from B.1,  $\operatorname{Dim}_{\mathbb{H}}(E)$  is upper bounded by  $\rho$  if  $\rho$ -dimensional Hausdorff content of E often computed as  $\mathbb{E}[\sum_{n=1}^{\infty} \nu_{\rho}^{n}(E)]$  is finite. See the paragraph before Definition B.1 for  $\nu_{\rho}^{n}(E)$ . Bounding the above expected value requires one more important ingredient which is (c) to bound the tail probability of the supremum of the solution of PAM in a small ball.

We obtain those ingredients via combinations of different tools that we develop throughout the paper. Inception of these tools and carrying out the rest of proof of our main results can be broadly divided into three steps: the first step is to show appropriate bounds on the solution of PAM in terms of spectrum of Anderson Hamiltonian, the second step is to derive some tail probability on the solution and the third is to integrate the first two steps with a series expansion (coming from Feynman-Kac representation of the PAM in higher dimension) of the solution to complete the proof. Below we discuss each steps in more details. Figure 2 describes schematic representation of where the different tools are introduced and how they are combined to prove Theorem 1.1 and 1.2.

Step 1. We derived appropriate bound on the solution of PAM in Section 5.1. We mainly use three tools to show such bounds on the solution. These three tools are respectively, the Feynman-Kac representation of the solution of PAM, transition kernel estimates and appropriate bounds on the noise. The Feynman-Kac representation of the solution is derived in Theorem 3.2 which shows that the solution of (1.1)  $u_{L,y}^{\phi}$  started from the initial data  $\phi$  and restricted on  $y + [-\frac{L}{2}, \frac{L}{2}]^d$  with Dirichlet boundary condition can be written as

$$u_{L,y}^{\phi}(t,x) = \mathbb{E}\Big[\exp\left(\int_{r}^{t} (Z_{L}^{y} + \eta Y_{L}^{y})(X_{s})ds + (Z_{L}^{y} + Y_{L}^{y})(X_{r}) - (Z_{L}^{y} + Y_{L}^{y})(X_{t})\right)\phi(X_{t})\mathbb{1}^{X}\Big] \quad (1.7)$$

where  $\eta > 0$  is a small number,  $\mathbbm{1}^X := \mathbbm{1}^X_{X_{[0,t]} \subset y + [-\frac{L}{2},\frac{L}{2}]^d}$  and  $X_t$  is a diffusion defined by

$$X_t = x + \int_0^t \nabla (Z_L^y + Y_L^y)(X_s) ds + B_t$$

such that  $B_t$  is Brownian motion independent of  $Z_L^y$  and  $Y_L^y$  where  $Z:=(1-\frac{1}{2}\Delta)^{-1}\xi\in\mathscr{C}^{\frac{1}{2}-}$  and Y solves

$$(\eta - \frac{1}{2})Y_L^y = \frac{1}{2}|\nabla Z_L^y|^2 + \nabla Y_L^y \cdot \nabla Z_L^y + \frac{1}{2}|\nabla Y_L^y|^2.$$

These two random processes are introduced in Proposition 3.1 of Section 3. The reason that the expression (1.7) differs from the classical Feynman-Kac representation is the roughness of the noise  $\xi$  (see (3.6) and the following discussion). We showed the equivalence between the classical form of Feynman-Kac and the modified form by using the Girsanov's theorem along the similar line as in [GP17]. Similar results have been shown for d=2 by [KPvZ20]. However the d=3 case requires handling of sufficient technical difficulties which has been overcome in the present paper using similar tools as in [CC18a] based on para-controlled distributions. See Remark 3.3 for more details.

The next main tool is the bound on the transition density of the diffusion  $X_t$ . To this end, the transition density of  $X_t$  is a solution of Cauchy problem as shown in (4.2) (see Theorem 4.5). Since  $\nabla(Z_L^y + Y_L^y)$  is distribution valued, it requires non-trivial fixed point argument to show that the transition density kernel exists. We first lift  $\nabla(Z_L^y + Y_L^y)$  in the space of rough distributions and then employ tools from para-controlled calculus to achieve this in Section 3. Similar problem had been considered before in [CC18a]. On the way of proving our result, we have extended their result (especially [CC18a, Theorem 3.10]) to cover the singular initial data case.

Once the existence is shown, the upper and lower bound on the transition kernel is derived using the ideas of [Str08]. The next main tool is to bound the mollified noise  $\xi_{\epsilon}$ , or more precisely  $(1-\frac{1}{2}\Delta)^{-1}\xi_{\epsilon}$  uniformly in  $\epsilon$  using hyper-contractivity of Gaussian noise (see Proposition 5.2 and 5.3).

**Step 2.** The second step is to derive the tail probabilities of the solution of PAM in (d+1)case where d=2,3. This is shown in Proposition 6.1 & 6.3. More precisely, we find the tail probability of the supremum value of PAM where the supremum is taken over a finite set of points. The main idea behind its proof lies in using a local 'representations' of the solution of PAM. Construction of such proxy is done via the Feynman-Kac representation obtained in Theorem 3.2. In more concrete words, the diffusion  $X_t$  in the Feynman-Kac formula could be restricted into a set of disjoint but space-filling boxes to write u(t,x) as a sum of local representations like  $u_{L,y}^{\phi}$  of (1.7). This is termed as series expansion of the solution of PAM on  $\mathbb{R}_{>0} \times \mathbb{R}^d$ . See Lemma 5.6 for more details. By construction, those local representations of u(t,x) are independent when they are taken from two separate far away boxes. Furthermore, those could be bounded from above and below by some functional of the largest point in the spectrum of Anderson Hamiltonian as shown in Proposition 5.5. Finally the tail probabilities of the solution of PAM restricted in finite boxes are found in terms of the tail probabilities of the largest point in the spectrum of the Anderson Hamiltonian and tail probabilities of  $(1-\frac{1}{2}\Delta)^{-1}\xi_{\epsilon}$  obtained from Lemma 5.2. Tail probabilities of the largest point in the spectrum were investigated before in many occasions in the past (see Proposition 2.2).

**Step 3.** The third step is to complete the proof of Theorem 1.1 and 1.2 using the tail probabilities and series expansion of u(t,x) of Lemma 5.6. This is mainly done in Section 6 and 7. As we have indicated earlier, the proof of the lower bound in fractal dimension goes by showing that the probability of the maximum of u(t,x) over a finite set of points (satisfying the conditions in Proposition 6.1 or 7.1) being less than a certain value decays fast to 0. Since the *local representations* of u(t,x) around those points (as described in **Step 2**) can be made independent and the tail probabilities of the local representations are determined through Proposition 5.5 and 2.2, these two

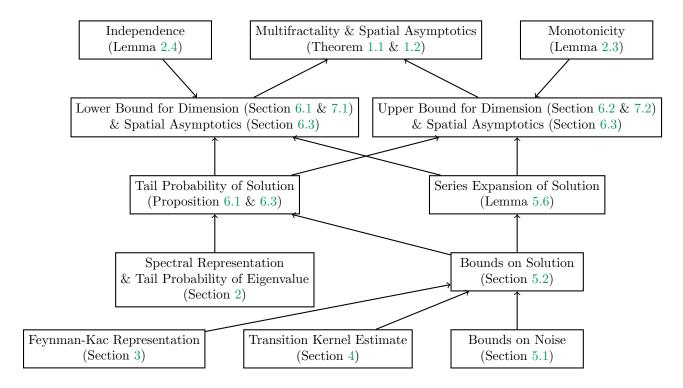


FIGURE 2. Flowchart of the proof of Theorem 1.1 and 1.2

tools are combined to bound the lower tail probability of the maximum of u(t, x) which finally leads to the lower bound in Hausdorff dimensions in Theorem 1.1 and 1.2. The upper bound part of Theorem 1.1 and 1.2 is proved using the set of tools from **Step 1** and **Step 2**. These tools provide the upper tail probability of the maximum of u(t, x) in compact sets which is used to control the expected value of Hausdorff contents of the given level sets.

**Acknowledgement.** PG was supported by NSF grant DMS-2153661 and JY was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1401-51. Part of the research of this paper was done when JY was visiting the department of Mathematics of MIT during the summer of 2022.

#### 2. Spectrum of the Anderson Hamiltonian

In this section, we discuss some preliminary facts about Anderson Hamiltonian and parabolic Anderson model which will be used throughout the rest of the paper. On our way, we introduce many notations, provide the context of their use in later sections and explain their roles in proving the main results of this paper.

We define  $Q_L(d) := [-\frac{L}{2}, \frac{L}{2}]^d \subset \mathbb{R}^d$ . We often use the symbol  $Q_L$  in place of  $Q_L(d)$  and the value d will be clear from the context. For any  $y \in \mathbb{R}^d$ , we set  $Q_L^y(d) := y + Q_L(d)$ . We consider the PAM with the Dirichlet boundary condition on  $Q_L^y(d)$  started from initial data  $\phi$  and with enhanced noise  $\xi_L$  as constructed in [CvZ21, Section 6]

$$\begin{cases} \frac{\partial}{\partial t} u_{L,y}^{\phi}(t,x) = \frac{1}{2} \Delta u_{L,y}^{\phi}(t,x) + u_{L,y}^{\phi}(t,x) \xi_{L,y}(x), & t > 0, x \in Q_L^y, \\ u(0,x) = \phi, & x \in Q_L^y, \quad \text{and} \quad u_{L,y}^{\phi} \mid_{\partial Q_L^y} = 0, \end{cases}$$

$$(2.1)$$

Construction of  $\xi_L$  goes by first projecting the white noise on the Neumann space of the box and then take the regularisation corresponding to a Fourier multiplier. Fix any even function

 $\tau \in C_c^{\infty}(^d, [0, 1])$  and define

$$\xi_{L,\varepsilon}^{y} = \sum_{k \in \mathbb{N}^{d}} \tau\left(\frac{\varepsilon}{L}k\right) \langle \xi, \mathfrak{n}_{k,L} \rangle \mathfrak{n}_{k,L}, \quad \mathfrak{n}_{k,L}(x) := 2^{-\frac{1}{2}\{i:k_{i}=0\}} \mathbb{1}\left(x \in Q_{L}^{y}(d)\right) \left(\frac{2}{L}\right)^{d} \prod_{i=1}^{d} \cos\left(\frac{\pi}{L}k_{i}x_{i}\right).$$

Theorem 6.7 of [CvZ21] shows that  $\xi_{L,\varepsilon}^y$  converges almost surely to the white noise  $\xi_L^y \in \mathscr{C}^{\alpha}$  as  $\varepsilon$  goes to 0 for  $\alpha < -\frac{d}{2}$  and the limit does not depend on the choice of  $\tau$ . The solution of (2.1) could be realized as the weak limit of the following system where we replace  $\xi_L^y$  by  $\xi_{L,\varepsilon}^y$ , i.e.,

$$\begin{cases}
\frac{\partial}{\partial t} u_{L,\varepsilon}^{\phi,y}(t,x) = \frac{1}{2} \Delta u_{L,\varepsilon}^{\phi,y}(t,x) + u_{L,\varepsilon}^{\phi,y}(t,x) (\xi_{L,\varepsilon}^{y}(x) - c_{\varepsilon}), & t > 0, x \in Q_{L}^{y}, \\
u(0,x) = \phi, & x \in Q_{L}^{y}(d), \text{ and } u_{L,y}^{\phi} \mid_{\partial Q_{L}^{y}(d)} = 0,
\end{cases} (2.2)$$

where  $c_{\varepsilon}$  denotes the renormalization constant which we set as  $c_{\varepsilon} = \frac{1}{2\pi} \log \frac{1}{\varepsilon}$ . It has been shown in Section 2 of [KPvZ20] for all T > 0,  $u_{L,\varepsilon}^{\phi,y}(t,x)$  converges in  $C([0,T], B_{\infty,\infty}^{\varrho,\beta}(Q_L^y(d)))$  uniformly on  $[0,T] \times Q_L^y(d)$  in probability to  $u_{L,y}^{\phi}$ . Here  $B_{\infty,\infty}^{\varrho,\beta}(Q_L^y(d))$  is the Dirichlet Besov space defined in [CvZ21, Section 4].

Our next goal is to introduce the spectral representation of  $u_{L,y}^{\phi}$  in  $B_{\infty,\infty}^{\varrho,\beta}(Q_L^y(d))$  in terms of the spectrum of Anderson Hamiltonian. To this end, we recall definition of the Anderson Hamiltonian operator from [CvZ21] and [Lab19]. Theorem 5.4 of [CvZ21] characterized the spectrum of Anderson Hamiltonian for d=2 case wheres [Lab19, Theorem 1] did the same for d=3. We summarize below their results. Denote the enhancement of  $B_{\alpha,\infty}^{\varrho,\beta}(Q_L^y(d))$  by  $\mathfrak{X}^{\alpha}(Q_L^y(d))$  and their respective Neumann extensions as  $B_{\alpha,\infty,\mathfrak{n}}^{\varrho,\beta}(Q_L^y(d))$  and  $\mathfrak{X}_{\mathfrak{n}}^{\alpha}(Q_L^y(d))$ .

Let L > 0,  $y \in \mathbb{R}^d$  and  $\xi$  be a d-dimensional spatial white noise. In dimension  $d \in \{2, 3\}$ , there exists  $\mathscr{H}_{\xi}$  that is densely defined on  $L^2(Q_L^y(d))$ , a closed and self-adjoint operator given by

$$\mathcal{H}_{\xi} = \Delta u + \xi u,$$

with values in  $L^2(Q_L^y(d))$ .  $\mathscr{H}_{\xi}$  has a pure spectrum consisting of eigenvalues  $\lambda_1(Q_L^y(d)) > \lambda_2(Q_L^y(d)) \geq \lambda_3(Q_L^y(d)) \geq \cdots$ . We let  $v_{n,L}^y$  be an eigenvector with eigenvalue  $\lambda_n(Q_L^y(d))$  such that  $\{v_{n,L}^y\}_{n\in\mathbb{N}}$  is an orthonormal basis of  $L^2(Q_L^y(d))$ . Due to the lack of regularity of  $\xi$ , the product  $\xi u$  is not well-defined in a classical sense. For the rigorous definition of the product, we refer the readers to [CvZ21, Theorem 5.4] and [Lab19, Theorem 1].

**Lemma 2.1** (Spectral representation, Lemma 2.11 and Theorem 2.12 of [KPvZ20]). For  $L, t > 0, y \in \mathbb{R}^d$  and  $\phi \in L^2(Q_L^y(d))$ , we have

$$u_{L,y}^{\phi}(t,\cdot) = \sum_{n \in \mathbb{N}} e^{t\lambda_n(Q_L^y(d))} \langle v_{n,L}^y, \phi \rangle v_{n,L}^y.$$
 (2.3)

Moreover, this representation holds for  $\phi = \delta_z$ . In other words,

$$u_{L,y}^{\delta_z}(t,x) = \sum_{n \in \mathbb{N}} e^{t\lambda_n(Q_L^y(d))} v_{n,L}^y(z) v_{n,L}^y(x), \quad \text{for } x, z \in Q_L^y(d). \tag{2.4}$$

*Proof.* We refer to [KPvZ20, Lemma 2.11, Theorem 2.12] for the proof in the case of d=2. For d=3, the proof follows from a similar argument as when d=2 combining [Lab19, Theorem 1.1] and Theorem 4.11.

Eigenvalues of the Anderson Hamiltonian play very important roles in proving our main results. In Section 5, we discuss how the supremum of the PAM restricted over a growing rectangle can be described in term of the largest eigenvalue of the Anderson Hamiltonian. In the next three results, we record some useful properties of the eigenvalues of  $\mathcal{H}_{\xi}$  such as the tail probabilities, monotonocity and the independence. These results will be instrumental in obtaining asymptotics of the solution of PAM in Section 5.

The first result is about the tail probabilities of the eigenvalues of Anderson Hamiltonian. We refer to [CvZ21, Theorem 2.17] for d = 2 case and [HL22, Theorem 2] for d = 3 case.

**Proposition 2.2.** Fix  $\epsilon \in (0,1)$ . There exist  $c_2 > c_1 > 0$  and  $s_0$  such that for all  $L \ge 1$  and  $s \ge s_0$ 

$$\mathbb{P}\left(\boldsymbol{\lambda}_1(Q_L^y(d))\right) \le s \le \exp\left(-c_2 s^{d/2} e^{d\log L - (1+\epsilon)\mathfrak{c}_d s^{2-d/2}}\right),\tag{2.5}$$

$$\mathbb{P}\left(\boldsymbol{\lambda}_1(Q_L^y(d))\right) \ge s \le c_1 s^{\frac{d}{2}} e^{d \log L - (1-\epsilon)\mathfrak{c}_d s^{2-d/2}},\tag{2.6}$$

where  $\mathfrak{c}_d$  is defined in (1.3).

The second result says that the eigenvalues of Anderson Hamiltonian grows monotonically as the size of the box grows. For the proof of this result, we refer to [CvZ21, Theorem 8.6] for d = 2 case and [HL22, Proposition 2.1] for d = 3 case.

**Proposition 2.3** (Monotonocity of eigenvalues). Let  $L \ge r \ge 1$ . For all  $x, y \in \mathbb{R}^d$  such that  $Q_r^y(d) \subseteq Q_L^x(d)$ ,

$$\lambda_1(Q_r^y(d)) \le \lambda_1(Q_L^x(d)). \tag{2.7}$$

The final result of this section is about the domain Markov property for the spectrum of the Anderson Hamiltonian with white-noise potential, i.e., the spectrum of  $\mathcal{H}_{\xi}$  restricted on two disjoint regions are independent of each other. We refer to Lemma 7.4 of [CvZ21] for the proof.

**Proposition 2.4. Independence of eigenvalues**] Suppose that  $y_1, ..., y_m \in \mathbb{R}^d$  satisfy  $\min_{1 \leq i \neq j \leq m} |y_i - y_j| \geq 3L$  and  $x_i \in Q_L^{y_i}$  for each i. For  $1 \leq i \neq j \leq m$ ,  $\lambda_n(Q_L^{y_i}(d))$  and  $\lambda_n(Q_L^{y_j}(d))$  are independent.

# 3. FEYNMAN-KAC REPRESENTATION

This section is devoted to proving the Feynman-Kac representation of the Anderson Hamiltonian in 3 dimension. The main purpose of deriving such Feynman-Kac representation is to derive useful upper bound on the tail probabilities of the eigenvalues of Anderson Hamiltonian. In the 2d case, [KPvZ20, Theorem 2.17] provides the modified version of the Feynman-Kac representation using Girsanov's transformation which has been used to derive tail probabilities. However, this upper bound works only when d=2. We slightly refine the representation to cover the 3-dimensional case. Before we present the representation, we introduce an equation related to the noise. For the rest of this section, we only consider the case of d=3.

**Proposition 3.1** (Resolvent equation). Let L>0  $y\in\mathbb{R}^d$ ,  $\alpha\in(\frac{2}{5},\frac{1}{2})$  and  $\xi_L^y$  be the spatial white noise on  $Q_L^y$ . Set  $Z_L^y:=(1-\frac{1}{2}\Delta)^{-1}\xi_L^y$ . Then there exists  $\eta_L>0$  such that for all  $\eta\geq\eta_L$  there exists a unique solution  $Y_L^y\in\mathscr{C}^{2\alpha}$  to the following resolvent equation

$$(\eta - \frac{1}{2}\Delta)Y_L^y = \frac{1}{2}|\nabla Z_L^y|^2 + \nabla Y_L^y \cdot \nabla Z_L^y + \frac{1}{2}|\nabla Y_L^y|^2. \tag{3.1}$$

Furthermore, if  $\{\xi_{L,\epsilon}^y\}_{\epsilon}$  is a mollification of  $\xi_L^y$  such that  $\xi_{L,\epsilon}^y \to \xi_L^y$  in  $\mathscr{C}^{\alpha-2}$ , then  $Y_{L,\epsilon}^y \to Y_L^y$  in  $\mathscr{C}^{2\alpha}$ .

Before proceeding to the proof of the above proposition, we state below the main result of this section which shows a Feynman-Kac representation of  $u_{L,\epsilon}^{\phi,y}$ .

Theorem 3.2 (Modified Feynman-Kac representation). For  $L, t > 0, y \in \mathbb{R}^d, \epsilon \in (0, 1]$  and  $\phi \in C_b(Q_L^y)$ , we have

$$u_{L,\epsilon}^{\phi,y}(t,x) = \mathbb{E}_{\mathbb{Q}_{L,\epsilon}^{x,y}} \left[ \mathscr{D}_{L,\epsilon}^{y}(0,t)\phi(X_{t}) \mathbb{1}_{X_{[0,t]} \subset Q_{L}^{y}} \right], \quad for \ x \in Q_{L}^{y}, \tag{3.2}$$

where  $\mathscr{D}_{L,\epsilon}^{y}(r,t)$  for  $r,t\in\mathbb{R}_{+}$  is defined by

$$\mathscr{D}_{L,\epsilon}^{y}(r,t) := \exp\left(\int_{r}^{t} (Z_{L,\epsilon}^{y} + \eta Y_{L,\epsilon}^{y})(X_{s})ds + (Z_{L,\epsilon}^{y} + Y_{L,\epsilon}^{y})(X_{r}) - (Z_{L,\epsilon}^{y} + Y_{L,\epsilon}^{y})(X_{t})\right)$$
(3.3)

with  $\eta > 0$  and  $\mathbb{Q}_{L,\epsilon}^{x,y}$  be the probability measure on  $C([0,\infty),\mathbb{R}^d)$  such that the coordinate process  $X_t$  satisfies  $\mathbb{Q}_{L,\epsilon}^{x,y}$ -almost surely

$$X_{t} = x + \int_{0}^{t} \nabla (Z_{L,\epsilon}^{y} + Y_{L,\epsilon}^{y})(X_{s})ds + B_{t}', \quad t \ge 0,$$
(3.4)

for a Brownian motion B'. Moreover, if  $\eta = \eta_L$  as in Proposition 3.1, we have

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mathbb{Q}_{L,\epsilon}^{x,y}} \left[ \mathscr{D}_{L,\epsilon}^{y}(0,t)\phi(X_{t}) \mathbb{1}_{X_{[0,t]} \subset Q_{L}^{y}} \right] = \mathbb{E}_{\mathbb{Q}_{L}^{x,y}} \left[ \mathscr{D}_{L}^{y}(0,t)\phi(X_{t}) \mathbb{1}_{X_{[0,t]} \subset Q_{L}^{y}} \right] = u_{L}^{\phi,y}(t,x). \tag{3.5}$$

Notice that the classical Feynman-Kac representation for the smooth mollifier  $\xi_{L,\epsilon}^y$  of  $\xi_L^y$  is different than (3.2). Let  $L \in (0,\infty)$  and  $y \in \mathbb{R}^d$ . For  $\phi \in C_b(Q_L^y)$ ,  $\epsilon > 0$ , and  $(t,x) \in \mathbb{R}_+ \times Q_L^y$ , the classical Feynman-Kac formula takes the form

$$u_{L,\epsilon}^{\phi,y}(t,x) = \mathbb{E}_x \left[ \exp\left( \int_0^t (\xi_{L,\epsilon}^y - c_\epsilon)(B_s) ds \right) \phi(B_t) \mathbb{1}_{B[0,t] \subset Q_L^y} \right]$$
(3.6)

where  $B[0,t] := \{B(s) : s \in [0,t]\}$ . Due to low regularity of the spatial white noise  $\xi_L^y$ , the  $L_{\infty}$ -norm of  $\xi_{L,\epsilon}^y - c_{\epsilon}$  blows up as  $\epsilon \to 0$ . We deal with this difficulty by adopting the ideas from the proof of [KPvZ20, Lemma 2.16, Theorem 2.17]. In a similar way as [KPvZ20], we use Girsanov's transform and Proposition 3.1 to obtain a modified version of the Feynman-Kac representation.

Remark 3.3. Theorem 3.2 is a variant of [KPvZ20, Theorem 2.17] in dimension 3. The difference in the Feynman-Kac representation of [KPvZ20, Theorem 2.17] and the one in (3.2) is the absence of the term  $\frac{1}{2}|\nabla Y|^2$  (see (24) of [KPvZ20]). Note that since the expected regularity of Y is 1<sup>-</sup>,  $\frac{1}{2}|\nabla Y|^2$  is not controlled in  $L_{\infty}$ -norm. Thus, we used the partial Girsanov theorem via solving the resolvent equation (3.1) which is different from the expression in (23) of [KPvZ20].

Without loss of generality, we drop the superscript y and let y=0 for convenience. All the results of this section can be extended for any  $y \in \mathbb{R}^d$  easily. We prove Theorem 3.2 after showing Proposition 3.1. To this end, that the products  $|\nabla Z_L|^2$  and  $\nabla Y_L \cdot \nabla Z_L$  in Proposition 3.1 are ill-defined due to the lack of the regularity. Indeed, for instance in d=3,  $Z_L \in \mathscr{C}^{\frac{1}{2}^-}$  and the expected regularity of  $Y_L$  is  $1^-$ . Thus the sum of the regularity of  $\nabla Z_L$  and  $\nabla Y_L$  is negative which makes the solution of (3.1) ill-posed. In order to overcome this difficulty, we use the idea of enhancing the noise following similar arguments as in [CC18a, Section 6].

**Definition 3.4** (Enhanced noise). Let L > 0 and  $\varrho < 1/2$ . For  $(a, b, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathscr{C}^2$ , define  $\mathfrak{Z}_L$  as

$$\mathfrak{Z}_{L} := \mathfrak{Z}_{L}(a,b,\theta) := (Z_{L}, Z_{L}^{\mathbf{Y}} - a, Z_{L}^{\mathbf{Y}}, Z_{L}^{\mathbf{Y}}, Z_{L}^{\mathbf{W}} - b, \nabla Q_{L} \circ \nabla Z_{L}),$$
 where  $\mathcal{I}_{\eta} := (\eta - \frac{1}{2}\Delta)^{-1},$  (3.7)

$$Z_{L} := \mathcal{I}_{\eta}(\theta), \quad Z_{L}^{\mathbf{Y}} := \mathcal{I}_{\eta}(|\nabla Z_{L}|^{2}),$$

$$Z_{L}^{\mathbf{Y}} := \mathcal{I}_{\eta}(\nabla Z_{L}^{\mathbf{Y}} \cdot \nabla Z), \quad Z_{L}^{\mathbf{Y}} := \mathcal{I}_{\eta}(\nabla Z_{L}^{\mathbf{Y}} \cdot \nabla Z_{L}),$$

$$Z_{L}^{\mathbf{W}} := \mathcal{I}_{\eta}(|\nabla Z_{L}^{\mathbf{Y}}|^{2}),$$

$$(3.8)$$

and

$$Q_L := \mathcal{I}(\nabla Z_L), \qquad \nabla Q_L \circ \nabla Z_L := (\partial_i (Q_L)^j \circ \partial_i Z_L)_{i,j=1,2,3}.$$

We define the space  $\mathcal{Z}_L^{\varrho}$  of enhanced noise as

$$\mathcal{Z}_L^{\varrho} := \operatorname{cl}_{\mathscr{H}^{\varrho}} \{ \mathfrak{Z}_L(\theta, a, b) : (a, b, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathscr{C}^2 \},$$

where  $\operatorname{cl}_{\mathcal{H}^\varrho}$  denotes the closure with respect to the topology of  $\mathcal{H}^\varrho := \mathscr{C}^\varrho_{\mathfrak{n}} \times \mathscr{C}^{2\varrho}_{\mathfrak{n}} \times \mathscr{C}^{3\varrho}_{\mathfrak{n}} \times \mathscr{C}^{\varrho+1}_{\mathfrak{n}} \times \mathscr{C}^{4\varrho}_{\mathfrak{n}} \times \mathscr{C}^{2\varrho-1}_{\mathfrak{n}}$  equipped the usual norm. We call  $\mathfrak{Z}$  a enhancement of  $\theta$ .

