Lecture Notes 12: Review Linear Algebra

CPSC 302: Numerical Computation for Algebraic Problems

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Goals of this Chapter

- Provide common background (no numerical algorithms) in linear algebra, necessary for developing numerical algorithms elsewhere.
- Collect several concepts and definitions for easy referencing.
- Ensure that those who have the necessary background can easily skip this chapter.

Chapter 4: Review Linear Algebra

- Basic Concepts: Linear Systems and Eigenvalue Problems
- Norms

- Special Matrix Classes
- Singular Value Decomposition (SVD)
- Examples in Applications

Outline

1. Basic Concepts: Linear Systems and Eigenvalue Problems
   - Linear Systems
   - Eigenvalues and Eigenvectors

2. Norms

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Linear Independence and Linear Space

- Consider m real-valued vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, each of length n, i.e., $\mathbf{v}_i \in \mathbb{R}^n$.
- These vectors are linearly independent if $\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = \mathbf{0}$ necessarily implies that all (real, scalar) coefficients α_i are equal to zero.
- Assuming linear independence, the linear space $V = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \ldots \mathbf{v}_m\}$ consists of all linear combinations $\sum_{i=1}^{m} \alpha_i \mathbf{v}_i$.
- The linear space V has dimension m. It is obviously a subspace of $\mathbb{R}^n$. (Hence, necessarily, $m \leq n$.)

Example

Are

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

linear independent?

Consider

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \alpha_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(I) $2\alpha_1 + \alpha_2 = 0 \quad \Leftrightarrow \quad \alpha_2 = -2\alpha_1$

(II) $5\alpha_1 + \alpha_2 = 5\alpha_1 - 2\alpha_1 = 3\alpha_1 = 0 \quad \Leftrightarrow \quad \alpha_1 = 0$

(I)' $\alpha_2 = 0$

Hence, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear independent.

Linear System of Equations

- Find $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ which satisfies

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1, \\ a_{21}x_1 + a_{22}x_2 &= b_2, \end{aligned}$$

  or $A\mathbf{x} = \mathbf{b}$ with $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.
- Unique solution **iff** lines are not parallel.
- Generalize to $n \times n$ systems $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

  and the elements $a_{ij} \equiv a_{i,j}$ are real numbers.

Linear System of Equations (cont.)

In general, for a square $n \times n$ system with $A \in \mathbb{R}^{n \times n}$ there is a unique solution if one of the following equivalent statements hold:

- A is nonsingular;
- $\det(A) \neq 0$;
- A has linearly independent columns or rows;
- there exists an inverse A^{-1} satisfying $AA^{-1} = I = A^{-1}A$;
- $\mathrm{range}(A) = \mathbb{R}^n$;
- $\mathrm{null}(A) = \{\mathbf{0}\}$.

Example

What is the nullspace of $A = \begin{pmatrix} 2 & 8 \\ 5 & 20 \end{pmatrix}$?

Bring A to triangular form:

$$A = \begin{pmatrix} 2 & 8 \\ 5 & 20 \end{pmatrix} \begin{array}{|l} \cdot 5 \\ \cdot (-2) \end{array} \; + \quad \rightsquigarrow \quad \tilde{A} = \begin{pmatrix} 2 & 8 \\ 0 & 0 \end{pmatrix}$$

Hence

$$2x_1 + 8x_2 = 0 \Leftrightarrow x_1 = -4x_2$$

and

$$\text{null}(A) = \left\{ \alpha \begin{pmatrix} -4 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}.$$

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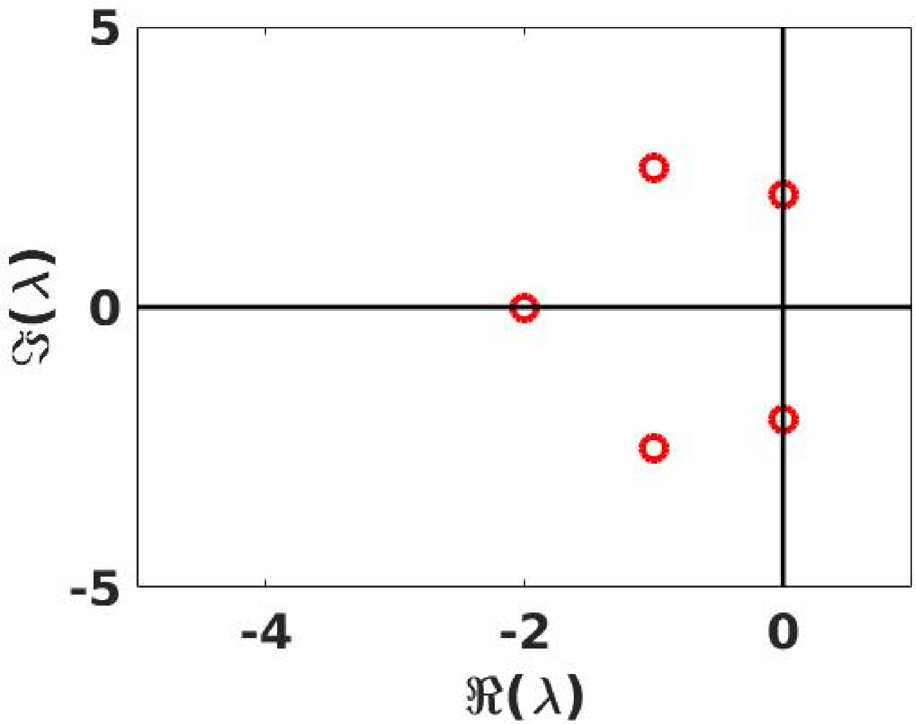
Eigenvalue Problems

