Homework 1

Tristan Rice, 25886145, q7w9a

A.1(3,4,6)

A.1.3

Fibonacci: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10| 11| 12| 13| 14| 15| |–|–|–|–|–|–|–|–|–|–|–|–|–|–|–| | 1 | 1 | 2 | 3 | 5 | 8 | 13| 21| 34| 55| 89|144|233|377|610|

$F_nF_{n+3} - F_{n+1}F_{n+2}$ 1. $1*3-1*2=1$ 2. $1*5-2*3=-1$ 3. $2*8-3*5=1$ 4. $3*13-5*8=-1$ 5. $5*21-8*13=1$

Theorem: The output of $f(n) = F_nF_{n+3} - F_{n+1}F_{n+2}$ is 1 when n is even and -1 when n is odd.

Proof by induction.

Case: n is odd.

Base cases: - $n=1$: $1*3-1*2=1$ - $n=2$: $1*5-2*3=-1$

Induction step ($n \geq 3$):

$f(n) = f(n+2)$

$F_nF_{n+3} - F_{n+1}F_{n+2} = F_{n+2}F_{n+5} - F_{n+3}F_{n+4}$

$= F_{n+2}(F_{n+1} + F_{n+5})$

$= (F_{n+1} + F_n)(F_{n+3} + F_{n+2} + F_{n+3}) - (F_{n+2} + F_{n+1})(F_{n+3} + F_{n+2})$

$$= 2*F_{n+1}F_{n+3} + F_{n+1}F_{n+2} + 2*F_nF_{n+3} + F_nF_{n+2} - F_{n+2}F_{n+3} - F_{n+2}F_{n+2} - F_{n+1}F_{n+3} - F_{n+1}F_{n+2}$$

$$= F_{n+1}F_{n+3} + 2*F_nF_{n+3} + F_nF_{n+2} - F_{n+2}F_{n+3} - F_{n+2}F_{n+2}$$

$$= F_{n+1}F_{n+3} + 2*F_nF_{n+3} + F_nF_{n+2} - F_{n+1}F_{n+3} - F_nF_{n+3} - F_{n+1}F_{n+2} - F_nF_{n+2}$$
$$= F_nF_{n+3} - F_{n+1}F_{n+2}$$

Thus, since $f(n) = f(n+2)$ we prove via induction that the output will be 1 for all odd values of n and -1 for all even values of n.

A.1.4

```
In [15]: def F(n):
    ...:         return int(0.5+((1+sqrt(5))**n-(1-sqrt(5))**n)/(2**n*sqrt(5)))
In [16]: [(n, F(n) * F(n+8) - F(n+1)*F(n+7)) for n in range(1,6)]
Out[16]: [(1, 13), (2, -13), (3, 13), (4, -13), (5, 13)]
```

For all even values of n, $f(n) = -13$. For all odd values of n, $f(n) = 13$.

Proof by induction.

Base cases: * $n=1$: $f(1) = 13$ * $n=2$: $f(2) = -13$

Induction step:

We must show that for all $n > 2$, $f(n) = f(n+2)$.

$$F_nF_{n+8} - F_{n+1}F_{n+7} = F_{n+2}F_{n+10} - F_{n+3}F_{n+9}$$

$$= F_{n+1}F_{n+10} + F_nF_{n+10} - F_{n+2}F_{n+9} - F_{n+1}F_{n+9}$$

$$= F_{n+1}F_{n+9} + F_{n+1}F_{n+8} + F_nF_{n+9} + F_nF_{n+8} - F_{n+0}F_{n+9} - F_{n+1}F_{n+9} - F_{n+1}F_{n+9}$$

$$= F_{n+1}F_{n+8} + F_nF_{n+8} - F_{n+1}F_{n+9}$$

$$= F_{n+1}F_{n+8} + F_nF_{n+8} - F_{n+1}F_{n+8} - F_{n+1}F_{n+7}$$

$$= F_nF_{n+8} - F_{n+1}F_{n+7}$$

Thus, by the property of induction it holds for all $n \geq 1$, $f(n) = 13$ for odd numbers and $f(n) = -13$ for even numbers.

