CPSC 302 - Assignment 1

2018-09-27, Tristan Rice, 25886145, q7w9a

Question 1

(a)

Proof by induction.

$$f(n) = 3n^3 - n^2, g(n) = n^3$$

$$c = 6, n_0 = 10$$

Base case.

$$|3(10)^3 - (10)^2| \le 6(10)^3$$
$$|3000 - 100| \le 6000$$
$$2900 \le 6000$$

Induction step.

Assume

$$f(n-1) \le cg(n-1)$$

.

We must show that $f(n) - f(n-1) \le c(g(n) - g(n-1))$ since if f grows slower than g it must be smaller.

$$|3n^3 - n^2 - 3(n-1)^3 + (n-1)^2| \le 6(n^3 - (n-1)^3)$$

$$|(n-1)^2 - n^2| \le 3(n^3 - (n-1)^3)$$
$$|n^2 - 2n + 1 - n^2| \le 3(n^3 - n^3 + 3n^2 - 3n + 1)$$
$$|2n + 1| \le 9n^2 - 9n + 9$$

This holds since $|2n+1|$ is trivially smaller than $9n^2 - 9n + 9$.

Thus, $f(n) = O(g(n))$.

(b)

Proof by induction.

$$f(n) = 3n^3 + n^2, g(n) = n^3$$

Show $f(n) = \Theta(g(n))$.

$$c_1 = 1, c_2 = 6, n_0 = 10$$

Base case: $n = 10$

$$(10)^3 \le 3(10)^3 + (10)^2 \le 6(10)^3$$
$$1000 \le 3100 \le 6000$$

Induction step.

Assume

$$f(n-1) \leq cg(n-1)$$

.

We must show that $c_1(g(n)-g(n-1)) \leq f(n)-f(n-1) \leq c_2(g(n)-g(n-1))$ since if f grows slower than g it must be smaller.

$$\begin{gathered}
n^3-(n-1)^3 \leq 3n^3+n^2-3(n-1)^3-(n-1)^2 \leq 6n^3-6(n-1)^3 \\
n^3-(n^3-3n^2+3n-1) \leq 3n^3+n^2-3(n^3-3n^2+3n-1)-(n^2-2n+1) \leq 6n^3-6(n^3-3n^2+3n-1) \\
3n^2-3n+1 \leq 3n^3+n^2-3n^3+9n^2-9n+9-n^2+2n-1 \leq 6n^3-6n^3+18n^2-18n+6 \\
3n^2-3n+1 \leq 9n^2-7n+8 \leq 18n^2-18n+6 \\
3n^2 \leq 9n^2-4n+7 \leq 18n^2-15n+5
\end{gathered}$$

We can ignore the smaller terms since they are $= o(n^2)$. Thus leaving us with:

$$3n^2 \leq 9n^2 \leq 18n^2$$

This clearly holds and thus $f(n) = \Theta(g(n))$.

Question 2

(a)

If you ever find a number less than 0, or greater than the number before you know there's an error.

(b)

True value $= 0.0024$.

```
u25ErrAbs = 2.3363e+12
u25ErrRel = 9.6964e+14
```

(c)

$$\hat{u}_0 = u_0 + \epsilon_0$$

$$\hat{u}_n = -k\hat{u}_{n-1} + \frac{1}{n}$$

$$\hat{u}_1 = -k\hat{u}_0 + \frac{1}{1} - u_1$$

$$\begin{gathered}
\hat{u}_1 = -k(u_0+\epsilon_0) + \frac{1}{1} \\
\hat{u}_1 - u_1 = -k(u_0+\epsilon_0) + \frac{1}{1} - u_1 \\
\hat{u}_1 - u_1 = -k(u_0+\epsilon_0) + \frac{1}{1} + ku_0 - 1 \\
\epsilon_1 = \hat{u}_1 - u_1 = -k\epsilon_0
\end{gathered}$$

(d)

$$\epsilon_2 = \hat{u}_2 - u_2 = -k(\hat{u}_1) + \frac{1}{2} + k(u_1) - \frac{1}{2}$$
$$\epsilon_2 = -k(\hat{u}_1) + k(u_1)$$
$$\epsilon_2 = -k(-k(u_0 + \epsilon_0) + 1) + k(-ku_0 + 1)$$
$$\epsilon_2 = k^2 u_0 + k^2 \epsilon_0 - k - k^2 u_0 + k$$
$$\epsilon_2 = k^2 \epsilon_0$$

General rule:

$$\epsilon_n = (-k)^n \epsilon_0$$

Since the error is exponential, every iteration gets $-k$ times as much error. Thus at $n = 25$, the error is $(15)^n = 2.5 * 10^{29}$ times worse than at $n = 0$.

