DIRECTED GRAPHS AND ASYMPTOTIC TESTS

JOEL FRIEDMAN

Contents

Disclaimer: The material may sketchy and/or contain errors, which I will elaborate upon and/or correct in class. For those not in CPSC 421/501: use this material at your own risk...

The reference [Sip] is to the course textbook, *Introduction to the Theory of Computation* by Michael Sipser, 3rd Edition.

1. Introduction

This article reviews matertial prerequisite to CPSC 421. It can be viewed as a bridge between Chapter 0 and Chapters 1-9 of [Sip].

In addition, we describe what we call *asymptotic tests*; when we cover Chapter 1 of [Sip], we will see that asymptotic tests give the most direct and simplest way to show that (some) languages are not *regular languages*. [I have never seen such tests in any introductory CS theory textbook; this omission has baffled me for decades.]

Here we formally describe *directed graphs*, to prepare students for formal descriptions of computing machines such as *finite automata* and *Turing machines*. Many such machines are most simply described by drawing a directed graph—whose vertices are the *states* of the machine—and adding some information. We will describe *asymptotic tests* in the language of directed graphs.

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A lot of examples and some extra material is developed in the exercises at the end of this article. Linear algebra is not a prerequisite for this course, but can be used to explain why asymptotic tests work and to give more powerful tests; this will be explained in an appendix (not required for CPSC 421 this year).

2. Review: Basic Notation and Asymptotics

The following material is mostly from Chapter 0 and Section 7.1 of [Sip].

2.1. **Basic Notation.** We use $\mathbb{N} = \{1, 2, \ldots\}$ to denote the natural numbers, $\mathbb{Z}$ to denote the integers, $\mathbb{R}$ to denote the real numbers. We use $\mathbb{R}_{\geq 0}$ to denote the non-negative real numbers, $\mathbb{R}_{>0}$ the positive reals, $\mathbb{N}_{>33}$ the integers greater than 33, etc. [Sip] uses $\mathcal{N}, \mathcal{R}, \mathcal{Z}$ for $\mathbb{N}, \mathbb{R}, \mathbb{Z}$; [Sip] uses $\mathcal{R}^+$ for $\mathbb{R}_{\geq 0}$, not $\mathbb{R}_{>0}$ (ouch!).

Consult Section 0.2 of [Sip] for common conventions regarding sets and sequences. A function $f \colon \mathbb{N} \to \mathbb{R}$ can be viewed as an infinite sequence of its values, $f(1), f(2), \ldots$, and an infinite sequence of real numbers, $a_1, a_2, \ldots$ can be viewed as a function $\mathbb{N} \to \mathbb{R}$. These notions are interchangeable, but sometimes one form is more convenient than the other.

If f, g are functions $\mathbb{N} \to \mathbb{R}$, we may write formulas like $f(n)/g(n)$, $\log(g(n))$, etc., that may not defined for all $n \in \mathbb{N}$ (e.g., when $g(0) = 0$) and therefore have a smaller domain. To make sense of the limit

$$\lim_{n\to\infty} f(n)/g(n)$$

it is permissible to have $g(n) = 0$ for finitely many n, and we allow this. In our limits we allow for infinite values such as $+\infty$ and $-\infty$, unless we insist that the limit is *finite*.

If $x \in \mathbb{R}$ is a real number, the *floor of* x, denoted $\lfloor x \rfloor$, is the largest integer no larger than x; the *ceiling of* x, denoted $\lceil x \rceil$, is the smallest integer no smaller than x.

Let $f = f(n)$ and $g = g(n)$ be functions $\mathbb{N} \to \mathbb{R}_{\geq 0}$.

(1) We write $f(n) = O(g(n))$ if there exists $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}_{\geq 0}$ for which
$$f(n) \leq c\, g(n)$$
provided that $n \geq n_0$.

