

# CPSC 421 - Homework 1

Tristan Rice, 25886145, q7w9a

## A.1(3,4,6)

### A.1.3

Fibonacci:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f(n)	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

$$F_n F_{n+3} - F_{n+1} F_{n+2} \quad 1. 1 * 3 - 1 * 2 = 1 \quad 2. 1 * 5 - 2 * 3 = -1 \quad 3. 2 * 8 - 3 * 5 = 1 \quad 4. 3 * 13 - 5 * 8 = -1 \quad 5. 5 * 21 - 8 * 13 = 1$$

Theorem: The output of  $f(n) = F_n F_{n+3} - F_{n+1} F_{n+2}$  is 1 when  $n$  is even and  $-1$  when  $n$  is odd.

Proof by induction.

Case:  $n$  is odd.

Base cases:  $n = 1: 1 * 3 - 1 * 2 = 1$   $n = 2: 1 * 5 - 2 * 3 = -1$

Induction step ( $n \geq 3$ ):

$$f(n) = f(n+2)$$

$$F_n F_{n+3} - F_{n+1} F_{n+2} = F_{n+2} F_{n+5} - F_{n+3} F_{n+4}$$

$$= F_{n+2}(F_{n+1} + F_{n+5})$$

$$= (F_{n+1} + F_n)(F_{n+3} + F_{n+2} + F_{n+3}) - (F_{n+2} + F_{n+1})(F_{n+3} + F_{n+2})$$

$$= 2 * F_{n+1} F_{n+3} + F_{n+1} F_{n+2} + 2 * F_n F_{n+3} + F_n F_{n+2} - F_{n+2} F_{n+3} - F_{n+2} F_{n+2} - F_{n+1} F_{n+3} - F_{n+1} F_{n+2}$$

$$= F_{n+1} F_{n+3} + 2 * F_n F_{n+3} + F_n F_{n+2} - F_{n+2} F_{n+3} - F_{n+2} F_{n+2}$$

$$= F_{n+1} F_{n+3} + 2 * F_n F_{n+3} + F_n F_{n+2} - F_{n+1} F_{n+3} - F_n F_{n+3} - F_{n+1} F_{n+2} - F_n F_{n+2}$$

$$= F_n F_{n+3} - F_{n+1} F_{n+2}$$

Thus, since  $f(n) = f(n+2)$  we prove via induction that the output will be 1 for all odd values of  $n$  and  $-1$  for all even values of  $n$ .

### A.1.4

In [15]: `def F(n):`

`...: return int(0.5+((1+sqrt(5))**n-(1-sqrt(5))**n)/(2**n*sqrt(5)))`

In [16]: `[(n, F(n) * F(n+8) - F(n+1)*F(n+7)) for n in range(1,6)]`

Out[16]: `[(1, 13), (2, -13), (3, 13), (4, -13), (5, 13)]`

For all even values of  $n$ ,  $f(n) = -13$ . For all odd values of  $n$ ,  $f(n) = 13$ .

Proof by induction.

Base cases:  $n = 1: f(1) = 13$   $n = 2: f(2) = -13$

Induction step:

We must show that for all  $n > 2$ ,  $f(n) = f(n+2)$ .

$$F_n F_{n+8} - F_{n+1} F_{n+7} = F_{n+2} F_{n+10} - F_{n+3} F_{n+9}$$

$$\begin{aligned}
&= F_{n+1}F_{n+10} + F_nF_{n+10} - F_{n+2}F_{n+9} - F_{n+1}F_{n+9} \\
&= F_{n+1}F_{n+9} + F_{n+1}F_{n+8} + F_nF_{n+9} + F_nF_{n+8} - F_{n+0}F_{n+9} - F_{n+1}F_{n+9} - F_{n+1}F_{n+9} \\
&= F_{n+1}F_{n+8} + F_nF_{n+8} - F_{n+1}F_{n+9} \\
&= F_{n+1}F_{n+8} + F_nF_{n+8} - F_{n+1}F_{n+8} - F_{n+1}F_{n+7} \\
&= F_nF_{n+8} - F_{n+1}F_{n+7}
\end{aligned}$$

Thus, by the property of induction it holds for all  $n \geq 1$ ,  $f(n) = 13$  for odd numbers and  $f(n) = -13$  for even numbers.