In the sequel, we fix  $\theta = \theta_{L,\epsilon} = \xi_{L,\epsilon}$  so that  $Z_{L,\epsilon} := \mathcal{I}_{\eta}(\xi_{L,\epsilon})$  where  $\{\xi_{L,\epsilon}\}_{\epsilon \in (0,1]}$  is a mollification of the spatial white noise  $\xi_L$  restricted on  $Q_L$ . The following theorem ensures that  $Z_L = (1 - \frac{1}{2}\Delta)^{-1}\xi_L$  can be enhanced.

**Theorem 3.5** (Theorem 6.12 of [CC18a]). Let  $\varrho < \frac{1}{2}$  and  $\xi_L$  is the spatial white noise on  $Q_L$  on some probability space  $(\Omega, \mathcal{F}, \mathbb{P}_{\xi})$ . Then there exists a mollification  $\{\xi_{L,\epsilon}\}_{\epsilon \in (0,1]}$  such that there exist the renormalizing constants  $c_{\epsilon}^{\mathsf{v}}, c_{\epsilon}^{\mathsf{v}} \in \mathbb{R}$  (not depending on L) and the sequence

$$\mathfrak{Z}_{\epsilon} := (Z_{L,\epsilon}, Z_{L,\epsilon}^{\mathbf{v}} - c_{L,\epsilon}^{\mathbf{v}}, Z_{L,\epsilon}^{\mathbf{v}}, Z_{L,\epsilon}^{\mathbf{v}}, Z_{L,\epsilon}^{\mathbf{v}} - c_{\epsilon}^{\mathbf{v}}, \nabla \mathcal{Q}_{L,\epsilon} \circ \nabla Z_{L,\epsilon})$$

$$(3.9)$$

which converges to a limit  $\mathfrak{Z}_L := (Z_L, Z_L^{\mathbf{Y}}, Z_L^{\mathbf{Y}}, Z_L^{\mathbf{Y}}, Z_L^{\mathbf{Y}}, \nabla \mathcal{Q}_L \circ \nabla Z_L) \in \mathscr{H}^{\varrho}$  in  $L^p(\Omega, \mathscr{H}^{\varrho})$  for every p > 1.

In order to prove Proposition 3.1, we will prove a fixed point problem by using the theory of paracontrolled distributions. This is an analogous result of [CC18a, Proposition 6.8]. At this moment, we omit the subscripts for simplicity like  $Z = Z_{L,\epsilon}$ . We rewrite (3.1) as

$$Y = \mathcal{I}_{\eta} \left( \frac{1}{2} |\nabla Z|^2 - c + \nabla Y \cdot \nabla Z + \frac{1}{2} |\nabla Y|^2 \right). \tag{3.10}$$

Now set

$$v = Y - \frac{1}{2}Z^{\mathbf{v}} - \frac{1}{2}Z^{\mathbf{v}}. (3.11)$$

Substituting (3.11) into (3.10), we observe that (3.10) is equivalent to

$$v = \frac{1}{2} Z^{\bullet} + \mathcal{I}_{\eta} (\nabla v \cdot \nabla Z) + R^{v}, \tag{3.12}$$

where  $R^v$  denotes

$$R^{v} = \frac{1}{8} Z^{\nabla} + \frac{1}{2} \mathcal{I}_{\eta} (\nabla (v + \frac{1}{2} Z^{\nabla}) \cdot \nabla Z^{\nabla}) + \frac{1}{2} \mathcal{I}_{\eta} (|\nabla (v + \frac{1}{2} Z^{\nabla})|^{2}).$$
 (3.13)

Note that v has a higher regularity  $\alpha + 1$  than Y by (3.12) and the definition of  $\mathfrak{Z}_L$  in Theorem 3.5. However, this is not sufficient for the well-definedness of  $\nabla v \cdot \nabla Z$  in (3.12) since its expected regularity is  $\alpha + \alpha - 1 < 0$ . The key idea is to introduce a new object  $v^{\#}$  with a paracontrolled term  $v' \prec \mathcal{Q}$  where v' denotes the pseudo derivative of v.

**Definition 3.6 (Paracontrolled distributions).** Let  $\alpha \in (\frac{2}{5}, \frac{1}{2})$ . Recall the notations of paraproduct from Section A. For  $Q \in \mathcal{C}^{\alpha+1}$ , we define the space of paracontrolled distributions  $\mathcal{D}_Q^{\alpha}$  as the set of  $(v, v') \in \mathcal{C}^{\alpha+1} \times \mathcal{C}_{\mathbb{R}^3}^{\alpha}$  such that

$$v^{\#} := v - (v' \prec \mathcal{Q}) \in \mathscr{C}^{4\alpha}. \tag{3.14}$$

We equip  $\mathcal{D}_{\mathcal{O}}^{\alpha}$  with the norm

$$\|(v,v')\|_{\mathcal{D}^{\alpha}_{\mathcal{O}}} := \|v\|_{3\alpha} + \|v'\|_{3\alpha-1} + \|v^{\#}\|_{\alpha+\beta+1}, \tag{3.15}$$

where  $\beta \in (0, 3\alpha - 1)$ .

By Proposition 6.6 and 6.7 of [CC18a], for  $\mathfrak{Z} \in \mathcal{Z}^{\varrho}$  we have  $(v, v') \in \mathcal{D}_{\mathcal{Q}}^{\alpha}$ ,  $\nabla \mathcal{Q} \circ \nabla Z \in \mathscr{C}^{2\alpha - 1}$  and hence,  $\nabla v \circ \nabla Z$  are well-defined. Note that v solves (3.12) if and only if  $v^{\#}$  solves

$$v^{\#} = \mathcal{I}_{\eta} \left( \nabla \left( \frac{1}{2} Z^{\nabla} + v \right) \prec \nabla z \right) - v' \prec Q + \mathcal{I}_{\eta} (\nabla v \circ \nabla Z) + R^{v}.$$
 (3.16)

Proposition 3.7 (Proposition 6.7 of [CC18a]). Suppose that

$$v' := \nabla v + \frac{1}{2} \nabla Z^{\checkmark}.$$

Then for  $\beta \in (0, 3\alpha - 1)$  and  $\epsilon > 0$ , we have

$$\|\mathcal{I}(v' \prec \nabla Z) - v' \prec Q\|_{\alpha + \beta + 1} \lesssim \eta^{-\epsilon} \|v'\|_{3\alpha - 1} \|\|Z\|_{\varrho}$$
 (3.17)

for the first coordinate of Z of  $\mathfrak{Z} \in \mathcal{Z}^{\varrho}$  with  $5/2 < \alpha < \varrho < 1/2$ .

Now we present the key proposition to obtain the contractivity of the solution map of (3.16) which is a variation of [CC18a, Proposition 6.8]

**Proposition 3.8.** Let  $2/5 < \alpha < \varrho < 1/2$  and  $\mathfrak{Z} \in \mathcal{Z}^{\varrho}$ . For  $(v, v') \in \mathcal{D}_{\mathcal{Q}}^{\alpha}$ , let  $\mathcal{G} : \mathcal{D}_{\mathcal{Q}}^{\alpha} \to \mathscr{C}^{\alpha+1} \times \mathscr{C}^{\alpha}$  be the map defined by  $\mathcal{G}(v, v') = (\tilde{v}, \tilde{v}')$  where

$$\tilde{v} := \frac{1}{2} Z^{\mathring{\mathbf{Y}}} + \mathcal{I}_{\eta} (\nabla v \cdot \nabla Z) + R^{v}, \qquad \tilde{v}' := \nabla v + \frac{1}{2} \nabla Z^{\mathring{\mathbf{Y}}}. \tag{3.18}$$

Then  $\mathcal{G}(v,v') \in \mathcal{D}_Q^{\alpha}$  and there exists  $\vartheta > 0$  such that

$$\|\mathcal{G}(v,v')\|_{\mathcal{D}^{\alpha}_{\mathcal{Q}}} \lesssim (1+\|\mathfrak{Z}\|_{\mathcal{Z}^{\varrho}})^{2}(1+\eta^{-\vartheta}\|(v,v')\|_{\mathcal{D}^{\alpha}_{\mathcal{Q}}})^{2},$$
 (3.19)

and for  $(v_1, v_1'), (v_2, v_2') \in \mathcal{D}_{\mathcal{O}}^{\alpha}$ ,

$$d_{\mathcal{D}_{\mathcal{Q}}^{\alpha}}(\mathcal{G}(v_{1}, v_{1}'), \mathcal{G}(v_{2}, v_{2}')) \\ \lesssim \eta^{-\vartheta} d_{\mathcal{D}_{\mathcal{Q}}^{\alpha}}((v_{1}, v_{1}'), (v_{2}, v_{2}')) (1 + \|(v_{1}, v_{1}')\|_{\mathcal{D}_{\mathcal{Q}}^{\alpha}} + \|(v_{2}, v_{2}')\|_{\mathcal{D}_{\mathcal{Q}}^{\alpha}}) (1 + \|\mathfrak{Z}\|_{\mathcal{Z}^{\varrho}})^{2},$$

$$(3.20)$$

where  $d_{\mathcal{D}_{\mathcal{O}}^{\alpha}}((v_1, v_1'), (v_2, v_2')) := \|(v_1 - v_2, v_1' - v_2')\|_{\mathcal{D}_{\mathcal{O}}^{\alpha}}$ .

*Proof.* The proof is very similar to the proof of [CC18a, Proposition 6.8] where the transition density operator  $\mathcal{I}$  is used instead of  $\mathcal{I}_{\eta}$ . Since  $(\eta - \frac{1}{2}\Delta)^{-1} = \int_{0}^{\infty} e^{-\eta t} P_{t} dt$ , the estimations in [CC18a, Proposition 6.8] can be applied to our setting with replacing  $T^{\vartheta}$  [CC18a, Proposition 6.8] into  $\eta^{-\vartheta}$ . We omit the detail.

We are now in the position to prove Proposition 3.1.

Proof of Proposition 3.1. Define  $M_L := \|\mathfrak{Z}_L\|_{\mathcal{Z}^{\varrho}}$  and take

$$\eta_L := A(1 + M_L)^\aleph, \tag{3.21}$$

where  $\aleph := \frac{3}{2\vartheta}$  and A > 0. Here  $\vartheta$  is the same constant which appear in Proposition 3.8. By taking A sufficiently large, for  $(v, v') \in \mathcal{D}_Q^{\alpha}$  with  $\|(v, v')\|_{\mathcal{D}_Q^{\alpha}} \leq M_L$ , we have

$$\|\mathcal{G}(v,v')\|_{\mathcal{D}^{\alpha}_{\Omega}} \leq \text{const.} \cdot A^{-2\vartheta} M_L \leq M_L$$

using Proposition 3.8. Using a similar argument with (3.20), we can conclude that for all  $\eta \geq \eta_L$ ,  $\mathcal{G}$  is a contraction mapping in  $\mathcal{D}_{\mathcal{Q}}^{\alpha}$ , hence there exists a unique solution  $v \in \mathscr{C}^{\alpha+1}$  (with a proper  $v' \in \mathscr{C}^{\alpha}$ ) to (3.12). Since we have set before  $Y = v + \frac{1}{2}(Z^{\mathbf{v}} + Z^{\mathbf{v}})$ , we get the unique solution  $Y \in \mathscr{C}^{2\alpha}$  to (3.10).

*Proof of Theorem 3.2.* By Proposition 3.1, we know that for  $\epsilon \in [0,1]$ , L > 0, there exists a unique solution  $Y_{L,\epsilon}$  to

$$(\eta_L - \frac{1}{2}\Delta)Y_{L,\epsilon} = \frac{1}{2}|\nabla Z_{L,\epsilon}|^2 - c_\epsilon + \nabla Y_{L,\epsilon} \cdot \nabla Z_{L,\epsilon} + \frac{1}{2}|\nabla Y_{L,\epsilon}|^2, \tag{3.22}$$

where we define  $Z_{L,0} := Z_L = \lim_{\epsilon \to 0} Z_{L,\epsilon}$  and  $Y_{L,0} := Y_L = \lim_{\epsilon \to 0} Y_{L,\epsilon}$ . For simplicity, we write Z for  $Z_{L,\epsilon}$  and  $Y, \xi, \eta, c$  similarly. Since  $Z - \frac{1}{2}\Delta Z = \xi$ , we have

$$\xi - c = Z + \eta Y - \frac{1}{2}\Delta(Z + Y) - \frac{1}{2}|\nabla(Z + Y)|^2,$$

or equivalently,

$$\frac{1}{2}\Delta(Z+Y) = -(\xi-c) + Z + \eta Y - \frac{1}{2}|\nabla(Z+Y)|^2. \tag{3.23}$$

On the other hand, by Ito's formula, we obtain

$$(Z+Y)(B_t) = (Z+Y)(B_0) + \frac{1}{2} \int_0^t \Delta(Z+Y)(B_s) ds + \int_0^t \nabla(Z+Y)(B_s) ds, \tag{3.24}$$

where B is a Brownian motion. By substituting (3.23) into (3.24), we have

$$\exp\left(\int_0^t [\xi(B_s) - c]ds\right) = N_t \cdot \exp\left(\int_0^t (Z + \eta Y)(B_s)ds + (Z + Y)(B_0) - (Z + Y)(B_t)\right), \quad (3.25)$$

where

$$N_t := \exp\left(\int_0^t \nabla(Z+Y)(B_s) \cdot dB_s - \frac{1}{2} \int_0^t |\nabla(Z+Y)(B_s)|^2 ds\right). \tag{3.26}$$

Notice that the right hand side of (3.25) is equal to  $N_t \cdot \mathscr{D}_{L,y}(r,t)$ . Applying Girsanov's theorem shows that the right hand side of (3.25) is equal to  $\mathbb{E}_{\mathbb{Q}_{L,\epsilon}^x}[\mathscr{D}_{L,0}(0,t)\phi(X_t)\mathbb{1}_{X_{[0,t]}\subset Q_L}]$  Therefore, (3.2) holds when  $\epsilon \in (0,1]$ .

In order to prove (3.5), we first show that  $\mathbb{Q}_{L,\epsilon}^x$  converges weakly to  $\mathbb{Q}_L^x$ . To this end, note that by Lemma 4.12, we can apply Theorem 4.5 to the martingale problem associated with (3.4). By Theorem 4.5 and Theorem 3.4, we obtain the weak convergence of  $\mathbb{Q}_{L,\epsilon}^x$  by the fact that  $\nabla(Z_{L,\epsilon} + Y_{L,\epsilon}) \to \nabla(Z_L + Y_L)$  in  $L_{\infty}$  as  $\epsilon \to \infty$ . Now the proof of (3.5) follows from that the set  $\{X[0,t] \subset Q_L\}$  is a  $\mathbb{Q}_L^x$ -continuity set, which was proven in the proof of [KPvZ20, Theorem 2.17]. This completes the proof.

#### 4. Existence of transition kernel & its estimate

The goal of this section is two-fold. We first show that the existence of transition kernel for the diffusion  $X_t$  of (3.4) of Theorem 3.2. Secondly we derive bounds on the transition kernel. This bound will later be used to derive suitable bound on the escape probability such as the probability of the event  $\{X_{[0,t]} \not\subset Q_L\}$ .

4.1. Existence of the transition kernel. We fix T > 0. Recall that X is the solution to the SDE with a distributional drift: For any  $x \in \mathbb{R}^d$ 

$$X_t = x + \int_0^t \mu(X_s)ds + B_t, \quad t \in [0, T]$$
 (4.1)

where  $\mu := \nabla(Z+Y)$  and Z,Y are defined in Section 3 (see Section 3.1). Throughout this section, we assume that  $Z \in \mathscr{C}^{\alpha}, Y \in \mathscr{C}^{2\alpha}$  where  $\alpha < 1/2$  as in the case d=3. Our goal is to show that X has a transition density  $\Gamma_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$  for all t>0. The following is the main result of this section.

**Theorem 4.1.** Suppose  $\mu = \nabla(Z+Y)$  in (4.1). Then the solution X of the SDE (4.1) admits the transition density  $\Gamma_t(x,y) = w^{\delta_y,\mu}(t,x)$  for  $(t,x,y) \in (0,T) \times \mathbb{R}^{2d}$  where  $w^{\delta_y,\mu}$  is the solution to the Cauchy problem (4.2) associated with  $\mu$  and the terminal condition  $\delta_y$ .

**Remark 4.2.** It is worthwhile to mention that Y defined via (3.1) in d=2 case belongs to  $\mathscr{C}^{\frac{3}{2}-}$ . In this occasion, [PvZ22, Proposition 2.9] has proven the existence of the heat kernel. However the results in [PvZ22] can only be applied when  $Y \in \mathscr{C}^{2\alpha}$  for some  $\alpha > 1/2$  which is not true for d=3. Therefore, Theorem 4.1 is quintessential for d=3.

In fact, we prove a more general result than Theorem 4.1. We show that any solution of (4.1) with a suitable  $\mu \in \mathcal{C}^{\beta}$  with  $\beta \in (-\frac{2}{3}, -\frac{1}{2}]$  admits a transition density. Later we will show that the suitable class for  $\mu$  is the space of *rough distributions* (see Definition 4.4) and  $\mu = \nabla(Z + Y)$  can be lifted to a rough distribution  $\mu$ .

In order to study the SDE (4.1), we consider the martingale problem associated with the following Cauchy problem on  $[0, T] \times \mathbb{R}^d$ : Define the differential operator

$$\mathcal{L}w := \frac{1}{2}\Delta + \mu \cdot \nabla$$

and the differential equation

$$\begin{cases} \partial_t w(t,x) + \mathcal{L}w(t,x) = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ w(T,x) = \phi, & x \in \mathbb{R}^d \end{cases}$$
(4.2)

where  $\phi : \mathbb{R}^d \to \mathbb{R}$  is the terminal conditions. We denote the solution to (4.2) with the terminal function  $\phi$  by  $w^{\phi}$ . We recall the definition of the martingale problem associated to (4.2).

**Definition 4.3.** Let us denote  $\Omega = C([0,T],\mathbb{R}^d)$ . A stochastic process  $X = \{X_t\}_{t \in [0,T]}$  on the probability space  $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$  endowed with the canonical filtration  $\{\mathfrak{F}_t\}_{t \geq 0}$  is said to be a solution to the SDE (4.1) if  $X_0 = x$  and it satisfies the martingale problem for  $(\mathcal{L}, x)$ : for all  $f \in C([0,T], L^{\infty}(\mathbb{R}^d))$  and all  $\phi \in C_c^{\infty}(\mathbb{R}^d)$ ,  $w = w^{\phi}$  is the solution to the Cauchy problem 4.2 and

$$\left\{ w(t, X_t) - \int_0^t f(s, X_s) ds \right\}_{t \in [0, T]}$$

is a martingale.

Before we mention the well-posedness of the martingale problem, we introduce the definition of the rough distribution ([CC18a, Definition 3.6]) of a suitable class of the drift  $\mu$  for the well-posedness.

**Definition 4.4 (Rough distribution).** Let  $\beta \in (-\frac{2}{3}, -\frac{1}{2})$ ,  $\gamma < \beta + 2$ . Set  $\mathcal{H}^{\gamma} := \mathscr{C}_{\mathbb{R}^d}^{\gamma-2} \times \mathscr{C}_{\mathbb{R}^d}^{2\gamma-3}$ . We define the space of *rough distributions* as

$$\mathscr{X}^{\gamma} = \operatorname{cl}_{\mathcal{H}^{\gamma}} \left\{ \mathcal{K}(\theta) := (\theta, \mathcal{I}(\partial_{j}\theta^{i}) \circ \theta^{j})_{i,j=1,\dots,d}, \theta \in \mathscr{C}^{\infty}_{\mathbb{R}^{d}} \right\}.$$

We denote by  $\boldsymbol{\mu} = (\mu, \mu')$  a generic element of  $\mathscr{X}^{\gamma}$  and we say that  $\boldsymbol{\mu}$  is a lift (or enhancement) of  $\mu$ .

We now present the well-posedness of the martingale problem associated to (4.2) proven in [CC18a, Theorem 1.2].

**Theorem 4.5** (Theorem 1.2 of [CC18a]). Let  $\beta \in (-\frac{2}{3}, -\frac{1}{2}]$  and  $\mu \in \mathscr{C}^{\beta}_{\mathbb{R}^d}$ . We assume that  $\mu$  can be enhanced to a rough distribution  $\mu \in \mathscr{X}^{\gamma}$  for some  $\gamma < \beta + 2$ . Then there exists a (stochastically) unique solution to the martingale problem for  $(\mathcal{L}, x)$  in the sense that there is a unique probability measure  $\mathbb{P}_x$  on  $\Omega = C([0, T], \mathbb{R}^d)$  such that the coordinate process  $X_t(\omega) = \omega(t)$  satisfies the martingale problem for  $(\mathcal{L}, x)$ . Moreover, X is a strong Markov process under  $\mathbb{P}_x$  and the  $\mathbb{P}_x$  depends (weakly) continuously on the drift  $\mu$ .

**Remark 4.6.** In [CC18a, Theorem 1.2], the continuity of  $\mathbb{P}_x$  in the drift  $\mu$  is not given. However, the proof of [CC18a, Theorem 1.2] directly implies this fact. See the proof of [CC18a, Theorem 4.3].

As in [PvZ22, Section 2], we will show that there exists a solution  $w^{\delta_y}$  to the Cauchy problem (4.2) with the terminal condition  $\delta_y$ . We will then show that the transition density  $\Gamma$  of X is defined as  $\Gamma_t(x,y) = w^{\delta_y}(t,x)$ . In order to deal with (4.2), we employ the paracontrolled distributions and extend [CC18a, Theorem 3.10] for the delta initial data. Recall that for any  $(t,x) \in [0,T] \times^d$  and function  $\psi$ ,  $\mathcal{J}(\psi)(t)$  is defined as  $\int_t^T P_{r-t}\psi(r)dr$  where  $P_t$  is the usual heat flow, i.e.,  $P_t = e^{\frac{1}{2}t\Delta}$ .

**Definition 4.7.** Let  $T>0, \ \frac{4}{3}<\alpha<\theta<\beta+2$  and  $\rho>\frac{\theta-1}{2}$ . For  $\bar{T}\in(0,T),\ p\in[1,\infty]$  and  $\delta>0$ , we define the space of *paracontrolled distributions*  $\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}$  as the set of pairs of distributions  $(w,w')\in C_{\bar{T},T}\mathscr{C}_p^{\theta}\times C_{\bar{T},T}\mathscr{C}_p^{\alpha-1}$  (see (A.6)) such that

$$w^{\#}(t) := w(t) - w'(t) \prec \mathcal{J}(\mu)(t) \in \mathscr{C}_{p}^{2\alpha - 1}$$
(4.3)

for all  $t \in (T - \bar{T}, T]$ . Here ' $\prec$ ' denotes the paraproduct which is defined in Section A. We equip  $\mathcal{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}$  with the norm

$$\|(w,w')\|_{\mathscr{D}^{\alpha,\theta,\delta,p}_{\rho,\bar{T},T}} := \|w\|_{C^{\delta}_{T,T}\mathscr{C}^{\theta}_{p}} + \|\nabla w\|_{C^{\delta}_{\rho,\bar{T},T}L^{\infty}} + \|w'\|_{C^{\delta}_{T,T}\mathscr{C}^{\alpha-1}_{p}} + \|w^{\#}\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{2\alpha-1}_{p}}. \tag{4.4}$$

and the metric  $d_{\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}$  defined as

$$d_{\mathcal{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}((w,w'),(\tilde{w},\tilde{w}')) := \|(w-\tilde{w},w'-\tilde{w}')\|_{\mathcal{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}.$$
(4.5)

Equipped with this metric, the space  $(\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p},d_{\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}})$  is complete metric space, thus it is closed.

We now introduce a solution map for the construction of a fixed point problem for (4.2).

**Definition 4.8.** Let  $\frac{4}{3} < \alpha < \theta < \gamma < \beta + 2$ ,  $p \in [1, \infty]$ ,  $\rho \in (\frac{\theta - 1}{2}, \frac{\gamma - 1}{2})$ , T > 0 and  $\mu \in \mathscr{X}^{\theta}$  be an enhancement of  $\mu$ . For  $\bar{T} \in (0, T)$ , define the map  $M_{\bar{T}} : \mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p} \to C_{\bar{T}, T} \mathscr{C}_{p}^{\alpha}$  by

$$M_{\bar{T}}(w, w')(t) := P_{T-t}\phi + \mathcal{J}(\nabla w \cdot \mu)(t), \tag{4.6}$$

for  $(u, u') \in \mathscr{D}_{\rho, \overline{T}, T}^{\alpha, \theta, \delta, p}$  and  $\phi \in \mathscr{C}_p^{\gamma}$ . We also define the map

$$\mathcal{M}_{\bar{T}}: \mathcal{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p} \to C_{\bar{T},T} \mathcal{C}_p^{\alpha} \times C_{\bar{T},T} \mathcal{C}_p^{\alpha-1}$$

$$(w,w') \mapsto (M_{\bar{T}}(w,w'), \nabla w).$$

$$(4.7)$$

The following proposition is a generalization of [CC18a, Proposition 3.9] in the sense that we have used different norms for (w, w') in (4.4) to allow a blowup at time T, which depends on  $\delta > 0$  (see (A.6)). We will then present a key estimate of the solution map for the rough terminal data which lies in  $\mathscr{C}_p^{-\epsilon}$  for some  $\epsilon > 0$ .

