Lecture Notes 09: Nonlinear Equations in One Variable

CPSC 302: Numerical Computation for Algebraic Problems

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Outline

1. Fixed Point Iteration
    - Family of One-Step Methods
    - Theory
    - Convergence

Fixed Point Iteration

This is an intuitively appealing approach which often leads to simple algorithms for complicated problems.

1. Write given problem
$$f(x) = 0$$
as
$$g(x) = x,$$
so that $f(x^*) = 0$ iff $g(x^*) = x^*$.
2. Iterate:
$$x_{k+1} = g(x_k), \quad k = 0, 1, \ldots,$$
starting with guess x_0.

It's all in the choice of the function g.

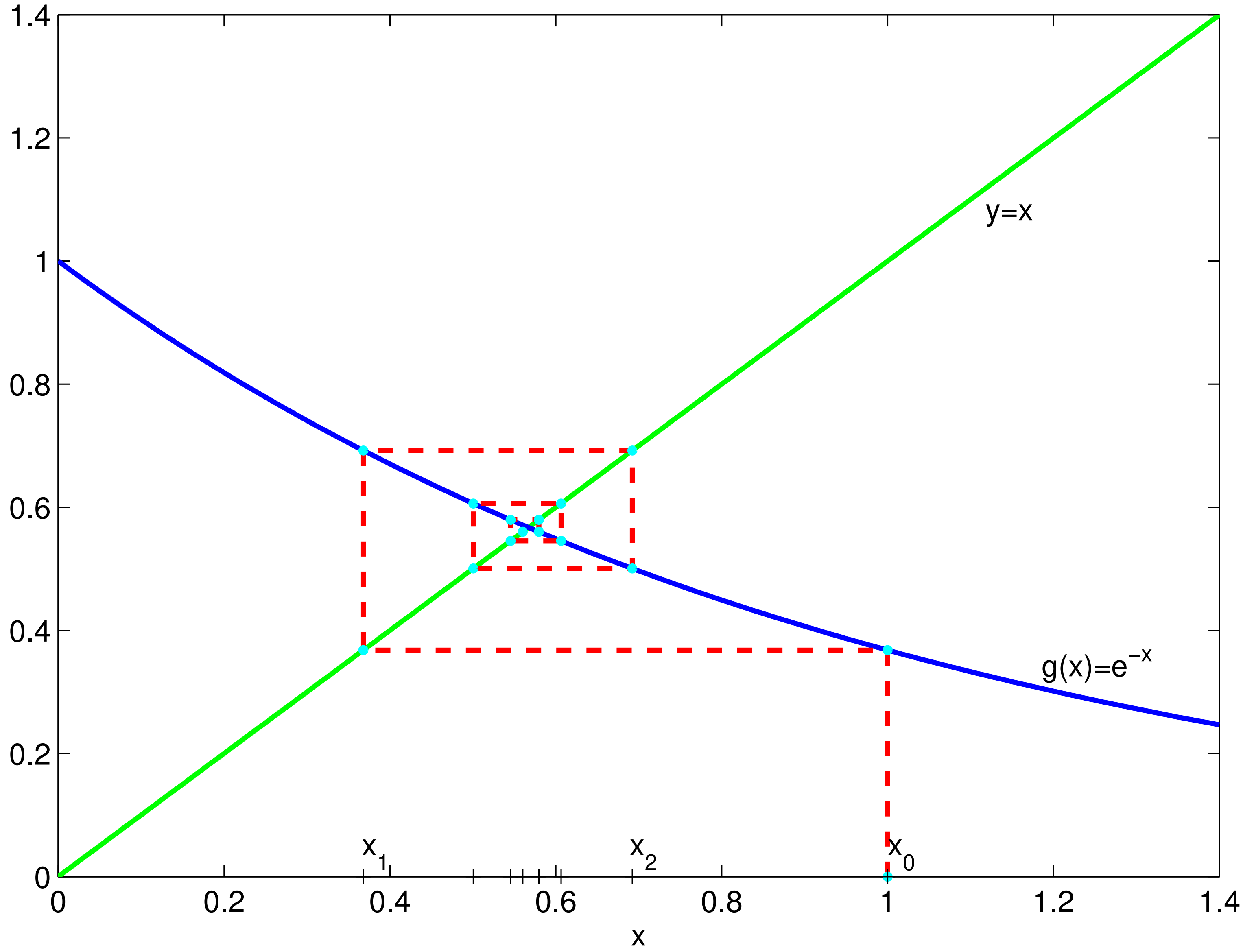
Choosing the Function g

- Note: there are many possible choices g for the given f: this is a family of methods.
- Examples:

$$
\begin{aligned}
g(x) &= x - f(x), \\
g(x) &= x + 2f(x), \\
g(x) &= x - f(x)/f'(x) \quad \text{(assuming f}' \text{ exists and f}'\text{(x)} \neq 0\text{)}.
\end{aligned}
$$

  The first two choices are simple, the last one has potential to yield fast convergence (we'll see later).
- Want resulting method to
  - be simple;
  - converge; and
  - do it rapidly.

Graphical Illustration, $x = e^{-x}$, Starting from $x_0 = 1$



Fixed Point Theorem

Existence:
If $g \in C[a, b],\ g(a) \geq a$ *and* $g(b) \leq b$*, then there is a fixed point* x^* *in the interval* $[a, b]$.

Uniqueness:
If, in addition, the derivative g' *exists and there is a constant* $\rho < 1$ *such that the derivative satisfies*

