CPSC 302 - Assignment 3

1. Linear Algebra

1.a

$$||A||_1 = max_j(a+b, b+a+b, \dots, b+a) = a+2b$$
$$||A||_\infty = max_j(a+b, b+a+b, \dots, b+a) = a+2b$$

$$||A||_2 = \sqrt{\Lambda_{max}(A^TA)} = \sigma_{max}(A)$$

The largest singular value is the square root of the largets eigen vector.

$$\Lambda_{max} = a+2b$$
$$\sigma_{max} = \sqrt{a+2b}$$

$$||A||_2 = \sqrt{a+2b}$$

1.b

The matrix A is defined to be symmetric.

Claim: If A is strictly diagonally dominant then it is symmetric positive definite.

TODO

1.c

We first need to figure out for which j is the maximum and minimum of $\cos(\frac{\pi j}{n+1}$. a, b are both positive so they don't affect the sign.

$$\cos(\frac{\pi}{n+1}) \approx 1$$
$$\cos(\frac{n\pi}{n+1}) \approx -1$$

$$\max_{j=1,\dots,n} = a+2b$$
$$\min_{j=1,\dots,n} = a-2b$$

$$\kappa_2(A) = \frac{a+2b}{a-2b}$$

2. Tridiagonal Systems of Equations

2.a

2.b

3. Cholesky Decomposition

3.a

If you set

$$A = \begin{bmatrix}1\end{bmatrix}, \alpha = -1, B = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$$

We see that the scalar α is negative, it violates the definition of positive definite: $z^T M z$ is positive for every non-zero column vector z. Thus α must be positive.

If we construct the reverse

$$A = \begin{bmatrix} -1 \end{bmatrix}, \alpha = 1, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1$$

We see that if A is not positive definite B is not positive definite. Thus, A must be positive definite.

3.b

3.c

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 1 & \sqrt{\frac{1}{5}} & \sqrt{\frac{19}{5}} \end{bmatrix}$$

4. LDL^T Decomposition

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & 1 \\ 0 & \frac{11}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{7}{4} \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & 1 \\ 0 & \frac{11}{4} & \frac{1}{4} \\ 0 & 0 & \frac{19}{11} \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{11} & 1 \end{bmatrix}$$

$$U = DL^T$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{11}{4} & \frac{1}{4} \\ 0 & 0 & \frac{19}{11} \end{bmatrix}$$

5. Hessenberg Matrix

5.a

5.b

5.c

5.d