A.1.6

$$x^2 = x + 1$$

$$x = \{\frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 + \sqrt{5})\}$$

TODO(d4l3k): Finish this!

A.2(1,2)

A.2.1

Prove $\sum_{m=1}^{n} m = \binom{n+1}{2}$ for any $n \in \mathbb{N}$.

Proof via induction.

Base case $n = 1$.

$\sum_{m=1}^{1} m = \binom{2}{2}$

$1 = 1$

Identity.

Induction step.

Must prove that:

$$\sum_{m=1}^{n+1} m - \sum_{m=1}^{n} = \binom{n+2}{2} - \binom{n+1}{2}$$

$$n = \binom{n+2}{2} - \binom{n+1}{2}$$

$$n = \frac{(n+2)!}{2!(n+2-2)!} - \frac{(n+1)!}{2!(n+1-2)!}$$

$$n = \frac{(n+2)!}{2n!} - \frac{(n+1)!}{2(n-1)!}$$

$$n = \frac{(n+2)(n+1)}{2} - \frac{(n+1)n}{2}$$

$$2n = (n+2)(n+1) - (n+1)n$$
$$2n = n^2 + 3n + 2 - n^2 - n$$
$$2n = 2n + 2$$
$$n = n + 1$$

Since $\lim_{n\to\infty} \frac{n+1}{n} = 1$ we can ignore the 1 term.

TODO(d4l3k): Verify this.

A.2.2

Prove $\sum_{m=1}^{n}\binom{m}{k}=\binom{n+1}{k+1}$.

Proof by induction.

Let $k\in\mathbb{N}$ be fixed and arbitrary.

Base case $n=1$:

$$\binom{1}{k}=\binom{1+1}{k+1}$$

If $k=1$, $\binom{1}{1}=\binom{2}{2}=1$. If $k>1$, $\binom{1}{k}=\binom{2}{1+k}=0$.

Inductive step:

We must show that $\binom{n+1}{k}=\binom{n+2}{k+1}-\binom{n+1}{k+1}$.

$$\frac{(n+1)!}{k!(n+1-k)!}=\frac{(n+2)!}{(k+1)!(n+2-k-1)!}-\frac{(n+1)!}{(k+1)!(n+1-k+1)!}$$

$$\frac{(n+1)!}{k!(n-k+1)!}=\frac{(n+2)!}{(k+1)!(n-k+1)!}-\frac{(n+1)!}{(k+1)!(n-k)!}$$

$$\frac{(n+1)!(k+1)!}{k!}=\frac{(n+2)!(n-k+1)!}{(n-k+1)!}-\frac{(n+1)!(n-k+1)!}{(n-k)!}$$

$$(n+1)!(k+1)=(n+2)!-(n+1)!(n-k+1)$$

$$\begin{gathered}(k+1)=\frac{(n+2)!}{(n+1)!}-(n-k+1)\\ k+1=n+2-n+k-1\\ k+1=k+1\end{gathered}$$

Identity.

Thus, $\sum_{m=1}^{n}\binom{m}{k}=\binom{n+1}{k+1}$ holds for all $n\geq 1$.

A.3(1)

We aim to show $|A_1\cup A_2|=|A_1|+|A_2|-|A_1\cap A_2|$.

Proof by induction.

Base case:

$$A_1=A_2=\varnothing$$

$$|\varnothing\cup\varnothing|=|\varnothing|+|\varnothing|-|\varnothing\cap\varnothing|$$

Inductive step:

Consider adding an element a to A_1.

Assume $|A_1\cup A_2|=|A_1|+|A_2|-|A_1\cap A_2|$ for A_1,A_2.

Since a only occurs in A_1, only the $|A_1|$ term changes, and the union is maintained. This holds for adding a only to A_2 since set addition, union and intersection are symmetric operations.

Consider adding an element a to A_1 and A_2.

Since a occurs in A_1 and A_2, the right side becomes $|A_1|+a+|A_2|+a-|A_1\cap A_2|-a$. Since a is in both, it is included in the intersection and cancels out. Thus, it is only added once.

Thus, $|A_1\cup A_2|=|A_1|+|A_2|-|A_1\cap A_2|$ holds for all A_1,A_2.