**Proposition 4.9.** Let 0 < T < 1,  $\frac{4}{3} < \alpha < \theta < \gamma < \beta + 2$ ,  $\rho \in (\frac{\theta - 1}{2}, \frac{\gamma - 1}{2})$  and  $\delta > 2\alpha - 1$ . Let the terminal data  $\phi \in \mathscr{C}_p^{\gamma}$ ,  $\mu \in \mathscr{C}^{\beta}$  and  $\mu \in \mathscr{X}^{\gamma}$  be an enhancement of  $\mu$ . Then, there exists  $\kappa > 0$  which depends only on  $\alpha, \theta, \rho, \gamma, p$  such that for any for  $(w, w'), (\tilde{w}, \tilde{w}') \in \mathscr{D}_{\rho, \overline{T}, T}^{\alpha, \theta, \delta, p}$  and  $\overline{T} \in (0, T)$ ,

$$\|\mathcal{M}_{\bar{T}}(w, w')\|_{\mathscr{D}^{\alpha, \theta, \delta, p}_{\rho, \bar{T}, T}} \lesssim (1 + \|\boldsymbol{\mu}\|_{\mathscr{X}^{\gamma}})^{2} (1 + \bar{T}^{\kappa}\|(w, w'))\|_{\mathscr{D}^{\alpha, \theta, \delta, p}_{\rho, \bar{T}, T}} + \|\phi\|_{\mathscr{C}^{\gamma}_{p}}, \tag{4.8}$$

and

$$d_{\mathcal{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}(\mathcal{M}_{\bar{T}}(w,w'),\mathcal{M}_{\bar{T}}(v,v')) \lesssim \bar{T}^{\kappa}(1+\|\boldsymbol{\mu}\|_{\mathcal{X}^{\gamma}})^{2}d_{\mathcal{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}((w,w'),(\tilde{w},\tilde{w}')). \tag{4.9}$$

Proof. The proof follows similar line of ideas as in [CC18a, Proposition 3.9]. The difference is that we use two more parameters  $p \in [1, \infty]$  and  $\delta > 2\alpha - 1$  than in [CC18a, Proposition 3.9]. However, with minor modifications we can get (4.8) and (4.9). Indeed, since Bony's estimate (Proposition A.2) holds for all  $p \in [1, \infty]$ , we can bound the  $\|\cdot\|_{\mathscr{C}_p^a}$ -norm as in the same way with the  $\|\cdot\|_{\mathscr{C}_{\bar{T},T}\mathscr{C}_p^a}$ -norm for any  $a \in \mathbb{R}$ . Moreover, by multiplying  $(T-t)^{\delta}$  with  $\delta > 2\alpha - 1$  on the  $\|\cdot\|_{C_{\bar{T},T}\mathscr{C}_p^a}$ -norm, we can proceed every estimation similarly as in the proof of [CC18a, Proposition 3.9].

The next lemma bounds the distance between two fixed points of the map  $\mathcal{M}_{\bar{T}}$  starting from two distinct initial data.

**Lemma 4.10.** Let 0 < T < 1,  $\frac{4}{3} < \alpha < \theta < \gamma < \beta + 2$ ,  $\rho \in (\frac{\theta-1}{2}, \frac{\gamma-1}{2})$ ,  $\epsilon > 0$  and  $\delta > \frac{2\alpha-1+\epsilon}{2}$ . Let  $\mu \in \mathscr{C}^{\beta}$  and  $\mu \in \mathscr{X}^{\gamma}$  be an enhancement of  $\mu$ . Then, there exists  $\kappa > 0$  which depends only on  $\alpha, \theta, \rho, \gamma, p$  such that for any  $\bar{T} \in (0,T)$  and  $(w,w'), (\tilde{w},\tilde{w}') \in \mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}$  being fixed points of the map  $\mathcal{M}_{\bar{T}}$  with the terminal data  $\phi, \tilde{\phi} \in \mathscr{C}_p^{\gamma}$  respectively, we have

$$d_{\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}((w,w'),(\tilde{w},\tilde{w}')) \lesssim \bar{T}^{\kappa}(1+\|\boldsymbol{\mu}\|_{\mathscr{X}^{\gamma}})^{2}d_{\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}((w,w'),(\tilde{w},\tilde{w}'))+\|\phi-\tilde{\phi}\|_{\mathscr{C}_{p}^{-\epsilon}}.$$

$$(4.10)$$

*Proof.* Note that since (w, w') is a fixed point of  $\mathcal{M}_{\bar{T}}$  associated with a terminal function  $\phi$ , we know that  $M_{\bar{T}}(w, w') = w$  and  $M_{\bar{T}}(w, w')' = \nabla w$ . Moreover,

$$w(t) = P_{T-t}\phi + \mathcal{J}(\nabla w \cdot \mu)(t) \tag{4.11}$$

holds for all  $t \in (T - \bar{T}, T]$  and the same facts hold for  $\tilde{w}$ . Since (4.11) is linear, we prove the lemma for w for simplicity. By Proposition 4.9, we have the first term on the r.h.s in (4.10). For the second term, it suffices to obtain the upper bound (4.8) for w with  $\|\phi\|_{\mathscr{C}_p^{-\epsilon}}$  instead of  $\|\phi\|_{\mathscr{C}_p^{\gamma}}$ . Using (4.11), we need to bound

$$\|(w,w')\|_{\mathscr{D}^{\alpha,\theta,\delta,p}_{\rho,\bar{T},T}} = \|w\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{\theta}_{p}} + \|\nabla w\|_{C^{\delta}_{\rho,\bar{T},T}L^{\infty}} + \|\nabla w\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{\alpha-1}_{p}} + \|w^{\#}\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{2\alpha-1}_{p}}$$
(4.12)

to extract the bound by  $\|\phi\|_{\mathscr{C}_p^{-\epsilon}}$ . The first term on the r.h.s. of the above display, i.e.,  $\|w\|_{C_{\overline{T},T}^{\delta}\mathscr{C}_p^{\theta}}$  is the upper bound by  $\|\phi\|_{\mathscr{C}_p^{-\epsilon}}$  since  $\delta > \frac{2\alpha - 1 + \epsilon}{2} > \frac{\theta + \epsilon}{2}$  and

$$||P_{T-\cdot}\phi||_{C^{\delta}_{\bar{T},T}\mathscr{C}^{\theta}_{p}} = \sup_{t \in (T-\bar{T},T]} (T-t)^{\delta} ||P_{T-t}\phi||_{\mathscr{C}^{\theta}_{p}} \lesssim ||\phi||_{\mathscr{C}^{\theta-2\delta}_{p}} \leq ||\phi||_{\mathscr{C}^{-\epsilon}_{p}}, \tag{4.13}$$

where we used Lemma A.3 in the first inequality. For  $\|\nabla w\|_{C^{\delta}_{\rho,\bar{T},T}L^{\infty}}$ , we can see that for  $t < s \in (T - \bar{T}, T]$ 

$$\|\nabla P_{T-t}\phi - \nabla P_{T-s}\phi\|_{L^{\infty}} = \|(P_{s-t} - \operatorname{Id})P_{T-s}\nabla\phi\|_{L^{\infty}}$$

$$\lesssim |t-s|^{\rho}\|P_{T-s}\nabla\phi\|_{\mathscr{C}_{p}^{2\rho}}$$

$$\lesssim |t-s|^{\rho}(T-s)^{-\delta}\|\phi\|_{\mathscr{C}_{p}^{2\rho+1-2\delta}}.$$

This implies that

$$\|\nabla P_{T-\cdot}\phi\|_{C^{\delta}_{\rho,\bar{T},T}L^{\infty}} \lesssim \|\phi\|_{\mathscr{C}^{2\rho+1-2\delta}_{p}} \leq \|\phi\|_{\mathscr{C}^{-\epsilon}_{p}},\tag{4.14}$$

since  $2\alpha - 1 > 2\rho + 1$  by the conditions on the parameters. For  $\|\nabla w\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{\alpha-1}_p}$ , it is enough to see that  $\|\nabla w\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{\alpha-1}_p} \lesssim \|w\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{\alpha}_p}$ . To bound  $\|w^{\#}\|_{C^{\delta}_{\bar{T},T}\mathscr{C}^{2\alpha-1}_p}$ , note that

$$w^{\#}(t) = w(t) - (\nabla w \prec \mathcal{J}(\mu))(t) \tag{4.15}$$

by (4.3). The only contribution of  $\phi$  appears in the first term on the r.h.s. We use Lemma A.3 to get

$$(T-t)^{\delta} \|P_{T-t}\phi\|_{\mathscr{C}_{p}^{2\alpha-1}} \lesssim \|\phi\|_{\mathscr{C}_{p}^{2\alpha-1-2\delta}} \lesssim \|\phi\|_{\mathscr{C}_{p}^{-\epsilon}}. \tag{4.16}$$

Putting all together and replacing w by  $w - \tilde{w}$ , we have

$$\|(w-\tilde{w},w'-\tilde{w}')\|_{\mathscr{D}^{\alpha,\theta,\delta,p}_{\rho,\bar{T},T}} \lesssim \bar{T}^{\kappa}(1+\|\boldsymbol{\mu}\|_{\mathscr{X}^{\gamma}})^{2}\|(w-\tilde{w},w'-\tilde{w}')\|_{\mathscr{D}^{\alpha,\theta,\delta,p}_{\rho,\bar{T},T}} + \|\phi-\tilde{\phi}\|_{\mathscr{C}^{-\epsilon}_{p}},$$

which completes the proof.

Now we are ready to present one of the main results of this section which extends [CC18a, Theorem 3.10] for a more larger class of terminal data including Dirac delta terminal data. We write  $w^{\phi}_{\mu}$  for the solution w of (4.2) with the terminal function  $\phi$  and the drift  $\mu$ .

**Theorem 4.11.** Let  $p \in [1, \infty], \beta \in (-\frac{2}{3}, -\frac{1}{2}], \frac{4}{3} < \theta < \gamma < \beta + 2 \text{ and } \epsilon, T > 0.$  For  $\phi \in \mathscr{C}_p^{-\epsilon}$  and  $\mu \in \mathscr{C}^{\beta}$  that can be enhanced to a lift  $\mu \in \mathscr{X}^{\gamma}$ , the Cauchy problem (4.2) has a unique mild solution  $w^{\phi}_{\mu}$  in  $C([0,T],\mathscr{C}^{-2\epsilon \wedge \beta})$  such that  $w^{\phi}_{\mu}(t) \in \mathscr{C}^{\alpha}$  for all  $t \in [0,T)$ . Moreover, for all t > 0 the map  $\mathscr{C}_p^{-\epsilon} \times \mathscr{C}^{\beta} \to \mathscr{C}^{\alpha}$  given by  $(\phi,\mu) \mapsto w^{\phi}_{\mu}(t,\cdot)$  is locally Lipshitz.

*Proof.* Fix  $\phi \in \mathscr{C}_p^{-\epsilon}$ . Suppose  $\{\phi_n\}_{n\in\mathbb{N}}$  be a sequence of functions in  $\mathscr{C}_p^{\gamma}$  such that

$$\|\phi_n - \phi\|_{\mathscr{C}_n^{-\epsilon}} \to 0 \tag{4.17}$$

as  $n \to \infty$ . Then by Proposition 4.10, for each n, and  $\bar{T} \in (0,T)$ , we have a unique solution  $(w_n, w'_n) \in \mathcal{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}$  of the fixed point problem of (4.7). Since  $(w_n, w'_n)$  are fixed point of the map  $\mathcal{M}_T$ , we have  $\mathcal{M}_T(u_n, u'_n) = (u_n, u'_n)$  for all n and use Lemma 4.10 to obtain

$$\|(w_n, w'_n) - (w_m, w'_m)\|_{\mathscr{D}^{\alpha, \theta, \delta, p}_{\rho, \bar{T}, T}} \lesssim \bar{T}^{\kappa} (1 + \|\boldsymbol{\mu}\|_{\mathscr{X}^{\gamma}})^2 \|(w_n, w'_n) - (w_m, w'_m)\|_{\mathscr{D}^{\alpha, \theta, \delta, p}_{\rho, \bar{T}, T}} + \|\phi_n - \phi_m\|_{\mathscr{C}^{-\epsilon}_{p}}.$$
(4.18)

We choose  $\bar{T} > 0$  (not depending on n) small such that we may rewrite the above inequality as

$$\|(w_n, w_n') - (w_m, w_m')\|_{\mathscr{D}_{\alpha, \bar{T}, T}^{\alpha, \theta, \delta, p}} \lesssim \|\phi_n - \phi_m\|_{\mathscr{C}_p^{-\epsilon}}.$$

The above display implies that  $\{(w_n, w'_n)\}_{n\in\mathbb{N}}$  is a Cauchy sequence in  $\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}$  due to (4.17). Since  $\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}$  is closed, there is a unique limit point  $(w, w') \in \mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}$  of  $\{(w_n, w'_n)\}_{n\in\mathbb{N}}$ . Since (w, w') is a limit of the set of fixed points under the map  $\mathcal{M}_{\bar{T}}$ , (w, w') is also a fixed point under of the same map. As a result, we get

$$w = M_{\bar{T}}(w, w') = \mathcal{J}(\nabla w \cdot \mu) + P_{T-}\phi$$
(4.19)

in  $C_{T,T}^{\delta}\mathscr{C}_p^{\theta}$ . We now verify that w solves this equation started from the initial data  $\phi \in \mathscr{C}_p^{-\epsilon}$ . This follows from the following estimates:

$$||P_{T-t}\phi - \phi||_{\mathscr{C}_p^{-2\epsilon}} \lesssim (T-t)^{\epsilon} ||u_0||_{\mathscr{C}_p^{-\epsilon}},$$
 (4.20)

and

$$\|\mathcal{J}(\nabla u \cdot \mu)(t)\|_{\mathscr{C}_{p}^{-2\epsilon}} \lesssim (T-t)^{\vartheta} \sup_{s \in (T-\bar{T},T]} (T-s)^{\gamma} \|\nabla w \cdot \mu\|_{\mathscr{C}^{\alpha+\beta-1}}, \quad \vartheta := \frac{\alpha+\beta-1}{2} + \epsilon - \gamma + 1, \tag{4.21}$$

for some  $\gamma \in (0,1)$ . We obtain the first inequality by applying the second Schauder's estimates from Lemma A.3. The second inequality is obtained by applying part (1) of Corollary 2.5 from [CC18a]. Recall that  $\alpha > \frac{4}{3}$ ,  $\beta \in (-\frac{2}{3}, -\frac{1}{2}]$ . We choose  $\gamma$  to be very small such that  $\vartheta > 0$ . Therefore the right hand side of the inequalities of the above display tends to 0 as t goes to  $\infty$ . This shows  $w \in C([T - \bar{T}, T], \mathscr{C}_p^{-2\epsilon}) \cap C_{\bar{T}, T}^{\delta} \mathscr{C}_p^{\theta}$  is the solution to the Cauchy problem

$$w(t) = \mathcal{J}(\nabla w \cdot \mu)(t) + P_{T-t}u_0, \quad t \in [0, T)$$

$$(4.22)$$

and  $w(T, \cdot) = \phi$ , or equivalently to (4.2).

Now we claim that for t>0 the map  $\mathscr{C}_p^{-\epsilon}\times\mathscr{X}^\gamma\to\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}$  given by  $(\phi,\mu)\mapsto (u_{\mu_1}^\phi(t,\cdot),\nabla u_{\mu_2}^\phi(t,\cdot))$  is locally Lipschitz. If we let  $\phi_1,\phi_2\in\mathscr{C}_p^{-\epsilon},\ \mu_1,\mu_2\in\mathscr{C}^\beta,\ w_i=w_{\mu_i}^{\phi_i}$  and  $w_i'=\nabla w_{\mu_i}^{\phi_i}$  be the solution for each i=1,2, then we have

$$\|(w_{1},w_{1}')-(w_{2},w_{2}')\|_{\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}} \leq \|\mathcal{M}_{\bar{T}}^{\mu_{1}}(w_{1},w_{1}')-\mathcal{M}_{\bar{T}}^{\mu_{1}}(w_{2},w_{2}')\|_{\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}} + \|\mathcal{M}_{\bar{T}}^{\mu_{1}}(w_{2},w_{2}')-\mathcal{M}_{\bar{T}}^{\mu_{2}}(w_{2},w_{2}')\|_{\mathscr{D}_{\rho,\bar{T},T}^{\alpha,\theta,\delta,p}}$$

$$(4.23)$$

Due to (4.8) of Proposition 4.9, we have

$$\|\mathcal{M}_{\bar{T}}^{\boldsymbol{\mu}_{1}}(w_{1}, w_{1}') - \mathcal{M}_{\bar{T}}^{\boldsymbol{\mu}_{1}}(w_{2}, w_{2}')\|_{\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}} \leq \|\phi_{1} - \phi_{2}\|_{\mathscr{C}_{p}^{-\epsilon}} + \bar{T}^{\kappa} (1 + \|\boldsymbol{\mu}_{1}\|_{\mathscr{X}^{\gamma}})^{2} \|(w_{1}, w_{1}') - (w_{2}, w_{2}')\|_{\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}}$$

$$(4.24)$$

Furthermore, notice that  $\mathcal{M}_{\bar{T}}^{\mu_1}(w_2, w_2') = (P_{T-} \phi_2 + \mathcal{J}(\nabla w_2 \cdot \mu_1), \nabla w_2)$  and  $\mathcal{M}_{\bar{T}}^{\mu_2}(w_2, w_2') = (P_{T-} \phi_2 + \mathcal{J}(\nabla w_2 \cdot \mu_1), \nabla w_2)$  $(P_{T-}.\phi_2 + \mathcal{J}(\nabla w_2 \cdot \boldsymbol{\mu}_2), \nabla w_2)$ . As a result, we get

$$\mathcal{M}_{\bar{T}}^{\boldsymbol{\mu}_1}(w_2, w_2') - \mathcal{M}_{\bar{T}}^{\boldsymbol{\mu}_2}(w_2, w_2') = (\mathcal{J}(\nabla w_2 \cdot (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)), 0).$$

Using Corollary 2.5 of [CC18a], we have

sing Corollary 2.5 of [CC18a], we have
$$\|(\mathcal{J}(\nabla w_2 \cdot (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)), 0)\|_{\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}} \lesssim \bar{T}^{\kappa} (1 + \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathscr{X}^{\gamma}}) \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathscr{X}^{\gamma}} (1 + \|(w_2, w_2')\|_{\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}}). \tag{4.25}$$

Applying the bounds from (4.24) and (4.25) into the right hand side of (4.23) yields

r.h.s. of (4.23) 
$$\lesssim \|\phi_1 - \phi_2\|_{\mathscr{C}_p^{-\epsilon}} + \bar{T}^{\kappa} (1 + \|\boldsymbol{\mu}_1\|_{\mathscr{X}^{\gamma}})^2 \|(w_1, w_1') - (w_2, w_2')\|_{\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}}$$
  
  $+ \bar{T}^{\kappa} (1 + \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathscr{X}^{\gamma}}) \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathscr{X}^{\gamma}} (1 + \|(w_2, w_2')\|_{\mathscr{D}_{\rho, \bar{T}, T}^{\alpha, \theta, \delta, p}})$ 

The above inequality proves  $(\phi, \mu) \mapsto u^{\phi}_{\mu}(t, \cdot)$  is locally Lipschitz. This completes the proof. 

To apply Proposition 3.8 and Theorem 4.11 to the SDE in Theorem 3.2, we need to verify that  $\nabla(Z+Y)$  can be lifted to the rough distribution which is done as follows.

**Lemma 4.12.** Let  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ . Suppose that  $\mu = \nabla(Z + Y)$  where  $Z = (1 - \frac{1}{2}\Delta)^{-1}\xi \in \mathscr{C}^{\alpha}$  and  $Y \in \mathscr{C}^{2\alpha}$ . Then  $\mu$  can be lifted a rough distribution  $\mu \in \mathscr{X}^{\gamma}$  in the sense of Definition 4.4 with  $\frac{1}{3} < \gamma < \alpha$ .

*Proof.* We need to verify that the second component of the rough distribution  $\mathcal{J}(\mu) \circ \mu$  is welldefined, i.e., can be approximated by a suitable sequence in the sense of Definition 4.4. Recalling  $Y \in \mathscr{C}^{2\alpha}$  and  $Z \in \mathscr{C}^{\alpha}$ , we know that  $\mathcal{J}(\nabla \cdot \nabla Z) \circ \nabla Y$ ,  $\mathcal{J}(\nabla \cdot \nabla Y) \circ \nabla Z$  and  $\mathcal{J}(\nabla \cdot \nabla Y) \circ \nabla Y$  are well defined since their regularities are at least  $3\alpha - 1 > 0$ . Furthermore, since  $\mathcal{J}$  and  $\nabla$  commute, we have

$$\mathcal{J}(\nabla \cdot \nabla Z) \circ \nabla Z = \nabla \mathcal{J}(\nabla Z) \circ \nabla Z.$$

Note that Theorem 3.5 implies that  $\nabla \mathcal{I}(\nabla Z) \circ \nabla Z$  is well defined. Since  $\mathcal{I}$  and  $\mathcal{J}$  have no significant difference on their estimation, Theorem 3.5 ensures that the above term is well defined. Therefore, we can conclude that  $\mathcal{J}(\mu) \circ \mu$  given by

$$\mathcal{J}(\nabla \cdot \nabla Z) \circ \nabla Z + \mathcal{J}(\nabla \cdot \nabla Z) \circ \nabla Y + \mathcal{J}(\nabla \cdot \nabla Y) \circ \nabla Z + \mathcal{J}(\nabla \cdot \nabla Y) \circ \nabla Y$$

is well defined. 

Since  $(\phi, \mu) \mapsto u_{\mu}^{\phi}(t, \cdot)$  is locally Lipschitz by Theorem 4.11, we can get the following proposition.

**Proposition 4.13.** Let  $\frac{4}{3} < \alpha < \gamma < \beta + 2$  with  $\beta \in (-\frac{2}{3}, -\frac{1}{2}]$ . Let  $\mu \in \mathscr{X}^{\gamma}$  and  $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \mathscr{C}^{\infty} \times \mathscr{C}^{\infty}_{\mathbb{R}^d}$  be a sequence which converges to  $\mu \in \mathscr{X}^{\gamma}$  in  $\mathscr{H}^{\gamma}$  (see Definition 4.4). Let  $\Gamma_t(x, y) = w^{\delta_y, \mu}(t, x)$  and  $\Gamma_t^{(n)}(x, y) = w^{\delta_y, \mu_n}(t, x)$ . Then  $\Gamma_t(x, y)$  and  $\Gamma_t^{(n)}(x, y)$  are continuous on  $\mathbb{R}^d \times \mathbb{R}^d$  and for all  $t \in [0,T)$ , we have

$$\sup_{x,y\in\mathbb{R}^d} |\Gamma_t(x,y) - \Gamma_t^{(n)}(x,y)| \to 0, \tag{4.26}$$

as  $n \to \infty$ .

Now we are in the position to prove Theorem 4.1, which is the main result of this section.

**Proof of Theorem 4.1.** Lemma 4.12 shows that  $\mu = \nabla(Z+Y) \in \mathscr{C}^{\alpha}$  can be lifted to a rough distribution  $\mu \in \mathcal{X}^{\gamma}$ . Thus, we can apply Theorem 4.11 to get the unique solution  $w^{\delta_y,\mu}$  for  $y \in \mathbb{R}^d$ since  $\delta_y \in \mathscr{C}_p^{-d(1-1/p)}$  for any  $p \in [1,\infty]$ . Since  $\mu$  is a rough distribution, we can take a sequence  $\mu_n := (\mu_n, \mu'_n) \in \mathscr{C}^\infty \times \mathscr{C}_{\mathbb{R}^d}^\infty$  for  $n \in \mathbb{N}$ , which converges to  $\mu = (\mu, \mu')$  in  $\mathscr{H}^\gamma$  for  $\alpha < \gamma < \beta + 2$  as in Proposition 4.13. It is known that for each  $n \in \mathbb{N}$ , there exists the transition density  $\Gamma_t^{(n)}(x,y) = w^{\delta_y, \boldsymbol{\mu}_n}(t,x)$  for all  $(t,x,y) \in (0,T) \times \mathbb{R}^{2d}$  (see for instance [Fri10, Theorem 6.5.4]). Let

 $\mathbb{P}_x^{(n)}$  be the unique probability measure on  $C([t,T],\mathbb{R}^d)$  such that the coordinate process  $X^{(n)}$  is the solution to the martingale problem for  $(\mathcal{L}^{(n)},x)$  where  $\mathcal{L}^{(n)}:=\frac{1}{2}\Delta+\mu_n\cdot\nabla$ . By Theorem 4.5, we know that  $\mathbb{P}_x^{(n)}$  weakly converges to  $\mathbb{P}_x$  that appears in Theorem 4.5. Suppose  $\Gamma_t(\cdot,\cdot)$  is the transition kernel of  $\mathbb{P}_x$ . Therefore, with Proposition 4.13, we can see that for all  $\psi\in C_c(\mathbb{R}^d)$ 

$$\mathbb{E}_x[\psi(X_T)] = \lim_{n \to \infty} \mathbb{E}_x[\psi(X_T^{(n)})] = \lim_{n \to \infty} \int_{\mathbb{R}^d} \psi(y) \Gamma_t^{(n)}(x, y) dy = \int_{\mathbb{R}^d} \psi(y) \Gamma_t(x, y) dy.$$

This shows  $\Gamma_t(x,y) = w^{\delta_y,\mu}(t,x)$  and hence, completes the proof.

4.2. **Transition kernel estimate.** We now provide the upper and lower bounds for the estimate of  $\Gamma_t(x,y) := w^{\delta_y,\mu}(t,x)$ . We first consider  $\mu$  is the form of  $\nabla U$  where U denotes a smooth and bounded function. Note that Proposition 4.13 allows us to extend the estimates to the case  $\mu = \nabla(Z+Y)$  as in Section 4.1.

**Theorem 4.14.** Suppose that  $\mu$  is the form of  $\nabla U$  where  $U : \mathbb{R}^d \to \mathbb{R}^d$  is a smooth and bounded function. Then, for the transition density  $\Gamma_t(x,y)$  of the solution X to the SDE (4.1), there exist constants  $C_1, C_2, C_3 > 0$  such that for all  $t > 0, x, y \in \mathbb{R}^d$ , we have

$$\Gamma_t(x,y) \le \frac{1}{t^{d/2}} \exp\left(C_1 \|U\|_{\infty} - \frac{|y-x|^2}{4t}\right),$$
(4.27)

$$\Gamma_t(x,y) \ge \frac{1}{t^{d/2}} \exp\left(-C_2 \|U\|_{\infty} - \frac{C_3 \|U\|_{\infty} |y-x|^2}{t}\right).$$
 (4.28)

In addition, the same estimates hold when  $U = Z_L^y + Y_L^y$  where  $Z_L^y$  and  $Y_L^y$  are defined in Section 3.