(2) We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, i.e., if there exists $n_0 \in \mathbb{N}$ and $c, c' \in \mathbb{R}_{\geq 0}$ for which
$$c' g(n) \leq f(n) \leq c\, g(n)$$
provided that $n \geq n_0$.

(3) We write $f(n) = o(g(n))$ if for any $c \in \mathbb{R}_{>0}$ there is an $n_0 \in \mathbb{N}$ for which $f(n) < c\, g(n)$ provided that $n \geq n_0$; this condition is equivalent to
$$\lim_{n\to\infty} f(n)/g(n) = 0$$
whenever $g(n) \neq 0$ for sufficiently large n.

(4) We write $f(n) \sim g(n)$ if
$$\lim_{n\to\infty} f(n)/g(n) = 1 \ .$$

These definitions also make sense and work well if f, g are functions $\mathbb{N} \to \mathbb{R}$ such that $f(n) > 0$ and $g(n) > 0$ for n sufficiently large. (Compare (1)–(3) above with Section 7.1 of [Sip], starting page 276.)

Chapter 0 of [Sip] has a lot of important notation regarding sets and graphs, and gives a review of proofs (by induction, by contradiction, etc.); you should review all of this. We will use this material in the exercises at the end of this article.

In class we will also discuss the $OO(f(n))$ notation[1].

2.2. **Determining Asymptotic Relationships.** Given two functions $\mathbb{N} \to \mathbb{R}_{\geq 0}$, f, g, it is not necessarily easy to determine when $f(n)$ is $O(g(n))$, $o(g(n))$, etc. However, in CPSC 421 we will only need a few simple relationships between logarithms, polynomials, and exponential functions: some examples are

$$2n^3 - n^2 = O(n^3),\ n^4 = o(2^n),\ (\log n)^{12} = o\big(n^{.01}\big)\ .$$

As another example, since $0 \leq x - \lfloor x \rfloor < 1$ for all $x \in \mathbb{R}$, we have

$$\lfloor \sqrt{n} \rfloor = \sqrt{n} + O(1) = \sqrt{n} + o\big(\sqrt{n}\big) \tag{1}$$

and similarly with $\sqrt{n}$ replaced with other functions, and similarly with the ceiling function. At times students will be provided with necessary asymptotics, such as $\pi(n) \sim n/\log_e n$, where $\pi(n)$ is the number of primes less than or equal to n.

3. The Asymptotic Ratio

Definition 3.1. Let $f \colon \mathbb{N} \to \mathbb{R}$ be a function. If

$$\rho \stackrel{\text{def}}{=} \lim_{n\to\infty} f(n+1)/f(n)$$

exists, we say that f has *asymptotic ratio* ρ (we allow for $\rho = \pm\infty$); otherwise we say that f *does not have an asymptotic ratio*. The ratio ρ is *finite* if $\rho \neq \pm\infty$.

We remind the reader that $f(n+1)/f(n)$ may be undefined for finitely many n, according to our convention on limits. Asymptotic limits exist for many sequences arising in complexity theory. This definition is used in the *asymptotic test for convergence* of the infinite sum $\sum_{n=1}^{\infty} f(n)$, e.g.,

$$\frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \cdots + \frac{n}{5^n} + \cdots$$

converges since the asymptotic ratio of the terms is $1/5$.

Here are some standard examples:

(1) The asymptotic ratio of $f(n) = 3^n$ is 3.
(2) The asymptotic ratio of $f(n) = n^2$ is 1, since
$$\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \big(1 + 2n^{-1} + n^{-2}\big) = 1.$$
(3) The asymptotic ratio of $f(n) = \log n$ is 1 (see the exercises).
(4) If $f(n)$ and $g(n)$ have respective asympotic ratios ρ and ρ', then the asymptotic ratio of $h(n) = f(n)g(n)$ is $\rho\,\rho'$; similarly for the product of three functions or of any finite number of functions.