A.4(1b,1d,2d)

Show that $f(n) = O(g(n))$...

A.4.1b

$f(n) = 100 * 3^n, g(n) = 4^n$.

Pick $C = 100, n_0 = 0$.

Proof by induction.

Base case: $n = 0$

$$100 * 3^0 \leq 100 * 4^0$$
$$100 \leq 100$$

Inductive step:

Must show that

$$100 * 3^{n+1} - 100 * 3^n \leq 100 * 4^{n+1} - 100 * 4^n$$
$$3^{n+1} - 3^n \leq 4^{n+1} - 4^n$$

$$3^n(3-1) \leq 4^n(4-1)$$

$$3^n(2) \leq 4^n(3)$$

$$n * \log(3) + \log(2) \leq n * \log(4) + \log(3)$$

The left side is clearly less than the right side. Thus, via induction we show that $f(n) = O(g(n))$ since $f(n) \leq Cg(n) \forall n \geq n_0$.

A.4.1d

Show that $f(n) = O(g(n)), f(n) = n^2 + 3n + 1, g(n) = n(n-1)$.

$g(n) = n^2 - n$

$C = 4, n_0 = 4$

Proof by induction.

Base case $n = 4$:

$$4^2 + 3 * 4 + 1 \leq 4 * 4^2 - 4 * 4$$
$$16 + 12 + 1 \leq 64 - 16$$
$$29 \leq 48$$

Inductive step:

We must show that $(n+1)^2 + 3(n+1) + 1 - n^2 - 3n - 1 \leq 4(n+1)^2 - 4(n+1) - 4n^2 + 4n$.

$$n^2 + 2n + 1 + 3n + 4 - n^2 - 3n - 1 \leq 4(n^2 + 2n + 1) - 4n + 4 - 4n^2 + 4n$$
$$2n + 4 \leq 4n^2 + 8n + 8 + 4 - 4n^2$$
$$2n + 4 \leq 8n + 12$$
$$n + 2 \leq 8n + 6$$

This holds for $n \geq 0$.

Thus, by induction $f(n) = O(g(n)), f(n) = n^2 + 3n + 1, g(n) = n(n-1)$ since $f(n) \leq Cg(n) \forall n \geq n_0$.

A.4.2d

Compute limit $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ to show that $f(n) = o(g(n))$ for $f(n) = n^2 + 3n + 1, g(n) = n(n-1)(n-2)$.

$$\lim_{n\to\infty} \frac{n^2+3n+1}{n(n-1)(n-2)}$$
$$= \lim_{n\to\infty} \frac{n^2+3n+1}{n(n^2-3n+6)}$$
$$= \lim_{n\to\infty} \frac{n^2+3n+1}{n^3-3n^2+6n)}$$
$$= \lim_{n\to\infty} \frac{n^2+3n+1}{n^3-3n^2+6n)}$$

Using L'Hospital's rule.

$$= \lim_{n\to\infty} \frac{2n+3}{3n^2-6n+6)}$$
$$= \lim_{n\to\infty} \frac{2}{3n-6)} = 0$$

Since the limit goes to 0 as $n \to \infty$, $f(n) = o(g(n))$.

A.5(2)

Prove that if f, g are functions $\mathbb{Z} \to \mathbb{R}^+$, then $f(n) = o(g(n)) \iff \log g(n) - \log f(n)$ tends to infinity (as $n \to \infty$).

Definition: We know that $f(n) = o(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$$\lim_{n\to\infty} \log g(n) - \log f(n) = \infty$$
$$= \lim_{n\to\infty} \log(\frac{g(n)}{f(n)}) = \infty$$

If $\lim_{x\to\infty} \log x = \infty$, then $\lim_{x\to\infty} x = \infty$, since $x >> \log x$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$$

Taking the reciprocal of this statement gets us to the definition for $f(n) = o(g(n))$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1}{\infty} = 0$$

Thus, if f, g are functions $\mathbb{Z} \to \mathbb{R}^+$, then $f(n) = o(g(n)) \iff \log g(n) - \log f(n)$ tends to infinity (as $n \to \infty$).