*Proof.* The proof is based on ideas from [Str08, Section 4.3]. It was shown in (4.3.6) of [Str08] that if  $\mu$  is of the form  $\nabla U$  for some  $U \in C^{\infty}(d)$ , then

$$\Gamma_t(x,y) \le \frac{K_d(U)}{t^{d/2}} \exp\left(\frac{|x-y|^2}{4t}\right).$$

where  $K_d(U)$  is a constant which is upper bounded by  $\kappa_d \exp((d+4)\delta(U)/4)$  and  $\delta(U) := \max_{x \in d} U(x) - \min_{x \in d} U(x)$ . Note that  $\delta(U) \leq 2\|U\|_{\infty}$ . Thus the upper bound (4.27) follows. We use [Str08, Lemma 4.3.8] to show the lower bound. It is shown in the paragraph before Lemma 4.3.8 of [Str08] that there exists  $C_2$  such that

$$\Gamma_t(x,y) \ge \frac{2^{d/2}}{t^{d/2}} \exp(-C_2 ||U||_{\infty}), \quad \text{whenever} |x-y| \le \sqrt{t}.$$

This bound is further used in Lemma 4.3.8 of [Str08] to show that

$$\Gamma_t(x,y) \ge \frac{2^{d/2}}{t^{d/2}} \exp\left(-C_2 \|U\|_{\infty} - \frac{C_3 \|U\|_{\infty} |y-x|^2}{t}\right)$$

for some constant  $C_3 > 0$ . From this bound, (4.28) follows.

We also obtain the estimate for the escape probability of X.

**Corollary 4.15.** Suppose that  $\mu$  is the form of  $\nabla U$  where  $U : \mathbb{R}^d \to \mathbb{R}^d$  is a smooth and bounded function. Then, for the solution X to the SDE (4.1),  $x \in \mathbb{R}^d$ , K > 0, and  $T \ge 1$ ,

$$\mathbb{P}\left(\sup_{t\in[0,T]}|X_t - x| \ge K\right) \le C \exp\left(CT\|U\|_{\infty} - \frac{K^2}{CT}\right). \tag{4.29}$$

Furthermore, this same upper bound holds for  $U = Z_L^y + Y_L^y$  where  $Z_L^y$  and  $Y_L^y$  are defined in Section 3.

*Proof.* Note that from (4.27), we know that  $\Gamma_t(x,y) \lesssim p_t(x,y)e^{C_1||U||_{\infty}}$  where  $p_t$  denotes the usual heat kernel density in d-dimension. Then the proof follows from the same line with the proof of [PvZ22, Corollary 5.2].

#### 5. Asymptotic bounds for the PAM

This section is devoted to estimating the asymptotic bounds for the solution u to the PAM (1.1). We aim to showcase two main things in this section. We first derive bounds on the moments of the enhanced noise (see Definition 3.8). We use these estimates to find bounds on the Feynman-Kac formula found in Section 3. We first recall few facts about the white noise.

**Definition 5.1.** On a probability space  $(\Omega, \mathbb{P})$ , we define a (spatial) white noise on  $\mathbb{R}^d$  by a random variable  $\xi : \Omega \to \mathscr{S}'(\mathbb{R}^d)$  such that for all  $f \in \mathscr{S}(\mathbb{R}^d)$ ,  $\langle \xi, f \rangle$  is a centered Gaussian random variable with covariance  $\mathbb{E}[\langle \xi, f \rangle \langle \xi, g \rangle] = \langle f, g \rangle$  for all  $f, g \in \mathscr{S}(\mathbb{R}^d)$ .

Note that since  $f \mapsto \langle \xi, f \rangle$  is linear and  $\|\langle \xi, f \rangle\|_{L^2(\Omega)} = \|f\|_{L^2(\mathbb{R}^d)}$ , we can extend  $\xi$  to a bounded linear operator  $W: L^2(\mathbb{R}^d) \to L^2(\Omega, \mathbb{P})$  such that W(f) is also a centered Gaussian random variable with  $\mathbb{E}[W(f)W(g)] = \langle f, g \rangle$  for all  $f, g \in L^2(\mathbb{R}^d)$ . We will abuse the notation using  $\langle \xi, f \rangle$  for W(f) when  $f \in L^2(\mathbb{R}^d)$ . Now we define a radially symmetric and nonnegative function  $\psi \in C_c^{\infty}(\mathbb{R}^d)$  such that  $\int_{\mathbb{R}^d} \psi(x) dx = 1$ . We define  $\psi_{\epsilon}(x) = \frac{1}{\epsilon^d} \psi(\frac{x}{\epsilon})$ . Let  $\xi_{\epsilon} = \psi_{\epsilon} * \xi$  be the mollification of  $\xi$ . Recall that the mollification of the noise on the box  $[-\frac{L}{2}, \frac{L}{2}]^d$  by  $\xi_{L,\epsilon}$  given as

$$\xi_{L,\epsilon} = \sum_{k \in \mathbb{N}^d} \tau(\frac{\epsilon}{L} k) \langle \xi, \mathfrak{n}_{k,L} \rangle \mathfrak{n}_{k,L}, \tag{5.1}$$

where  $\tau \in C_c^{\infty}(\mathbb{R}^d, [0, 1])$  is an even function such that  $\tau(x) = 1$  for  $|x| \leq \frac{1}{2}$  and  $\{\mathfrak{n}_{k,L}\}_{k \in \mathbb{N}^d}$  is the Neumann basis for the  $Q_L = [-\frac{L}{2}, \frac{L}{2}]^d$  as shown in Section 2. We also define  $\xi_{L,y}^{\epsilon}$  on  $Q_L^y := y + Q_L$  for  $y \in \mathbb{R}^d$  by a shifted white noise

$$\xi_{L,\epsilon}^{y} := \mathcal{T}_{y} \left( \sum_{k \in \mathbb{N}^{d}} \tau(\frac{\epsilon}{L} k) \langle \xi, \mathcal{T}_{y} \mathfrak{n}_{k,L} \rangle \mathfrak{n}_{k,L} \right)$$
(5.2)

where  $\mathcal{T}_y f(\cdot) = f(\cdot + y)$  for all  $f \in C(\mathbb{R}^d)$ . Define  $\sigma_{\eta}(x) = \frac{1}{\eta + \pi^2 |x|^2}$  for all  $x \in \mathbb{R}^d$ . Then, for  $u \in \mathscr{S}'_{\mathfrak{n}}(Q_L)$ ,

$$(\eta - \Delta)^{-1}u =: \sigma_{\eta}(D)u := \sum_{k \in \mathbb{N}^d} \sigma_{\eta}(\frac{k}{L}) \langle u, \mathfrak{n}_{k,L} \rangle \mathfrak{n}_{k,L}, \tag{5.3}$$

We write  $\sigma(D) := \sigma_1(D)$ . For any  $\delta > 0$ , it is known that almost surely  $\xi_{L,\epsilon}^y \to \xi_L^y$  in  $\mathscr{C}^{-d/2-\delta}$  for all  $y \in \mathbb{R}^d$  from [CvZ21, Theorem 6.7].

5.1. Bound on the enhanced noise. From the Feynman-Kac representation (see (3.2) and (3.3) in Theorem 3.2), we need to bound the terms Z, Y and  $\eta$  therein in order to estimate the solution. From the view of Proposition 3.1 and Definition 3.4, these terms are bounded in terms of enhanced white noise.

Recall the following notations from the previous sections: For  $L \in (1, \infty]$  and  $\epsilon \in [0, 1]$ , we denoted  $\xi_{L,\epsilon}$  to be the mollification of the spatial white noise in d-dimension restricted on  $Q_L$  with  $Q_{\infty} := \mathbb{R}^d$ . Set  $Z_{L,\epsilon} = (1 - \frac{1}{2}\Delta)^{-1}\xi_{L,\epsilon}$ . Let  $\eta_{L,\epsilon} > 0$  be  $\eta_0$  in Proposition 3.1 which will be specified later and  $Y_{L,\epsilon}$  be the solution to (3.1) associated with  $Z_{L,\epsilon}$  and  $\eta = \eta_{L,\epsilon}$ .  $\mathfrak{Z}_{L,\epsilon}$  is defined by Theorem 3.5 with  $\xi_{L,\epsilon}$  instead of  $\xi_{\epsilon}$ .

The following proposition says that  $\mathfrak{Z}_{L,\epsilon}$  is bounded by the logarithm of the length L, which enables us to bound  $Z_{L,\epsilon}$ ,  $Y_{L,\epsilon}$ , and  $\eta_{L,\epsilon}$ . The following result is for d=3. We refer to [KPvZ20, Lemma 6.15] for d=2 case.

**Proposition 5.2.** Let  $\frac{2}{5} < \varrho < \frac{1}{2}$ . Let  $\epsilon \in [0,1]$  and define

$$\mathfrak{a}_{\epsilon} := \max \left\{ 1, \sup_{L > e, L \in \mathbb{N}} \frac{\|\mathfrak{Z}_{L,\epsilon}\|_{\mathcal{Z}^{\varrho}}}{(\log L)^2} \right\}. \tag{5.4}$$

Then  $\mathfrak{a}_{\epsilon}$  is almost surely finite. Moreover, there exists  $h_0 > 0$  such that for all  $h \in [0, h_0]$  we have  $\sup_{\epsilon \in [0,1]} \mathbb{E}[e^{h\sqrt{\mathfrak{a}_{\epsilon}}}] < \infty$ .

*Proof.* Before proceeding to the main body of the proof, let us briefly explain the main idea. Below, we first show how to express  $\mathfrak{a}_{\epsilon}$  with the help of inhomogeneous terms (of bounded order) from the Wiener chaos expansion of the white noise. Due to the hyper-contractivity of the Wiener chaos (see [Nua06]), for every random variable X in the k-th inhomogeneous Wiener chaos generated by the white noise and for any p > 2 we have

$$\mathbb{E}[|X|^p] \le C_{k,p} \mathbb{E}[|X|^2]^{p/2},\tag{5.5}$$

where  $C_{k,p} = (p-1)^{pk/2}$ . This allows us to bound higher moments  $\mathbb{E}[(\sqrt{\mathfrak{a}_{\epsilon}})^p]$  by some constant multiple time  $\mathbb{E}[\mathfrak{a}_{\epsilon}]^p$  which finally leads to the desired results. Details of this argument is given as follows. Recall that  $\Xi_{L,\epsilon} := \{Z_{L,\epsilon}, Z_{L,\epsilon}^{\mathbf{v}} - c_{\epsilon}^{\mathbf{v}}, Z_{L,\epsilon}^{\mathbf{v}}, Z_{L,\epsilon}^{\mathbf{v}}, Z_{L,\epsilon}^{\mathbf{v}} - c_{\epsilon}^{\mathbf{v}}, \nabla Q_{L,\epsilon} \circ \nabla Z_{L,\epsilon}\}$ . We know that

$$\|\mathfrak{Z}_{L,\epsilon}\|_{\mathcal{Z}^{\varrho}} = \sum_{\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}} \|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\alpha}} \tag{5.6}$$

where  $\alpha > 0$  depends on each  $\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}$ . We bound each term of the right hand side of the above identity to derive the desired bound. Let  $\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}$ . Observe that  $\zeta_{L,\epsilon}$  is in the k-th Wiener chaos where  $k = k(\zeta)$  denotes the number of the occurrences of the noise  $\xi_{L,\epsilon}$  in  $\zeta_{L,\epsilon}$ . For instance, k(Z) = 1,  $k(Z^{\overset{\checkmark}{\nu}}) = 3$ , and  $k(Z^{\overset{\checkmark}{\nu}}) = 4$ . Note also that  $\max_{\zeta} k(\zeta) = 4$ . At this moment, we assume that for every  $\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}$ , there exist some constant  $a \in \mathbb{R}$  and  $C_0 > 0$  (not depending on  $L,\epsilon$ ) such that all  $i \in \mathbb{N}_{-1}, x \in [-L/2, L/2]^d$ , we have

$$\mathbb{E}[|\Delta_i \zeta_{L,\epsilon}(x)|^2] \le C_0 2^{ai}. \tag{5.7}$$

We prove these bounds towards the end of the proof of this proposition. Set  $C_{\kappa} = \sum_{i=-1}^{\infty} 2^{-\kappa i}$ . Observe that we can use the definition of the Besov space (see Section A) and (5.5) to obtain

$$\mathbb{E}\Big[\|\zeta_{L,\epsilon}\|_{B^{n,-\frac{a}{2}-\kappa}_{p,p}}^p\Big] = \sum_{i=-1}^{\infty} 2^{(-\frac{a}{2}-\kappa)pi} \mathbb{E}\Big[\|\Delta_i \zeta_{L,\epsilon}\|_{L^p}^p\Big] \le C_0^{\frac{p}{2}} p^{\frac{pk}{2}} L^d \Big(\sum_{i=-1}^{\infty} 2^{-p\kappa i}\Big) \le C_{\kappa} (C_0 p^k)^{\frac{p}{2}} L^d.$$

By the embedding property of the Besov space, there exists C>0 such that  $\|\cdot\|_{\mathscr{C}^{-\frac{a}{2}-\kappa-\frac{2}{p}}}\leq C\|\cdot\|_{B^{n,-\frac{a}{2}-\kappa}_{p,p}}$ . Combining this with the observation in the above display implies that for all  $\kappa>0$  there exists C>0 independent of  $\zeta$  such that for all  $p\geq 1$ ,

$$\mathbb{E}[\|\zeta_{L,\epsilon}\|_{\omega^{-\frac{3}{2}-\kappa-\frac{2}{2}}}^p] \le C_{\kappa}C^p L^d p^{\frac{pk}{2}}.$$
(5.8)

Choose  $p_0 \in \mathbb{N}$  large so that  $\frac{2}{p_0} < \kappa$  and  $\frac{2p_0}{k} > 2$ . Then there exists  $C_k > 0$  such that for  $p \geq p_0$ 

$$\mathbb{E}[\|\zeta_{L,\epsilon}\|_{\mathscr{L}^{-\frac{3}{2}-2\kappa}}^{\frac{2p}{k}}] \le C_{\kappa} C_k^p L^d p^p. \tag{5.9}$$

For  $h \geq 0$ , by the above upper bounds, we have

$$\mathbb{E}[\exp(h\|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\frac{n}{2}-2\kappa}}^{\frac{2}{k}})] = \sum_{n=0}^{\infty} \frac{h^n}{n!} \mathbb{E}[\|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\frac{n}{2}-2\kappa}}^{\frac{2n}{k}}] \leq \sum_{n=0}^{p_0} \frac{h^n}{n!} \mathbb{E}[\|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\frac{n}{2}-2\kappa}}^{\frac{2p_0}{k}}]^{\frac{n}{kp_0}} + \sum_{n=p_0}^{\infty} \frac{h^n}{n!} \mathbb{E}[\|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\frac{n}{2}-2\kappa}}^{\frac{2n}{k}}]$$

$$\leq \sum_{n=0}^{p_0-1} \frac{h^n}{n!} (C_{\kappa} C^{2p_0} L^d (2p_0)^{p_0 k})^{\frac{n}{kp_0}} + \sum_{n=p_0}^{\infty} \frac{h^n}{n!} C_{\kappa} C_k^n L^d n^n$$

$$\leq C_{\kappa}^k L^{\frac{d}{k}} \exp(2hC^{\frac{2}{k}}p_0) + C_{\kappa} L^d \sum_{n=p_0}^{\infty} (heC_k)^n,$$

where we have used Jensen's inequality and  $\frac{1}{n!} \leq (\frac{e}{n})^n$ . Then, there exists  $h_0$  such that for all  $h \in [0, h_0]$  we have

$$\mathbb{E}[\exp(h\|\zeta_{L,\epsilon}\|_{\mathscr{L}^{-\frac{a}{2}-2\kappa}}^{\frac{2}{k}})] \le AL^{\frac{2}{k}\vee d}$$

for some A>0 which is independent of k. We can choose  $b>\frac{d}{k}\vee d+1$  such that

$$\sum_{L \in \mathbb{N}} L^{-b} \mathbb{E}[\exp(h_0 \| \zeta_{L,\epsilon} \|_{\mathscr{C}^{-\frac{\alpha}{2} - 2\kappa}}^{\frac{2}{k}})] < \infty.$$

This implies that for  $\tilde{\zeta}_{\epsilon} := \sum_{L \in \mathbb{N}} L^{-b} \exp(h_0 \| \zeta_{L,\epsilon} \|_{\mathscr{C}^{-\frac{a}{2}-2\kappa}}^{\frac{2}{k}})$ , we have

$$\frac{\|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\frac{a}{2}-2\kappa}}^{\frac{2}{k}}}{\log L} \le \frac{1}{h_0} (b + \log \tilde{\zeta}_{\epsilon})$$

for all  $L \in \mathbb{N}$  with L > e. We can rewrite this as

$$\|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\frac{\alpha}{2}-2\kappa}} \le \mathfrak{a}_{\zeta,\epsilon}(\log L)^{\frac{k}{2}},\tag{5.10}$$

where  $\mathfrak{a}_{\zeta,\epsilon} := (\frac{b + \log \tilde{\zeta}_{\epsilon}}{h_0})^2$  since  $k(\zeta) \leq 4$ . Moreover, Jensen's inequality implies that

$$\sup_{\epsilon \in [0,1]} \mathbb{E}[e^{h\sqrt{\mathfrak{a}_{\zeta,\epsilon}}}] \le e^{\frac{hb}{h_0}} \sup_{\epsilon \in [0,1]} \mathbb{E}[\tilde{\zeta}_{\epsilon}]^{\frac{h}{h_0}} < \infty \tag{5.11}$$

for all  $h \in [0, h_0]$ . From (5.10), we have

$$\|\mathfrak{Z}_{L,\epsilon}\|_{\mathcal{Z}^{\varrho}} = \sum_{\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}} \|\zeta_{L,\epsilon}\|_{\mathscr{C}^{-\frac{\alpha}{2}-2\kappa}} \le (\log L)^2 \sum_{\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}} \mathfrak{a}_{\zeta,\epsilon} =: (\log L)^2 \tilde{\mathfrak{a}}_{\epsilon}, \tag{5.12}$$

where  $\tilde{\mathfrak{a}}_{\epsilon} := \sum_{\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}} \mathfrak{a}_{\zeta,\epsilon}$ . Since  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$  for  $x,y \geq 0$ , we can take  $\tilde{h}_0 > 0$  such that for all  $\tilde{h} \in [0,\tilde{h_0}]$ 

$$\sup_{\epsilon \in [0,1]} \mathbb{E}[e^{\tilde{h}\sqrt{\tilde{\mathfrak{a}}_{\epsilon}}}] \leq \sup_{\epsilon \in [0,1]} \mathbb{E}\Big[\exp\Big(\tilde{h}\sum_{\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}} \sqrt{\mathfrak{a}_{\zeta,\epsilon}}\Big)\Big] \leq \sup_{\epsilon \in [0,1]} \prod_{\zeta_{L,\epsilon} \in \Xi_{L,\epsilon}} (\mathbb{E}[e^{5\tilde{h}_0\sqrt{\mathfrak{a}_{\zeta,\epsilon}}}])^{\frac{1}{5}} < \infty, \quad (5.13)$$

where we used Hölder's inequality and (5.11). Since  $\mathfrak{a}_{\epsilon}$  in (5.4) satisfies  $\mathfrak{a}_{\epsilon} \leq \tilde{\mathfrak{a}}_{\epsilon}$ , we get the desired result. Now it remains to show (5.7). In particular, we can show that for any fixed  $\delta \in (0,1)$ ,

$$\mathbb{E}[|\Delta_i Z_{L,\epsilon}(x)|^2] \lesssim 2^{(-1+\delta)i}, \quad \mathbb{E}[|\Delta_i (Z_{L,\epsilon}^{\mathbf{v}}(x) - c_{\epsilon}^{\mathbf{v}})|^2] \lesssim 2^{(-2+\delta)i}, \quad \mathbb{E}[|\Delta_i Z_{L,\epsilon}^{\mathbf{v}}(x)|^2] \lesssim 2^{(-3+\delta)i}$$

$$\mathbb{E}[|\Delta_i Z_{L,\epsilon}^{\mathbf{v}}(x)|^2] \lesssim 2^{(-3+\delta)i}, \quad \mathbb{E}[|\Delta_i (Z_{L,\epsilon}^{\mathbf{v}}(x) - c_{\epsilon}^{\mathbf{v}})|^2] \lesssim 2^{(-4+\delta)i}, \quad \mathbb{E}[|\Delta_i (\nabla Q_{L,\epsilon} \circ \nabla Z_{L,\epsilon})(x)|^2] \lesssim 2^{\delta i}.$$

Indeed, these estimates were shown for the parabolic case on the torus in the proof of [CC18a, Theorem 6.12]. With a similar argument as in the proof of Proposition 3.8, we can obtain the same estimates for the  $L_2$  bound of the Wiener-chaos components for the elements in  $\Xi_{L,\epsilon}$  with minor modifications. We demonstrate below the bound on  $\mathbb{E}[|\Delta_i Z_{L,\epsilon}(x)|^2]$ . Other cases follow by similar arguments.

Recall from Appendix A that for  $f \in \mathcal{S}'$  and for any  $\delta > 0$ ,

$$\Delta_i f = \sum_{k \in \mathbb{N}_0^d} \langle f, \mathfrak{n}_{k,L} \rangle \varrho_i \left(\frac{k}{L}\right) \mathfrak{n}_{k,L}, \quad \text{where} \quad \varrho_i(x) \lesssim \left(\frac{2^i}{1+|x|}\right)^{\frac{3+3\delta}{2}}.$$

Since  $\varrho_i(x) = 0$  if  $|x| \leq 2^i$ , we have

$$\sigma\left(\frac{k}{L}\right)\varrho_i\left(\frac{k}{L}\right) \lesssim 2^{-2i}\varrho_i\left(\frac{k}{L}\right) \lesssim 2^{\frac{-1+3\delta}{2}i} \left(\frac{1}{1+\left|\frac{k}{L}\right|}\right)^{\frac{3+3\delta}{2}},$$

where  $\sigma(x) = (1 + \pi |x|^2)^{-1}$ . Using similar computation as in [KPvZ20, Lemma 6.11], one gets  $\mathbb{E}[\langle \xi_{L,\epsilon}, \mathfrak{n}_{k,L} \rangle \langle \xi_{L,\epsilon}, \mathfrak{n}_{l,L} \rangle] \leq \prod_{i=1}^{3} (\epsilon \wedge ((k_i + \ell_i) \vee 1)^{-1})$ . Since  $\|\mathfrak{n}_{k,L}\|_{L^{\infty}} \lesssim L^{-3/2}$  and noting that  $Z_{L,\epsilon}(x) = \mathcal{I}(\xi_{L,\epsilon})(x) = (1 - \frac{1}{2}\Delta)^{-1}\xi_{L,\epsilon}(x)$ , we have

$$2^{-(-1+3\delta)i} \mathbb{E}[|\Delta_{i} Z_{L,\epsilon}(x)|^{2}] \lesssim L^{-3} \sum_{k,l \in \mathbb{N}_{0}^{3}} \frac{1}{(1+|\frac{k}{L}|)^{\frac{3+3\delta}{2}}} \frac{1}{(1+|\frac{l}{L}|)^{\frac{3+3\delta}{2}}} |\mathbb{E}[\langle \xi_{L,\epsilon}, \mathfrak{n}_{k,L} \rangle \langle \xi_{L,\epsilon}, \mathfrak{n}_{l,L} \rangle]|$$

$$\lesssim \left( \sum_{k,l \in \frac{1}{L} \mathbb{N}_{0}} L^{-3} \frac{1}{(1+k)^{\frac{1+\delta}{2}}} \frac{1}{(1+l)^{\frac{1+\delta}{2}}} \frac{1}{(k+l) \vee 1} \right)^{3}$$

$$\lesssim \left( \sum_{k \in \frac{1}{L} \mathbb{N}_{0}} L^{-1} \frac{1}{(1+k)^{1+\frac{\delta}{2}}} \right)^{9} \leq C,$$

where C is an universal constant. The second inequality is obtained by symmetrizing the sum. The above display shows the bound on  $\mathbb{E}[|\Delta_i Z_{L,\epsilon}(x)|^2]$ . This completes the proof.

Using Proposition 5.2, we can obtain the following bounds on  $Z_{L,\epsilon}$ ,  $Y_{L,\epsilon}$ , and  $\eta_{L,\epsilon}$ .

**Proposition 5.3.** Let  $\frac{2}{5} < \alpha < \frac{1}{2}$  and  $\aleph > 0$  be the constant in (3.21). Then for any  $\epsilon \in [0,1]$ , we have

$$||Z_{L,\epsilon}||_{\mathscr{C}^{\alpha}} \leq \mathfrak{a}_{\epsilon}(\log L)^{2}, \quad ||Y_{L,\epsilon}||_{\mathscr{C}^{2\alpha}} \leq 2\mathfrak{a}_{\epsilon}(\log L)^{2}, \quad \eta_{L,\epsilon} \leq C\mathfrak{a}_{\epsilon}^{\aleph}(\log L)^{2\aleph}$$
where  $\mathfrak{a}_{\epsilon}$  is defined in Proposition 5.2. (5.14)

Proof. From the definition of  $\mathfrak{a}_{\epsilon}$ , we know  $\|Z_{L,\epsilon}\|_{\mathscr{C}^{\alpha}} \leq \mathfrak{a}_{\epsilon}(\log L)^2$ . Furthermore, it also clear that if we take  $\eta_{L,\epsilon} := C\|\mathfrak{Z}\|_{\mathscr{Z}^{\varrho}}^v$ , then  $\eta_{L,\epsilon} \leq C^{\aleph}\mathfrak{a}_{\epsilon}^{\aleph}(\log L)^{2\aleph}$ . We are left to show the bound on  $\|Y_{L,\epsilon}\|_{\mathscr{C}^{2\alpha}}$ . Recall from Section 3 that  $Y_{L,\epsilon} = v + \frac{1}{2}(Z_{L,\epsilon}^{\mathsf{v}} + Z_{L,\epsilon}^{\mathsf{v}})$  where v is the fixed point solution of the equation (3.12). Recall the map  $\mathcal{G}: \mathcal{D}_{\mathcal{Q}}^{\alpha} \to \mathscr{C}^{\alpha+1} \times \mathscr{C}^{\alpha}$  from Proposition 3.8. Note that (v,v') is the fixed point of the map  $\mathcal{G}$ . By choosing  $\eta$  large in (3.19) of Proposition 3.8, it follows that  $\|v\|_{\mathscr{C}^{\alpha+1}} \leq C\|\mathfrak{Z}\|_{\mathscr{L}^{\varrho}}$  for some C > 0. Since  $Y_{L,\epsilon} = v + \frac{1}{2}(Z_{L,\epsilon}^{\mathsf{v}} + Z_{L,\epsilon}^{\mathsf{v}})$ , we have for  $\alpha < \varrho < \frac{1}{2}$ 

$$||Y_{L,\epsilon}||_{\mathscr{C}^{2\alpha}} \le (C+1)||\mathfrak{Z}||_{\mathcal{Z}^{\varrho}} \le (C+1)\mathfrak{a}_{\epsilon}(\log L)^2.$$

This completes the proof.