[1] To the best of the author's knowledge, this notation is due to Udi Manbar; it is not precise: an algorithm running in time $f(n) = 10^{1000}(n-1)+3$ runs in time order n or $O(n)$, but the factor of 10^{1000}—hidden in the $O(n)$ notation—means that the algorithm may be wildly impractical for $n \geq 2$. We write $f(n) = OO(n)$ (the $OO(n)$ is pronounced *uh-oh of* n).

(5) The asymptotic ratio of $f(n) = 3^n n^2 \log n$ is, in view of (1)–(4) above, $3 \cdot 1 \cdot 1 = 3$.
(6) If $f(n) \sim g(n)$, then f, g have the same asymptotic ratio, i.e., either both ratios exist and are equal, or both ratios don't exist.
(7) If $f(n) = o(g(n))$, then $f(n) + g(n)$ and $g(n)$ have the same asymptotic ratios (indeed, $g \sim f + g$).
(8) For example, the asymptotic ratio of n^2 is the same as that of $n^2+5n+100$, since $5n + 100 = o(n^2)$. Hence the asymptotic ratio of $n^2 + 5n + 100$ is one.
(9) More generally, the asymptotic ratio of any nonzero polynomial is 1.

Here are examples of functions without asymptotic ratios:

(1) The function $f(n)$ which is 0 if n is even and 1 if n is odd does not have an asymptotic ratio, since $f(n + 1)/f(n)$ is undefined at infinitly many values of n.
(2) The function $f(n)$ which is 2 if n is even and 1 if n is odd does not have an asymptotic ratio, since $f(n + 1)/f(n)$ alternates between 2 and 1/2.

As in Subsection 2.2, finding the asymptotic ratio of a function is more of an art rather than a deterministic procedure. For example, the asymptotic ratio of $f(n) = \lfloor\sqrt{n}\rfloor$ is, by definition,

$$\lim_{n\to\infty} \frac{\lfloor\sqrt{n+1}\rfloor}{\lfloor\sqrt{n}\rfloor} = \lim_{n\to\infty} \mathrm{MessyExpression}(n).$$

However, in view of (1) and item 7 above, $f(n)$ and $\sqrt{n}$ have the same asymptotic ratio, and

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{1 + (1/n)} = 1.$$

4. Directed Graphs

Directed graphs are described informally in [Sip] Section 0.2, page 12. Let us give a formal definition.

Definition 4.1. A directed graph is a 4-tuple $G = (V, E, t, h)$ where V and E are sets—called the *vertex set* and *edge set*—and t, h are maps $E \to V$ called the *tail map* and *head map*.

There is a lot of terminology that is derived from this definition; for example, if $v \in V$ (i.e., v is an element of V), we say that *v is a vertex* of G. On page 12 of [Sip], edges are called *arrows*. If $e \in E$, $t(e) = v_1$, $h(e) = v_2$, we sometimes say that *e points from v_1*, *points to v_2*, *e is (runs) from v_1 to v_2*, etc. *Indegree* and *outdegree* of a vertex are defined on page 12 of [Sip].

Example 4.2. The Fibonacci graph: $V = \{v_1, v_2\}$, $E = \{e_1, e_2, e_3\}$; t, h are described in one of three ways:

(1) e_1 is from v_1 to v_2, e_2 is from v_2 to v_1, e_3 is from v_1 to itself;
(2) $t(e_1) = v_1$, $t(e_2) = v_2$, $t(e_3) = v_1$ and $h(e_1) = v_2$, $h(e_2) = v_1$, $h(e_1) = v_1$;
(3) the simple picture we draw in class.

An edge whose tail equals its head is called a *self-loop*; in the Fibonacci graph above, e_3 is a self-loop.

There is an immense literature on directed graphs; they are often used as models, such as for the world-wide web, tournaments, preferences, tasks with a chronological order, etc. The rest of this article and the exercises have more examples of digraphs.

5. Directed Graphs and [Sip]

All of the examples of DFA's, NFA's, GNFA's in Chapter 1 of [Sip] are depicted there as directed graphs with some additional information.

