Computer Graphics Notes

HOMOGENEOUS COORINDATE SYSTEMS

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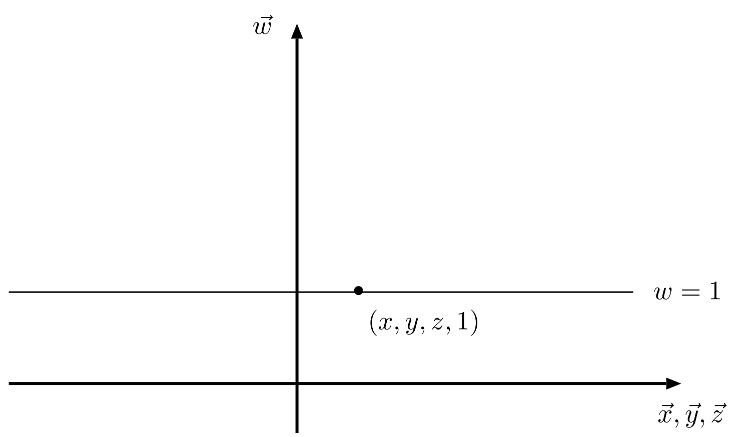
A 4-dimensional homogeneous coordinate system is frequently useful in representing the projective operations inherent in computer graphics systems. It has two principal uses:

- it allows the use of 4×4 matrices to represent general 3-dimensional transformations
- it allows a simplified representation of mathematical functions – the rational form – in which rational polynomial functions can be simply represented.

We will concentrate on the first of these in these notes.

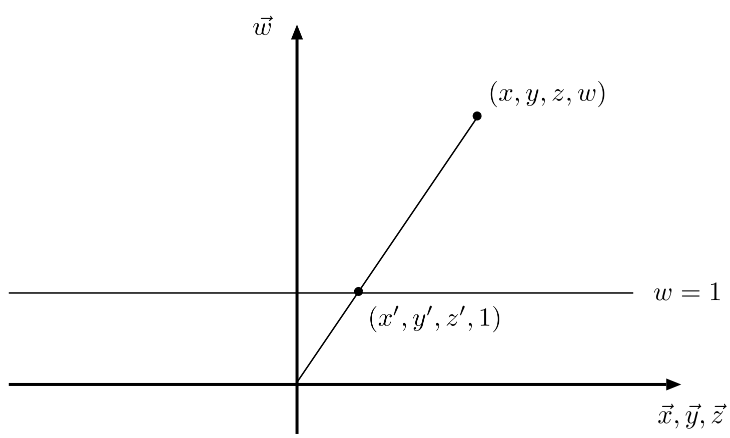
Points in four-dimensional homogeneous space are usually specified by (x, y, z, w). This is an extension of the usual coordinate representation that we utilize with frames $(x, y, z, 1)$. If we look at the coordinates $(x, y, z, 1)$ in four dimensions, we notice that they all lie on the space $w = 1$. This is illustrated in the figure below.



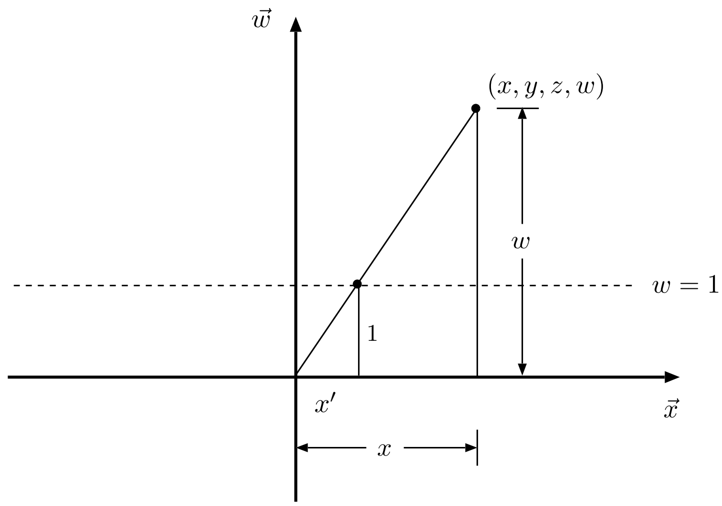
The transformations that we have defined all transform the fourth coordinate to one, thereby limiting the movement of the points to the $w = 1$ space. However, it will be necessary to utilize transformations that move the points away from this space creating a conflict – as long as the points are in the $w = 1$ space we can consider them as being in an exact copy of three-dimensional space ; if they move away from this space, how do we identify them with three-dimensional points any longer?

