Lecture Notes 10: Nonlinear Equations in One Variable

CPSC 302: Numerical Computation for Algebraic Problems

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Chapter 2: Nonlinear Equations in One Variable

- Bisection method
- Fixed point iteration
- Newton's method and variants
- Minimizing a function in one variable

Outline

1. Newton's Method and Variants
   - Newton's Method
   - Secant Method

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Newton's Method

This fundamentally important method is everything that bisection is not, and vice versa:

- Not so simple
- Not very safe or robust (local convergence)
- Requires more than continuity on f
- Fast
- Automatically generalizes to systems

Derivation

- By Taylor series ($h = x - x_k$)

  $$f(x) = f(x_k + h) = f(x_k) + f'(x_k)(x - x_k) + f''(\xi(x))(x - x_k)^2/2.$$

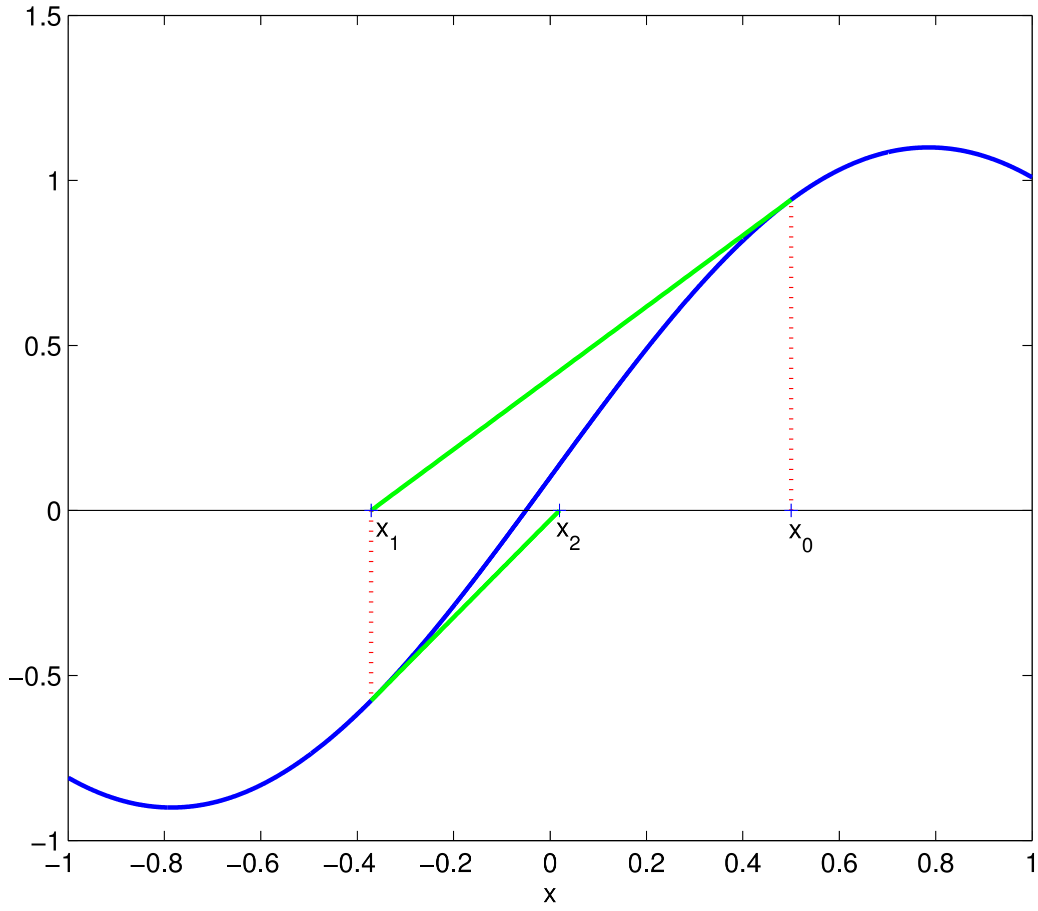
- So, for $x = x^*$

  $$0 = f(x_k) + f'(x_k)(x^* - x_k) + \mathcal{O}\left((x^* - x_k)^2\right).$$

- The method is obtained by neglecting nonlinear term, defining $0 = f(x_k) + f'(x_k)(x_{k+1} - x_k)$, which gives the iteration step

  $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots.$$

A Geometric Interpretation



Next iterate is x-intercept of the tangent line to f at current iterate.

Example: Cosh with Two Roots

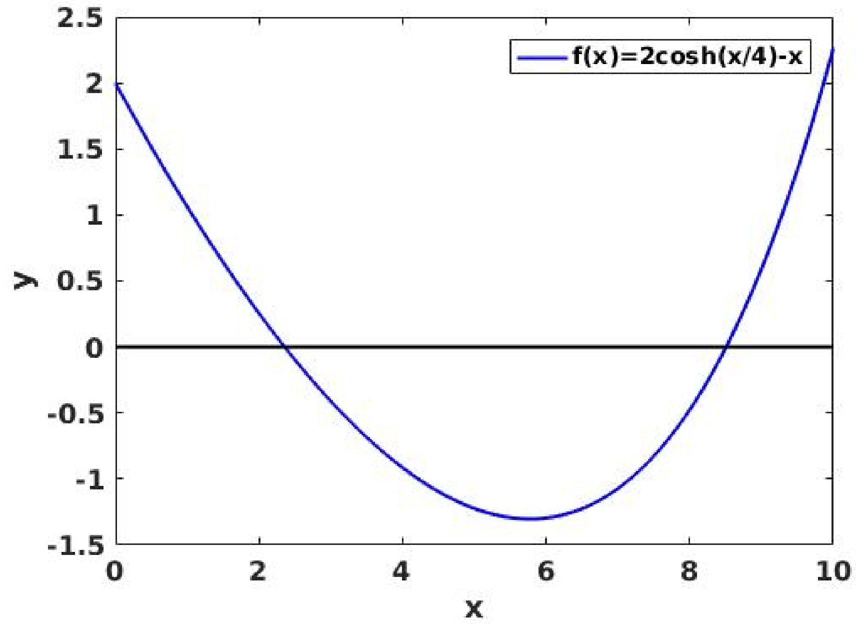
- The function

$$f(x) = 2\cosh(x/4) - x$$

has two solutions in the interval $[2, 10]$.