Here is an alternate definition of directed graphs that more closely resembles what you'll see in Chapter 1 of [Sip].

Definition 5.1 (Alternate definition of a directed graph)**.** A *directed graph* is a 3-tuple $G = (V, E, \delta)$, where V, E are sets, and $\delta\colon E \times \{\mathrm{H}, \mathrm{T}\} \to V$.

The product of sets, such as $E \times \{\mathrm{H}, \mathrm{T}\}$ above, is explained in Chapter 0 of [Sip]; it appears in all definitions of finite automata and Turing machines. This alternate definition is equivalent to the original one: in the alternate definition V, E are still the vertex and edge sets, and $\delta(e, \mathrm{H})$ is the head of e, and similarly for T and tail.

6. Walks Counts in Directed Graphs

Definition 6.1. Let $k \geq 0$ be an integer, and $G = (V, E, t, h)$ a directed graph. A *walk of length k in G* is an alternating sequence of vertices and edges

$$w = (v_0, e_1, v_1, \ldots, e_k, v_k),$$

such that e_i is an edge from v_{i-1} to v_i for all $i = 1, \ldots, k$. We say that w *begins in* v_0 and *ends in* v_k.

Example 6.2. In the Fibonacci graph (Example 4.2), let $f = f(n)$ be the number of walks of length n beginning and ending in v_1. Then the values of f, i.e., $f(1), f(2), \ldots$, form the usual Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Example 6.3. Consider the Fibonacci graph (Example 4.2) with e_2 omitted. Let $f(n)$ be the number of walks of length n from v_1 to v_2. Then $f(n) = 1$ for all n, since a walk from v_1 to v_2 of length n must traverse e_3 $n-1$ times and then traverse e_1 at the end. If we add to this graph a single self-loop at v_2, then $f(n) = n$ for all n, since now we may loop at v_1 using e_3 i times, traverse e_1 to v_2, and loop at v_2 $n - i + 1$ times for any $i = 0, 1, \ldots, n - 1$; if we add two self-loops at v_2 instead of only one, then $f(n)$ is

$$1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$$

since once we arrive at v_2 we have 2^{n-i+1} ways to loop $n - i + 1$ times instead of a single way; if instead of adding two self-loops at v_2 we add three self-loops, then $f(n)$ becomes

$$1 + 3 + 3^2 + \cdots + 3^{n-1} = (3^n - 1)/2.$$

Example 6.4. Consider the Fibonacci graph (Example 4.2) with e_3 omitted. Let $f(n)$ be the number of walks of length n from v_1 to v_2. Then $f(n) = 1$ if n is odd, and $f(n) = 0$ if n is even.

Definition 6.5. A *walk counting function*, or simply a *walk count*, is any $f(n)$ such that there is a finite graph G with subsets of vertices V_1 and V_2 such that $f(n)$ is the number of walks of length n beginning in (some vertex of) V_1 and ending in (some vertex of) V_2.

[A finite graph is one whose vertex and edge sets are finite.] For example, all the functions the previous examples are walk counts. If $f(n)$ is the number of walks of length n in the Fibonacci graph beginning in v_1 and ending in $\{v_1, v_2\}$, then $f(n)$ is a walk count given by $f(n) = F_n + F_{n-1}$, where F_n is the n-th Fibonacci number.

Example 6.6. If $f(n), g(n)$ are walk counts, then it is not hard to see that $f(n) + g(n)$ and $f(n)g(n)$ are walk counts; see the exercises.

7. Asymptotic Tests

If $f : \mathbb{N} \to \mathbb{Z}$ has an asymptotic ratio, there are many situations where we can tell that $f(n)$ is not a walk count.

Theorem 7.1 (Asymptotic Tests)**.** *Let $f(n)$ have finite asymptotic ratio ρ. Then $f(n)$ is not a walk count if any of the following hold:*

(1) $f(n) = o(\rho^n)$,
(2) $f(n) = \Theta(\rho^n n^r)$ for a real r that is not an integer,
(3) ρ is a rational number that is not an integer.