Well there are many ways to get a point away from the $w = 1$ space to identify with a point in the space. We will utilize a *projective* approach in which we will identify points out of the $w = 1$ space with points in the space as follows:

> *Given a coordinate* (x, y, z, w)*, we identify the point* $(x', y', z', 1)$ *in the* $w = 1$ *space with* (x, y, z, w) *if the point* $(x', y', z', 1)$ *is the unique point on the line connecting* (x, y, z, w) *with the origin.*



It is fairly easy to calculate the simple formula for identification of these points. If we look at the following figure, where we have focussed on two axes (the w and the x axis),



we can utilize a similar triangle argument to obtain that

$$x' = \frac{x}{w}$$

and by considering projections onto the other axes, we can also obtain that

$$y' = \frac{y}{w}$$

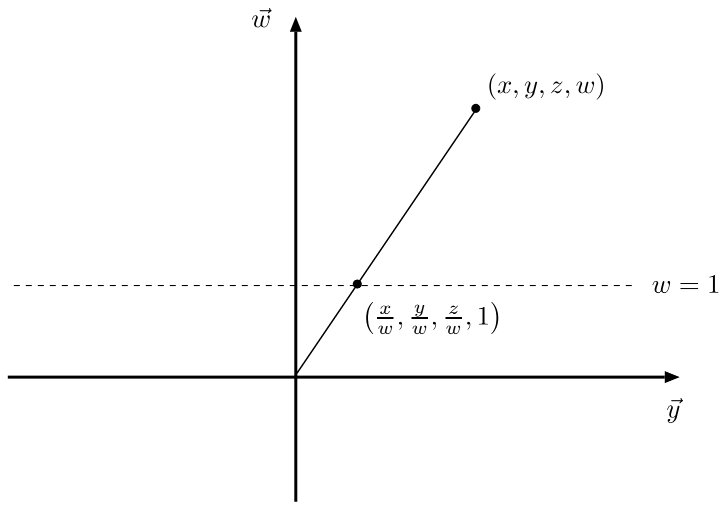
and

$$z' = \frac{z}{w}$$

Thus we have the identification

$$(x, y, z, w) \text{ identifies with } \left[\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1\right]$$

which is shown in the following figure



This mapping has the following properties:

- The space $w = 0$ is not in the domain of this mapping. Clearly if we consider the point $(x, y, z, 0)$, the line through this point and the origin does not intersect the space $w = 1$.
- The mapping is many-to-one, in that any point (xw, yw, zw, w) also maps to the point (x, y, z) for $w \neq 0$.
- Points (x, y, z, w) with $w < 0$ are in the domain of this mapping, as the line through this point and the origin does intersect the space $w = 1$.

The correspondence between 3-dimensional space and 4-dimensional homogeneous space is frequently utilized to apply 4-dimensional functions that define operators on 3-dimensional points. If we are given a coordinate (x, y, z) in 3-dimensional space, we apply a 4×4 matrix to the coordinate by multiplying

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \left[\begin{array}{c} \\ \quad 4 \times 4 \quad \\ \\ \end{array} \right] = \begin{bmatrix} x' & y' & z' & w \end{bmatrix}$$

The point (x', y', z', w') in homogeneous space is then projects to the space $w = 1$ by dividing by the fourth coordinate

$$\left[\frac{x'}{w}, \frac{y'}{w}, \frac{z'}{w}, 1 \right]$$

and is then identified with the coordinate $(\frac{x'}{w}, \frac{y'}{w}, \frac{z'}{w})$ in 3-dimensional space, which is considered to be the result of the transformation.