- Newton's iteration is

$$x_{k+1} = x_k - \frac{2\cosh(x_k/4) - x_k}{0.5\sinh(x_k/4) - 1}.$$



For absolute tolerance 1.e-8:

- Starting from $x_0 = 2$ requires 4 iterations to reach x_1^*.
- Starting from $x_0 = 4$ requires 5 iterations to reach x_1^*.
- Starting from $x_0 = 8$ requires 5 iterations to reach x_2^*.
- Starting from $x_0 = 10$ requires 6 iterations to reach x_2^*.

Example (cont.): Cosh with Two Roots

- Tracing the iteration's progress for $x_0 = 8$:

k	0	1	2	3	4	5
$f(x_k)$	-4.76e-1	8.43e-2	1.56e-3	5.65e-7	7.28e-14	1.78e-15

- Note that the number of significant digits essentially doubles at each iteration (until the 5th, when roundoff error takes over).

Speed of Convergence

A given method is said to be

- **linearly convergent** if there is a constant $\rho < 1$ such that

$$|x_{k+1} - x^*| \leq \rho |x_k - x^*| \ ,$$

  for all k sufficiently large;

- **superlinearly convergent** if there is a sequence of constants $\rho_k \to 0$ such that

$$|x_{k+1} - x^*| \leq \rho_k |x_k - x^*|,$$

  for all k sufficiently large.

- **quadratically convergent** if there is a constant M such that

$$|x_{k+1} - x^*| \leq M |x_k - x^*|^2 \ ,$$

  for all k sufficiently large;

Convergence Theorem for Newton's Method

If $f \in C^2[a, b]$ and there is a root x^* in $[a, b]$ such that $f(x^*) = 0,\ f'(x^*) \neq 0$, then there is a number δ such that, starting with x_0 from anywhere in the neighborhood $[x^* - \delta, x^* + \delta]$, Newton's method converges quadratically.

Idea of proof:

- Expand $f(x^*)$ in terms of a Taylor series about x_k;
- divide by $f'(x_k)$, rearrange, and replace $x_k - \frac{f(x)}{f'(x_k)}$ by x_{k+1};
- find the relation between $e_{k+1} = x_{k+1} - x^*$ and $e_k = x_k - x^*$.

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Secant Method

- One potential disadvantage of Newton's method is the need to *know and evaluate* the derivative of f.
- The secant method circumvents the need for explicitly evaluating this derivative.
- Observe that near the root (assuming convergence)

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

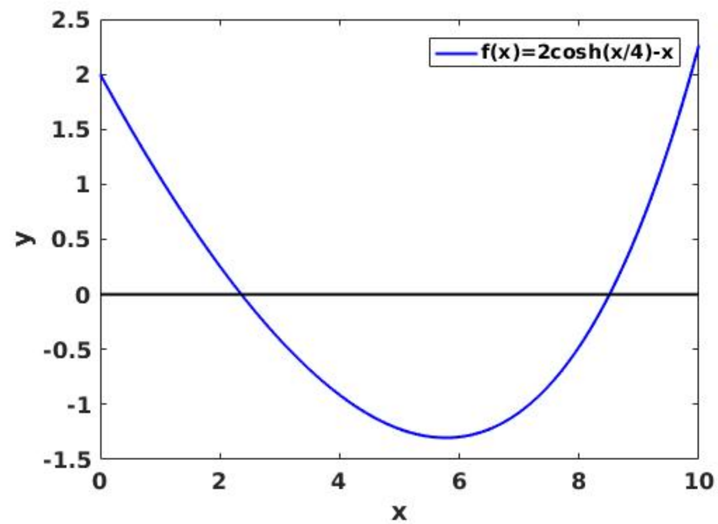
- So, define Secant iteration

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad k = 1, 2, \ldots.$$

- Note the need for two initial starting iterates x_0 and x_1: a *two-step method*.

Example: Cosh with Two Roots

$$f(x) = 2\cosh(x/4) - x.$$



Same absolute tolerance 1.e-8 and initial iterates as before:

- Starting from $x_0 = 2$ and $x_1 = 4$ requires 7 iterations to reach x_1^*.
- Starting from $x_0 = 10$ and $x_1 = 8$ requires 7 iterations to reach x_2^*.

k	0	1	2	3	4	5	6
$f(x_k)$	2.26	-4.76e-1	-1.64e-1	2.45e-2	-9.93e-4	-5.62e-6	1.30e-9

Observe superlinear convergence: much faster than bisection and simple fixed point iteration, yet not quite as fast as Newton's iteration.

Newton's Method as a Fixed Point Iteration

- If $g'(x^*) \neq 0$ then fixed point iteration converges linearly, as discussed before, as $\rho > 0$.
- Newton's method can be written as a fixed point iteration with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

  From this we get $g'(x^*) = 0$.
- In such a situation the fixed point iteration may converge faster than linearly: indeed, Newton's method converges quadratically under appropriate conditions.