There are many related tests, most of which we will not use in CPSC 421: for example, if f is a walk count then ρ must be an algebraic integer[2]. We discuss this and other tests in Appendix B.

Example 7.2. Let $f(n)$ be the number of strings in $\{0, 1\}$ of length n of the form $0^i 1^j$ where i is a perfect square. Then $f(n) = \lfloor \sqrt{n} \rfloor$, which has asymptotic ratio 1. But

$$f(n) = \lfloor \sqrt{n} \rfloor = \sqrt{n} + O(1) = \Theta(n^{1/2}) \ ,$$

so $f(n)$ cannot be a walk count.

Example 7.3. Let $f(n)$ be the number of strings in $\{0, 1\}$ of length n that represent prime numbers written in base 2. It is a classical fact that $f(n) \sim 2^n / \log_e(2^n)$. Hence $f(n) \sim \gamma 2^n/n$ where for the constant $\gamma = \log_2 e = 1.4426\ldots$ Since the asypmtotic ratios of the functions γ, 2^n, and $1/n$, are respectively 1, 2, and 1, the asymptotic ratio of $f(n)$ is 2. But

$$f(n) \sim \gamma 2^n/n = o(2^n),$$

$f(n)$ is not a walk count.

8. Conclusion

We have reviewed a lot of the material prerequisite to CPSC 421; you will need some set theory, which we will review in another handout before we begin Chapter 1 of [Sip] covering regular languages. When we wish to prove that certain languages are not regular, we will have some tools such as (1) asymptotic tests, (2) the Pumping Lemma, and (3) the Myhill-Nerode theorem. All of this material is a good warmup to the main goal of this course, which is to study computability and the complexity of algorithms, beginning with Chapter 3 of [Sip].

[2] i.e., must satisfy a polynomial equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ where the $a_i \in \mathbb{Z}$.

Appendix A. Exercises form This Article and Chapter 0 of [Sip]

These exercises are based on the material in this article and parts of Chapter 0 in [Sip] not covered here, including set notation and proofs of various kinds.

I AM IN THE PROCESS OF ADDING MORE EXERCISES TO THIS SECTION AND/OR CHANING THEIR ORDER; THE EXERCISE NUMBERS WILL CHANGE.

Exercise A.1. Let F_n denote the n-th Fibonacci number, i.e., $F_1 = 1$, $F_2 = 1$, and for $n \geq 3$ we have $F_n = F_{n-1} + F_{n-2}$. Give a proof by induction in the following exercises.

(1) Prove that for all $n \in \mathbb{Z}$, the number of walks of length n from v_1 to itself in the Fibonacci graph (Example 4.2) equals F_n.
(2) Compute $F_n F_{n+2} - F_{n+1}^2$ for $n = 1, \ldots, 5$; guess a simple formula for this expression, and prove that your guess is correct.
(3) Compute $F_n F_{n+3} - F_{n+1} F_{n+2}$ for $n = 1, \ldots, 5$; guess a simple formula for this expression, and prove that your guess is correct.
(4) Compute $F_n F_{n+8} - F_{n+1} F_{n+7}$ for $n = 1, \ldots, 5$; you can write your answer in terms of the Fibonacci numbers rather than writing out the actual integer; guess a simple formula for this expression, and prove that your guess is correct.
(5) Compute $F_n F_{n+100} - F_{n+1} F_{n+99}$ for $n = 1, \ldots, 5$; you can write your answer in terms of the Fibonacci numbers rather than writing out the actual integer; guess a simple formula for this expression, and prove that your guess is correct.
(6) Prove that if $\xi_+ = (1+\sqrt{5})/2$ and $\xi_- = (1-\sqrt{5})/2$, then $F_n = (\xi_+^n - \xi_-^n)/\sqrt{5}$; use this formula to determine the asymptotic ratio of the function F_n. [Hint: first find the roots of $x^2 = x + 1$.]
(7) Prove that the GCD (greatest common divisor) of F_n and F_{n+1} is 1.

