CPSC 302 - Assignment 5

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Discussed with Bryan Chiu and Jerome Rasky

1. Stationary Method

1.a

$$A = M - N$$
$$Mx_{k+1} = Nx_k + b$$

$$x_{k+1} = x_k + \alpha(b - Ax_k)$$
$$x_{k+1} = x_k + \alpha b - \alpha Ax_k$$
$$x_{k+1} = (I - \alpha A)x_k + \alpha b$$
$$\frac{I}{\alpha}x_{k+1} = (\frac{I}{\alpha} - A)x_k + b$$

$$M = \frac{I}{\alpha}$$

$$T = (I - \alpha A)$$

1.b

1.b.i Converges only if $\rho(T) < 1$

Thus, only converges if $\rho(I - \alpha A) < 1$.

This converges if max$_i$ $|1 - \alpha\lambda_i| < 1$.

1.b.ii To maximize the speed of convergence we need to minimize $\rho(T)$.

Thus, we need to find the values α that minimizes max$_i$ $|1 - \alpha\lambda_i|$.

If $1 = \alpha\lambda_i$ for some i, that means it won't maximize $|1 - \alpha\lambda_i|$, since all eigenvalues are distinct and thus at least one must be further away. Assuming one of the eigenvalues equals 1 and $\alpha = 1$, the smallest possible convergence rate would be $|\lambda_1 - \lambda_n|$.

For the best possible rate, we want to minimize both $|1 - \alpha\lambda_1|, |1 - \alpha\lambda_n|$ in order to have the fastest possible rate.

$$1 - \alpha\lambda_1 + 1 - \alpha\lambda_n = 0$$
$$\alpha\lambda_1 + \alpha\lambda_n = 2$$
$$\alpha(\lambda_1 + \lambda_n) = 2$$

$$\alpha = \frac{2}{\lambda_1 + \lambda_n}$$

Thus, we end up at the same best value for the step size in terms of maximizing the speed of convergence.

The spectral radius works out to be

$$\rho(T) = \max_i |1 - \frac{2\lambda_i}{\lambda_1 + \lambda_n}|$$

The closest eigenvalue to 1 will either be λ_1 or λ_n due to the scaling factor.

$$\rho(T) = \max\{1 - \frac{2\lambda_n}{\lambda_1 + \lambda_n}, \frac{2\lambda_1}{\lambda_1 + \lambda_n} - 1\}$$

$$\kappa_2(A) = \frac{\lambda_1}{\lambda_n}$$

$$\rho(T) = \max\{1 - \frac{2}{\kappa_2(A) + 1}, \frac{2}{1 + \frac{1}{\kappa_2(A)}} - 1\}$$

1.c

Using the convergence condition above, for the statement to hold, there must be no strictly diagonally dominant matrices with $\alpha = 1$ such that $\max_i |1 - \alpha\lambda_i| < 1$.

A diagonal matrix is by definition strictly diagonally dominant and the eigenvalues are just the values on the diagonal.

Thus we can construct a matrix with diagonal and eigenvalues $\lambda = \{5, 4, 3\}$. Thus, the $\max_i |1 - \lambda_i| = |1 - 5| = 4$ and this statement is contradicted since our convergence condition fails. Thus, this statement is false.

2. Consider the two-dimensional partial differential equation …

2.a

Since B is symmetric we can express the condition number of it in terms of:

$$\kappa_2(B) = \frac{|\lambda|_{max}}{|\lambda|_{min}}$$

Since A is also symmetric, we can find the eigenvalues of B by adding $(\alpha h)^2 I$ to each eigenvalue of A.

$$\lambda_{l,m} = 4 - 2(\cos(l\pi h) + \cos(m\pi h)) + (\alpha h)^2, 1 \le l, m \le N$$

We need to find the maximum and minimum eigenvalues to determine the condition number.

Since $\cos(0) = 1, \cos(\pi) = -1$ we see that having $l = m = 1$ minimizes the second term and results in the smallest eigenvector.

