Lecture Notes 13: Review Linear Algebra

CPSC 302: Numerical Computation for Algebraic Problems

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Chapter 4: Review Linear Algebra

- Basic Concepts: Linear Systems and Eigenvalue Problems
- Norms

- Special Matrix Classes
- Singular Value Decomposition (SVD)
- Examples in Applications

Outline

1. Special Matrix Classes
   - Symmetric Positive Definite
   - Orthogonality
2. Singular Value Decomposition (SVD)
3. Examples in Applications

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Symmetric Positive Definite Matrices

Extend notion of positive scalar to matrices:

$$A = A^T, \quad \mathbf{x}^T A\mathbf{x} > 0, \quad \forall\ \mathbf{x} \neq \mathbf{0}.$$

A symmetric matrix is positive definite if and only if all its eigenvalues are positive:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0.$$

Useful Facts about Matrices

Consider a real square $n \times n$ matrix $A = (a_{ij})$.

- If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then the matrix is nonsingular iff $d = det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$. If $d \neq 0$ then

$$A^{-1} = \frac{1}{d} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

- The matrix is strictly diagonally dominant if for all $i,\ i = 1, 2, \ldots, n$,

$$|a_{ii}| > \sum_{j \neq i} |a_{i,j}|.$$

- Let A be a symmetric, strictly diagonally dominant matrix whose diagonal elements are all positive. Then A is symmetric positive definite.

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1. Special Matrix Classes
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Orthogonal and Orthonormal Vectors

- **Orthogonal vectors**: two vectors $\mathbf{u}$ and $\mathbf{v}$ of the same length are orthogonal if

$$\mathbf{u}^T\mathbf{v} = 0.$$

- **Orthonormal vectors:** if *also* $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$.

Orthogonal Matrices

- Square matrix Q is **orthogonal** if its columns are pairwise orthonormal, i.e.,

$$Q^T Q = I. \quad \text{Hence also } Q^{-1} = Q^T.$$

- Important properties: for any orthogonal matrix Q and vector $\mathbf{x}$

$$\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2. \quad \text{Hence } \|Q\|_2 = \|Q^{-1}\|_2 = 1.$$

- For any symmetric matrix A there is a real orthogonal eigenvector matrix X, so that $X^{-1}AX$ is diagonal.

Outline

Singular Value Decomposition

Let A be real $m \times n$ (rectangular in general). Then there are orthogonal matrices U $(m \times m)$ and V $(n \times n)$ such that

$$A = U\Sigma V^T,$$

where

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \mathrm{diag}\{\sigma_1, \ldots, \sigma_r\},$$

with the **singular values** satisfying

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, \quad \sigma_{r+1} = \cdots = \sigma_n = 0.$$

Connection to eigenvalues: $\sigma_i = \sqrt{\lambda_i}$, where λ_i are eigenvalues of $A^T A$.