### A.1.6

$$x^2 = x + 1$$

$$x = \left\{ \frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 + \sqrt{5}) \right\}$$

TODO(d4l3k): Finish this!

### A.2(1,2)

#### A.2.1

Prove  $\sum_{m=1}^n m = \binom{n+1}{2}$  for any  $n \in \mathbb{N}$ .

Proof via induction.

Base case  $n = 1$ .

$$\sum_{m=1}^1 m = \binom{2}{2}$$

$$1 = 1$$

Identity.

Induction step.

Must prove that:

$$\sum_{m=1}^{n+1} m - \sum_{m=1}^n m = \binom{n+2}{2} - \binom{n+1}{2}$$

$$n = \binom{n+2}{2} - \binom{n+1}{2}$$

$$n = \frac{(n+2)!}{2!(n+2-2)!} - \frac{(n+1)!}{2!(n+1-2)!}$$

$$n = \frac{(n+2)!}{2n!} - \frac{(n+1)!}{2(n-1)!}$$

$$n = \frac{(n+2)(n+1)}{2} - \frac{(n+1)n}{2}$$

$$2n = (n+2)(n+1) - (n+1)n$$

$$2n = n^2 + 3n + 2 - n^2 - n$$

$$2n = 2n + 2$$

$$n = n + 1$$

Since  $\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$  we can ignore the 1 term.

TODO(d4l3k): Verify this.

### A.2.2

Prove  $\sum_{m=1}^n \binom{m}{k} = \binom{n+1}{k+1}$ .

Proof by induction.

Let  $k \in \mathbb{N}$  be fixed and arbitrary.

Base case  $n = 1$ :

$$\binom{1}{k} = \binom{1+1}{k+1}$$

If  $k = 1$ ,  $\binom{1}{1} = \binom{2}{2} = 1$ . If  $k > 1$ ,  $\binom{1}{k} = \binom{2}{1+k} = 0$ .

Inductive step:

We must show that  $\binom{n+1}{k} = \binom{n+2}{k+1} - \binom{n+1}{k+1}$ .

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+2)!}{(k+1)!(n+2-k-1)!} - \frac{(n+1)!}{(k+1)!(n+1-k+1)!}$$

$$\frac{(n+1)!}{k!(n-k+1)!} = \frac{(n+2)!}{(k+1)!(n-k+1)!} - \frac{(n+1)!}{(k+1)!(n-k)!}$$

$$\frac{(n+1)!(k+1)!}{k!} = \frac{(n+2)!(n-k+1)!}{(n-k+1)!} - \frac{(n+1)!(n-k+1)!}{(n-k)!}$$

$$(n+1)!(k+1) = (n+2)! - (n+1)!(n-k+1)$$

$$(k+1) = \frac{(n+2)!}{(n+1)!} - (n-k+1)$$

$$k+1 = n+2 - n + k - 1$$

$$k+1 = k+1$$

Identity.

Thus,  $\sum_{m=1}^n \binom{m}{k} = \binom{n+1}{k+1}$  holds for all  $n \geq 1$ .

### A.3(1)

We aim to show  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ .

Proof by induction.

Base case:

$$A_1 = A_2 = \emptyset$$

$$|\emptyset \cup \emptyset| = |\emptyset| + |\emptyset| - |\emptyset \cap \emptyset|$$

Inductive step:

Consider adding an element  $a$  to  $A_1$ .

Assume  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$  for  $A_1, A_2$ .

Since  $a$  only occurs in  $A_1$ , only the  $|A_1|$  term changes, and the union is maintained. This holds for adding  $a$  only to  $A_2$  since set addition, union and intersection are symmetric operations.

Consider adding an element  $a$  to  $A_1$  and  $A_2$ .

Since  $a$  occurs in  $A_1$  and  $A_2$ , the right side becomes  $|A_1| + a + |A_2| + a - |A_1 \cap A_2| - a$ . Since  $a$  is in both, it is included in the intersection and cancels out. Thus, it is only added once.

Thus,  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$  holds for all  $A_1, A_2$ .

## A.4(1b,1d,2d)

Show that  $f(n) = O(g(n))$ ...

### A.4.1b

$$f(n) = 100 * 3^n, g(n) = 4^n.$$

Pick  $C = 100, n_0 = 0$ .

