Grading	Delivery	Tasks below	Total
Points	/3	/48	/51

Assignment 3 – Direct Methods for Linear Systems (and a little bit of Linear Algebra)
Due Nov 1, 2017, beginning of class

Reminder:

1. When submitting your hardcopy assignment, you must also include all your code (m-files) and plots. **In addition** to the hardcopy, all code and plots should be submitted electronically using the `handin` command. **Codes/plots/outputs that have not been submitted in hard copy will NOT be graded.**

2. For both your hardcopy and electronic submission, make sure to **include your name and UBC student number**.

3. If you cannot hand in your hardcopy assignment in class, you may hand it in to the Assignment Box marked "CPSC 302" in ICCS X235 **before** lecture on the due date.

4. For a guideline on the grading scheme, please see the file `Rubrics_CPSC_302.pdf` on Piazza. We assume you are aware of it. This means also that, for instance, we will not explicitly state in a task that a legible plot should be titled, contain a legend, contain labeled axes, use different line styles and/or markers. There is a rubric on that in the mentioned file, and you can read out the resulting points if you don't create legible plots. The same holds for all other rubrics. **From now on, we scale the grading scheme more towards the written part. Since everybody should be familiar with programming standards by now, we will group some rubrics into one category. Nevertheless, careless coding will lead to points deduction.**

5. You can access the documentation for any MATLAB command by typing `help COMMAND` (printed to screen) or `doc COMMAND` (documentation GUI).

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1. **Linear Algebra.** Given that a and b are two real positive numbers, the eigenvalues of the symmetric $n \times n$ tridiagonal matrix

$$A = \begin{pmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & b & a & b \\ & & & & b & a \end{pmatrix}$$

are $\lambda_j = a + 2b\cos\left(\frac{\pi j}{n+1}\right)$, $j = 1, \ldots, n$.

   (a) (3 points, Calculations) Determine by hand $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$.

   (b) (3 points, Correctness of Proof) Show that if A is strictly diagonally dominant, then it is symmetric positive definite.

   (c) (3 points, Calculations) Suppose A is symmetric positive definite. Determine the condition number
   $$\kappa_2(A) = \frac{\max_{j=1,\ldots,n} \lambda_j}{\min_{j=1,\ldots,n} \lambda_j}$$
   for large n. It should depend only on a and b, not on j or n.
   Hint: You can use the approximation $\cos(x) \approx 1$ for $x \approx 0$. You may also assume that $a \neq 2b$.

2. **Tridiagonal Systems of Equations.**

Grading	Coding Standards, Documentation, Runtime & Output	Efficiency	Correctness	**Total**
Points	/3	/3	/3	/9

Write a MATLAB function that solves tridiagonal systems of equations of size n. Assume that no pivoting is needed, but do not assume that the tridiagonal matrix A is symmetric. Your program should expect as input four vectors of size n (or $n-1$): one right-hand side $\mathbf{b}$ and the three nonzero diagonals of A. It should calculate and return $\mathbf{x} = A^{-1}\mathbf{b}$ using a Gaussian elimination variant that requires $\mathcal{O}(n)$ flops and consumes no additional space as a function of n (i.e., in total $5n$ storage locations are required).

   (a) Try your program on the matrix defined by $n = 10{,}000$, $a_{i-1,i} = a_{i+1,i} = -i$, and $a_{i,i} = 3i$, for all i such that the relevant indices fall in the range 1 to n. Invent a right-hand side vector $\mathbf{b}$.

   (b) Apply your program to the problem described in Example 4.17 in the course textbook using the second set of boundary conditions, $v(0) = v'(1) = 0$, for $g(t) = (\frac{\pi}{2})^2 \sin(\frac{\pi}{2} t)$ and $N = 100$. Compare the results to the vector $\mathbf{u}$ composed of $u(ih) = \sin(\frac{\pi}{2} ih)$, $i = 1, \ldots, N$, by recording $\|\mathbf{v} - \mathbf{u}\|_\infty$.

3. **Cholesky Decomposition.** Suppose that the symmetric matrix

$$B = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{a}^T & \alpha \end{pmatrix}$$

of order $n+1$ is positive definite.

   (a) (3 points, Calculations) Show that the scalar α must be positive and the $n \times n$ matrix A must be positive definite.

   (b) (3 points, Calculations) Determine the Cholesky decomposition of B in terms of α, $\mathbf{a}$, and the Cholesky factor of A?

   (c) (3 points, Calculations) Determine by hand the Cholesky factorization of

$$B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

   as $B = CC^T$. Find the matrix C.

4. (3 points, Calculations) LDL^T **Decomposition.** In some circumstances it may be advantageous to express the Cholesky decomposition in the form $A = LDL^T$, where $A \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times n}$ is a unit lower triangular matrix, and $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive entries on the diagonal. Determine by hand the LDL^T factorization of

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

as $A = LDL^T$. Find the matrices L and D.

5. **Hessenberg Matrix.**

Grading (a)	Coding Standards, Documentation, Runtime & Output	Efficiency	Correctness	**Total**
Points	/3	/3	/3	/9

An $n \times n$ matrix A is said to be in *Hessenberg* or *upper Hessenberg* form if all its elements below the first subdiagonal are zero, so that

$$a_{ij} = 0\,, \quad i > j+1\,.$$

Consider the LU decomposition of such a matrix, assuming that no pivoting is needed: $A = LU$.

   (a) Write a MATLAB function that computes this LU decomposition in $\mathcal{O}(n^2)$ flops. Try your function on a 5×5 Hessenberg matrix.

   (b) (3 points, Justification) What is the sparsity structure of the resulting matrix L (i.e., where are its nonzeros)?

   (c) (3 points, Justification) How many operations (to a leading order) does it take to solve a linear system $A\mathbf{x} = \mathbf{b}$, where A is upper Hessenberg?

   (d) (3 points, Justification) Suppose now that partial pivoting is applied. What are the sparsity patterns of the factors of A?