Outline

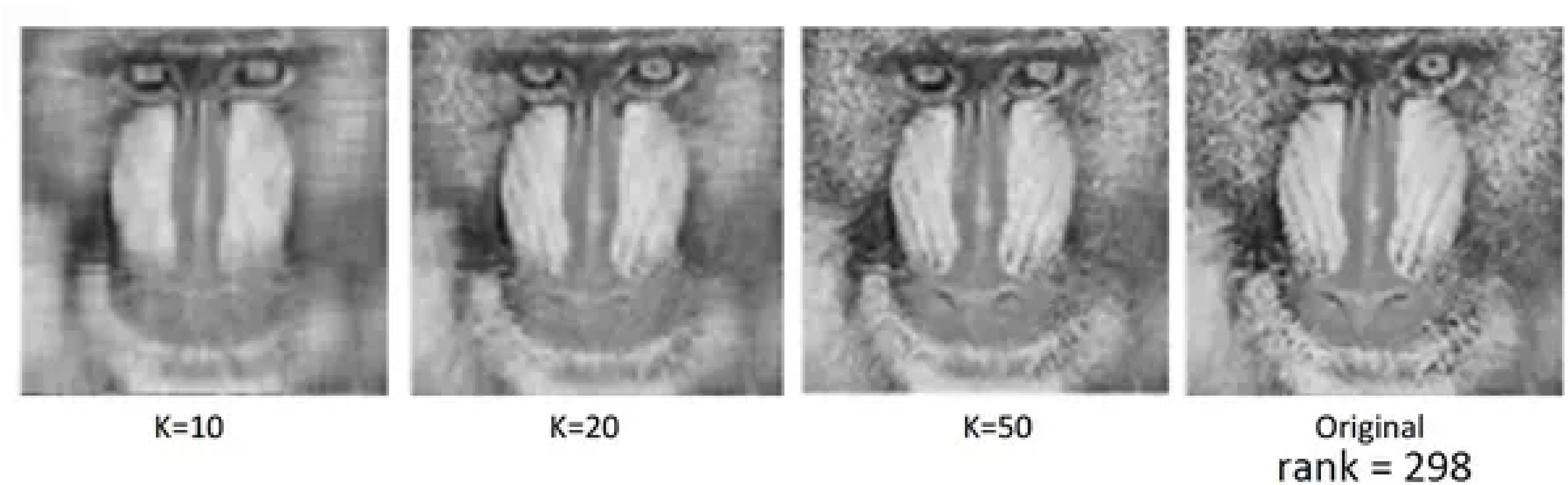
1. Special Matrix Classes

2. Singular Value Decomposition (SVD)

3. Examples in Applications
   - Low Rank Approximation
   - PCA
   - Data Fitting
   - Differential Equation

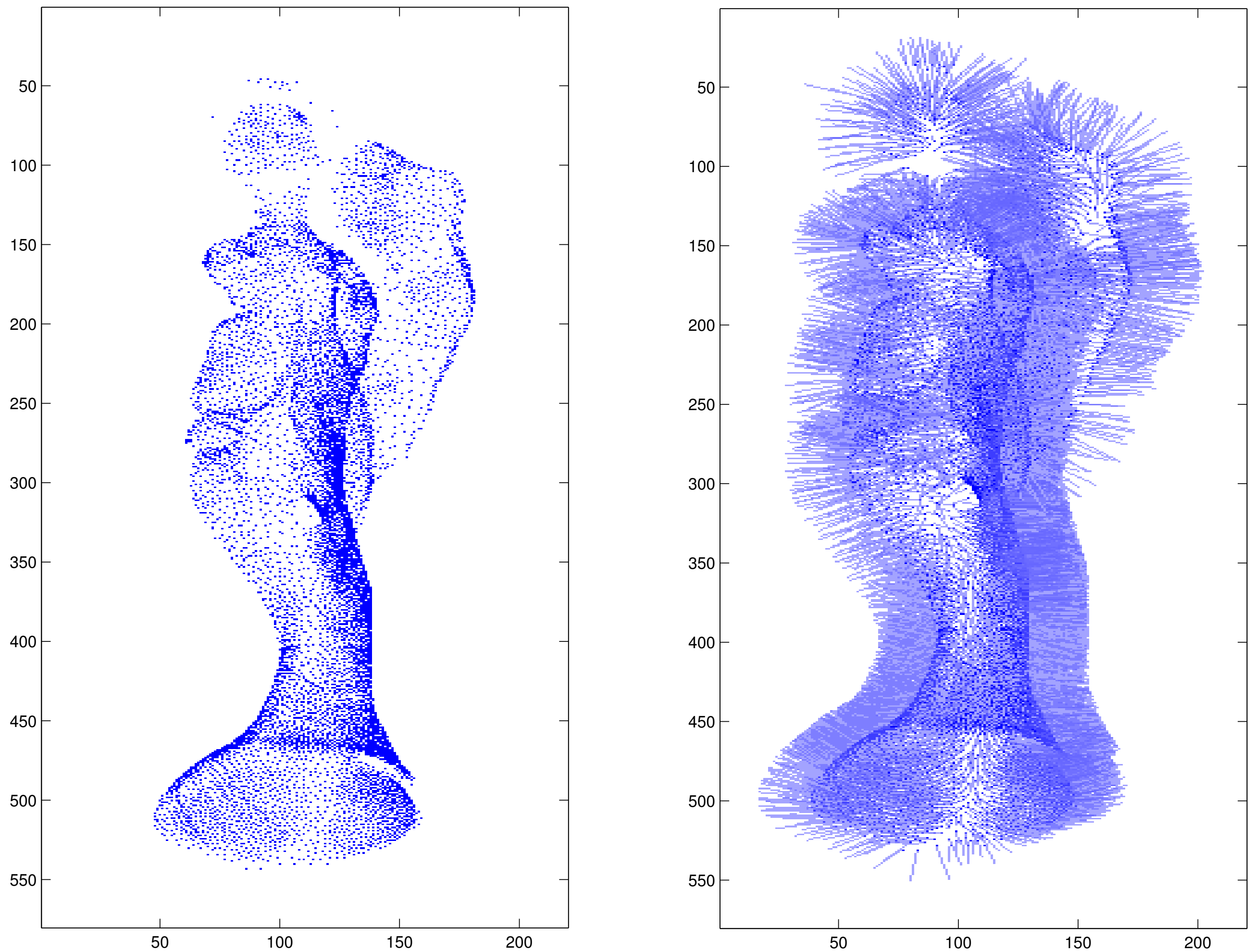
Outline

Low Rank Approximation



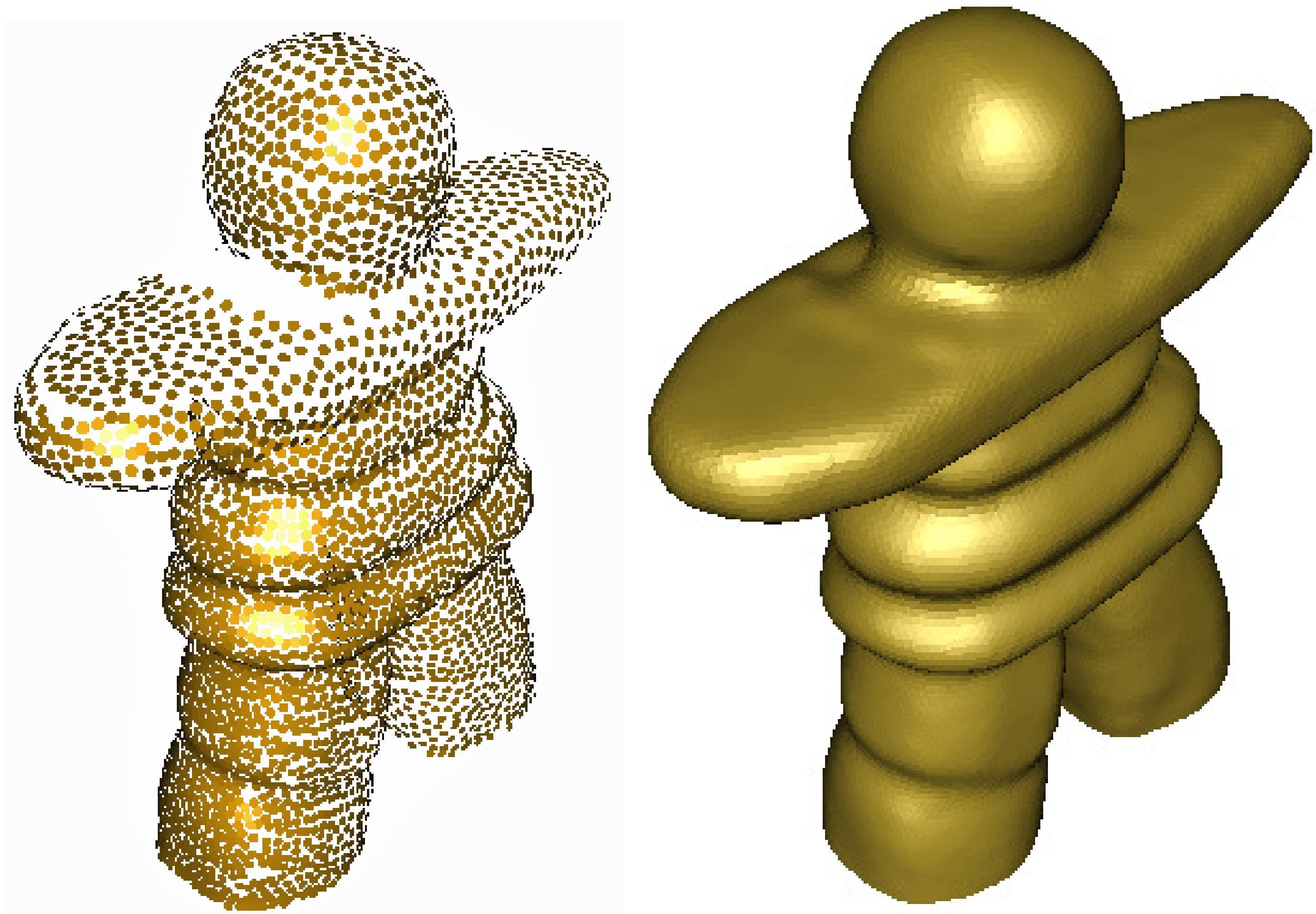
Outline

Instance: Point Cloud



Instance: RBF Interpolation

Left: consolidated point cloud. Right: RBF surface.



Outline

Data Fitting

Given measurements, or observations

$$(t_1, b_1), (t_2, b_2), \ldots, (t_m, b_m) = \{(t_i, b_i)\}_{i=1}^{m},$$

want to fit a function

$$v(t) = \sum_{j=1}^{n} x_j \phi_j(t),$$

For example, polynomial fit:

$$v(t) = x_1 + x_2 t + x_3 t^2 + \ldots + x_n t^{n-1}, \quad \text{so } \phi_j(t) = t^{j-1}.$$