The largest eigenvector is found when $l\pi h, m\pi h$ are close to π since that maximizes the second term.

$$\lambda_{min} = 4 - 2(\cos(1\pi h) + \cos(1\pi h)) + (\alpha h)^2$$
$$\lambda_{max} = 4 - 2(\cos(\lfloor N \rfloor \pi h) + \cos(\lfloor N \rfloor \pi h)) + (\alpha h)^2$$

$$\kappa_2(B) = \frac{4 - 2(\cos(1\pi h) + \cos(1\pi h)) + (\alpha h)^2}{4 - 2(\cos(\lfloor N \rfloor \pi h) + \cos(\lfloor N \rfloor \pi h)) + (\alpha h)^2}$$

$$\kappa_2(B) = \frac{4 - 2(\cos(\lfloor N \rfloor \pi \frac{1}{N+1}) + \cos(\lfloor N \rfloor \pi \frac{1}{N+1})) + (\alpha \frac{1}{N+1})^2}{4 - 2(\cos(1\pi \frac{1}{N+1}) + \cos(1\pi \frac{1}{N+1})) + (\alpha \frac{1}{N+1})^2}$$

```
> n = 1:20; N = sqrt(n)
N =

   1.0000   1.4142   1.7321   2.0000   2.2361   2.4495   2.6458   2.8284
   3.0000   3.1623   3.3166   3.4641   3.6056   3.7417   3.8730   4.0000
   4.1231   4.2426   4.3589   4.4721

> k2 = (4 -2 * (cos(floor(N) * pi /(N+1)) + cos(floor(N) * pi / (N+1))) +
> (alpha./(N+1)).^2)./(4-2*(cos(pi./(N+1)) + cos(pi./(N+1))) +
> (alpha./(N+1)).^2)

   1.2381   1.5050   1.7508   1.9843   2.2095   2.4285   2.6428   2.8532
   3.0605   3.2650   3.4673   3.6676   3.8660   4.0629   4.2584   4.4526
   4.6456   4.8376   5.0286   5.2186
```

We see that increasing n causes the condition number to increase since the cos terms get closer to $1, -1$.

