CPSC 302 - Assignment 5

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1. Stationary Method

1.a

$$A = M - N$$
$$Mx_{k+1} = Nx_k + b$$

$$x_{k+1} = x_k + \alpha(b - Ax_k)$$
$$x_{k+1} = x_k + \alpha b - \alpha A x_k$$
$$x_{k+1} = (I - \alpha A)x_k + \alpha b$$
$$\frac{I}{\alpha} x_{k+1} = (\frac{I}{\alpha} - A)x_k + b$$

$$M = \frac{I}{\alpha}$$

$$T = (I - \alpha A)$$

1.b

Converges only if $\rho(T) < 1$

Thus, only converges if $\rho(I - \alpha A) < 1$.

This converges if $|1 - \alpha\lambda| < 1$ where λ is the eigenvalue that maximizes the left hand statement. That happens to be λ_n since the eigenvalues are all positive and λ_n is the smallest eigenvalue.

$-1 < 1 - \alpha\lambda_n < 1$ $-2 < -\alpha\lambda_n < 0$ $2 > \alpha\lambda_n > 0$

We know that λ_n won't ever be 0 since it's defined as such.

Thus, the condition for convergence is just $\alpha\lambda_n < 2$

1.c

Using the above convergence conditions, we get $\lambda_n < 2$. Thus, the smallest eigenvalue must always be smaller than 2 for the statement to hold.

A diagonal matrix is by definition strictly diagonally dominant and the eigenvalues are just the values on the diagonal.

Thus we can construct a matrix with diagonal and eigenvalues $\lambda = \{5, 4, 3\}$. Thus, the smallest eigenvalue is 3 and this statement is contradicted since our convergence condition fails.

2. Consider the two-dimensional partial differential equation ...

2.a

Since B is symmetric we can express the condition number of it in terms of:

$$\kappa_2(B) = \frac{|\lambda|_{max}}{|\lambda|_{min}}$$

Since A is also symmetric, we can find the eigenvalues of B by adding $(\alpha h)^2 I$ to each eigenvalue of A.

$$\lambda_{l,m} = 4 - 2(\cos(l\pi h) + \cos(m\pi h)) + (\alpha h)^2, 1 \leq l, m \leq N$$

We need to find the maximum and minimum eigenvalues to determine the condition number.

Since $\cos(0) = 1, \cos(\pi) = -1$ we see that having $l = m = 1$ minimizes the second term and results in the smallest eigenvector.

The largest eigenvector is found when $l\pi h, m\pi h$ are close to π since that maximizes the second term.

$$\lambda_{min} = 4 - 2(\cos(1\pi h) + \cos(1\pi h)) + (\alpha h)^2$$
$$\lambda_{max} = 4 - 2(\cos(\lfloor N \rfloor \pi h) + \cos(\lfloor N \rfloor \pi h)) + (\alpha h)^2$$

$$\kappa_2(B) = \frac{4 - 2(\cos(1\pi h) + \cos(1\pi h)) + (\alpha h)^2}{4 - 2(\cos(\lfloor N \rfloor \pi h) + \cos(\lfloor N \rfloor \pi h)) + (\alpha h)^2}$$

$$\kappa_2(B) = \frac{4 - 2(\cos(\lfloor N \rfloor \pi \frac{1}{N+1}) + \cos(\lfloor N \rfloor \pi \frac{1}{N+1})) + (\alpha \frac{1}{N+1})^2}{4 - 2(\cos(1\pi \frac{1}{N+1}) + \cos(1\pi \frac{1}{N+1})) + (\alpha \frac{1}{N+1})^2}$$

```
octave:45> n = 1:20; N = sqrt(n)
N =

 Columns 1 through 18:

   1.0000    1.4142    1.7321    2.0000    2.2361    2.4495    2.6458    2.8284    3.0000    3.1623    3.3166    3.4641

 Columns 19 and 20:

   4.3589    4.4721

octave:46> k2 = (4 -2 * (cos(floor(N) * pi /(N+1)) + cos(floor(N) * pi / (N+1))) + (alpha./(N+1)).^2)./(4-2*(cc
k2 =

 Columns 1 through 18:

   1.2381    1.5050    1.7508    1.9843    2.2095    2.4285    2.6428    2.8532    3.0605    3.2650    3.4673    3.6676

 Columns 19 and 20:

   5.0286    5.2186
```

