Question 1

(a) Prove that this is equivalent to finding a flow f from s to t that minimizes…

The objective function $\min \sum_e l_e f_e$ computes the sum of every edge times the flow. Since $size(f) = 1$, this is equivalent to $\min \sum_e l_e$ for every edge along the flow.

Proof by contradiction.

Assume that there is a flow such that it is not the shortest path, and yet minimizes $\sum_e l_e f_e$. Since this is a network flow with $size(f) = 1$, there is only one path of edges through the graph. The minimizer is the sum of the distances along this path. Since this flow is not the shortest path, there must be a shorter path. The shorter path must have a smaller sum. However, since we've minimized the distance of the path this must be the shortest. Contradiction. Therefore, finding the flow f that minimizes $\sum_e l_e f_e$ subject to $size(f) = 1$ is equivalent to finding the shortest path from s to t.

(b) Write the shortest path problem as a linear program.

minimize $\sum_e l_e f_e$ where f is a vector with a value for all edges such that it is 1 if the flow goes through that edge and 0 if it does not.

The value of all edges inputing to a vertex must be equal to those outputting from a vertex.

$f_{inputs} - f_{outputs} = 0$ for all edges pointing to the vertex (inputs) and all edges pointing away from a vertex (outputs) for all vertexes.

Sum of all inputs into $t = 1$

Sum of all outputs from $s = 1$

(c) Explain why the dual LP can be written as …

The duality theorem tells us that the optimal values of the primal and the dual will be equal. This flips the condition from maximizing to be minimizing.

The dual is in terms of distance between s and t. It tries to maximize the distance between them, while also limiting how far each node can be apart corresponding to edge length. This works as a dual since the primal is minimizing sums of selected edges with constraints on how many edges must be included, the dual is maximizing the number of edges with constraints on the length.