(e)

The observed error is very close to what we calculated. When comparing the error for two sequential u_n values we see that the error increases by almost exactly $-k = -15$.

```
>> 2336290021325.815918/-155752668088.387726
ans = -15
```

(f)

Emperically we see ϵ_0 as

```
>> 2336290021325.815918/(-15)^25
ans = -9.2522e-18
```

Since floating points are represented using exponentials, we should be able to divide the rounding error by k to get the error adjusted by the exponent. Which matches our empirical answer.

$$|\epsilon_0| \leq \frac{2^{-53}}{k}$$

$$|\epsilon_n| = k^n (\frac{2^{-53}}{k})$$

(g)

$$\hat{u}_n = u_n + \epsilon_n$$

i.

$$\hat{u}_{n-1} = -\frac{\hat{u}_n}{k} + \frac{1}{kn}$$
$$\hat{u}_{n-1} = -\frac{u_n}{k} - \frac{\epsilon_n}{k} + \frac{1}{kn}$$
$$\hat{u}_{n-1} - u_{n-1} = -\frac{u_n}{k} - \frac{\epsilon_n}{k} + \frac{1}{kn} - u_{n-1}$$
$$\hat{u}_{n-1} - u_{n-1} = -\frac{u_n}{k} - \frac{\epsilon_n}{k} + \frac{1}{kn} + \frac{u_n}{k} - \frac{1}{kn}$$
$$\hat{u}_{n-1} - u_{n-1} = -\frac{\epsilon_n}{k}$$

ii.

$$\hat{u}_{n-2} = -\frac{\hat{u}_{n-1}}{k} + \frac{1}{k(n-1}$$
$$\hat{u}_{n-2} = -\frac{-\frac{u_n}{k} - \frac{\epsilon_n}{k} + \frac{1}{kn}}{k} + \frac{1}{k(n-1)}$$
$$\hat{u}_{n-2} - u_{n-2} = -\frac{-\frac{u_n}{k} - \frac{\epsilon_n}{k} + \frac{1}{kn}}{k} + \frac{1}{k(n-1)} - u_{n-2}$$
$$\hat{u}_{n-2} - u_{n-2} = \frac{u_n}{k^2} + \frac{\epsilon_n}{k^2} - \frac{1}{k^2 n} + \frac{1}{k(n-1)} - u_{n-2}$$
$$\hat{u}_{n-2} - u_{n-2} = \frac{u_n}{k^2} + \frac{\epsilon_n}{k^2} - \frac{1}{k^2 n} + \frac{1}{k(n-1)} - \frac{u_n}{k^2} + \frac{1}{k^2 n} - \frac{1}{k(n-1)}$$
$$\hat{u}_{n-2} - u_{n-2} = \frac{\epsilon_n}{k^2}$$

iii.

$$\epsilon_n = \frac{\epsilon_N}{k^{N-n}}$$

iv.

Backwards recursion has a rounding error that is divided by k for each step so the total error is actually very small compared to the inverse.

(h)

Forwards:

```
u25True = 0.0024
u25ErrAbs = 2.3363e+12
u25ErrRel = 9.6964e+14
```

Backwards:

```
u25True = 0.0024
u25ErrAbs = 9.6602e-05
u25ErrRel = 0.0401
```

The error is significantly less doing backwards recursion.