Exercise A.2. Recall that the number of subsets of size k from a fixed set of n elements is

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 1} = \frac{n!}{k!(n-k)!},$$

where ! denotes the "factorial," e.g., $k! = k(k-1)\ldots 1$.

(1) Prove that for any $n \in \mathbb{N}$ we have that $\sum_{m=1}^{n} m = 1 + 2 + \cdots + n$ equals $\binom{n+1}{2}$; use induction on n.
(2) Prove that for any $n, k \in \mathbb{N}$ we have that

$$\binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1};$$

prove this by fixing an integer k and using induction on n.

Exercise A.3. Let $k \in \mathbb{N}$ (i.e., k is a positive integer) and $A_1, \ldots, A_k$ be finite sets.

(1) Prove that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

[Hint: each element of $x \in A_1 \cup A_2$ is counted once on the LHS (left-hand side). What about the RHS (right-hand side)? You may need to consider a few cases.]

(2) Prove that

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

(3) Prove that for any $m \in \mathbb{N}$ that

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} + \cdots + (-1)^m \binom{m}{m} = 0$$

[Hint: you may use induction on m, or you may use the binomial theorem $(x+y)^m = \sum_{j=0}^{m} x^{m-j} y^j \binom{m}{j}$ and cleverly choose x, y.]

(4) Show that

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{0 \le i \le k} |A_i| - \sum_{0 \le i_1 < i_2 \le k} |A_{i_1} \cap A_{i_2}|$$
$$+ \sum_{0 \le i_1 < i_2 < i_3 \le k} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \cdots + (-1)^k |A_1 \cap A_2 \cap \ldots \cap A_k|.$$

[Hint: you can use part (3) of this exercise; or you can ignore part (3) and use induction.]

Exercise A.4. Exercises 7.1 and 7.2 of [Sip] are good to make sure you have the basic idea of big-O and little-o notation. However, rather than just answering true or false, you might be asked to justify your answer.

(1) Show that $f(n) = O(g(n)$ in the following examples, by finding C, n_0 such that $f(n) \le Cg(n)$ provided that $n \ge n_0$ for:
  (a) $f(n) = 3^n$, $g(n) = 4^n$;
  (b) $f(n) = 100 \cdot 3^n$, $g(n) = 4^n$;
  (c) $f(n) = 3n$, $g(n) = n(n-1)$;
  (d) $f(n) = n^2 + 3n + 1$, $g(n) = n(n-1)$.
(2) Compute the limit $\lim_{n\to\infty} f(n)/g(n)$ to show that $f(n) = o(g(n))$ in the following examples:
  (a) $f(n) = 3^n$, $g(n) = 4^n$;
  (b) $f(n) = 100 \cdot 3^n$, $g(n) = 4^n$;
  (c) $f(n) = 3n$, $g(n) = n(n-1)$;
  (d) $f(n) = n^2 + 3n + 1$, $g(n) = n(n-1)(n-2)$.

Exercise A.5. Prove that if f, g are functions $\mathbb{Z} \to \mathbb{R}^+$, then:

(1) $f(n) = O(g(n))$ iff for some C we have $\log f(n) - \log g(n) \le C$ for all n;
(2) $f(n) = o(g(n))$ iff $\log g(n) - \log f(n)$ tends to infinity (as $n \to \infty$);
(3) $f(n) \sim g(n)$ iff $\log g(n) - \log f(n)$ tends to zero (as $n \to \infty$).

Exercise A.6. Let f_1, f_2, g_1, g_2 be functions $\mathbb{N} \to \mathbb{R}^+$ such that $f_2 = o(f_1)$ (which we write instead of $f_2(n) = o(f_1(n))$ for brevity) and $g_2 = o(g_1)$. Let $f = f_1 + f_2$ and $g = g_1 + g_2$. Prove that:

(1) $f = o(g)$ iff $f_1 = o(g_1)$.
(2) $f = O(g)$ iff $f_1 = O(g_1)$.
(3) $f = \Theta(g)$ iff $f_1 = \Theta(g_1)$.
(4) $f \sim g$ iff $f_1 \sim g_1$.