5.2. Asymptotics of PAM started from constant initial data. In this section, we reveal how the solution of the parabolic Anderson model started from constant initial data is related to the largest point of the spectrum of Anderson Hamiltonian. We achieve this goal in Lemma 5.7 and 5.8. It is worthwhile to note that similar claims have been proved for d = 2 case in [KPvZ20]. Showing those results for d = 3 needs new estimates which are provided in Lemma 5.4 and Proposition 5.5.

**Lemma 5.4.** Recall  $\mathbb{Q}_{L,\epsilon}^{x,y}$  and the diffusion X from Section 2. Furthermore, recall  $\mathscr{D}_{L,\epsilon}^{y}$  defined in terms of  $Z_{L,\epsilon}^{y}$ ,  $Y_{L,\epsilon}^{y}$  and  $\eta$  from Section 4. Then, we have

$$\mathbb{Q}_{L,\epsilon}^{x,y}(X[0,t] \not\subset Q_L^y) \le C \exp\left(Ct\mathfrak{a}_{\epsilon}(\log L)^2 - \frac{L^2}{Ct}\right),\tag{5.15}$$

and

$$\mathbb{E}_{\mathbb{Q}_{L,\epsilon}^{x,y}} \left[ \mathbb{1}_{X[0,t] \not\subset Q_r^y, X[0,t] \subset Q_L^y} \cdot \mathscr{D}_{L,\epsilon}^y(0,t) \right] \le C \exp\left( C \mathfrak{a}_{\epsilon}^{\aleph+1} (\log L)^{2\aleph+2} - \frac{r^2}{Ct} \right)$$
 (5.16)

where  $\mathfrak{a}_{\epsilon}$  is same as in the one defined in (5.4).

*Proof.* In what follows, we use the upper bound on  $||Z_{L,\epsilon}^y||_{\mathscr{C}^{\alpha}}$ ,  $||Y_{L,\epsilon}^y||_{\mathscr{C}^{2\alpha}}$  and  $\eta_{L,\epsilon}^y$  from Proposition 5.3 to derive upper and lower bound on  $\mathscr{D}_{L,\epsilon}^y$ . Combining (5.14) with the definition of  $\mathscr{D}_{L,\epsilon}^y$  (3.3), for L > 1,  $s, t \geq 0$  and s < t, we have

$$e^{-C\mathfrak{a}_{\epsilon}^{\aleph+1}(t-s)(\log L)^{2\aleph+2}} \le \mathbb{1}_{X[s,t]\subset Q_{\tau}^{y}} \cdot \mathscr{D}_{L,\epsilon}^{y}(s,t) \le e^{C\mathfrak{a}_{\epsilon}^{\aleph+1}(t-s)(\log L)^{2\aleph+2}},\tag{5.17}$$

where C>0 is an absolute constant and  $\mathfrak{a}_{\epsilon}$  is same as in (5.4). To complete the proof of the inequalities in (5.15) and (5.17), we further need bound on the transition probability of the diffusion X as defined in (3.4). We derive those in the following using the estimates of Theorem 4.14. Recall the upper and lower bound on the transition kernel  $\Gamma_t^L(x,y)$  from (4.27) and (4.28) respectively. We may bound  $\|U_{L,\epsilon}\| = \|Z_{L,\epsilon} + Y_{L,\epsilon}\|_{\infty}$  by  $\|Z_{L,\epsilon}\|_{\mathscr{C}^{\alpha}} + \|Y_{L,\epsilon}\|_{2\mathscr{C}^{\alpha}}$  which is further bounded above by  $C\mathfrak{a}_{\epsilon}(\log L)^2$  due to (5.14). As a result, we get for L > r > e and  $L, r \in \mathbb{N}$ 

$$\Gamma_t^L(x,y) \ge \frac{1}{t^{d/2}} \exp\left(-C\mathfrak{a}_{\epsilon}(\log L)^2 (1 + \frac{r^2}{t})\right)$$

$$\Gamma_t^L(x,y) \le p_t(x,y)e^{\mathfrak{a}_{\epsilon}(\log L)^2},$$
(5.18)

where  $p_t$  is the transition density of the d-dimensional Brownian motion. Note that  $\mathbb{Q}_{L,y}^{x,\epsilon}(X[0,t] \not\subset Q_L^y)$  is bounded above by  $\mathbb{P}\left(\sup_{s\in[0,t]}|X_s-x|\geq L/2\right)$ . Corollary 4.15 bounds the last probability by  $C\exp(Ct\|Z_{L,\epsilon}+Y_{L,\epsilon}\|_{\infty}-\frac{L^2}{CT})$ . Due to Proposition 5.3, we may bound  $\|Z_{L,\epsilon}+Y_{L,\epsilon}\|_{\infty}$  by  $C\mathfrak{a}_{\epsilon}(\log L)^2$ . Plugging this into the bound  $C\exp(Ct\|Z_{L,\epsilon}+Y_{L,\epsilon}\|_{\infty}-\frac{L^2}{CT})$  yields the inequality (5.15). To prove (5.16), we first apply (5.17) to obtain

$$\mathbb{E}_{\mathbb{Q}^{x,y}_{L,\epsilon}} \left[ \mathbb{1}_{X[0,t] \not\subset Q^y_r, X[0,t] \subset Q^y_L} \cdot \mathscr{D}^y_{L,\epsilon}(0,t) \right] \leq \mathbb{Q}^{x,y}_{L,\epsilon}(X[0,t] \not\subset Q^y_L) e^{C\mathfrak{a}^{\aleph+1}_\epsilon(t-s)(\log L)^{2\aleph+2}}.$$

Substituting (5.15) into the right hand side of the above display yields (5.16).

Now we use the bounds of Lemma 5.4 to upper and lower bound of the solution of PAM (2.1) started from constant initial data. Our proof ideas will be similar in spirit to [KPvZ20, Lemma 3.2].

**Proposition 5.5.** Let L > 1. Recall that  $u_L^{1,y}$  is the solution of PAM restricted on box  $Q_L^y$  under Dirichlet boundary condition started from constant initial data 1. Then we have

$$u_L^{1,y}(t,x) \le C \exp\left(t\lambda_1(Q_L^y) + C\mathfrak{a}_0^{\aleph+1}(\log L)^{2\aleph+2}\right),$$
 (5.19)

where  $\mathfrak{a}_0 := \lim_{\epsilon \to 0} \mathfrak{a}_{\epsilon}$ . Moreover, for  $t > \delta > 1$ 

$$u_L^{1,y}(t,x) \ge \exp\left(-C\mathfrak{a}_0^{\aleph+1}\delta(\log L)^{2\aleph+2} - \frac{C\mathfrak{a}_0(\log L)^2r^2}{\delta} + (t-\delta)\boldsymbol{\lambda}_1(Q_r^y)\right)$$
(5.20)

$$-\exp\left(C\mathfrak{a}_0^{2\aleph+1}t(\log L)^{2\aleph+2} - \frac{L^2}{C\delta}\right) \tag{5.21}$$

where  $\aleph$  is the same constant as in Proposition 5.3.

*Proof.* We start by writing that

$$u_L^{1,y}(t,x) = \mathbb{E}_{\mathbb{Q}_L^{x,y}} \left[ \mathscr{D}_L^y(0,t) \mathbb{1}_{X[0,t] \subset Q_L^y} \right] = \int_{Q_L^y} u_L^{x,y}(t,z) dz.$$
 (5.22)

where  $u_r^{x,y}(t,z) := u_L^{\delta_x,y}(t,z)$ . Note that  $u_r^{x,y}(t,z) := u_L^{\delta_x,y}(t,z) = \lim_{\epsilon \to 0} u_L^{\psi_\epsilon^x,y}(t,z)$  where  $\psi_\epsilon^x(z) := \psi_\epsilon(z-x) \in C^\infty(Q_L)$  (see [Lab19]). For  $\delta \in (1,t), \epsilon \in [0,1]$  and 0 < r < L, we have

$$\begin{split} u_r^{x,y}(t,z) &= \lim_{\epsilon \to 0} \mathbb{E}_{\mathbb{Q}_{r,\epsilon}^{x,y}} \left[ \mathscr{D}_{r,\epsilon}^{y}(0,t) \psi_{\epsilon}^{x}(X_t) \mathbb{1}_{X[0,t] \subset Q_r^y} \right] \\ &\leq \lim_{\epsilon \to 0} e^{C\mathfrak{a}_{\epsilon}^{\aleph+1} \delta(\log r)^{2\aleph+2}} \mathbb{E}_{\mathbb{Q}_{r,\epsilon}^{x,y}} \left[ \mathscr{D}_{r,\epsilon}^{y}(0,t-\delta) \mathbb{1}_{X[0,t-\delta] \subset Q_r^y} \mathbb{E}_{\mathbb{Q}_{r,\epsilon}^{x,y}} [\psi_{\epsilon}^{x}(X_t) | X_{t-\delta}] \right] \\ &= \lim_{\epsilon \to 0} e^{C\mathfrak{a}_{\epsilon}^{\aleph+1} \delta(\log r)^{2\aleph+2}} \mathbb{E}_{\mathbb{Q}_{r,\epsilon}^{x,y}} \left[ \mathscr{D}_{r,\epsilon}^{y}(0,t-\delta) \mathbb{1}_{X[0,t-\delta] \subset Q_r^y} \int_{\mathbb{R}^3} \Gamma_{\delta}^{r}(X_{t-\delta},z') \psi_{\epsilon}^{x}(z') dz' \right] \\ &\leq C e^{C\mathfrak{a}_{0}^{\aleph+1} \delta(\log r)^{2\aleph+2} + \mathfrak{a}_{0}(\log r)^{2}} \mathbb{E}_{\mathbb{Q}_{r}^{x,y}} \left[ \mathscr{D}_{r}^{y}(0,t-\delta) \mathbb{1}_{X[0,t-\delta] \subset Q_r^y} \right] \\ &\leq C e^{C\mathfrak{a}_{0}^{\aleph+1} \delta(\log r)^{2\aleph+2}} u_r^{\mathbb{I},y}(t-\delta,z). \end{split}$$
(5.23)

The first inequality in the above display is obtained by splitting  $\mathscr{D}^{y}_{r,\epsilon}(0,t)\mathbb{1}_{X[0,t]\subset Q^{y}_{r}}$  as a product of  $\mathscr{D}^{y}_{r,\epsilon}(0,t-\delta)\mathbb{1}_{X[0,t-\delta]\subset Q^{y}_{r}}$  and  $\mathscr{D}^{y}_{r,\epsilon}(t-\delta,t)\mathbb{1}_{X[t-\delta,t]\subset Q^{y}_{r}}$  and (5.17) to bound  $\mathscr{D}^{y}_{r,\epsilon}(t-\delta,t)\mathbb{1}_{X[t-\delta,t]\subset Q^{y}_{r}}$ . The second to last inequality is obtained by applying the bound on the transition kernel  $\Gamma^{r}_{\delta}(X_{t-\delta},z')$  as shown in (5.18). From (5.22) and (5.17) again, we have for  $q\in[1,\infty)$ 

$$\left(\int_{Q_r^y} |u_r^{\mathbb{T},y}(t-\delta,z)|^q dz\right)^{\frac{1}{q}} \lesssim e^{C\mathfrak{a}_0^{\aleph+1}(t-\delta)(\log r)^{2\aleph+2}}.$$

This implies that for  $q \in [1, \infty]$ ,

$$||u_r^{x,y}(t,\cdot)||_{L^q} \le Ce^{C\mathfrak{a}_0^{\aleph+1}t(\log r)^{2\aleph+2}}.$$
(5.24)

Now observe that for  $\phi \in C(Q_L)$ , using Hölder inequality,

$$\int_{Q_L^y} u_L^{\phi, y}(t, x) dx = \sum_{n \in \mathbb{N}} e^{t \lambda(Q_L^y)} \langle v_{n, L}^y, \phi \rangle \langle v_{n, L}^y, \mathbb{1}_{Q_L^y} \rangle \le e^{t \lambda(Q_L^y)} \|\phi\|_{L^2} \|\mathbb{1}_{Q_L^y}\|_{L^2}.$$
 (5.25)

Set  $\phi = u_L^{x,y}(1,\cdot)$ . Then by the Chapman-Kolmogorov equation, we know  $u_L^{1,y}(t,x) = \int_{Q_L^y} u_L^{\phi,y}(t-1,z)dz$ . Applying (5.22) to the right hand side of the latter equation in conjuction with (5.23) and (5.24) yields

$$u_L^{1,y}(t,x) \lesssim \exp(t\lambda(Q_L^y) + C\mathfrak{a}_0^{\aleph+1}(\log L)^{2\aleph+2}). \tag{5.26}$$

Now we proceed to prove the lower bound. Using Markov property at time  $\delta \in (1, t)$  and lower bound on  $\mathbb{1}_{X[0,\delta] \subset Q_L^y} \cdot \mathscr{D}_{L,\epsilon}^y(0,\delta)$  from (5.17), we have

$$u_L^{\mathbb{1},y}(t,x) \ge e^{-C\mathfrak{a}_0^{\aleph+1}\delta(\log L)^{2\aleph+2}} \mathbb{E}_{\mathbb{Q}_L^{x,y}} \Big[ \mathbb{1}_{X[0,\delta] \subset Q_L^y} u_L^{\mathbb{1},y}(t-\delta,X_\delta) \Big]. \tag{5.27}$$

Then for  $r \in (0, L)$ , we have

$$\mathbb{E}_{\mathbb{Q}_{L}^{x,y}} \left[ \mathbb{1}_{X[0,\delta] \subset Q_{L}^{y}} u_{L}^{\mathbb{1},y}(t-\delta, X_{\delta}) \right] \\
\geq \mathbb{E}_{\mathbb{Q}_{L}^{x,y}} \left[ \mathbb{1}_{X_{\delta} \in Q_{r}^{y}} u_{r}^{\mathbb{1},y}(t-\delta, X_{\delta}) \right] - \mathbb{E}_{\mathbb{Q}_{L}^{x,y}} \left[ \mathbb{1}_{X[0,\delta] \not\subset Q_{L}^{y}} \mathbb{1}_{X_{\delta} \in Q_{r}^{y}} u_{r}^{\mathbb{1},y}(t-\delta, X_{\delta}) \right].$$
(5.28)

The second term on the r.h.s. can be bounded as

$$\mathbb{E}_{\mathbb{Q}_{L}^{x,y}} \left[ \mathbb{1}_{X[0,\delta] \not\subset Q_{L}} \mathbb{1}_{X_{\delta} \in \mathbb{Q}_{r}^{y}} u_{r}^{\mathbb{1},y}(t-\delta, X_{\delta}) \right] \leq \mathbb{P}(X[0,\delta] \not\subset Q_{L}) \sup_{z \in \mathbb{Q}_{r}^{y}} u_{r}^{\mathbb{1},y}(t-\delta, z) 
\leq C \exp\left( C \mathfrak{a}_{0} \delta (\log L)^{2} - \frac{L^{2}}{C\delta} + C \mathfrak{a}_{0}^{\aleph+1}(t-\delta) (\log r)^{2\aleph+2} \right) 
\leq C \exp\left( C \mathfrak{a}_{0}^{\aleph+1} t (\log L)^{2\aleph+2} - \frac{L^{2}}{C\delta} \right),$$
(5.29)

where we used (5.15) and (5.17) in the second inequality. Now we bound the first term in the r.h.s. of (5.28). Using (5.18), we have

$$\mathbb{E}_{\mathbb{Q}_{L}^{x,y}} \left[ \mathbb{1}_{X_{\delta} \in Q_{r}^{y}} u_{r}^{\mathbb{1},y}(t-\delta, X_{\delta}) \right] = \int_{Q_{r}^{y}} \Gamma_{\delta}^{L}(x, z) u_{r}^{\mathbb{1},y}(t-\delta, z) dz$$

$$\geq \frac{C}{t^{d/2}} e^{-C\mathfrak{a}_{0}(\log L^{2}) - C \frac{\mathfrak{a}_{0}(\log L)^{2} r^{2}}{\delta}} \int_{Q_{r}^{y}} u_{r}^{\mathbb{1},y}(t-\delta, z) dz.$$
(5.30)

Note that

$$u_r^{1,y}(t-\delta,z) \ge C^{-1} e^{-C\mathfrak{a}_0^{\aleph+1} \delta(\log r)^{2\aleph+2}} u_r^{\delta_x,y}(t-\delta,z)$$
(5.31)

from (5.23). This gives

$$\mathbb{E}_{\mathbb{Q}_L^{x,y}} \left[ \mathbb{1}_{X_{\delta} \in Q_r^y} u_r^{\mathbb{1},y}(t-\delta,X_{\delta}) \right] \geq \frac{C}{t^{d/2}} e^{-C\mathfrak{a}_0(\log L^2) - C\frac{\mathfrak{a}_0(\log L)^2 r^2}{\delta} - C\mathfrak{a}_0^{\aleph+1} \delta(\log r)^{2\aleph+2}} \int_{Q_r^y} u_r^{\delta_x,y}(t-\delta,z) dz$$

We pose to observe that using the spectral decomposition of the solution (Lemma 2.1).

$$u_L^{\delta_z,y}(t,x) = \sum_{n \in \mathbb{N}} e^{t \mathbf{\lambda}_n(Q_L^y)} v_{n,L}^y(z) v_{n,L}^y(x), \quad \text{for } x,z \in Q_L^y.$$

Using this, we can derive  $\int_{Q_L^y} u_L^{\delta_z,y}(t,z)dz \ge e^{t\lambda_n(Q_L^y)}$ . Combining these with the above lower bound, we obtain

$$\mathbb{E}_{\mathbb{Q}_{L}^{x,y}} \left[ \mathbb{1}_{X_{\delta} \in Q_{r}^{y}} u_{r}^{\mathbb{1},y}(t-\delta, X_{\delta}) \right] \ge \frac{C}{t^{d/2}} \exp\left( -C \mathfrak{a}_{0}^{\aleph+1} (\log L)^{2\aleph+2} \right) - \frac{C \mathfrak{a}_{0} (\log L)^{2} r^{2}}{\delta} + (t-\delta) \lambda_{1}(Q_{r}^{y}) \right). \tag{5.32}$$

Putting (5.29) and (5.32) together, we have (5.20). This completes the proof.

The following lemma says that the solution u can be represented as the sum of the localized solution on  $A_k^y$  where  $A_k^y := \left\{ X[0,t] \subset (Q_{L_t^{k+1}}^y \setminus Q_{L_t^k}^y) \right\}$ . This helps us to employ Proposition 5.5 for the solution u on  $\mathbb{R}^d$ .

**Lemma 5.6.** Let  $L_t := \lfloor t^b \rfloor$  for b > 1. With probability one, for all  $x \in Q_{L_t}^y$  and  $y \in \mathbb{R}^d$ , we have for  $\epsilon \in [0,1]$ ,

$$u_{\epsilon}^{\mathbb{1},y}(t,x) = \sum_{k \in \mathbb{N}_0} \mathcal{U}_{k,\epsilon}^y(t,x),$$

$$\label{eq:where upper problem} \text{where } \mathcal{U}^y_{k,\epsilon}(t,x) := \mathbb{E}_{\mathbb{Q}^{x,y}_{L^{k+1}_t,\epsilon}} \left[ \mathscr{D}^y_{L^{k+1}_t,\epsilon}(0,t) \mathbb{1}_{A^y_k} \right].$$

*Proof.* For simplicity, we prove this result for x=y=0. Let us denote  $u_t^{\epsilon}:=u_{\epsilon}^{1,0}(t,0)$ . The general case follows easily from the stationarity of the solution u. We let  $\mathcal{U}_{k,\epsilon}(t):=\mathcal{U}_{k,\epsilon}^0(t,0)$  for  $\epsilon\in[0,1]$ . When  $\epsilon\in(0,1]$ , the lemma is proved from the classical Feynmann-Kac representation in the proof of Theorem 3.2. To deal with the case of  $\epsilon=0$ , we first estimate  $\mathcal{U}_{k,\epsilon}(t)$ . Recall that  $\mathcal{U}_{k,\epsilon}(t)$  is equal to

$$\mathbb{E}_{\mathbb{Q}^{0,0}_{L^{k+1}_t,\epsilon}} \left[ \mathbb{1}_{X[0,t] \not\subset Q^0_{L^k_t}, X[0,t] \subset Q^0_{L^{k+1}_t}} \cdot \mathscr{D}^0_{L^{k+1}_t,\epsilon}(0,t) \right].$$

Using (5.16) to bound the above display yields

$$\mathcal{U}_{k,\epsilon}(t) \leq C \exp\left(Ct\mathfrak{a}_{\epsilon}^{\aleph+1}((k+1)\log L_t)^{2\aleph+2} - \frac{L_t^{2k}}{Ct}\right) \\
\leq C \exp\left(Ct\mathfrak{a}_{\epsilon}^{\aleph+1}(b(k+1)\log t)^{\aleph+2} - \frac{t^{2bk}}{Ct}\right) \tag{5.33}$$

where the last inequality is obtained by substituting  $L_t = \lfloor t^b \rfloor$ . For a small  $\delta > 0$ , define the following event:

$$\Upsilon_{\epsilon} := \left\{ \mathfrak{a}_{\epsilon} \le C_b t^{\frac{2bk - 2 - \delta_0}{v + 1}} \right\}.$$

Under the event  $\Upsilon_{\epsilon}$ , we have that for all  $t \geq t_0$  and all  $k \geq 1$ ,

$$\mathcal{U}_{k,\epsilon}(t) \le C \exp\left(-\frac{t^{2bk-1}}{2C}\right). \tag{5.34}$$

By the union bound of the probability to obtain that for all  $K \in \mathbb{N}$  and for all  $\epsilon \in (0,1]$ 

$$\mathbb{P}\left(u_{t}^{0} - \sum_{k=0}^{K} \mathcal{U}_{k}(t) > \delta\right) \leq \mathbb{P}\left(|u_{t}^{0} - u_{t}^{\epsilon}| > \frac{\delta}{3}\right) + \mathbb{P}\left(|u_{t}^{\epsilon} - \sum_{k=0}^{K} \mathcal{U}_{k,\epsilon}(t)| > \frac{\delta}{3}\right) + \sum_{k=0}^{K} \mathbb{P}\left(|\mathcal{U}_{k,\epsilon}(t) - \mathcal{U}_{k}(t)| > \frac{\delta}{3K}\right)$$
(5.35)

By [HL18b, Theorem 1.1], we know  $u_t^{\epsilon}$  converges in probability to  $u_t$  as  $\epsilon \to 0$ . This shows the first term in the right hand side of (5.35) converges to 0 as  $\epsilon \to 0$ . Theorem 3.2 shows that the convergence of  $u^{\epsilon}$  in every box  $Q_{L_t}$ . As a result, the third term in the right hand side of the above display also converges to 0. Therefore, we need to show that the second term goes to zero taking K large. Note that  $u_t^{\epsilon} = \sum_{k=0}^{\infty} \mathcal{U}_{k,\epsilon}(t)$  by the classical Feynman-Kac representation. Then, we have

$$\mathbb{P}\Big(|u_t^{\epsilon} - \sum_{k=0}^K \mathcal{U}_{k,\epsilon}(t)| > \frac{\delta}{3}\Big) \leq \mathbb{P}\Big(\Big\{\Big|\sum_{k=K+1}^{\infty} \mathcal{U}_{k,\epsilon}(t)\Big| > \frac{\delta}{3}\Big\} \cap \Upsilon_{\epsilon}\Big) + \mathbb{P}(\neg \Upsilon_{\epsilon}).$$

Due to (5.34),  $\sum_{k=K+1}^{\infty} \mathcal{U}_{k,\epsilon}(t)$  is bounded above by  $\exp(-t^{2b(K-1)}/C)$  for all K > 1 on the event  $\Upsilon_{\epsilon}$ . This shows that the first term of the right hand side of the above display converges to 0 as K approaches to  $\infty$ . We now seek to bound  $\mathbb{P}(\neg \Upsilon_{\epsilon})$ . By Markov's inequality, we have

$$\mathbb{P}(\Upsilon_{\epsilon}) = \mathbb{P}\left(\mathfrak{a}_{\epsilon} > C_b t^{\frac{2bK - 2 - \delta_0}{\aleph + 1}}\right) \leq \mathbb{E}[e^{h_0\sqrt{\mathfrak{a}_{\epsilon}}}] \cdot \exp\left(-h_0\sqrt{C_b} t^{\frac{2bK - 2 - \delta_0}{2(\aleph + 1)}}\right).$$

Recall that  $\mathbb{E}[e^{h_0\sqrt{a_{\epsilon}}}]$  is uniformly bounded as shown in Proposition 5.2. Letting  $K \to \infty$  sends the right hand side of the above display to 0. This shows that the middle term of the right hand side of (5.35) also converges to 0 as  $\epsilon \to 0$  and  $K \to \infty$ . As a result, we get  $u(t,0) = \sum_{k=0}^{\infty} \mathcal{U}_k(t)$  in probability. Since each term of the series is non-negative, the latter identity also holds in the almost sure sense. This completes the proof.

