CPSC 302 - Assignment 3

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1. Linear Algebra

1.a

$$||A||_1 = max_j(a+b, b+a+b, \dots, b+a) = a+2b$$
$$||A||_\infty = max_j(a+b, b+a+b, \dots, b+a) = a+2b$$

$$||A||_2 = \sqrt{\lambda_{max}(A^TA)} = \sigma_{max}(A)$$

The largest singular value is the square root of the largets eigen vector.

$$\lambda_{max} = a+2b$$
$$\sigma_{max} = \sqrt{a+2b}$$

$$||A||_2 = \sqrt{a+2b}$$

1.b

The matrix A is defined to be symmetric.

Claim: If A is strictly diagonally dominant then it is symmetric positive definite.

$$v \in \mathbb{R}^n$$

$$Av = \begin{bmatrix} av_1 + bv_2 \\ bv_1 + av_2 + bv_3 \\ bv_2 + av_3 + bv_4 \\ \dots \\ bv_{n-2} + av_{n-1} + bv_n \\ bv_{n-1} + av_n \end{bmatrix}$$

$$v^TAv = v_1(av_1 + bv_2) + v_2(bv_1 + av_2 + bv_3) + v_3(bv_2 + av_3 + bv_4) + \dots v_{n-1}(bv_{n-2} + av_{n-1} + bv_n) + v_n(bv_{n-1} + av_n)$$

$$v^TAv = av_1^2 + bv_1v_2 + bv_1v_2 + av_2^2 + bv_3v_2 + bv_2v_3 + av_3^2 + bv_4v_3 + \dots bv_{n-2}v_{n-1} + av_{n-1}^2 + bv_nv_{n-1} + bv_{n-1}v_n + av_n^2$$

$$v^TAv = av_1^2 + 2bv_1v_2 + av_2^2 + 2bv_2v_3 + av_3^2 + 2bv_3v_4 + \dots av_{n-1}^2 + 2bv_{n-1}v_n + av_n^2$$

Since A is strictly diagonally dominant, that means $a > 2b$. Thus, $v^TAv > 0$ and A is symmetric positive definite.

1.c

We first need to figure out for which j is the maximum and minimum of $\cos(\frac{\pi j}{n+1}$. a, b are both positive so they don't affect the sign.

$$\cos(\frac{\pi}{n+1}) \approx 1$$
$$\cos(\frac{n\pi}{n+1}) \approx -1$$

$$\max_{j=1,\ldots,n} = a + 2b$$
$$\min_{j=1,\ldots,n} = a - 2b$$

$$\kappa_2(A) = \frac{a+2b}{a-2b}$$

2. Tridiagonal Systems of Equations

2.a

Seems to work just fine.

2.b

We find $||v - u||_\infty = 0.98998$. The computation of v matches the solution that matlab's operator returns.

3. Cholesky Decomposition

3.a

If you set

$$A = \begin{bmatrix} 1 \end{bmatrix}, \alpha = -1, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$$

We see that the scalar α is negative, it violates the definition of positive definite: $z^T M z$ is positive for every non-zero column vector z. Thus α must be positive.

If we construct the reverse

$$A = \begin{bmatrix} -1 \end{bmatrix}, \alpha = 1, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1$$

We see that if A is not positive definite B is not positive definite. Thus, A must be positive definite.

3.b

$$A = \begin{bmatrix} A & a \\ a^T & \alpha \end{bmatrix}$$

Let's define G to be the Cholesky decomposition of A.