- $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ are known linearly independent **basis functions**
- $x_1, \ldots, x_n$ are **coefficients** to be determined s.t. (hopefully)

$$v(t_i) = b_i, \quad i = 1, 2, \ldots, m.$$

Data Fitting (cont.)

Define $a_{ij} = \phi_j(t_i)$. Want $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Assume that A has full column rank n.

1. If $m = n$ get **interpolation problem**. Use methods of Chapters 5 or 7 to solve
$$A\mathbf{x} = \mathbf{b}.$$
2. If $m > n$ want, e.g., $\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|_2$. Get **least squares data fitting**. Use methods of Chapter 6 to solve
$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2.$$

Outline

Differential Equation

Given $g(t), \;\; 0 \le t \le 1$, recover $v(t)$ satisfying $-v'' = g$.

Require two boundary conditions

1. $v(0) = v(1) = 0$, or
2. $v(0) = 0$, $v'(1) = 0$.

Discretize on mesh $t_i = ih, \; i = 0, 1, \ldots, N$:

$$-\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = g(t_i), \quad i = 1, 2, \ldots, N-1.$$

Note $h = 1/(N+1)$. So, smaller h means larger number of linear equations.

With BC $v(0) = v(1) = 0$, require $v_0 = v_N = 0$.

Linear System for Differential Equation

Need to solve $A\mathbf{v} = \mathbf{g}$, where

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix}, \ \mathbf{g} = \begin{pmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_{N-2}) \\ g(t_{N-1}) \end{pmatrix}, \ A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

Thus, A is **tridiagonal**.