**Lemma 5.7.** Let  $L_t := t^b$  for some  $b \in (\frac{1}{2}, 1]$ . With probability one, for all  $y \in \mathbb{R}^d$  with d = 2, 3,

$$\lim_{t \to \infty} \sup_{x \in B(y,1)} \frac{\log u_{L_t}^{\mathbb{I},y}(t,x)}{t \lambda_1(Q_{L_t}^y)} = 1.$$
 (5.36)

*Proof.* We only prove the lemma when d = 3. The d = 2 case follows from [KPvZ20, Lemma 3.6]. By (5.19), we have

$$\sup_{x \in B(y,1)} \frac{\log u_{L_t}^{1,y}(t,x)}{t \lambda_1(Q_{L_t}^y)} \le 1 + \frac{C \mathfrak{a}_0^{\aleph+1} (\log L_t)^{2\aleph+2}}{t \lambda_1(Q_{L_t}^y)}.$$

By Section 2, we know that for enough large t > 0,  $\lambda_1(Q_{L_t}^y) > 0$  almost surely. Since  $\mathfrak{a}_0$  is almost surely finite, we have the upper bound

$$\limsup_{t \to \infty} \sup_{x \in B(y,1)} \frac{\log u_{L_t}^{1,y}(t,x)}{t \lambda_1(Q_{L_t}^y)} \le 1.$$
 (5.37)

Let  $b \in (\frac{1}{2}, 1], b_1 \in (0, b)$ , and  $b_2 \in (2b_1 - 1, 2b - 1)$ . For the lower bound, by (5.20) with  $L_t := t^b, r_t = t^{b_1}$ , and  $\delta_t := t^{b_2}$ , we have

$$\sup_{x \in B(y,1)} u_{L_t}^{1,y}(t,x) \ge \log A_t + \log(1 - B_t)$$

where

$$A_t := \operatorname{const} \cdot \exp\left((t - \delta_t) \lambda_1(Q_{r_t}^y) - \frac{\mathfrak{a}_0 r^2 (\log L_t)^2}{C \delta_t} - C \mathfrak{a}_0^{v+1} \delta_t (\log L_t)^{2v+2}\right)$$

and

$$B_t := \operatorname{const} \cdot \exp\Big(-(t - \delta_t) \boldsymbol{\lambda}_1(Q_{r_t}^y) + \frac{\mathfrak{a}_0 r_t^2 (\log L_t)^2}{C \delta_t} - \frac{L_t^2}{C \delta_t} - C \mathfrak{a}_0^{\upsilon + 1} \delta_t (\log L_t)^{2\upsilon + 2} + C \mathfrak{a}_0^{\upsilon + 1} t (\log L_t)^{2\upsilon + 2} \Big).$$

Since  $2b - b_2 > 1$ ,  $b > b_1$  and  $\lambda(Q_{r_t}^y) > 0$  for all large t, we have  $B_t \to 0$ . Furthermore, since  $2b_1 - b_2 < 1$  and  $b_2 < 2b - 1 \le 1$ , we have

$$\liminf_{t \to \infty} \log A_t \ge \liminf_{t \to \infty} \frac{\lambda_1(Q_{r_t}^y)}{\lambda_1(Q_{I_t}^y)}.$$

Note that  $\lim_{L\to\infty} \frac{\lambda_1(Q_L^y)}{(\log L)^2} = \chi$  for some constant  $\chi > 0$  (see [HL22, Theorem 1]). This shows that

$$\liminf_{t\to\infty}\sup_{x\in B(y,1)}\frac{\log u_{L_t}^{\mathbb{1},y}(t,x)}{t\boldsymbol{\lambda}_1(Q_{L_t}^y)}\geq \liminf_{t\to\infty}\frac{\boldsymbol{\lambda}_1(Q_{r_t}^y)}{\boldsymbol{\lambda}_1(Q_{L_t}^y)}\geq \liminf_{t\to\infty}\frac{\chi(\log r_t)^2}{\chi(\log L_t)^2}\geq \left(\frac{b_1}{b}\right)^2.$$

Letting  $b_1 \uparrow b$ , we have the lower bound. This completes the proof.

**Lemma 5.8.** Let  $L_t := t(\log t)^{2\aleph+2}$ . With probability one, for all  $y \in \mathbb{R}^d$  with d = 2, 3,

$$\lim_{t \to \infty} \sup_{x \in B(y,1)} \frac{\log u^{1,y}(t,x)}{t \lambda_1(Q_{L_t}^y)} - \sup_{x \in B(y,1)} \frac{\log u_{L_t}^{1,y}(t,x)}{t \lambda_1(Q_{L_t}^y)} = 0, \tag{5.38}$$

and

$$\lim_{t \to \infty} \sup_{x \in B(y,1)} \frac{\log u^{1,y}(t,x)}{t \lambda_1(Q_{L_t}^y)} = 1.$$
 (5.39)

*Proof.* As in the previous lemma, we only prove the case when d=3 since the case of d=2 follows from [KPvZ20, Proposition 4.5]. Due to the fact  $\lim_{L\to\infty}\frac{\lambda_1(Q_L^y)}{(\log L)^2}=\chi$ , it suffices to show

$$\lim_{t \to \infty} \sup_{x \in B(y,1)} \frac{\log u^{1,y}(t,x)}{t(\log L_t)^2} - \sup_{x \in B(y,1)} \frac{\log u_{L_t}^{1,y}(t,x)}{t(\log L_t)^2} = 0.$$
 (5.40)

Letting  $\epsilon \to 0$  in Lemma 5.6, we have  $u^{\mathbb{I},y} = \sum_{k \in \mathbb{N}_0} \mathcal{U}_k^y$ . By simple observation, it follows that

$$\lim_{t \to \infty} \left| \sup_{x \in B(y,1)} \frac{\log \sum_{k \in \mathbb{N}_0} \mathcal{U}_k^y(t,x)}{t(\log L_t)^2} - \max \left\{ \sup_{x \in B(y,1)} \frac{\log \mathcal{U}_0^y(t,x)}{t(\log L_t)^2}, \sup_{x \in B(y,1)} \frac{\log \sum_{k \ge 1} \mathcal{U}_k^y(t,x)}{t(\log L_t)^2} \right\} \right| = 0.$$
(5.41)

Since  $\mathcal{U}_0^y = u_{L_t}^{1,y}$ , applying Lemma 5.7 yields that

$$\sup_{x \in B(y,1)} \frac{\log \mathcal{U}_0^y(t,x)}{t(\log L_t)^2} = \sup_{x \in B(y,1)} \frac{\log \mathcal{U}_0^y(t,x)}{t \lambda_1(Q_{L_t}^y)} \cdot \frac{\lambda_1(Q_{L_t}^y)}{(\log L_t)^2} \ge \frac{\chi}{2}$$
 (5.42)

for all large t. Moreover, from (5.33), we have

$$\sup_{x \in B(y,1)} \mathcal{U}_k^y(t,x) \le C \exp\left(Ct((k+1)\log L_t)^{2\aleph+2} \left[\mathfrak{a}_0^{\aleph+1} - \frac{L_t^{2k}}{C^2 t^2 ((k+1)\log L_t)^{2\aleph+2}}\right]\right).$$

Since  $L_t = t(\log t)^{2\aleph+2}$ , we have for all  $k \geq 1$ 

$$\sup_{x \in B(y,1)} \mathcal{U}_k^y(t,x) \le e^{-Ct(\log t)^{2\aleph + 2}}$$

$$\tag{5.43}$$

for all large t. This implies that for all large t,

$$\sup_{x \in B(y,1)} \log \left( \sum_{k \ge 1} \mathcal{U}_k^y(t,x) \right) \le 0. \tag{5.44}$$

By combining (5.42) and (5.44) with (5.41), we can conclude (5.40), which is the first part of the lemma. The second part of the lemma follows immediately from the first part and Lemma 5.7.  $\Box$ 

#### 6. Spatial Multifractality and Asymptotics of the PAM: Proof of Theorem 1.1

In the remaining sections, we will show the multifractality of the solution to (1.1). We only consider the solution with flat initial data and write u for  $u^{1}$ .

6.1. **Proof of the lower bound in Theorem 1.1.** In this section, we prove the lower bound of the dimension in Theorem 1.1. The following proposition is one of the key tools for proving such lower bound.

**Proposition 6.1.** Let  $\epsilon > 0$  and  $\theta > 0$ . There exists  $n_0 > 0$  such that for all  $n \geq n_0$  and  $x_1, ..., x_m \in (e^n, e^{n+1}]^d$  satisfying  $\min_{i \neq j} ||x_i - x_j||_{\infty} > e^{n\theta}$ , we have

$$\mathbb{P}\left(\max_{1\leq j\leq m}\frac{\log u(t,x_j)}{(\log \|x_j\|_{\infty})^{\frac{2}{4-d}}}\leq \alpha t\right)\leq \exp\left(-cm(\alpha+\epsilon)^{\frac{d}{2}}n^{\frac{d}{4-d}}e^{d\log r_n-\mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}}n}\right)+e^{-cm\log n},\tag{6.1}$$

where  $r_n := n^{\frac{1}{2} \log t}$ .

*Proof.* Since  $x_1, \ldots x_m \in (e^n, e^{n+1}]^d$ , we have

$$\mathbb{P}\left(\max_{1 \le j \le m} \frac{\log u(t, x_j)}{(\log \|x_j\|_{\infty})^{\frac{2}{4-d}}} \le \alpha t\right) \le \mathbb{P}\left(\max_{1 \le j \le m} u(t, x_j) \le e^{\alpha t(n+1)^{\frac{2}{4-d}}}\right). \tag{6.2}$$

Let  $L:=L_n:=t^{\log n}$ . By Proposition 5.6, there exists  $t_0>0$  such that for all  $t\geq t_0$  the series  $\sum_{k\in\mathbb{N}_0}\mathcal{U}_k^y=u$  for all  $y\in\mathbb{R}^d$ . Moreover, since  $0\leq\mathcal{U}_0^y\leq\sum_{k\in\mathbb{N}_0}\mathcal{U}_k^y=u$ , we have

$$\mathbb{P}\left(\max_{1\leq j\leq m} \frac{\log u(t,x_j)}{(\log \|x_j\|_{\infty})^{\frac{2}{4-d}}} \leq \alpha t\right) \leq \mathbb{P}\left(\max_{1\leq j\leq m} \mathcal{U}_0^{x_j}(t,x_j) \leq e^{\alpha t(n+1)^{\frac{2}{4-d}}}\right) \\
= \prod_{j=1}^{m} \mathbb{P}\left(\mathcal{U}_0^{x_j}(t,x_j) \leq e^{\alpha t(n+1)^{\frac{2}{4-d}}}\right), \tag{6.3}$$

where the last equality follows from the independence of  $\mathcal{U}_0^{x_j}(t,x_j)$  shown in Lemma 2.4 thanks to the fact that  $\min_{i\neq j}\|x_i-x_j\|_{\infty}>e^{n\theta}\gg 3L_t=3t^{\log n}$ . We also set  $r:=r_n:=t^{\frac{1}{2}\log n}$ ,. Let  $\epsilon>0$ . Observe that for any fixed  $t\geq t_0$ , there exists  $n_0>0$  such that for all  $n\geq n_0$ 

$$\epsilon tn \gg (\mathfrak{a}_0 \log L_n)^{2\aleph+2} + \frac{\mathfrak{a}_0 (\log L_n)^2 r_n^2}{C\delta}, \quad \frac{L_n^2}{C\delta} \gg t (\mathfrak{a}_0 \log L_n)^{2\aleph+2} + \frac{\mathfrak{a}_0 (\log L_n)^2 r_n^2}{C\delta} \tag{6.4}$$

on the event  $\Upsilon_n := \{\mathfrak{a}_0 \leq (\log n)^2\}$ . Using this observation, when d = 3, there exist  $t_0 > 0$  and  $n'_0 > 0$  such that for all  $n \geq n'_0$  and all  $t \geq t_0$ 

$$\begin{split} & \mathbb{P}\Big(\mathcal{U}_{0}^{x_{j}}(t,x_{j}) \leq e^{\alpha t(n+1)^{2}}\Big) \\ & \leq \mathbb{P}\Big(e^{-C\mathfrak{a}_{0}^{\aleph+1}\delta(\log L_{n})^{2\aleph+2} - \frac{C\mathfrak{a}_{0}(\log L_{n})^{2}r_{n}^{2}}{\delta} + (t-\delta)\boldsymbol{\lambda}_{1}(Q_{r_{n}}^{y})} - e^{C\mathfrak{a}_{0}^{2\aleph+1}t(\log L_{n})^{2\aleph+2} - \frac{L_{n}^{2}}{C\delta}} \leq e^{\alpha t(n+1)^{2}}\Big) \\ & \leq \mathbb{P}\Big(\Big\{\frac{1}{2}\exp\Big((t-\delta)\boldsymbol{\lambda}_{1}(Q_{r_{n}}^{x_{j}}) - C\mathfrak{a}_{0}^{\aleph+1}\delta(\log L_{n})^{2\aleph+2} - \frac{C\mathfrak{a}_{0}(\log L_{n})^{2}r^{2}}{\delta}\Big) \leq e^{\alpha t(n+1)^{2}}\Big\} \cap \Upsilon_{n}\Big) + \mathbb{P}(\neg \Upsilon_{n}) \\ & \leq \mathbb{P}\left(\boldsymbol{\lambda}_{1}(Q_{r_{n}}^{x_{j}}) \leq (\alpha+\epsilon)n^{2}\right) + \mathbb{P}(\neg \Upsilon_{n}) \end{split}$$

The first inequality in the above display is obtained by using the lower bound on  $\mathcal{U}_0^{x_j}(t, x_j)$  from Proposition 5.5. While the second inequality is just consequence of the union bound, the last inequality utilizes (6.4). When d=2, we use the similar argument except the fact that the lower bound on  $\mathcal{U}_0^{x_j}(t, x_j)$  is now provided by Lemma 5.2 of [KPvZ20]. We obtain similarly

$$\mathbb{P}(\mathcal{U}_{0}^{x_{j}}(t, x_{j}) \leq e^{\alpha t(n+1)})$$

$$\leq \mathbb{P}\left(\exp\left((t - \delta)\boldsymbol{\lambda}_{1}(Q_{r_{n}}^{x_{j}}) - \frac{r_{n}^{2}}{C\delta} - C\delta(\mathfrak{a}_{0}\log L_{n})^{5}\right) - \exp(Ct(\mathfrak{a}_{0}\log L_{n})^{5} - \frac{L_{n}^{2}}{C\delta}\right) \leq e^{\alpha t(n+1)}\right)$$

$$\leq \mathbb{P}\left(\left\{\frac{1}{2}\exp\left((t - \delta)\boldsymbol{\lambda}_{1}(Q_{r_{n}}^{x_{j}}) - \frac{r_{n}^{2}}{C\delta} - C\delta(\mathfrak{a}_{0}\log L_{n})^{5}\right) \leq e^{\alpha t(n+1)}\right\} \cap \Upsilon_{n}\right) + \mathbb{P}(\neg \Upsilon_{n})$$

$$\leq \mathbb{P}(\boldsymbol{\lambda}_{1}(Q_{r_{n}}^{x_{j}}) \leq (\alpha + \epsilon)n) + \mathbb{P}(\neg \Upsilon_{n})$$

Here we note that  $t_0$  can be chosen independently of  $\alpha$ . Now we use Lemma 2.2 to obtain

$$\max_{1 \leq j \leq m} \mathbb{P}\left(\boldsymbol{\lambda}_1(Q_{r_n}^{x_j}) \leq (\alpha + \epsilon)n^{\frac{2}{4-d}}\right) \leq \exp\left(-c_2(\alpha + \epsilon)^{\frac{d}{2}}n^{\frac{d}{4-d}}e^{d\log r_n - \mathfrak{c}_d(1+\epsilon)(\alpha + \epsilon)^{\frac{4-d}{2}}n}\right)$$

Substituting this into (6.3) and (6.2), we have

$$\mathbb{P}\left(\max_{1\leq j\leq m} \frac{\log u(t,x_j)}{(\log \|x_j\|_{\infty})^{\frac{2}{4-d}}} \leq \alpha t\right) \\
\leq 2^{m-1} \exp\left(-c_2 m(\alpha+\epsilon)^{\frac{d}{2}} n^{\frac{d}{4-d}} e^{d\log r_n - \mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}} n}\right) + 2^{m-1} \mathbb{P}(\neg \Upsilon_n)^m$$

On the other hand, since  $\mathbb{E}[e^{h_0\sqrt{a_0}}] < \infty$  (see 5.4), we use Markov's inequality to get  $\mathbb{P}(\neg \Upsilon_n) \leq e^{-h_0 \log n}$ . This shows that

$$\mathbb{P}\left(\max_{1\leq j\leq m}\frac{\log u(t,x_j)}{(\log\|x_j\|_{\infty})^{\frac{2}{4-d}}}\leq \alpha t\right)\leq \exp\left(-cm(\alpha+\epsilon)^{\frac{d}{2}}n^{\frac{d}{4-d}}e^{d\log r_n-\mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}}n}\right)+e^{-cm\log n}$$

for some constant c > 0 and hence, completes the proof.

Now we proceed to prove Theorem 1.1. Recall that

$$\mathcal{P}_t^d(\alpha) = \left\{ x \in \mathbb{R}^d : u(t, x) \ge e^{\alpha t (\log|x|)^{\frac{2}{4-d}}} \right\}.$$

The following result proves a lower bound to the macroscopic Hausdorff dimension of the set  $\mathcal{P}_t^d(\alpha)$ . The proof of the upper bound is deferred to the subsection after the following result.

**Theorem 6.2.** There exists a non-random finite constant  $t_0 > 0$  such that for all  $t \ge t_0$ ,

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_t^d(\alpha)] \ge d - \alpha^{\frac{4-d}{2}} \mathfrak{c}_d, \quad a.s.$$

*Proof.* Let us choose  $\alpha > 0$  satisfying  $\alpha^{\frac{4-d}{2}} \mathfrak{c}_d < d$ . Fix  $\epsilon > 0$  such that  $\mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}} < d$ . Define

$$\widetilde{\mathcal{P}}_t^d(\alpha) := \mathcal{P}_t^d(\alpha) \cap \bigcup_{n=0}^{\infty} (e^n, e^{n+1}]^d.$$

Then it suffices to show that

$$\operatorname{Dim}_{\mathbb{H}}[\widetilde{\mathcal{P}}_{t}^{d}(\alpha)] \ge d - \alpha^{\frac{4-d}{2}} \mathfrak{c}_{d},$$

with probability one. We now choose  $\gamma \in (\mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}}/d,1)$  and define, for all integers  $n \geq 0$ ,

$$a_{j,n}(\gamma) := e^n + je^{n\gamma}, \quad j \in [0, e^{n(1-\gamma)}) \cap \mathbb{Z},$$

and

$$I_n(\gamma) := \bigcup_{j \in [0, e^{n(1-\gamma)}) \cap \mathbb{Z}} \{a_{j,n}(\gamma)\}, \qquad \mathcal{I}_n(\gamma) := \prod_{k=1}^d I_n^k(\gamma),$$

where  $I_n^k(\gamma)$  is a copy of  $I_n(\gamma)$  for all  $1 \leq j \leq d$ . We choose  $x \in \mathcal{I}_n(\gamma)$  and  $\theta \in (0, \gamma - \frac{\mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}}}{d})$ . By the construction of the set  $\mathcal{I}_n(\gamma)$ , we first find the points  $\{x_i\}_{i=1}^{m(n)}$  satisfying the followings: (a.1)  $x_i \in \mathcal{I}_n(\gamma) \cap B(x, e^{n\gamma})$  for all i = 1, ..., m(n); (a.2)  $||x_i - x_j||_{\infty} \geq e^{n\theta}$  whenever  $1 \leq i < j \leq m(n)$ ; (a.3)  $d^{-1}e^{dn(\gamma-\theta)} \leq m(n) \leq de^{dn(\gamma-\theta)}$ . Then by Proposition 6.1, we have

$$\mathbb{P}\left(\max_{1\leq j\leq m(n)} \frac{\log u(t,x_j)}{(\log \|x_j\|_{\infty})^{\frac{2}{4-d}}} \leq \alpha t\right) \\
\leq \exp\left(-cm(n)(\alpha+\epsilon)^{\frac{d}{2}} n^{\frac{d}{4-d}} e^{d\log r_n - \mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}}n}\right) + e^{-cm(n)\log n} \\
\leq \exp\left(-\frac{c}{2}(\alpha+\epsilon)^{\frac{d}{2}} n^{\frac{d}{4-d}} e^{c\log n + [d(\gamma-\theta) - \mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}}]n}\right) + e^{-\frac{c}{2}} e^{dn(\gamma-\theta)\log n}.$$

By our choice of  $\gamma$  and  $\theta$ ,  $\kappa := d(\gamma - \theta) - \mathfrak{c}_d(1 + \epsilon)(\alpha + \epsilon)^{\frac{4-d}{2}} > 0$ . Therefore,

$$\mathbb{P}\left(\max_{\{x_i\}_{i=1}^{m(n)}\subseteq\mathcal{I}_n(\theta)\cap B(x,e^{n\gamma})}\frac{\log u(t,x_i)}{(\log \|x_i\|_{\infty})^2} \le \alpha t\right) \le \exp\left(-C_1 e^{\kappa n - C_2 \log n}\right) + e^{-\frac{c}{2}e^{dn(\gamma - \theta)} \log n},$$

for some constant  $C_1, C_2 > 0$ . Then we have

$$\begin{split} \sum_{n=0}^{\infty} \mathbb{P} \left( \min_{x \in \mathcal{I}_n(\gamma)} \max_{\{x_i\}_{i=1}^{m(n)} \subseteq \mathcal{I}_n(\theta) \cap B(x, e^{n\gamma})} \frac{\log u(t, x_i)}{(\log \|x_i\|_{\infty})^2} \le \alpha t \right) \\ \le \sum_{n=0}^{\infty} \sum_{x \in \mathcal{I}_n(\gamma)} \mathbb{P} \left( \max_{1 \le i \le m(n)} \frac{\log u(t, x_i)}{(\log \|x_i\|_{\infty})^2} \le \alpha t \right) \\ \le \sum_{n=0}^{\infty} C e^{2n(1-\gamma)} \left( \exp \left( -C_1 e^{\kappa n - C_2 \log n} \right) + e^{-\frac{c}{2} e^{3n(\gamma - \theta)} \log n} \right) < \infty. \end{split}$$

for some constant C>0. Hence, the Borel-Cantelli lemma implies that  $\mathcal{P}^d$  contains a  $\gamma$ -thick set (see Definition B.2) almost surely. By Proposition B.3, We get  $\mathrm{Dim}_{\mathbb{H}}(\widetilde{\mathcal{P}}^d_t(\alpha)) \geq d(1-\gamma)$  with probability one. Letting  $\gamma \downarrow \frac{\mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{d-d}{2}}}{d}$  and  $\theta \downarrow 0$  with  $0<\theta<\gamma-\frac{\mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{d-d}{2}}}{d}$ , we get

 $\operatorname{Dim}_{\mathbb{H}}(\widetilde{\mathcal{P}}_t^d(\alpha)) \geq d - \mathfrak{c}_d(1+\epsilon)(\alpha+\epsilon)^{\frac{4-d}{2}}$ . Since  $\epsilon > 0$  is arbitrary, by the monotonicity of the macroscopic Hausdorff dimension, we conclude that

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_t^d(\alpha)] \ge \operatorname{Dim}_{\mathbb{H}}[\widetilde{\mathcal{P}}_t^d(\alpha)] \ge d - \alpha^{\frac{4-d}{2}}\mathfrak{c}_d,$$

almost surely.

6.2. **Proof of the upper bound in Theorem 1.1.** In this section, we prove the upper bound in Theorem 1.1. The following proposition will be used to complete the proof.

**Proposition 6.3.** Let  $\epsilon > 0$  and M > 0. There exist b := b(M) > 1 and  $n_0 := n_0(M, b, \epsilon) > 0$  such that for all  $n \ge n_0$  and  $t \ge 1$ 

$$\mathbb{P}\left(\sup_{x\in\mathcal{B}(y,1)}\frac{\log u(t,x)}{(\log\|x\|_{\infty})^{\frac{2}{4-d}}}\geq \alpha t, \mathfrak{a}_0\leq M\right)\leq c_1(\alpha-\epsilon)^{\frac{d}{2}}n^{\frac{d}{4-d}}e^{d\log L_t-(1-\epsilon)\mathfrak{c}_d(\alpha-\epsilon)^{\frac{4-d}{2}}n},\tag{6.5}$$

where  $L_t := t^b$  and  $y \in \mathbb{S}_n$  for  $n \in \mathbb{N}$ .

*Proof.* Since we can write  $u(t,x) = \sum_{k=0}^{\infty} \mathcal{U}_k^y(t,x)$  due to Lemma 5.6, we have

$$\mathbb{P}\Big(\sup_{x\in\mathcal{B}(y,1)}\frac{\log u(t,x)}{(\log\|x\|_{\infty})^{\frac{2}{4-d}}}\geq \alpha t, \mathfrak{a}_0\leq M\Big)\leq \mathbb{P}\Big(\sup_{x\in B(y,1)}u(t,x)\geq \frac{1}{2}e^{\alpha tn^{\frac{2}{4-d}}}, \mathfrak{a}_0\leq M\Big)\leq (\mathbf{C_1})+(\mathbf{C_1}),$$

where

$$(\mathbf{C_1}) := \mathbb{P}\Big(\sup_{x \in B(y,1)} \mathcal{U}_0^y(t,x) \ge \frac{1}{2} e^{\alpha t n^{\frac{2}{4-d}}}, \mathfrak{a}_0 \le M\Big),$$

$$(\mathbf{C_2}) := \mathbb{P}\Big(\sup_{x \in B(y,1)} \sum_{k=1}^{\infty} \mathcal{U}_k^y(t,x) \ge \frac{1}{2} e^{\alpha t n^{\frac{2}{4-d}}}, \mathfrak{a}_0 \le M\Big)$$

We first bound (C<sub>2</sub>). When d = 3, using (5.33), on the event  $\{a_0 \leq M\}$  we have

$$\mathcal{U}_k^y(t,x) \le C \exp\left(CtM^{\aleph+1}(b(k+1)\log t)^{2\aleph+2} - \frac{t^{2bk}}{Ct}\right).$$

Therefore, we can choose a large b := b(M) > 0 such that for all  $t \ge e$  and  $n \ge 1$ ,

$$\sum_{k=1}^{\infty} \mathcal{U}_{k}^{y}(t,x) \le \sum_{k=1}^{\infty} C \exp\left(-C_{1} t^{2bk-1}\right) \le \frac{1}{2} e^{\alpha t n^{\frac{2}{4-d}}}.$$

for some constant  $C_1 > 0$ . This implies that  $(\mathbf{C_2}) = 0$  for all  $t \ge e$ . For d = 2 case, one can use Lemma 5.2 of [KPvZ20] to get the same result.