Exercise A.7. By taking a limit, find the asymptotic ratios, when they exist, of the following sequences $\{a_n\}$ where:

(1) $a_n = 5^n$;

(2) $a_n = \rho^n$ for some real $\rho > 0$;
(3) $a_n = n^2$;
(4) $a_n = 2^{n^2}$.

Exercise A.8. Find the asymptotic ratios, when they exist, of the following sequences $\{a_n\}$ where:

(1) $a_n = 5^n + 1$;
(2) $a_n = 3n^2 - 4$ (make a computation; don't just quote the fact that any nonzero polynomial has asymptotic ratio 1);
(3) $a_n = 7n^6 + 5n^2 + n$;
(4) $a_n = 2^{n^2} + 2^n + n^2 + 3$.

[Hint: use the fact that $f(n)$ and $f(n) + g(n)$ have the same asymptotic ratio if $f(n) = o(g(n))$]

Exercise A.9. Let $\{a_n\}$ be a sequence of asymptotic ratio ρ. Find the asymptotic ratios of the following sequences $\{b_n\}$ by taking a limit.

(1) $b_n = 3a_n$.
(2) $b_n = a_{n+1}$.
(3) $b_n = (a_n)^3$.

Exercise A.10. Describe $f(n)$ defined to be the number of walks of length n beginning in V_1 and ending in V_2 in the graph $G = (V, E, h, t)$ where G, V_1, V_2 are given below; justify your answer (give an explanation, but not a formal proof):

(1) $V_1 = V_2 = V$, G has one vertex and m edges (i.e., $|V| = 1$ and $|E| = m$).
(2) $V_1 = V_2 = V$, G has five vertices and each vertex has outdegree four.
(3) $V_1 = V_2 = \{v_1\}$, G has $V = \{v_1, v_2\}$, E has 2 self-loops at v_1, 2 edges from v_1 to v_2, 2 edges from v_2 to v_1; your answer can involve the Fibonacci numbers.
(4) $V_1 = V_2 = \{v_1, v_3\}$, G has $V = \{v_1, v_2, v_3\}$, E one self-loop at v_1, one self-loop at v_3, one edge from v_1 to v_2, one edge from v_3 to v_2, one edge from v_2 to v_1, one edge from v_2 to v_3; your answer can involve the Fibonacci numbers.
(5) $V_1 = \{v_1\}$, $V_2 = \{v_3\}$, G has $V = \{v_1, v_2, v_3\}$, E has one self-loop at v_1, one self-loop at v_2, one edge from v_1 to v_2, and one edge from v_2 to v_3.

Exercise A.11. Let $G = (V, E, h, t)$ be the following "star" graph with one "center" and seven "outer vertices" (this is an informal description): $V = \{c, v_1, \ldots, v_7\}$, and E has one edge from c to each v_i with $1 \leq i \leq 7$, and one edge from each v_i to c with $1 \leq i \leq 7$. Describe $f(n)$ defined to be the number of walks of length n beginning in V_1 and ending in V_2 in the graph G with V_1, V_2 given below; justify your answer (give an explanation, but not a formal proof):

(1) $V_1 = V_2 = \{c\}$.
(2) $V_1 = \{c\}$, $V_2 = \{v_1\}$.
(3) $V_1 = \{v_1\}$, $V_2 = \{c\}$.
(4) $V_1 = V_2 = \{v_1\}$.
(5) $V_1 = \{c, v_1, v_2\}$, $V_2 = \{c, v_2, v_3\}$.