Consider $A \in \mathbb{R}^{n \times n}$. A scalar $\lambda \in \mathbb{C}$ and a vector $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$ are an eigenvalue-eigenvector pair (or eigenpair) of A if

$$A\mathbf{x} = \lambda \mathbf{x}.$$

Example: Spectrum of A

$$A = \begin{pmatrix} -0.9880 & 1.8000 & -1.1777 & 0.5662 & -0.1852 \\ -1.9417 & -0.5835 & -0.2830 & -0.6202 & 1.0868 \\ 0.7845 & 1.0422 & -0.5945 & -1.9064 & -0.4689 \\ 0.1628 & 0.9532 & 1.7775 & -0.1745 & 0.5301 \\ -0.7548 & -0.4258 & 0.9639 & -0.1903 & -1.6595 \end{pmatrix}$$



See: spectrum.ipynb.

Eigenvalue Problems (cont.)

- For a **diagonalizable** $A \in \mathbb{R}^{n\times n}$ there are n (generally complex-valued) eigenpairs $(\lambda_j, \mathbf{x}_j)$, with $X = \left[\mathbf{x}_1, \ldots, \mathbf{x}_n\right]$ nonsingular, and $X^{-1}AX$ is a diagonal matrix with the eigenvalues on the main diagonal.

- **Similarity transformation:** Given a nonsingular matrix S, the matrix $S^{-1}AS$ has the same eigenvalues as A. (Exercise: what about the eigenvectors?)

Symmetric Matrices

$A \in \mathbb{R}^{n\times n}$ is symmetric if $A^T = A$.

- **Examples**:
  1. $A = \begin{pmatrix} 4 & 1.01 \\ 1.01 & -2 \end{pmatrix}$
  2. $A = B^T B$ for any real rectangular B.

- **Properties**:
  1. The eigenvalues of A are all real.
  2. The matrix A is always diagonalizable, i.e., it has a full set of n linearly independent eigenvectors.
  3. The eigenvector matrix X can be scaled so that it is real and

$$X^{-1} = X^T.$$

(orthogonal)

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2. Norms
   Vector Norms
   Matrix Norms

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Vector Norms

A **vector norm** is a function "$\|\cdot\|$" from $\mathbb{R}^n$ to $\mathbb{R}$ that satisfies:

1. $\|\mathbf{x}\| \geq 0; \quad \|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$,
2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\| \ \forall\ \alpha \in \mathbb{R}$,
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

- This generalizes absolute value or magnitude of a scalar.
- The distance between vectors $\mathbf{x}$ and $\mathbf{y}$ can be measured by $\|\mathbf{x} - \mathbf{y}\|$.

Famous Vector Norms

- ℓ_2-*norm*

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

- ℓ_∞-*norm*

$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$$

- ℓ_1-*norm*

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

Example

- Problem: Find the distance between

$$\mathbf{x} = \begin{pmatrix} 11 \\ 12 \\ 13 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 12 \\ 14 \\ 16 \end{pmatrix}.$$

- Solution: let

$$\mathbf{z} = \mathbf{y} - \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

  and find $\|\mathbf{z}\|$.

- Calculate

$$\begin{aligned} \|\mathbf{z}\|_1 &= 1 + 2 + 3 = 6, \\ \|\mathbf{z}\|_2 &= \sqrt{1 + 4 + 9} \approx 3.7417, \\ \|\mathbf{z}\|_\infty &= 3. \end{aligned}$$

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Matrix Norms

Induced matrix norm of $m \times n$ matrix A for a given vector norm:

$$\|A\| = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \quad = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Then consistency properties hold,

$$\|AB\| \leq \|A\|\|B\|, \quad \|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|,$$

in addition to the previously stated three norm properties.

Famous Matrix Norms

- ℓ_2-*norm*

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sigma_{\max}(A),$$

where ρ is **spectral radius**

$$\rho(B) = \max\{|\lambda|;\ \ \lambda \text{ is an eigenvalue of } B\}.$$

(A relatively painful calculation for such a simple task.)

- ℓ_∞-*norm*

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|.$$

- ℓ_1-*norm*

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$