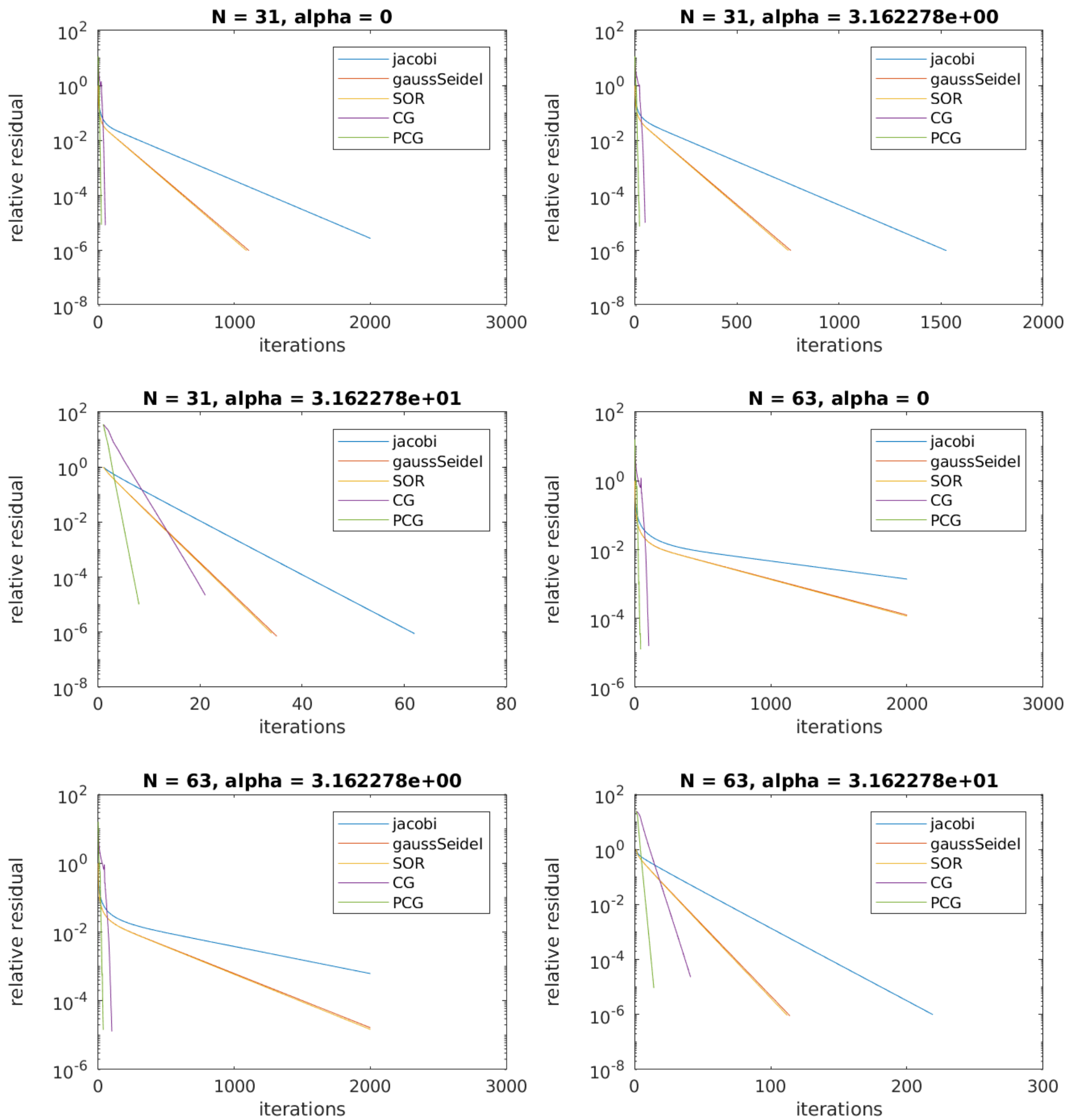
2.b

```
N = 31, n = 961, k2(B) = 4.143451e+02
jacobi: 2000 iters
gaussSeidel: 1108 iters
SOR: 1090 iters
cg: 52 iters
pcg: 23 iters

N = 31, n = 961, k2(B) = 2.752810e+02
jacobi: 1525 iters
gaussSeidel: 764 iters
SOR: 752 iters
cg: 50 iters
pcg: 22 iters

N = 31, n = 961, k2(B) = 8.994868e+00
jacobi: 61 iters
gaussSeidel: 34 iters
SOR: 33 iters
cg: 20 iters
pcg: 7 iters

N = 63, n = 3969, k2(B) = 1.659380e+03
jacobi: 2000 iters
gaussSeidel: 2000 iters
SOR: 2000 iters
cg: 102 iters
pcg: 42 iters
```

```
N = 63, n = 3969, k2(B) = 1.101665e+03
jacobi: 2000 iters
gaussSeidel: 2000 iters
SOR: 2000 iters
cg: 100 iters
pcg: 37 iters

N = 63, n = 3969, k2(B) = 3.309512e+01
jacobi: 218 iters
gaussSeidel: 113 iters
SOR: 111 iters
cg: 40 iters
pcg: 13 iters
```

We see that the speed of convergence is: Jacobi > Gauss-Seidel > SOR > CG > PCG.

The difference in speed of convergence makes sense since that's what the theoretical convergence rates of the different methods tells us. We also see that with varying N, α params that the speed of convergence is correlated with the condition number. Problems that are better conditioned converge faster.



3. Suppose we wish to solve Ax = b ...

3.a

Increasing `diagonalIncrement` causes gsMorph to converge faster.

Increasing the diagonal entries of A increases the spectral radius of A. This increase causes the spectral radius of the convergence matrix to go down since the iteration matrix is $I - M^{-1}A$.

3.b

N/A

4. The smoothing factor ...

4.a

A is the discrete Laplacian.

$$x_{k+1} = x_k + \omega D^{-1}(b - Ax_k)$$
$$x_{k+1} = (I - \omega D^{-1}A)x_k + \omega D^{-1}b$$

$$T_\omega = I - \omega D^{-1}A$$

$$T_\omega v_{l,m} = (I - \omega D^{-1}A)v_{l,m}$$
$$T_\omega v_{l,m} = v_{l,m} - \omega D^{-1}Av_{l,m}$$

The definition of an eigenvector and eigenvalue pair is

$$Av = \lambda v$$

The diagonal values for A are all the same by definition, thus you can write $D = dI, D^{-1} = I/d$.

$$T_\omega v_{l,m} = v_{l,m} - \omega d^{-1}\lambda_{l,m}v_{l,m}$$

$$T_\omega v_{l,m} = (1 - \omega d^{-1}\lambda_{l,m})v_{l,m}$$

Thus for T_ω the eigen vectors are the same, and the eigenvalues are $1 - \omega d^{-1}\lambda_{l,m}$ where $\lambda_{l,m}$ are the eigenvalues of A.