Exercise A.12. Say that a function $f \colon \mathbb{N} \to \mathbb{Z}$ is *infinite walk counting* if it satisfies the criteria of Definition 6.5 except that the graph is allowed to be infinite. Prove that a function $f \colon \mathbb{N} \to \mathbb{Z}$ is infinite walk counting iff $f(n) \geq 0$ for all $n \in \mathbb{N}$.

Exercise A.13. Show that if f, g are walk counts, then so is $f+g$, i.e., the function $f(n)+g(n)$. Do so by taking two graphs, G_1 and G_2, involved in producing the walk counting functions f and g, and creating a new graph from G_1 and G_2.

Exercise A.14. Show that if f, g are walk counts, then so is fg, i.e., the function $f(n)g(n)$. Do so by taking two graphs, G_1 and G_2, involved in producing the walk counting functions f and g, and creating a new graph from G_1 and G_2.

Exercise A.15. Let a_n be the number of strings of length n in $\{0,1\}$ such that each 0 must immediately follow and immediately precede a 1; examples of such strings would be 11010111 and 1111, but not 110 or 1001.

(1) Write out the values of a_n for $n = 1, \ldots, 6$.
(2) Prove that $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$.
(3) What is the capacity of $\{a_n\}$.
(4) Show that $\{a_n\}$ is a walk count: draw or describe the graph and the set of beginning and of ending vertices.

Exercise A.16. Use Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ to show that $\binom{2n}{n} = (2n)!/(n!)^2$ is $\sim \gamma 2^{2n}/\sqrt{n}$ for a constant $\gamma > 0$. What is γ?

Exercise A.17. Use an asymptotic test to show that the following functions $f(n)$ are not walk functions of any (finite) directed graph:

(1) The number of strings of length n over $\{0,1\}$ representing a binary integer of size at most $n^{2/3}$.
(2) The number of strings of length n over $\{0,1\}$ representing a binary integer of size at most $n^{4/3}$.
(3) The number of strings of length $2n$ over $\{0,1\}$ with exactly n 0's and n 1's; you may use the result of Exercise A.16 (the value of γ is unimportant here).
(4) The number of strings of length $2n$ of "matching parenthesis" (this number is known to be $\binom{2n}{n}/(n+1)$, the n-th *Catalan number*).

Appendix B. Asymptotic Tests and Linear Algebra

The basic idea is that any walk function is of the form

$$f(n) = u^{\mathrm{T}} A^n w,$$

where A is the adjacency matrix of the graph, and u, w are vectors whose components are all 0's or 1's. From Jordan canonical form it follows that

$$f(n) = \sum_{i=1}^{s} \lambda_i^n p_i(n)$$

where $\lambda_1, \ldots, \lambda_s$ are the eigenvalues of A, and the $p_i = p_i(n)$ are polynomials. The Perron-Frobenius theorem[3] implies that A has at least one real eigenvalue, and that if $\lambda_1 \geq 0$ is the largest one, then every other eigenvalue is either (1) of absolute value less than λ_1, or (2) of the form $\zeta\lambda_1$, where ζ is a d-th root of unity for some $d \in \mathbb{N}$. This easily implies our "asymptotic tests."

[3] Sometimes the Perron-Frobenius theorem is stated requiring that the associated digraph is strongly connected; but one can order the strong components, whereupon the adjacency matrix becomes a block lower triangular matrix whose blocks are strongly connected.

Since A has integer coefficients, the λ_i must be algebraic integers. This gives a further test.

It follows from the Cayley-Hamilton theorem that $f(n)$ satisfies a linear recurrence equation

$$f(n) + c_1 f(n-1) + \ldots + c_t f(n-t) = 0$$

where the c_i are the coefficients of the characteristic polynomial (or, sometimes better yet, the minimal polynomial) of A.

You can probably use other aspects of linear algebra to give futher conditions that walk counting functions must satisfy.

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA, and Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, CANADA.

E-mail address: `jf@cs.ubc.ca or jf@math.ubc.ca`

URL: `http://www.math.ubc.ca/~jf`