(i)

Question 3

Part I.

$$fl(x + fl(y+z)) = 1.12345 * 10^0 + (3.12345 * 10^3 - 3.12345 * 10^3)$$
$$fl(x + fl(y+z)) = 1.12345 * 10^0 + (0.00000 * 10^3)$$
$$fl(x + fl(y+z)) = 1.12345 * 10^0$$

$$fl(x + fl(y+z)) = (1.12345 * 10^0 + 3.12345 * 10^3) - 3.12345 * 10^3$$
$$fl(x + fl(y+z)) = (0.00112 * 10^3 + 3.12345 * 10^3) - 3.12345 * 10^3$$
$$fl(x + fl(y+z)) = 3.12457 * 10^3 - 3.12345 * 10^3$$
$$fl(x + fl(y+z)) = 0.00112 * 10^3$$
$$fl(x + fl(y+z)) = 1.12000 * 10^0$$

$$1.12345 * 10^0 \neq 1.12000 * 10^0$$

Thus it holds $fl(x + fl(y + z)) \neq fl(fl(x + y) + z)$.

Relative error:

```
>> (1.12345-1.12)/1.12345
ans = 0.0031
```

Part II.

```
>> 7.45678*10^3 - 7.45631*10^3
ans = 0.4700
```

$$fl(x) = 7.457 * 10^3$$
$$fl(y) = 7.457 * 10^3$$

$$\hat{z} = fl(fl(x) - fl(y))$$
$$\hat{z} = 0.000 * 10^0$$

Absolute error $7.457 * 10^3 - 0 = 7457$.

Relative error $(7457 - 0)/7457 = 1$.