4.b

The eigenvalues for A are
$$\lambda_{l,m} = 4 - 2(\cos(l\pi h) + \cos(m\pi h)), 1 \leq l, m \leq N$$

Thus, the eigenvalues of the iteration matrix are:

$$\mu_{l,m} = 1 - \omega d^{-1}(4 - 2(\cos(l\pi h) + \cos(m\pi h)))$$

$d = 4$ for the 2D Laplacian

$$\mu_{l,m} = 1 - \omega(1 - \frac{1}{2}(\cos(l\pi h) + \cos(m\pi h)))$$

$|\mu_{l,m}|$ is maximized when $l = \frac{N+1}{2}, m = 1$ or when $l = m = N$.

Thus those factors are:

$$\mu_{(N+1)/2,1} = 1 - \omega(1 - \frac{1}{2}(\cos(\frac{(N+1)\pi}{2(N+1)}) + \cos(\frac{\pi}{N+1})))$$

$$\mu_{(N+1)/2,1} = 1 - \omega(1 - \frac{1}{2}(\cos(\frac{\pi}{2}) + \cos(\frac{\pi}{N+1})))$$

$$\mu_{(N+1)/2,1} = 1 - \omega(1 - \frac{1}{2}\cos(\frac{\pi}{N+1}))$$

$$\lim_{N\to\infty} \mu_{(N+1)/2,1} = 1 - \omega(1 - \frac{1}{2}\cos(0))$$

$$\lim_{N\to\infty} \mu_{(N+1)/2,1} = 1 - \omega(1 - \frac{1}{2})$$

$$\lim_{N\to\infty} \mu_{(N+1)/2,1} = 1 - \frac{\omega}{2}$$

$$\lim_{N\to\infty} \mu_{N,N} = 1 - \omega(1 - \frac{1}{2}(\cos(\pi) + \cos(\pi)))$$

$$\lim_{N\to\infty} \mu_{N,N} = 1 - \omega(1 + 1) = 1 - 2\omega$$

Since these both are maxes we need to solve for a ω^* that maximizes them both.

$$1 - 2\omega^* + 1 - \frac{\omega^*}{2} = 0$$

$$2\omega^* + \frac{\omega^*}{2} = 2$$

$$5\omega^* = 4$$

$$\omega^* = \frac{4}{5}$$

We can plug this value back into the earlier eigenvalue expressions to get

$$\mu^* = |1 - 2(\frac{4}{5})| = \frac{3}{5}$$

4.c

$$\mu_{l,m} = 1 - \omega d^{-1}(4 - 2(\cos(l\pi h) + \cos(m\pi h)))$$

In this case, $\omega = 1, d = 4$.

$$\mu_{l,m} = 1 - (1 - \frac{1}{2}(\cos(l\pi h) + \cos(m\pi h)))$$

$$\mu_{l,m} = \frac{1}{2}(\cos(l\pi h) + \cos(m\pi h))$$

$$h = \frac{1}{N+1}$$

We're interested in the maximum eigenvalue of the iteration matrix. In this case, that's when $|\cos(\frac{l,m\pi}{N+1})|$ is largest. That means the insides is closest to 0.

We are constrained by $\frac{N+1}{2} \leq l \leq N, 1 \leq m \leq N$. Thus, $|\cos(...)|$ is maximized when $l = m = N$.

$$\mu_{l,m} = \frac{1}{2}(\cos(\frac{N\pi}{(N+1)}) + \cos(\frac{N\pi}{N+1}))$$

If we take the limit

$$\lim_{N\to\infty} \mu_{l,m} = \frac{1}{2}(\cos(\pi) + \cos(\pi))$$

$$\lim_{N\to\infty} \mu_{l,m} = -1$$

The scaling factor $\mu * (\omega = 1) = |-1| = 1$. Since the scaling factor is 1, that means no scaling occurs and that Jacobi is not an effective smoother.