CPSC 302 - Assignment 1

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Question 1

(a)

Proof by induction.

$$f(n) = 3n^3 - n^2, g(n) = n^3$$

$$c = 6, n_0 = 10$$

Base case.

$$|3(10)^3 - (10)^2| \leq 6(10)^3$$
$$|3000 - 100| \leq 6000$$
$$2900 \leq 6000$$

Induction step.

Assume

$$f(n-1) \leq cg(n-1)$$

.

We must show that $f(n) - f(n-1) \leq c(g(n) - g(n-1))$ since if f grows slower than g it must be smaller.

$$|3n^3 - n^2 - 3(n-1)^3 + (n-1)^2| \leq 6(n^3 - (n-1)^3)$$

$$|(n-1)^2 - n^2| \leq 3(n^3 - (n-1)^3)$$

This holds since $|(n-1)^2 - n^2|$ is trivially smaller than $3(n^3 - (n-1)^3)$.

Thus, $f(n) = O(g(n))$.

TODO: cleanup

(b)

Proof by induction.

$$f(n) = 3n^3 + n^2, g(n) = n^3$$

Show $f(n) = \Theta(g(n))$.

$$c_1 = 1, c_2 = 6, n_0 = 10$$

Base case: $n = 10$

$$(10)^3 \leq 3(10)^3 + (10)^2 \leq 6(10)^3$$
$$1000 \leq 3100 \leq 6000$$

Induction step.

Assume

$$f(n-1) \leq cg(n-1)$$

.

We must show that $c_1(g(n)-g(n-1)) \leq f(n)-f(n-1) \leq c_2(g(n)-g(n-1))$ since if f grows slower than g it must be smaller.

$$n^3-(n-1)^3 \leq 3n^3+n^2-3(n-1)^3-(n-1)^2 \leq 6n^3-6(n-1)^3$$

$$-2n^3+2(n-1)^3 \leq n^2-(n-1)^2 \leq 3n^3-3(n-1)^3$$

TODO: cleanup

Question 2

(a)

If you ever find a number less than 0, or greater than the number before you know there's an error.

(b)

True value $= 0.0024.$

```
u25ErrAbs = 2.3363e+12
u25ErrRel = 9.6964e+14
```

(c)

$$\hat{u}_0 = u_0 + \epsilon_0$$

$$\hat{u}_n = -k\hat{u}_{n-1} + \frac{1}{n}$$

$$\hat{u}_1 = -k\hat{u}_0 + \frac{1}{1} - u_1$$

$$\begin{aligned}
\hat{u}_1 &= -k(u_0+\epsilon_0) + \frac{1}{1} \\
\hat{u}_1 - u_1 &= -k(u_0+\epsilon_0) + \frac{1}{1} - u_1 \\
\hat{u}_1 - u_1 &= -k(u_0+\epsilon_0) + \frac{1}{1} + ku_0 - 1 \\
\epsilon_1 = \hat{u}_1 - u_1 &= -k\epsilon_0
\end{aligned}$$

(d)

$$\begin{aligned}
\epsilon_2 = \hat{u}_2 - u_2 &= -k(\hat{u}_1) + \frac{1}{2} + k(u_1) - \frac{1}{2} \\
\epsilon_2 &= -k(\hat{u}_1) + k(u_1) \\
\epsilon_2 &= -k(-k(u_0+\epsilon_0)+1) + k(-ku_0+1) \\
\epsilon_2 &= k^2u_0 + k^2\epsilon_0 - k - k^2u_0 + k \\
\epsilon_2 &= k^2\epsilon_0
\end{aligned}$$

General rule:

$$\epsilon_n = (-k)^n\epsilon_0$$

Since the error is exponential, every iteration gets $-k$ times as much error. Thus at $n = 25$, the error is $(15)^n = 2.5 * 10^{29}$ times worse than at $n = 0$.

(e)

The observed error is very close to what we calculated. When comparing the error for two sequential u_n values we see that the error increases by almost exactly $-k = -15$.

```
>> 2336290021325.815918/-155752668088.387726
ans = -15
```

(f)

Emperically we see ϵ_0 as

```
>> 2336290021325.815918/(-15)^25
ans = -9.2522e-18
```

Since floating points are represented using exponentials, we should be able to divide the rounding error by k to get the error adjusted by the exponent. Which matches our empirical answer.

$$|\epsilon_0| \leq \frac{2^{-53}}{k}$$

$$|\epsilon_n| = k^n(\frac{2^{-53}}{k})$$

(g)

(h)

(i)

Question 3

Question 4

Question 5