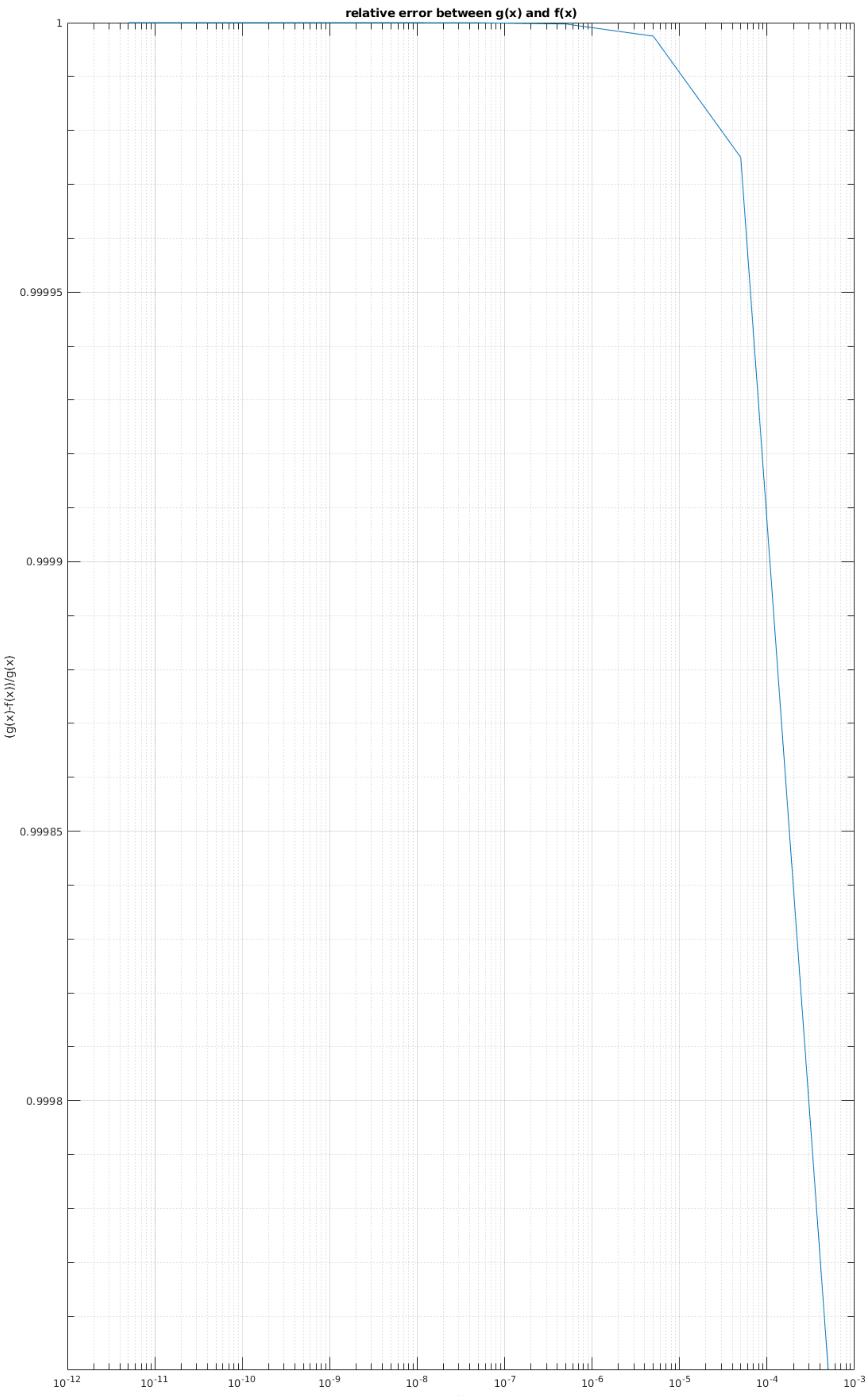
Question 4

$$f(x) = 1 - cos(x)$$

$$\sin^2(x) + \cos^2(x) = 1$$
$$1 - \cos(x) = 2\sin^2(x/2)$$

$$g(x) = 2\sin^2(x/2)$$

$$f(5 * 10^{-12}) = 0, g(5 * 10^{-12}) = 0.000000005000000$$

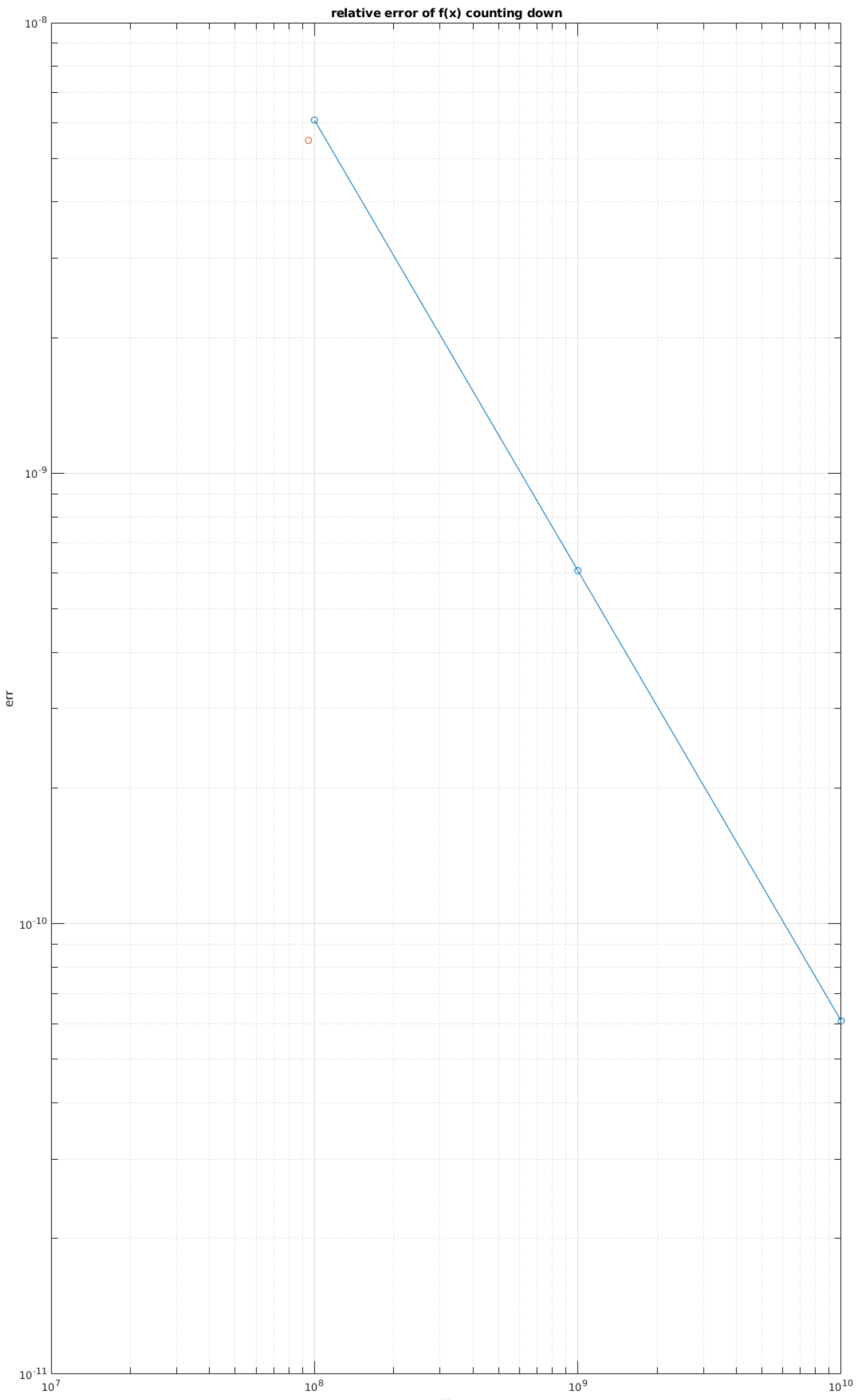

relative error between g(x) and f(x)
1
0.99995
0.9999
0.99985
0.9998
(g(x)-f(x))/g(x)
10^-12
10^-11
10^-10
10^-9
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
x

Question 5

```
>> run p5.m
y = 1.644934057834575
i = 94906267
```

We compute 94906266 elements before stopping.

$$e_1 = 5.479642961076798 * 10^{-09}$$



relative error of f(x) counting down
err
m
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{7}
10^{8}
10^{9}
10^{10}

```
% problem 2.a
k = 15
u0 = log((1+k)/k)

un = u0
for n = 1:25
  un = -k * un + 1/n
end

% problem 2.b

u25True = integral(@(x)x.^25./(x+15),0,1)

u25ErrAbs = un-u25True

u25ErrRel = u25ErrAbs/u25True

% problem 3.e

un = u0;
for n = 1:25
  uTrue = integral(@(x)x.^n./(x+15),0,1);
  un = -k * un + 1/n;
  err = un - uTrue;