Now we proceed to bound (C<sub>1</sub>). Applying Proposition 5.5 for d=3, there exists  $n_0=n_0(M,b,\epsilon)>0$  such that for all  $n\geq n_0$ 

$$(\mathbf{C_1}) \leq \mathbb{P}\left(C \exp(t\lambda_1(Q_{L_t}^y) + CM^{v+1}(\log L_t)^{2v+2}) \geq \frac{1}{2}e^{\alpha t n^{\frac{2}{4-d}}}\right)$$

$$\leq \mathbb{P}\left(\lambda_1(Q_{L_t}^y) \geq \alpha n^{\frac{2}{4-d}} - \frac{\log(2C) + CM^{v+1}(b\log t)^{2v+2}}{t}\right)$$

$$\leq \mathbb{P}(\lambda_1(Q_{L_t}^y) \geq (\alpha - \epsilon)n^{\frac{2}{4-d}}),$$

The d=2 case follows similarly from using Lemma 5.2 of [KPvZ20]. By Lemma 2.2, we further have

$$(\mathbf{C_1}) \leq \mathbb{P}(\boldsymbol{\lambda}_1(Q_{L_t}^y) \geq (\alpha - \epsilon)n^{\frac{2}{4-d}}) \leq c_1(\alpha - \epsilon)^{\frac{d}{2}}n^{\frac{d}{4-d}}e^{db\log t - (1-\epsilon)\mathfrak{c}_d(\alpha - \epsilon)^{\frac{4-d}{2}}n}.$$

Summing the bounds  $(C_1)$  and  $(C_2)$ , we arrive at

$$(\mathbf{C_1}) + (\mathbf{C_2}) \le c_1(\alpha - \epsilon)^{\frac{d}{2}} n^{\frac{d}{4-d}} e^{db \log t - (1-\epsilon)\mathfrak{c}_d(\alpha - \epsilon)^{\frac{4-d}{2}} n},$$

which completes the proof.

**Theorem 6.4.** For all  $t \ge e$  and all  $\alpha \in (0, (d/\mathfrak{c}_d)^{\frac{2}{4-d}})$ ,

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_{t}^{d}(\alpha)] = \operatorname{Dim}_{\mathbb{H}}\left[\left\{x \in \mathbb{R}^{d} : u(t, x) \ge e^{\alpha t (\log \|x\|_{\infty})^{\frac{2}{4-d}}}\right\}\right] \le d - \alpha^{\frac{4-d}{2}} \mathfrak{c}_{d}, \tag{6.6}$$

with probability one.

*Proof.* Fix M > 0. By Markov's inequality and Proposition 5.2, we have

$$\mathbb{P}(\mathfrak{a}_0 \ge M) \le \delta_M := \mathbb{E}[e^{h_0\sqrt{\mathfrak{a}_0}}] \cdot e^{-h_0\sqrt{M}}.$$
(6.7)

Note that  $\delta_M \to 0$  as  $M \to \infty$ . Choose  $\epsilon \in (0, \alpha \wedge 1)$  and  $\rho \in (d - (1 - \epsilon)\mathfrak{c}_d(\alpha - \epsilon)^{\frac{4-d}{2}}, d)$ . Using Proposition 6.3, we have that for all  $t \geq 1$ 

$$\sum_{n=0}^{\infty} e^{-n\rho} \sum_{\substack{y \in \mathbb{N}^d \\ B(y,1) \subset \mathbb{S}_n}} \mathbb{P}\left(\sup_{x \in B(y,1)} \frac{\log u(t,x)}{(\log ||x||_{\infty})^{\frac{2}{4-d}}} \ge \alpha t, \mathfrak{a}_0 \le M\right)$$

$$\le \sum_{n=0}^{\infty} c_1(\alpha - \epsilon)^{\frac{d}{2}} n^{\frac{d}{4-d}} \exp\left(db \log t + \left[d - \rho - \mathfrak{c}_d \alpha_{\epsilon}^{\frac{4-d}{2}}\right] n\right) < \infty,$$
(6.8)

where b = b(M) > 0 is taken as in Proposition 6.3 and  $\alpha_{\epsilon} := (1 - \epsilon)^{\frac{2}{4-d}} (\alpha - \epsilon)$ . Recall the definition of Hausdorff content  $\nu_{\rho}^{n}(E)$  of any set E from Section B.1. The first line in (6.8) is an upper bound to  $\mathbb{E}[\sum_{n=0}^{\infty} \nu_{\rho}^{n}(\mathcal{P}_{t}^{d}(\alpha))\mathbb{1}(\mathfrak{a}_{0} \leq M)]$ . As a result, we have

$$\mathbb{P}\Big(\{\mathfrak{a}_0 \leq M\} \cap \Big\{\sum_{n=0}^{\infty} \nu_{\rho}^n(\mathcal{P}_t^d(\alpha)) < \infty\Big\}\Big) = \mathbb{P}(\{\mathfrak{a}_0 \leq M\}).$$

Combining (6.7) with the above display yields

$$\mathbb{P}\Big(\sum_{n=0}^{\infty} \nu_{\rho}^{n}(\mathcal{P}_{t}^{d}(\alpha)) < \infty\Big) \ge 1 - \mathbb{E}[e^{h_{0}\sqrt{\mathfrak{a}_{0}}}] \cdot e^{-h_{0}\sqrt{M}}.$$

Note that the l.h.s. of the above inequality does not depend on M. Thus, by letting  $M \to \infty$ , we have  $\sum_{n=0}^{\infty} \nu_{\rho}^{n}(\mathcal{P}_{t}^{d}(\alpha)) < \infty$  a.s., which implies that  $\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_{t}^{d}(\alpha)] \leq \rho$  for any  $\rho \in (d - \mathfrak{c}_{d}\alpha_{\epsilon}^{\frac{4-d}{2}}, d)$ , a.s. By taking  $\rho \downarrow d - \mathfrak{c}_{d}\alpha_{\epsilon}^{\frac{4-d}{2}}$  and  $\epsilon \downarrow 0$ , we get  $\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_{t}^{d}(\alpha)] \leq d - \mathfrak{c}_{d}\alpha^{\frac{4-d}{2}}$ .

6.3. Spatial asymptotics of the PAM. This section is devoted to proving the spatial asymptotics (1.5). This result is a consequence of the first part of Theorem 1.1, i.e., there exists  $t_0 > 0$  such that

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_{t}^{d}(\alpha)] = \operatorname{Dim}_{\mathbb{H}}\left(\left\{x \in \mathbb{R}^{d} : \frac{\log u(t,x)}{\log |x|} \ge \alpha t\right\}\right) = (d - \alpha^{\frac{4-d}{2}}\mathfrak{c}_{d}) \vee 0, \quad \text{a.s.}, \tag{6.9}$$

for  $t \geq t_0$  and  $\alpha > 0$ . We provide the details of the proof below.

**Proof of** (1.5). By the definition of the macroscopic Hausdorff dimension,  $\operatorname{Dim}_{\mathbb{H}}(A) > 0$ ,  $A \subset \mathbb{R}^d$  implies that A is an unbounded set. This fact and (6.9) imply that for any  $\alpha < (d/\mathfrak{c}_d)^{\frac{2}{4-d}}$ , the set  $\mathcal{P}_t^d(\alpha)$  is unbounded, hence in particular

$$\limsup_{\|x\|_{\infty} \to \infty} \frac{\log u(t, x)}{(\log \|x\|_{\infty})^{\frac{2}{4-d}}} \ge \alpha t, \quad \text{a.s.}$$

$$(6.10)$$

This implies that

$$\limsup_{\|x\|_{\infty} \to \infty} \frac{\log u(t, x)}{(\log \|x\|_{\infty})^{\frac{2}{4-d}}} \ge \left(\frac{d}{\mathfrak{c}_d}\right)^{\frac{2}{4-d}} t, \quad \text{a.s.}$$

$$(6.11)$$

Now we prove the upper bound. Fix  $\epsilon \in (0, \alpha \wedge 1)$ . Let M > 0 and  $\delta_M$  be defined as in (6.7). Note that we can rewrite (6.8) as

$$\sum_{n=0}^{\infty} \sum_{\substack{y \in \mathbb{N}^d \\ B(y,1) \subset \mathbb{S}_n}} \mathbb{P}\left(\sup_{x \in B(y,1)} \frac{\log u(t,x)}{\left(\log \|x\|_{\infty}\right)^{\frac{2}{4-d}}} \ge g(d,\epsilon)t, \mathfrak{a}_0 \le M\right) < \infty, \tag{6.12}$$

where  $g(d, \epsilon) := \left(d/\mathfrak{c}_d(1-\epsilon)\right)^{\frac{2}{4-d}} + \epsilon$ . The Borel-Cantelli lemma yields that with probability greater than  $1 - e^{-h\sqrt{M}}\mathbb{E}[e^{h\sqrt{a_0}}]$ ,

$$\limsup_{\|x\|_{\infty} \to \infty} \frac{\log u(t,x)}{(\log \|x\|_{\infty})^{\frac{2}{4-d}}} \le \left(\frac{d}{\mathfrak{c}_d(1-\epsilon)} + \epsilon\right)^{\frac{2}{4-d}} t,$$

for all  $t \geq t_0$ . Letting  $\epsilon \to 0$  and  $M \to \infty$ , we can conclude that for all  $t \geq t_0$ 

$$\limsup_{\|x\|_{\infty} \to \infty} \frac{\log u(t, x)}{(\log \|x\|_{\infty})^{\frac{2}{4-d}}} \le \left(\frac{d}{\mathfrak{c}_d}\right)^{\frac{2}{4-d}} t$$

with probability one. This completes the proof.

#### 7. Spatio-temporal Multifractality: Proof of Theorem 1.2

#### 7.1. Proof of the lower bound in Theorem 1.2.

**Proposition 7.1.** Let  $\epsilon > 0$  and  $\theta > 0$ . There exists  $c, t_0 > 0$  such that for all  $t \geq t_0$  and  $x_1, ..., x_m \in \mathbb{R}^d$  satisfying  $\min_{i \neq j} \|x_i - x_j\|_{\infty} > 3L_t$  where  $L_t := t$ , we have

$$\mathbb{P}\left(\max_{1\leq j\leq m}\log u(t,x_j)\leq \beta t^{\frac{6-d}{4-d}}\right)\leq \exp\left(-cm(\beta+\epsilon)^{\frac{d}{2}}t^{\frac{d}{4-d}}e^{d\log r_t-\mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}t}\right)+e^{-cm\log t},\tag{7.1}$$

where  $r_t := t^{\frac{1}{2}}$ .

*Proof.* The proof is similar to the proof of Proposition 6.1. By Lemma 5.6, we have

$$\mathbb{P}\left(\max_{1\leq j\leq m}\log u(t,x_j)\leq \beta t^{\frac{6-d}{4-d}}\right)\leq \mathbb{P}\left(\max_{1\leq j\leq m}\mathcal{U}_0^{x_j}(t,x_j)\leq e^{\beta t^{\frac{6-d}{4-d}}}\right)=\prod_{j=1}^m\mathbb{P}\left(\mathcal{U}_0^{x_j}(t,x_j)\leq e^{\beta t^{\frac{6-d}{4-d}}}\right),$$

whenever  $\min_{i\neq j} \|x_i - x_j\|_{\infty} > 3L_t$  where  $L_t := t$ . Let  $r := r_t := t^{1/2}$ . The last equality follows due to the independence between  $\{\mathcal{U}_0^{x_j}(t,x_j)\}_{1\leq j\leq m}$ . There exist  $t_0 > 0$  such that for all  $t \geq t_0$ 

$$\epsilon t^2 \gg (\mathfrak{a}_0 \log L_t)^{2\aleph+2} + \frac{\mathfrak{a}_0 (\log L_t)^2 r_t^2}{C\delta}, \quad \frac{L_t^2}{C\delta} \gg t (\mathfrak{a}_0 \log L_t)^{2\aleph+2} + \frac{\mathfrak{a}_0 (\log L_t)^2 r_t^2}{C\delta}$$

on the event  $\Upsilon_t := \{\mathfrak{a}_0 \leq (\log t)^2\}$ . For d = 3, by Proposition 5.5 there exists  $t_0 > 0$  such that for all  $t \geq t_0$ ,

$$\begin{split} & \mathbb{P}\left(\mathcal{U}_{0}^{x_{j}}(t,x_{j}) \leq e^{\beta t^{3}}\right) \\ & \leq \mathbb{P}\left(e^{-C\mathfrak{a}_{0}^{\aleph+1}\delta(\log L_{t})^{2\aleph+2} - \frac{C\mathfrak{a}_{0}(\log L_{t})^{2}r_{t}^{2}}{\delta} + (t-\delta)\lambda_{1}(Q_{r_{t}}^{y})} - e^{C\mathfrak{a}_{0}^{2\aleph+1}t(\log L_{t})^{2\aleph+2} - \frac{L_{t}^{2}}{C\delta}} \leq e^{\beta t^{3}}\right) \\ & \leq \mathbb{P}\left(\left\{\frac{1}{2}\exp\left((t-\delta)\lambda_{1}(Q_{r_{n}}^{x_{j}}) - C\mathfrak{a}_{0}^{\aleph+1}\delta(\log L)^{2\aleph+2} - \frac{C\mathfrak{a}_{0}(\log L)^{2}r^{2}}{\delta}\right) \leq e^{\beta t^{3}}\right\} \cap \Upsilon_{t}\right) + \mathbb{P}(\neg \Upsilon_{t}) \\ & \leq \mathbb{P}\left(\lambda_{1}(Q_{r_{t}}^{x_{j}}) \leq (\beta+\epsilon)t^{2}\right) + \mathbb{P}(\neg \Upsilon_{t}). \end{split}$$

For d=2, we can proceed similarly using Lemma 5.2 of [KPvZ20] to obtain

$$\mathbb{P}(\mathcal{U}_{0}^{x_{j}}(t, x_{j}) \leq e^{\beta t^{2}}) \\
\leq \mathbb{P}\left(\exp\left((t - \delta)\boldsymbol{\lambda}_{1}(Q_{r_{t}}^{x_{j}}) - \frac{r_{t}^{2}}{C\delta} - C\delta(\mathfrak{a}_{0}\log L_{t})^{5}\right) - \exp(Ct(\mathfrak{a}_{0}\log L_{t})^{5} - \frac{L_{t}^{2}}{C\delta}\right) \leq e^{\beta t^{2}}\right) \\
\leq \mathbb{P}\left(\left\{\frac{1}{2}\exp((t - \delta)\boldsymbol{\lambda}_{1}(Q_{r_{t}}^{x_{j}}) - \frac{r_{t}^{2}}{C\delta} - C\delta(\mathfrak{a}_{0}\log L_{t})^{5}\right) \leq e^{\beta t^{2}}\right\} \cap \Upsilon_{t}\right) + \mathbb{P}(\neg \Upsilon_{n}) \\
\leq \mathbb{P}(\boldsymbol{\lambda}_{1}(Q_{r_{t}}^{x_{j}}) \leq (\beta + \epsilon)t) + \mathbb{P}(\neg \Upsilon_{t})$$

By Lemma 2.2, we have

$$\max_{1 \le j \le m} \mathbb{P}\left(\lambda_1(Q_{r_t}^{x_j}) \le (\beta + \epsilon)t^{\frac{6-d}{4-d}}\right) \le \exp\left(-c_2(\beta + \epsilon)^{\frac{d}{2}}t^{\frac{d}{4-d}}e^{d\log r_t - \mathfrak{c}_d(1+\epsilon)(\beta + \epsilon)^{\frac{4-d}{2}}t}\right).$$

Moreover, we have  $\mathbb{P}(\neg \Upsilon_t) \leq e^{-h_0 \log t}$  for some  $h_0 > 0$  by the fact that  $\mathbb{E}[e^{h_0 \sqrt{\mathfrak{a}_0}}] < \infty$ . This yields that

$$\mathbb{P}\left(\max_{1\leq j\leq m}\log u(t,x_j)\leq \beta t^3\right)\leq 2^{m-1}\exp(-cm(\beta+\epsilon)^{\frac{d}{2}}t^{\frac{d}{4-d}}e^{d\log r_t-\mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}t})+2^{m-1}e^{-h_0m\log t}.$$

By choosing  $t_0$  large such that  $2^{m-1}e^{-h_0m\log t} < e^{-cm\log t}$  for some constant c, we achieve the bound in (7.1). This completes the proof.

Now we are ready to prove the lower bound in Theorem 1.2. Recall that

$$\mathcal{P}^{d}(\beta, v) = \left\{ (t, x) \in (1, \infty) : u(v \log t, x) > e^{\beta(v \log t)^{\frac{2}{4-d}}} \right\}$$

**Theorem 7.2.** For every  $\beta, v > 0$ , with probability one,

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}^{d}(\beta, v)] \ge (d + 1 - \beta^{\frac{4-d}{2}} v \mathfrak{c}) \lor d. \tag{7.2}$$

*Proof.* Choose  $\beta, \epsilon, v > 0$  such that  $(\beta + \epsilon)^{\frac{4-d}{2}} (1 + \epsilon) v \mathfrak{c}_d < d + 1$ . Let us define

$$\widetilde{\mathcal{P}}^d(\beta, v) := \mathcal{P}^d(\beta, v) \cap \bigcup_{n=0}^{\infty} (e^n, e^{n+1}]^{d+1}.$$

Since  $\widetilde{\mathcal{P}}^d(\beta, v) \subseteq \mathcal{P}^d(\beta, v)$ , it is enough to show  $\operatorname{Dim}_{\mathbb{H}}[\widetilde{\mathcal{P}}^d(\beta, v)] \geq (d+1-\beta^{\frac{4-d}{2}}v\mathfrak{c}_d) \vee d$  with probability one. We choose  $\gamma \in (\frac{(\beta+\epsilon)^{\frac{4-d}{2}}(1+\epsilon)v\mathfrak{c}_d}{d}, 1)$  and  $\theta \in (0, \gamma - \frac{(\beta+\epsilon)^{\frac{4-d}{2}}(1+\epsilon)v\mathfrak{c}_d}{d})$ . We borrow the notations  $a_{j,n}$  and  $I_n(\gamma)$  from Theorem 6.2. Suppose  $I_n^k(\gamma)$  are copies of  $I_k(n)$  for  $1 \leq k \leq d+1$ . We introduce further

$$\tilde{\mathcal{I}}_n(\gamma) := \prod_{k=1}^{d+1} I_n^k(\gamma),$$

where  $I_n^k(\gamma)$  is a copy of  $I_n(\gamma)$  for all  $1 \leq j \leq d+1$ . We choose  $x \in \tilde{\mathcal{I}}_n(\gamma)$  and take the points  $\{x_i\}_{i=1}^{m(n)}$  such that they satisfy the following conditions: (b.1)  $x_i \in \tilde{\mathcal{I}}_n(\gamma) \cap B(x, e^{n\gamma})$  for all i = 1, ..., m(n); (b.2)  $|x_i - x_j| \geq e^{n\theta}$  whenever  $1 \leq i < j \leq m(n)$ ; (b.3)  $d^{-1}e^{dn(\gamma-\theta)} \leq m(n) \leq de^{dn(\gamma-\theta)}$ .

Observe that  $e^{n\theta} \gg 3L_{v \log t} = 3(v \log t)$  for all  $t \in (e^n, e^{n+1}]$ . We now notice that there exists  $n_0 > 0$  such that for all  $n \ge n_0$ ,

$$\begin{split} & \mathbb{P}\Big(\mathcal{P}_1^d \cap (\{t\} \times B(x, e^{n\gamma})) = \varnothing \text{ for some } t \in (e^n, e^{n+1}] \text{ and } x \in \tilde{\mathcal{I}}_n(\gamma)\Big) \\ & \leq \mathbb{P}\Big(\min_{\substack{t \cap \mathbb{Z} \\ t \in (e^n, e^{n+1}]}} \min_{x \in \tilde{\mathcal{I}}_n(\gamma)} \max_{\{x_i\}_{i=1}^{m(n)} \subseteq \tilde{\mathcal{I}}_n(\theta) \cap B(x, e^{n\gamma})} \Big\{e^{-\beta(v \log t)^{\frac{6-d}{4-d}}} u(v \log t, x)\Big\} \leq 1\Big) \\ & \leq \sum_{\substack{t \cap \mathbb{Z} \\ t \in (e^n, e^{n+1}]}} \sum_{x \in \tilde{\mathcal{I}}_n(\gamma)} \mathbb{P}\left(\max_{\{x_i\}_{i=1}^{m(n)} \subseteq \tilde{\mathcal{I}}_n(\theta) \cap B(x, e^{n\gamma})} u(v \log t, x) \leq e^{\beta(v \log t)^{\frac{6-d}{4-d}}} \right) \\ & \leq Ce^{dn(1-\gamma)+n} \cdot \exp\left(-cm(n)(\beta+\epsilon)^{\frac{d}{2}}(v(n+1))^{\frac{d}{4-d}}e^{d \log r_{v(n+1)}-\mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}v(n+1)} \right) + e^{-cm(n)v(n+1)} \\ & \leq Ce^{dn(1-\gamma)+n} \cdot \left[\exp\left(-c(\beta+\epsilon)^{\frac{d}{2}}v(n+1)^{\frac{d}{4-d}}e^{\log[v(n+1)]+\kappa n - \mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}v}\right) \\ & + \exp\left(-\frac{c}{2}e^{dn(\gamma-\theta)}v(n+1)\right)\right], \end{split}$$

where  $\kappa := d(\gamma - \theta) - \mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}v > 0$  by the choice of  $\gamma$  and  $\theta$ . The first inequality is straightforward. The inequality follows by applying the union bound and the third inequality is obtained by applying Proposition 7.1. The right hand side of the above inequality is summable w.r.t. n. Hence, by the Borel-Cantelli lemma, there exists  $n_0 > 0$  such that for all  $n \ge n_0$ 

$$\widetilde{\mathcal{P}}^d(\beta, v) \cap (\{t\} \times B(x, e^{n\gamma})) \neq \emptyset \text{ for all } t \in (e^n, e^{n+1}] \text{ and all } x \in \mathcal{I}_n(\gamma).$$
 (7.3)

This implies that  $\mu_n(\widetilde{\mathcal{P}}^d(\beta, v)) \geq Ce^{dn(1-\gamma)+n}$  where  $\mu_n$  is defined in Proposition B.4. Therefore, by Proposition B.4 we can deduce that  $\sum_n \nu_{n,d+1-d\gamma}(\widetilde{\mathcal{P}}^d(\beta, v)) = \infty$  almost surely, which shows that  $\operatorname{Dim}_{\mathbb{H}}(\widetilde{\mathcal{P}}^d(\beta, v)) \geq 4 - 3\gamma$ . Letting  $\gamma \downarrow \frac{\mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}v}{d}$  and  $\theta \downarrow 0$  without violating  $\theta \in (0, \gamma - \frac{\mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}v}{d})$ , we get  $\operatorname{Dim}_{\mathbb{H}}(\widetilde{\mathcal{P}}^d(\beta, v)) \geq d+1 - \mathfrak{c}_d(1+\epsilon)(\beta+\epsilon)^{\frac{4-d}{2}}v$ . Since  $\epsilon > 0$  is arbitrary,  $\operatorname{Dim}_{\mathbb{H}}(\widetilde{\mathcal{P}}^d(\beta, v)) \geq d+1 - \beta^{\frac{4-d}{2}}v\mathfrak{c}_d$ .

Now it remains to show  $\operatorname{Dim}_{\mathbb{H}}(\widetilde{\mathcal{P}}^d(\beta, v)) \geq d$ , a.s. for any  $\beta, v > 0$ . First note that

$$\left\{ (s,x) \in (1,\infty) \times \mathbb{R}^d : u(v\log s,x) > e^{\beta(v\log s)^{\frac{6-d}{4-d}}} \right\} \supseteq \{t\} \times \left\{ x \in \mathbb{R}^d : u(v\log t,x) > e^{\beta(v\log t)^{\frac{6-d}{4-d}}} \right\},$$

for all  $t \geq 1$ . Let us define

$$\mathcal{P}^{(t)} := \left\{ x \in \mathbb{R}^d : u(v \log t, x) > e^{\beta(v \log t)^{\frac{6-d}{4-d}}} \right\},$$

for any t > 1. Then, it suffices to show  $\text{Dim}_{\mathbb{H}}(\mathcal{P}^{(t)}) \geq d$ , a.s. for some t > 1. Indeed,

$$\operatorname{Dim}_{\mathbb{H}}\left(\left\{t\right\} \times \left\{x \in \mathbb{R}^d : u(v \log t, x) > e^{\beta(v \log t)^{\frac{6-d}{4-d}}}\right\}\right) = \operatorname{Dim}_{\mathbb{H}}(\mathcal{P}^{(t)}),$$

for any fixed  $t \ge 1$  (see [BT92, Section 9]). Let  $t_0$  be the constant in Theorem 6.2. Let  $\beta, v > 0$ . Observe that for all  $(M, \alpha) \in \mathbb{R}^2$  such that

$$M \ge e$$
,  $\alpha \ge \frac{\beta(v \log t_0)^{\frac{6-d}{4-d}}}{t_0(\log M)^{\frac{2}{4-d}}}$ ,

we have

$$\mathcal{P}_{M}^{(t_0)} := \left\{ \|x\|_{\infty} \ge M : u(v \log t_0, x) > e^{\beta(v \log t_0)^{\frac{6-d}{4-d}}} \right\} \supseteq \left\{ \|x\|_{\infty} \ge M : u(v \log t_0, x) > e^{\alpha t_0(\log \|x\|_{\infty})^{\frac{2}{4-d}}} \right\}.$$

Note that  $\mathcal{P}_{M}^{(t_0)}$  and  $\mathcal{P}^{(t_0)}$  have the same macroscopic Hausdorff dimension since  $\operatorname{Dim}_{\mathbb{H}}[E] = 0$  for every bounded set  $E \in \mathbb{R}^d$ . Therefore, by Theorem 6.2, we have

$$\operatorname{Dim}_{\mathbb{H}}(\mathcal{P}^{(t_0)}) = \operatorname{Dim}_{\mathbb{H}}(\mathcal{P}_M^{(t_0)}) \ge d - \frac{\beta(v \log t_0)^{\frac{6-d}{4-d}}}{t_0(\log M)^{\frac{2}{4-d}}}.$$

By taking  $M \uparrow \infty$ , we can conclude that  $Dim_{\mathbb{H}}(\mathcal{P}^{(t_0)}) \geq d$ , a.s.

# 7.2. Proof of the upper bound in Theorem 1.2.

**Proposition 7.3.** Let  $0 < \epsilon < \beta$  and M > 0. There exist b := b(M) > 1 and  $n_0 := n_0(M, b, \epsilon) > 0$  such that for all  $n \ge n_0$  and  $t \ge 1$ 

$$\mathbb{P}\Big(\text{ For some } t \in (a, a+l] \text{ s.t. } \sup_{x \in B(y,1)} \log u(t,x) \ge \beta t^{\frac{6-d}{4-d}}, \mathfrak{a}_0 \le M\Big) \\
\le c_1(\beta - \epsilon)^{\frac{d}{2}} a^{\frac{d}{4-d}} e^{db \log(a+l) - (1-\epsilon)\mathfrak{c}_d(\beta - \epsilon)^{\frac{4-d}{2}} a}, \tag{7.4}$$

where b > 1 and  $y \in \mathbb{R}^d$ .