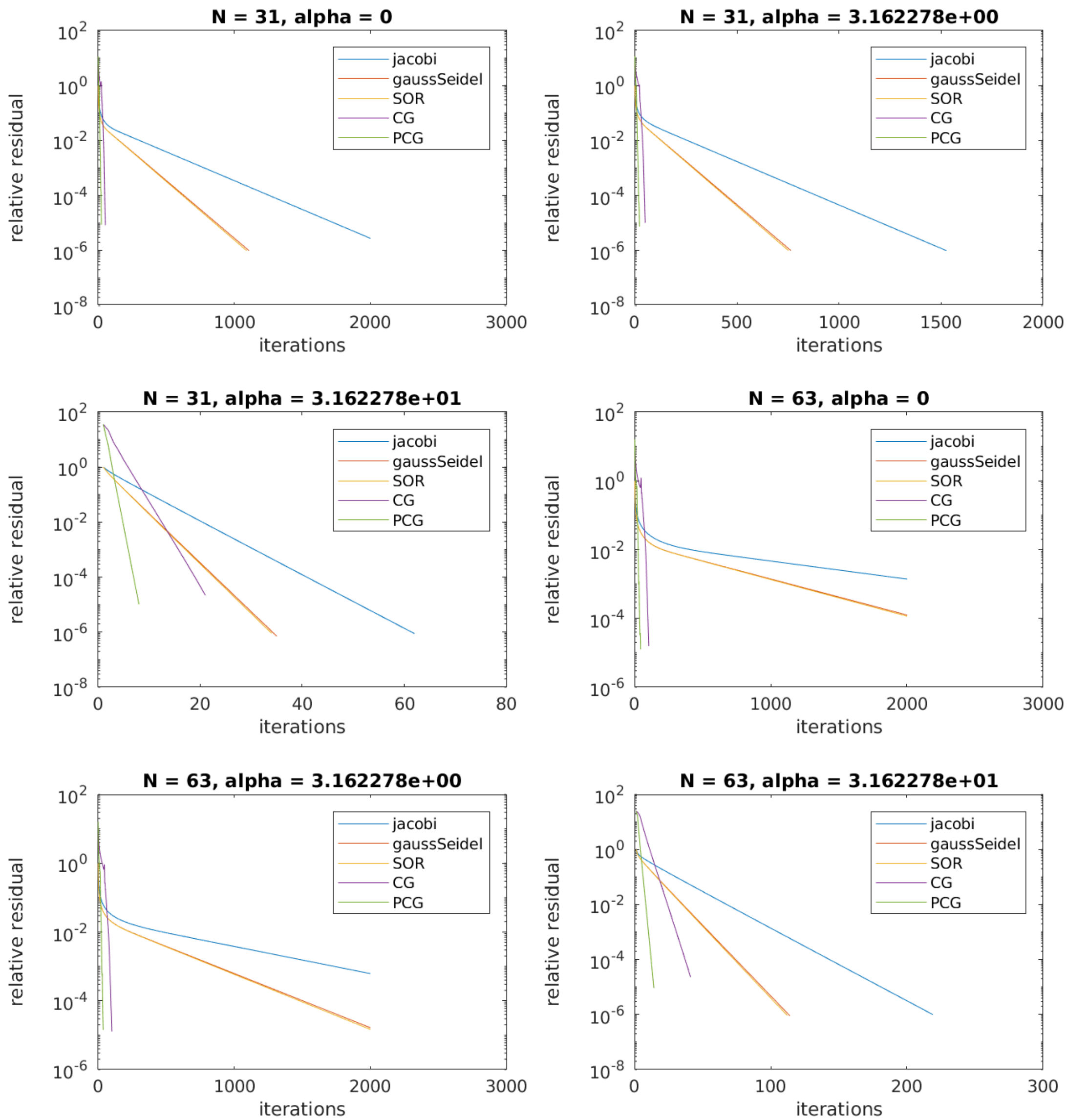
We see that increasing n causes the condition number to increase since the cos terms get closer to $1, -1$.

2.b

```
N = 31, n = 961, k2(B) = 4.143451e+02
jacobi: 2000 iters
gaussSeidel: 1108 iters
SOR: 1090 iters
cg: 52 iters
pcg: 23 iters

N = 31, n = 961, k2(B) = 2.752810e+02
jacobi: 1525 iters
gaussSeidel: 764 iters
SOR: 752 iters
cg: 50 iters
pcg: 22 iters

N = 31, n = 961, k2(B) = 8.994868e+00
jacobi: 61 iters
gaussSeidel: 34 iters
SOR: 33 iters
cg: 20 iters
pcg: 7 iters

N = 63, n = 3969, k2(B) = 1.659380e+03
jacobi: 2000 iters
gaussSeidel: 2000 iters
SOR: 2000 iters
cg: 102 iters
pcg: 42 iters

N = 63, n = 3969, k2(B) = 1.101665e+03
jacobi: 2000 iters
gaussSeidel: 2000 iters
SOR: 2000 iters
cg: 100 iters
pcg: 37 iters

N = 63, n = 3969, k2(B) = 3.309512e+01
jacobi: 218 iters
gaussSeidel: 113 iters
SOR: 111 iters
cg: 40 iters
pcg: 13 iters
```

We see that the speed of convergence is: Jacobi > Gauss-Seidel > SOR > CG > PCG.

The difference in speed of convergence makes sense since that's what the theoretical convergence rates of the different methods tells us. We also see that with varying N, α params that the speed of convergence is correlated with the condition number. Problems that are better conditioned converge faster.



3. Suppose we wish to solve Ax = b ...

3.a

Increasing `diagonalIncrement` causes gsMorph to converge faster.

Increasing the diagonal entries of A increases the spectral radius of A. This increase causes the spectral radius of the convergence matrix to go down since the iteration matrix is $I - M^{-1}A$.

3.b

N/A

4. The smoothing factor …

4.a

A is the discrete Laplacian.

4.b

4.c