  fprintf('n = %d, u_n = %f, err = %f\n', n, un, err)
end

% problem 3.h

u50 = integral(@(x)x.^50./(x+15),0,1)

un = u50;
for i = 1:25
  n = 50-i;
  uTrue = integral(@(x)x.^n./(x+15),0,1);
  un = -un/k + 1/(n*k);
  err = un - uTrue;
  fprintf('n = %d, u_n = %f, err = %f\n', n, un, err)
end

u25True = integral(@(x)x.^25./(x+15),0,1)

u25ErrAbs = un-u25True

u25ErrRel = u25ErrAbs/u25True
```

```
x = [5e-4, 5e-5, 5e-6,5e-7,5e-8,5e-9,5e-10,5e-11,5e-12]
f = 1-cos(x)
g = 2*sin(x./2);

err = (g-f)./g

loglog(x, err);
title('relative error between g(x) and f(x)');
xlabel('x');
ylabel('(g(x)-f(x))/g(x)');
grid on;
```

```
% Algorithm 1
y = 0;
last_y = 1;
i = 1;
while y ~= last_y
  last_y = y;
  y = y + 1/i^2;
  i = i + 1;
end

y
i

yTrue = pi^2/6
e1 = (yTrue - y)/yTrue

% Algorithm 2

ms = [10^8 10^9 10^10];
errs = [];

for m = ms
  m
  y = 0;
  while m > 0
    y = y + 1/m^2;
    m = m - 1;
  end
  errs = [errs, (yTrue - y)/yTrue];
end

loglog(ms, errs, '-o', [i], [e1], '-o');
title('relative error of f(x) counting down');
xlabel('m');
ylabel('err');
grid on;
```