Proof by induction.

Base case:  $n = 0$

$$\begin{aligned} 100 * 3^0 &\leq 100 * 4^0 \\ 100 &\leq 100 \end{aligned}$$

Inductive step:

Must show that

$$\begin{aligned} 100 * 3^{n+1} - 100 * 3^n &\leq 100 * 4^{n+1} - 100 * 4^n \\ 3^{n+1} - 3^n &\leq 4^{n+1} - 4^n \end{aligned}$$

$$3^n(3 - 1) \leq 4^n(4 - 1)$$

$$3^n(2) \leq 4^n(3)$$

$$n * \log(3) + \log(2) \leq n * \log(4) + \log(3)$$

The left side is clearly less than the right side. Thus, via induction we show that  $f(n) = O(g(n))$  since  $f(n) \leq Cg(n) \forall n \geq n_0$ .

### A.4.1d

Show that  $f(n) = O(g(n)), f(n) = n^2 + 3n + 1, g(n) = n(n - 1)$ .

$$g(n) = n^2 - n$$

$$C = 4, n_0 = 4$$

Proof by induction.

Base case  $n = 4$ :

$$\begin{aligned} 4^2 + 3 * 4 + 1 &\leq 4 * 4^2 - 4 * 4 \\ 16 + 12 + 1 &\leq 64 - 16 \\ 29 &\leq 48 \end{aligned}$$

Inductive step:

We must show that  $(n+1)^2 + 3(n+1) + 1 - n^2 - 3n - 1 \leq 4(n+1)^2 - 4(n+1) - 4n^2 + 4n$ .

$$n^2 + 2n + 1 + 3n + 4 - n^2 - 3n - 1 \leq 4(n^2 + 2n + 1) - 4n + 4 - 4n^2 + 4n$$

$$2n + 4 \leq 4n^2 + 8n + 8 + 4 - 4n^2$$

$$2n + 4 \leq 8n + 12$$

$$n + 2 \leq 8n + 6$$

This holds for  $n \geq 0$ .

Thus, by induction  $f(n) = O(g(n))$ ,  $f(n) = n^2 + 3n + 1$ ,  $g(n) = n(n-1)$  since  $f(n) \leq Cg(n) \forall n \geq n_0$ .

#### A.4.2d

Compute limit  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$  to show that  $f(n) = o(g(n))$  for  $f(n) = n^2 + 3n + 1$ ,  $g(n) = n(n-1)(n-2)$ .

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{n^2 + 3n + 1}{n(n-1)(n-2)} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + 3n + 1}{n(n^2 - 3n + 6)} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + 3n + 1}{n^3 - 3n^2 + 6n} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + 3n + 1}{n^3 - 3n^2 + 6n} \end{aligned}$$

Using L'Hospital's rule.

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{2n + 3}{3n^2 - 6n + 6} \\ &= \lim_{n \rightarrow \infty} \frac{2}{3n - 6} = 0 \end{aligned}$$

Since the limit goes to 0 as  $n \rightarrow \infty$ ,  $f(n) = o(g(n))$ .

#### A.5(2)

Prove that if  $f, g$  are functions  $\mathbb{Z} \rightarrow \mathbb{R}^+$ , then  $f(n) = o(g(n)) \iff \log g(n) - \log f(n)$  tends to infinity (as  $n \rightarrow \infty$ ).

Definition: We know that  $f(n) = o(g(n))$  iff  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ .

$$\begin{aligned} & \lim_{n \rightarrow \infty} \log g(n) - \log f(n) = \infty \\ &= \lim_{n \rightarrow \infty} \log\left(\frac{g(n)}{f(n)}\right) = \infty \end{aligned}$$

If  $\lim_{x \rightarrow \infty} \log x = \infty$ , then  $\lim_{x \rightarrow \infty} x = \infty$ , since  $x \gg \log x$ .

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \infty$$

Taking the reciprocal of this statement gets us to the definition for  $f(n) = o(g(n))$ .

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \frac{1}{\infty} = 0$$

Thus, if  $f, g$  are functions  $\mathbb{Z} \rightarrow \mathbb{R}^+$ , then  $f(n) = o(g(n)) \iff \log g(n) - \log f(n)$  tends to infinity (as  $n \rightarrow \infty$ ).