$$|g'(x)| \leq \rho \quad \forall\, x \in (a, b),$$

then the fixed point x^* *is unique in this interval.*

Convergence of Fixed Point Iteration

- Assuming $\rho < 1$ as for the fixed point theorem, obtain

$$|x_{k+1} - x^*| = |g(x_k) - g(x^*)| \leq \rho |x_k - x^*|.$$

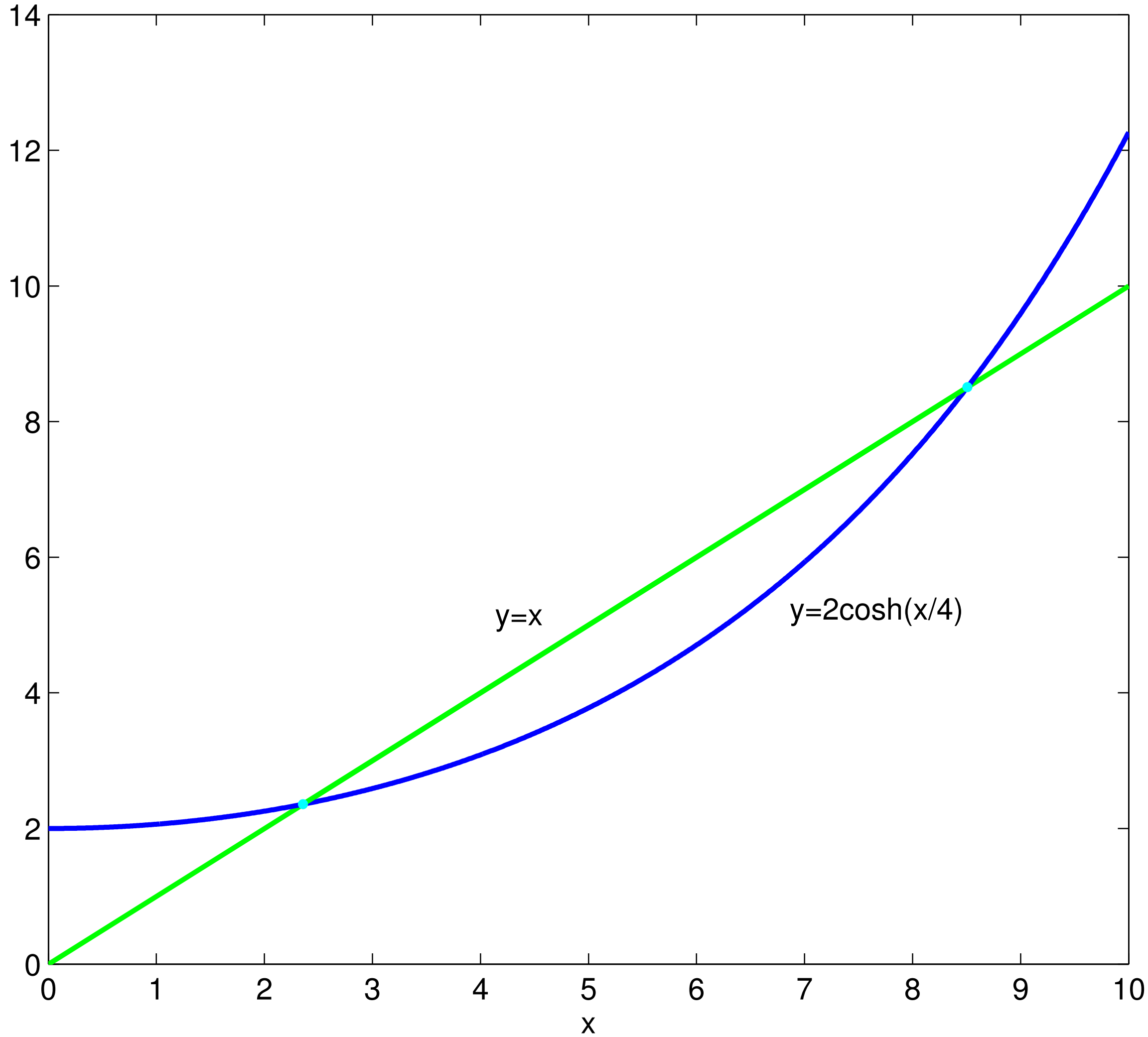
- This is a **contraction** by factor ρ.
- So

$$|x_{k+1} - x^*| \leq \rho |x_k - x^*| \leq \rho^2 |x_{k-1} - x^*| \leq \ldots \leq \rho^{k+1} |x_0 - x^*| \to 0.$$

- The smaller ρ the faster convergence is.

Example: Cosh with Two Roots

$$f(x) = g(x) - x, \quad g(x) = 2\cosh(x/4)$$



Fixed Point Iteration with g

For absolute error tolerance 10^{-8}:

- Starting at $x_0 = 2$ converge to x_1^* in 16 iterations.
- Starting at $x_0 = 4$ converge to x_1^* in 18 iterations.
- Starting at $x_0 = 8$ converge to x_1^* (even though x_2^* is closer to x_0).
- Starting at $x_0 = 10$ obtain **overflow** in 3 iterations.

Note: bisection yields both roots in 27 iterations.

Rate of Convergence

- Suppose we want $|x_k - x^*| \approx 0.1|x_0 - x^*|$.
- Since $|x_k - x^*| \le \rho^k |x_0 - x^*|$, want

$$\rho^k \approx 0.1,$$

  i.e., $k \log_{10} \rho \approx -1$.

- Define the **rate of convergence** as

$$rate = -\log_{10} \rho.$$

- Then it takes about $k = \lceil 1/rate \rceil$ iterations to reduce the error by more than an order of magnitude.

Return to Cosh Example

- Bisection: $rate = -\log_{10} 0.5 \approx 0.3 \quad \Rightarrow \quad k = 4$.

- For the root x_1^* of fixed point example, $\rho \approx 0.31$ so

$$rate = -\log_{10} 0.31 \approx 0.5 \quad \Rightarrow \quad k = 2.$$

- For the root x_2^* of fixed point example, $\rho > 1$ so

$$rate = -\log_{10}(\rho) < 0 \quad \Rightarrow \quad \text{no convergence.}$$