We want to find a new factorization of B in the form

$$\begin{bmatrix} G & 0 \\ h^T & i \end{bmatrix}$$

$$\begin{bmatrix} A & a \\ a^T & \alpha \end{bmatrix} = \begin{bmatrix} G & 0 \\ h^T & i \end{bmatrix} \begin{bmatrix} G^T & h \\ 0 & i \end{bmatrix} = \begin{bmatrix} GG^T & Gh \\ G^T h^T & h^T h + i^2 \end{bmatrix}$$

Thus,

$$a = Gh, \alpha = h^T h + i^2$$

Solving for h,

$$h = G^{-1} a$$

Solving for i,

$$i^2 = \alpha - (G^{-1}a)^T (G^{-1}a)$$
$$i = \sqrt{\alpha - (G^{-1}a)^T G^{-1} a}$$

Thus, the Cholesky factorization of B is,

$$chol(B) = \begin{bmatrix} G & 0 \\ (G^{-1}a)^T & \sqrt{\alpha - (G^{-1}a)^T G^{-1} a} \end{bmatrix}$$

3.c

$$B = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 1 & \frac{1}{\sqrt{5}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{19}{5} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \sqrt{5} & \frac{1}{\sqrt{5}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 1 & \frac{1}{\sqrt{5}} & \sqrt{\frac{19}{5}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \sqrt{5} & \frac{1}{\sqrt{5}} \\ 0 & 0 & \sqrt{\frac{19}{5}} \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 1 & \frac{1}{\sqrt{5}} & \sqrt{\frac{19}{5}} \end{bmatrix}$$

4. LDL^T Decomposition

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & 1 \\ 0 & \frac{11}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{7}{4} \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & 1 \\ 0 & \frac{11}{4} & \frac{1}{4} \\ 0 & 0 & \frac{19}{11} \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{11} & 1 \end{bmatrix}$$

$$U = DL^T$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{11}{4} & \frac{1}{4} \\ 0 & 0 & \frac{19}{11} \end{bmatrix}$$

5. Hessenberg Matrix

5.a

```
A =

   1   4   2   3   9
   3   4   1   7   9
   0   2   3   4   9
   0   0   1   3   4
   0   0   0   4   5

L =

   1.00000   0.00000   0.00000   0.00000   0.00000
   3.00000   1.00000   0.00000   0.00000   0.00000
   0.00000  -0.25000   1.00000   0.00000   0.00000
   0.00000   0.00000   0.57143   1.00000   0.00000
   0.00000   0.00000   0.00000   4.00000   1.00000

U =

    1.00000    4.00000    2.00000    3.00000    9.00000
    0.00000   -8.00000   -5.00000   -2.00000  -18.00000
    0.00000    0.00000    1.75000    3.50000    4.50000
    0.00000    0.00000    0.00000    1.00000    1.42857
    0.00000    0.00000    0.00000    0.00000   -0.71429

L * U =

   1   4   2   3   9
   3   4   1   7   9
   0   2   3   4   9
   0   0   1   3   4
   0   0   0   4   5
```

5.b

L has 1 values along the diagonal, and first subdiagonal is non-zero, but everything else is zero.

5.c

$$Ax = b, A = LU, LUx = b$$

If we solve this by first doing the LU decomposition as specified above and then doing backwards substitution and then forward subsitution we see that:

- LU decomposition $= O(n^2)$
- Forward subsitution with $L = O(n)$ since there's only one element other than the pivot per column.
- Backwards subsitution with $U = O(n^2)$

Thus, the number of operations is $= O(n^2)$.

5.d

The sparsity pattern changes with partial pivoting. We can see this if we use MatLab's `lu` function.

```
[L, U] = lu(A)

L =

   0.33333   1.00000   0.00000   0.00000   0.00000
   1.00000   0.00000   0.00000   0.00000   0.00000
   0.00000   0.75000   1.00000   0.00000   0.00000
   0.00000   0.00000   0.57143   0.25000   1.00000
   0.00000   0.00000   0.00000   1.00000   0.00000

U =

   3.00000   4.00000   1.00000   7.00000   9.00000
   0.00000   2.66667   1.66667   0.66667   6.00000
   0.00000   0.00000   1.75000   3.50000   4.50000
   0.00000   0.00000   0.00000   4.00000   5.00000
   0.00000   0.00000   0.00000   0.00000   0.17857
```

Since partial pivoting operates row by row and there's only one entry below the pivot per column that means each row only can be switched with the one below it. Thus, there can be some 1 values on the diagonal above the middle.