*Proof.* We use similar argument as in Proposition 6.3. Applying Lemma 5.6 with  $L_t := t^b$ , we can write  $u(t,x) = \sum_{k=0}^{\infty} \mathcal{U}_k^y(t,x)$  for any  $y \in \mathbb{R}^d$ . Then we have

$$\mathbb{P}\Big(\text{For some }t\in(a,a+l]\sup_{x\in B(y,1)}\log u(t,x)\geq\beta t^{\frac{6-d}{4-d}},\mathfrak{a}_0\leq M\Big)\leq (\mathbf{D_1})+(\mathbf{D_2}),$$

where

$$(\mathbf{D_1}) := \mathbb{P}\Big(\text{For some } t \in (a, a+l] \sup_{x \in B(y,1)} \mathcal{U}_0^y(t, x) \ge \frac{1}{2} e^{\beta t^{\frac{6-d}{4-d}}}, \mathfrak{a}_0 \le M\Big),$$

$$(\mathbf{D_2}) := \mathbb{P}\Big(\text{For some } t \in (a, a+l] \sup_{x \in B(y,1)} \sum_{k=1}^{\infty} \mathcal{U}_k^y(t, x) \ge \frac{1}{2} e^{\beta t^{\frac{6-d}{4-d}}}, \mathfrak{a}_0 \le M\Big).$$

For d = 3, on the event  $\{\mathfrak{a}_0 \leq M\}$ , we have

$$\mathcal{U}_k^y(t,x) \le C \exp\left(CtM^{\aleph+1}(b(k+1)\log t)^{2\aleph+2} - \frac{t^{2bk}}{Ct}\right).$$

by applying (5.33). For d=2, similar bounds follows again from Lemma 5.2 of [KPvZ20]. Now we can choose a large b:=b(M)>0 such that for all  $t\geq e$  and  $n\geq 1$ 

$$\sum_{k=1}^{\infty} \mathcal{U}_k^y(t, x) \le \sum_{k=1}^{\infty} C \exp\left(-C_1 t^{2bk-1}\right) \le \frac{1}{2} e^{\alpha t n^{\frac{2}{4-d}}},\tag{7.5}$$

which shows that  $(\mathbf{D_2}) = 0$  for all  $a \geq e$ . We now bound  $(\mathbf{D_1})$ . Fix  $\epsilon \in (0, \beta \wedge 1)$  and use Proposition 5.5 to obtain that there exists  $a_0 := a_0(b, M, \epsilon) > 0$  such that for all  $a \geq a_0$ 

$$(\mathbf{D_{1}}) \leq \mathbb{P}\Big(\text{For some } t \in (a, a + l], \ C \exp(t\lambda_{1}(Q_{L_{t}}^{y}) + CM^{v+1}(\log L_{t})^{2v+2}) \geq \frac{1}{2}e^{\beta t^{\frac{6-d}{4-d}}}\Big)$$

$$\leq \mathbb{P}\Big(\text{For some } t \in (a, a + l], \ \lambda_{1}(Q_{L_{t}}^{y}) \geq \beta t^{\frac{2}{4-d}} - \frac{\log(2C) + CM^{v+1}(b\log t)^{2v+2}}{t}\Big)$$

$$\leq \mathbb{P}(\text{For some } t \in (a, a + l], \ \lambda_{1}(Q_{L_{t}}^{y}) \geq (\beta - \epsilon)a^{\frac{2}{4-d}}).$$

$$(7.6)$$

By Lemma 2.2 and Lemma 2.3, we have

$$(\mathbf{D_1}) \le \mathbb{P}(\lambda_1(Q_{L_{a+l}}^y) \ge (\beta - \epsilon)a^{\frac{2}{4-d}})) \le c_1(\beta - \epsilon)^{\frac{d}{2}}a^{\frac{d}{4-d}}e^{db\log(a+l) - (1-\epsilon)\mathfrak{c}_d(\beta - \epsilon)^{\frac{4-d}{2}}a}. \tag{7.7}$$

This completes the proof.

Now we proceed to prove the upper bound of macroscopic Hausdorff dimension of the set  $\mathcal{P}^d(\beta, v)$ .

**Theorem 7.4.** For every v > 0 and  $\beta \in (0, (d/(v\mathfrak{c}_d))^{\frac{2}{4-d}})$ , with probability one,

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}^{d}(\beta, v)] \le (d + 1 - \beta^{\frac{4-d}{2}} v \mathfrak{c}_{d}) \lor d. \tag{7.8}$$

*Proof.* For  $\epsilon := (\epsilon_1, ..., \epsilon_d) \in \{-1, 1\}^d$ , define an (open) orthant as

$$\mathcal{O}_{\epsilon} := \left\{ (t, x) = (t, x_1, ..., x_d) \in (1, \infty) \times \mathbb{R}^d : \epsilon_1 x_1 > 0, \epsilon_2 x_2 > 0, ..., \epsilon_d x_d > 0 \right\}.$$

We then define

$$\mathcal{P}^d_{\epsilon}(\beta, v) := \mathcal{P}^d(\beta, v) \cap \mathcal{O}_{\epsilon}.$$

In order to prove this theorem, it suffices to prove that  $\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_{\epsilon_{+}}^{d}(\beta, v)](d + 1 - \beta^{\frac{4-d}{2}}v\mathfrak{c}_{d}) \vee d$  for any  $\epsilon \in \{-1, 1\}^{d}$ . Due to symmetry between different orthants, it further suffices to prove for  $\epsilon_{+} := (1, ..., 1) \in \{-1, 1\}^{d}$  that

$$\operatorname{Dim}_{\mathbb{H}}[\mathcal{P}_{\epsilon_{+}}^{d}(\beta, v)] \leq (d + 1 - \beta^{\frac{4-d}{2}} v \mathfrak{c}_{d}) \vee d. \tag{7.9}$$

For q > 1 and  $n \in \mathbb{N}$ , let us denote  $\mathcal{L}_n := \mathcal{L}_n(q, n, \beta, v, d) := \mathcal{P}_{\epsilon_+}^d(\beta, v) \cap \mathcal{I}_n^{(q)}$  where  $\mathcal{I}_n^{(q)} := (e^{n/q}, e^{n+1}]^{d+1}$ . By Lemma B.5, we have

$$\operatorname{Dim}_{\mathbb{H}} \left[ \mathcal{P}_{\epsilon_{+}}^{d}(\beta, v) \setminus \bigcup_{n=0}^{\infty} \mathcal{L}_{n} \right] \leq d.$$

Since  $\operatorname{Dim}_{\mathbb{H}}(A \cup B) = \max\{\operatorname{Dim}_{\mathbb{H}}(A), \operatorname{Dim}_{\mathbb{H}}(B)\}\$  for any two sets  $A, B \subseteq \mathbb{R}^d$ , we have

$$\operatorname{Dim}_{\mathbb{H}}\left[\mathcal{P}^{d}_{\epsilon_{+}}(\beta,v)\right] \leq \operatorname{Dim}_{\mathbb{H}}\left[\mathcal{P}^{d}_{\epsilon_{+}}(\beta,v) \setminus \bigcup_{n=0}^{\infty} \mathcal{L}_{n}\right] \vee \operatorname{Dim}_{\mathbb{H}}\left[\bigcup_{n=0}^{\infty} \mathcal{L}_{n}\right].$$

Let  $\bar{\mathcal{L}} := \bigcup_{n=0}^{\infty} \mathcal{L}_n$ . The above inequality implies that to show (7.9), it is enough to prove

$$\operatorname{Dim}_{\mathbb{H}}(\bar{\mathcal{L}}) \leq (d+1-\beta^{\frac{4-d}{2}}v\mathfrak{c}_d).$$

To this end, observe that Proposition 7.3 implies for all  $a \in (e^{n/q}, e^{n+1}]$ 

$$\mathbb{P}\left(\text{For some } t \in (a, a+1] \sup_{x \in B(y,1)} \log u(v \log t, x) \ge \beta(v \log t)^{\frac{6-d}{4-d}}, \mathfrak{a}_0 \le M\right) \\
= \mathbb{P}\left(\text{For some } t \in (v \log a, v \log(a+1)] \sup_{x \in B(y,1)} \log u(t, x) \ge \beta t^{\frac{6-d}{4-d}}, \mathfrak{a}_0 \le M\right) \\
\le c_1(\beta - \epsilon)^{\frac{d}{2}} \left(\frac{vn}{a}\right)^{\frac{d}{4-d}} \exp\left(db \log(2vn) - \frac{(1-\epsilon)\mathfrak{c}_d(\beta - \epsilon)^{\frac{4-d}{2}}vn}{a}\right), \tag{7.10}$$

For all sufficiently large  $n \in \mathbb{N}$ , we cover  $\mathcal{L}_n \subseteq \mathcal{I}_n^{(q)}$  with  $O(e^{d+1})$ -many boxes of the form  $(a, a+1] \times B(y,1)$  satisfying that for some  $t \in (a, a+1]$ 

$$\sup_{x \in B(u,1)} \log u(v \log t, x) \ge \beta(v \log t)^{\frac{6-d}{4-d}}. \tag{7.11}$$

on the event  $\Upsilon_M := \{\mathfrak{a}_0 \leq M\}$ . Choose

$$\rho \in \left(d+1 - \frac{\mathfrak{c}_d \beta_{\epsilon}^{\frac{4-d}{2}} vn}{q}, d+1\right].$$

By (7.10), we have that for all sufficiently large  $n \ge 1$  and for all  $\rho > 0$ ,

$$\mathbb{E}[\nu_{\rho}^{n}(\mathcal{L}_{n})\mathbb{1}_{\Upsilon_{M}}] \leq \mathbb{E}\Big[\sum_{\substack{(a,a+1] \times B(y,1) \subseteq \mathcal{I}_{n}^{q}:\\ (7,11) \text{ holds}}} e^{-n\rho} \cdot \mathbb{1}_{\Upsilon_{M}}\Big]$$

$$\leq C \exp\Big(\Big\{d+1-\rho-\frac{\mathfrak{c}_d\beta_\epsilon^{\frac{4-d}{2}}vn}{q}\Big\}n+Cb\log n\Big),$$

where C > 0 is a constant which depends only on  $(v, \beta)$  and  $\beta_{\epsilon} := (1 - \epsilon)^{\frac{2}{4-d}} (\beta - \epsilon)$ . This implies that

$$\mathbb{P}\Big(\Upsilon_M\cap\Big\{\sum_{n=0}^\infty\nu_\rho^n(\bar{\mathcal{L}})<\infty\Big\}\Big)=\mathbb{P}\big(\Upsilon_M\big).$$

Because  $\mathbb{P}(\Upsilon_M) \geq 1 - e^{-h\sqrt{M}}\mathbb{E}[e^{\sqrt{\mathfrak{a}_0}}]$  (see (6.7)), we have

$$\mathbb{P}\Big(\sum_{n=0}^{\infty}\nu_{\rho}^{n}(\bar{\mathcal{L}})<\infty\Big)\geq 1-e^{-h\sqrt{M}}\mathbb{E}[e^{\sqrt{\mathfrak{a}_{0}}}],$$

which in turn implies  $\mathbb{P}(\operatorname{Dim}_{\mathbb{H}}(\bar{\mathcal{L}}) \leq \rho) \geq 1 - e^{-h\sqrt{M}}\mathbb{E}[e^{\sqrt{\mathfrak{a}_0}}]$ . Since the definition of  $\bar{\mathcal{L}}$  does not depend on M and M>0 can be arbitrarily large, we have  $\operatorname{Dim}_{\mathbb{H}}(\bar{\mathcal{L}}) \leq \rho$  almost surely. Taking  $\rho \downarrow d+1-\frac{\mathfrak{c}_d\beta_\epsilon^{\frac{4-d}{2}}vn}{q}$ , we also have  $\operatorname{Dim}_{\mathbb{H}}(\bar{\mathcal{L}}) \leq d+1-\frac{\mathfrak{c}_d\beta_\epsilon^{\frac{4-d}{2}}v}{q}$ . Moreover, since q>1 and  $\epsilon \in (0,\beta\wedge 1)$  are arbitrary, we can let  $q\to 1$  and  $\epsilon\to 0$  to conclude that  $\operatorname{Dim}_{\mathbb{H}}(\bar{\mathcal{L}}) \leq d+1-\mathfrak{c}_d\beta^{\frac{4-d}{2}}v$ . This completes the proof.

### Appendix A. Besov space and paracontrolled generator

We start with introducing few notations about the function spaces and the paracontrolled calculus. Let  $\chi$  and  $\varrho$  be non-negative radial function such that

- (1) the support of  $\chi$  is contained in a ball and the support of  $\varrho$  is contained in an annulus  $\{x \in \mathbb{R}^d : 1 < |x| < 2\}.$
- (2)  $\chi(\xi) + \sum_{j \ge 0} \varrho(2^{-j}\xi) = 1 \text{ for all } \xi \in {}^d.$
- (3)  $\operatorname{Supp}(\chi) \cap \operatorname{Supp}(\varrho(2^{-j}\cdot)) = \emptyset$  for  $i \geq 1$  and  $\operatorname{Supp}(\varrho(2^{-i}\cdot)) \cap \operatorname{Supp}(\varrho(2^{-j}\cdot)) = \emptyset$  when |i-j| > 1. To this end,  $(\chi, \varrho)$  satisfying the above properties are said to form a dyadic partition of unity. For the existence, we refer to [BCD11, Proposition 2.10].

For any Schwartz distribution f, we define the Littlewood-Paley blocks by

$$\Delta_{i} f = \sum_{k \in \mathbb{N}_{0}^{d}} \langle f, \mathfrak{n}_{k,L} \rangle \varrho_{i} \left(\frac{k}{L}\right) \mathfrak{n}_{k,L} \tag{A.1}$$

where  $\{\mathfrak{n}_{k,L} : k \in \mathbb{N}_0\}$  forms an orthonormal basis of  $L^2([0,L]^d)$  given in  $[\operatorname{CvZ21}$ , Section 4] and  $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ . We also note that since  $\varrho_j$  is supported in a ball with radius  $2^j$  and  $\varrho_j \leq 1$ , for all  $j \in \mathbb{N}_0$ ,  $x \in \mathbb{R}^d$  and  $\gamma > 0$ 

$$\varrho_j(x) \lesssim \left(\frac{2^j}{1+|x|}\right)^{\gamma}.$$
(A.2)

For  $u \in \mathcal{S}'$ , we define  $(1 - \frac{1}{2}\Delta)^{-1}u$  by

$$(1 - \frac{1}{2}\Delta)^{-1}u := \sum_{k \in \mathbb{N}_0^d} \sigma\left(\frac{k}{L}\right) \langle u, \mathfrak{n}_{k,L} \rangle \mathfrak{n}_{k,L}, \tag{A.3}$$

where  $\sigma(x) := (1 + \pi |x|^2)^{-1}$ .

Denote the Fourier transform operator by  $\mathfrak{F}$  and let  $(\chi, \varrho)$  be the dyadic partition of unity. Then the Littlewood-Paley blocks are defined as

$$\Delta_{-1}u = \mathfrak{F}^{-1}(\chi \mathfrak{F}(u)), \quad \Delta_{j}u = \mathfrak{F}^{-1}(\varrho_{j}(\cdot)\mathfrak{F}(u)), \quad j \ge 0$$
(A.4)

where  $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$  and, for  $\alpha \in p, q \in [1, \infty]$ , the Besov space  $B_{p,q}^{\alpha}(d^n)$  is

$$B_{p,q}^{\alpha}(^{d},^{n}) := \{ u \in \mathcal{S}'(^{d},^{n}); \quad \|u\|_{B_{p,q}^{\alpha}(^{d},^{n})}^{q} = \sum_{j \ge -1} 2^{jq\alpha} \|\Delta_{j}u\|_{L^{p}(^{d},^{n})}^{q} < \infty \}$$
(A.5)

We often use the notation  $\mathscr{C}^{\alpha}_{p}({}^{d},{}^{n})$  to denote  $B^{\alpha}_{p,p}({}^{d},{}^{n})$  for  $p \in [1,\infty]$  and write  $\mathscr{C}^{\alpha}({}^{d},{}^{n})$  for  $\mathscr{C}^{\alpha}_{\infty}({}^{d},{}^{n})$ . This notation is consistent with the fact that  $\mathscr{C}^{\alpha}({}^{d},{}^{n})$  is indeed space of all  $\alpha$ -Hölder continuous function. For simplicity, we sometimes use the notation  $\mathscr{C}^{\alpha}_{p}$  for  $\mathscr{C}^{\alpha}_{p}(\mathbb{R}^{d},\mathbb{R})$ .

Let  $\delta, \rho > 0$ , T > 0, and  $\bar{T} \in [0, T)$ . Let  $(D, \| \cdot \|_D)$  be a Banach space and  $u, v : [T - \bar{T}, T] \to D$  be function (or distribution) valued processes. We say that  $u \in C^{\delta}_{\rho, \bar{T}, T}D$  if  $\|u\|_{C^{\delta}_{\rho, \bar{T}, T}D} < \infty$  and  $v \in C^{\delta}_{\bar{T}, T}D$  if  $\|v\|_{C^{\delta}_{\bar{T}, T}D} < \infty$  where

$$||u||_{C^{\delta}_{\rho,\bar{T},T}D} := \sup_{s < t \in (T-\bar{T},T]} (T-t)^{\delta} \frac{||u(t) - u(s)||_{D}}{|t-s|^{\rho}},$$

$$||v||_{C^{\delta}_{\bar{T},T}D} := \sup_{t \in (T-\bar{T},T]} (T-t)^{\delta} ||v(t)||_{D}.$$
(A.6)

In the case of  $\delta = 0$  or  $\bar{T} = T$ , then we drop the respective subscripts in the above definition.

A.1. Some properties of the Besov-Hölder continuous distributions. Let f and g be two distributions in  $\mathscr{S}'(^d)$ . Then the Paley-Littlewood decomposition of fg is written as

$$fq = f \prec q + f \circ q + f \succ q$$

where  $f \prec g$  and  $f \succ g$  are called *paraproducts* and  $f \circ g$  is called the *resonant terms* and they are defined as

$$f \prec g = f \succ g = \sum_{j \ge -1} \sum_{i < j - 1} \Delta_i \Delta_j g$$
, and  $f \circ g = \sum_{j \ge -1} \sum_{|i - j| \le 1} \Delta_i \Delta_j g$ .

In the following propositions, we note few useful properties of the paraproduct.

**Proposition A.1** (Bony's estimates (I), [BCD11]). Let  $\alpha, \beta \in \mathbb{R}$ . Let  $f \in \mathscr{C}^{\alpha}$  and  $f \in \mathscr{C}^{\beta}$ ,

- (1) If  $\alpha > 0$ , then  $f \prec g \in \mathscr{C}^{\beta}$  and  $||f \prec g||_{\beta} \le ||f||_{L^{\infty}} ||g||_{\beta}$
- (2) If  $\alpha < 0$ , then  $f \prec g \in \mathscr{C}^{\alpha+\beta}$  and  $\|f \prec g\|_{L^{\infty}} \lesssim \|f\|_{\alpha} \|g\|_{\beta}$ .
- (3) If  $\alpha + \beta > 0$ , then  $f \circ g \in \mathscr{C}^{\alpha + \beta}$  and  $||f \circ g||_{\alpha + \beta} \lesssim ||f||_{\alpha} ||g||_{\beta}$ .

**Proposition A.2** (Bony's estimates (II), [BCD11]). Let  $\alpha < 0, \beta > 0$  and  $\alpha + \beta > 0$ . Let  $p, p_1, p_2, q_1, q_2 \in [1, \infty]$  be satisfy  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ . Let  $f \in B^{\alpha}_{p_1, q_1}$  and  $g \in B^{\beta}_{p_2, q_2}$ . For  $q \ge q_1$ 

- $(1) \| f \prec g \|_{B^{\alpha+\beta}_{p,q}} \lesssim \| f \|_{B^{\alpha}_{p_1,q_1}} \| g \|_{B^{\beta}_{p_2,q_2}}.$
- (2)  $||f \succ g||_{B_{p,q}^{\alpha}} \lesssim ||f||_{B_{p_1,q_1}^{\alpha}} ||g||_{B_{p_2,q_2}^{\beta}}$
- $(3) \ \|f\circ g\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{\beta}_{p_2,q_2}}.$

**Proposition A.3** (Schauder's estimate, Lemma 2.5 of [CC18b], Lemma A.8 of [GIP15a]). Let  $P_t$  be the heat semigroup for  $\frac{1}{2}\Delta$ . Let  $\theta \geq 0, p \in [1, \infty]$  and  $\alpha \in \mathbb{R}$ . Then for  $\phi \in \mathscr{C}_p^{\alpha}$  and  $0 \leq s \leq t$  we have

$$||P_t \phi||_{\mathscr{C}_p^{\alpha+2\theta}} \lesssim t^{-\theta} ||\phi||_{\mathscr{C}_p^{\alpha}}, \qquad ||(P_{t-s} - Id)\phi||_{\mathscr{C}_p^{\alpha-2\theta}} \lesssim |t-s|^{\theta} ||\phi||_{\mathscr{C}_p^{\alpha}}. \tag{A.7}$$

# APPENDIX B. MACROSCOPIC HAUSDORFF DIMENSION

In this section, we introduce the notion of the macroscopic Hausdorff dimension given by Barlow and Taylor [BT89, BT92], and Khoshnevisan-Kim-Xiao [KKX17]. We also present a useful propositions that help to provide lower bound and upper bounds to the macroscopic Hausdorff dimension of any given set.

B.1. **Definition.** For all integers  $n \geq 1$ , we define the exponential cubes and shells as follows:

$$V_n := [-e^n, e^n)^d, \quad S_0 := V_0, \quad \text{and} \quad S_{n+1} := V_{n+1} \setminus V_n.$$
 (B.1)

Let  $\mathcal{B}$  be the collection of all cubes of the form

$$B(x,r) := \prod_{i=1}^{d} [x_i, x_i + r), \tag{B.2}$$

for  $x=(x_1,...,x_d)\in\mathbb{R}^d$ , and  $r\in[1,\infty)$ . For any subset  $E\subset\mathbb{R}^d$ ,  $\rho>0$ , and all integers  $n\geq 1$ , we define

$$\nu_{\rho}^{n}(E) := \inf \left\{ \sum_{i=1}^{m} \left( \frac{s(B_{i})}{e^{n}} \right)^{\rho} : B_{i} \in \mathcal{B}, B_{i} \subset \mathbb{S}_{n} \text{ and } E \cap \mathbb{S}_{n} \subset \cup_{i=1}^{m} B_{i} \right\},$$
 (B.3)

where s(B) := r denotes the side of B = B(x, r). We now introduce the definition of the macroscopic Hausdorff dimension.

**Definition B.1.** [[BT89, BT92]] The macroscopic Hausdorff dimension of  $E \subset \mathbb{R}^d$  is defined as

$$\operatorname{Dim}_{\mathbb{H}}(E) := \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} \nu_{\rho}^{n}(E) < \infty \right\}. \tag{B.4}$$

B.2. Useful bounds for macroscopic Hausdorff dimension. Choose and fix any  $\theta \in (0,1)$ . We define

$$a_{j,n}(\theta) := e^n + je^{n\theta}, \qquad 0 \le j < e^{n(1-\theta)}$$
$$I_n(\theta) := \bigcup_{\substack{0 \le j \le e^{n(1-\theta)}:\\ j \in \mathbb{Z}}} \{a_{j,n}(\theta)\},$$

and

$$\mathcal{I}_n(\theta) := \prod_{i=1}^d I_n^i(\theta),$$

where  $I_n^i(\theta)$  is a copy of  $I_n(\theta)$  for each i. We call  $\bigcup_{n=1}^{\infty} \mathcal{I}_n(\theta)$  a  $\theta$ -skeleton of  $\mathbb{R}^d$  (see [KKX17, Definition 4.2]). Note that  $\operatorname{Dim}_{\mathbb{H}} (\bigcup_{n=k}^{\infty} \mathcal{I}_n(\theta)) = d(1-\theta)$  for any integer  $k \geq 1$ .

**Definition B.2** (Definition 4.3 of [KKX17]). E is called  $\theta$ -thick if there exists a positive integer  $k = k(\theta)$  such that

$$E \cap Q(x, e^{n\theta}) \neq \emptyset,$$

for all  $x \in \mathcal{I}_n(\theta)$  and  $n \geq k$ .

By the monotonicity of the macroscopic Hausdorff dimension, we get the following lower bound.

**Proposition B.3** (Proposition 4.4 of [KKX17]). Let  $E \subset \mathbb{R}^d$ . If E contains a  $\theta$ -thick set for some  $\theta \in (0,1)$ , then

$$Dim_{\mathbb{H}}(E) \ge d(1-\theta).$$

For the set of spatio-temporal peaks, we separately provide a proposition for the lower bound of the macroscopic Hausdorff dimension.

**Proposition B.4** (Theorem 4.1 of [BT92]). Fix  $\gamma \in (0,d)$ . For any set E and  $n \in \mathbb{Z}$ , let us define

$$\mu_n(E) = \sum_{\substack{s \in \mathbb{Z} \\ e^n < s \le e^{n+1} \\ \vec{j} \in [0, e^{n(1-\gamma)})}} \mathbb{1}\{(s, j) \in E\}.$$
(B.5)

Then, there exists a constant C > 0 such that  $\nu_{n,d+1-d\gamma}(E) \ge Ce^{-nd(1-\gamma)-n}\mu_n(E)$ .

*Proof.* The proof follows from [BT92, Theorem 4.1]. For the condition of [BT92, Theorem 4.1], it suffices to verify that  $\mu_n((s, s+r] \times B(x, r)) \lesssim r^{d+1}$  for all  $r \geq 1$ , which is proven below (4.24) of [Yi22].

The following lemma helps us to compute the upper bound of the macroscopic Hausdorff dimension.

**Lemma B.5** (Lemma 4.2 of [Yi22]). For any q > 1 and  $k \in \{1, ..., d-1\}$ , define a set  $E \subseteq \mathbb{R}^d$  as

$$E := \bigcup_{n=0}^{\infty} E_n,$$

where

$$E_n := (0, e^{n/q})^k \times (e^{n/q}, e^{n+1})^{d-k}.$$

Then we have  $Dim_{\mathbb{H}}[E] \leq d - k$ .

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