CPSC 421 - Homework 5

Tristan Rice, q7w9a, 25886145

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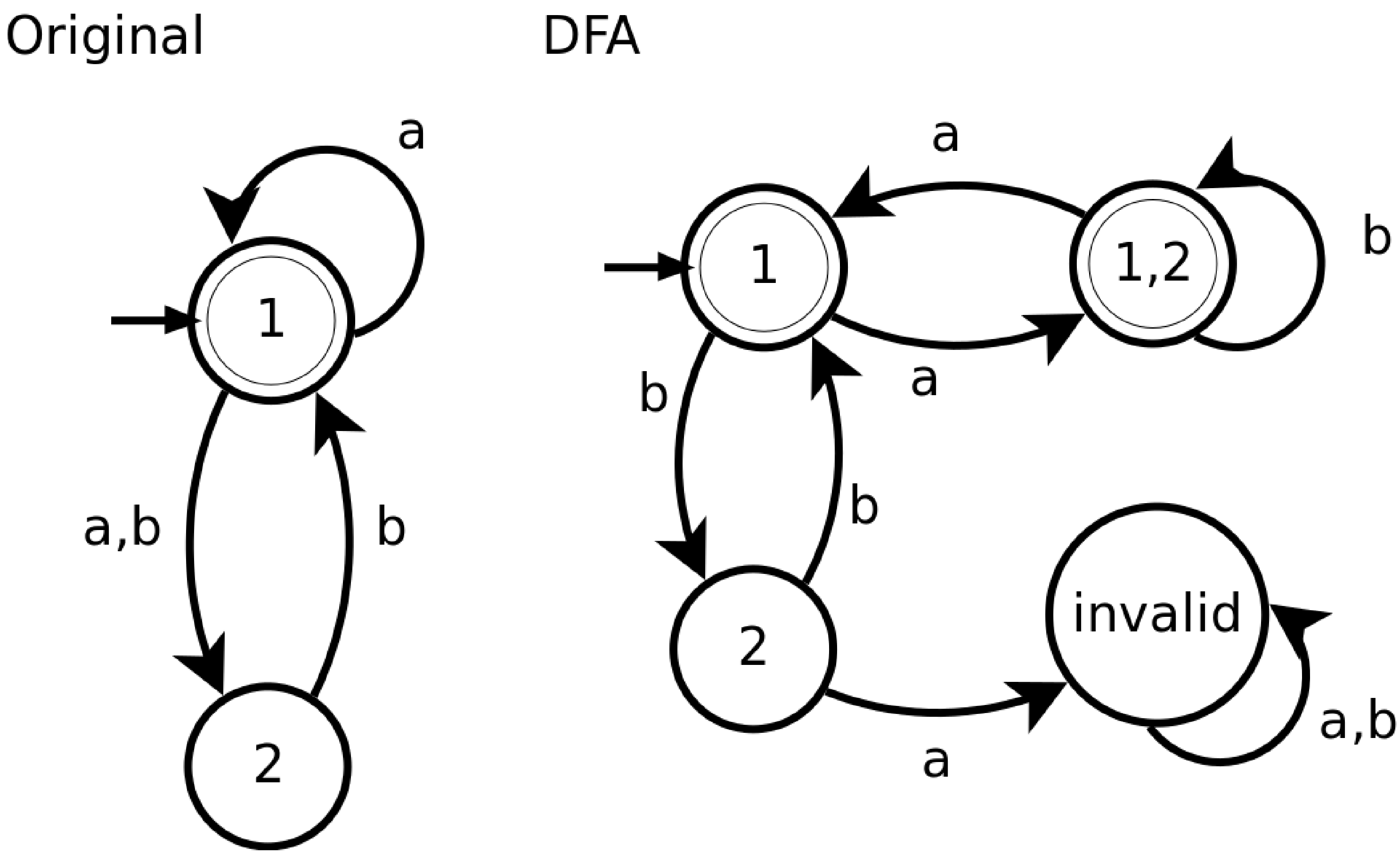
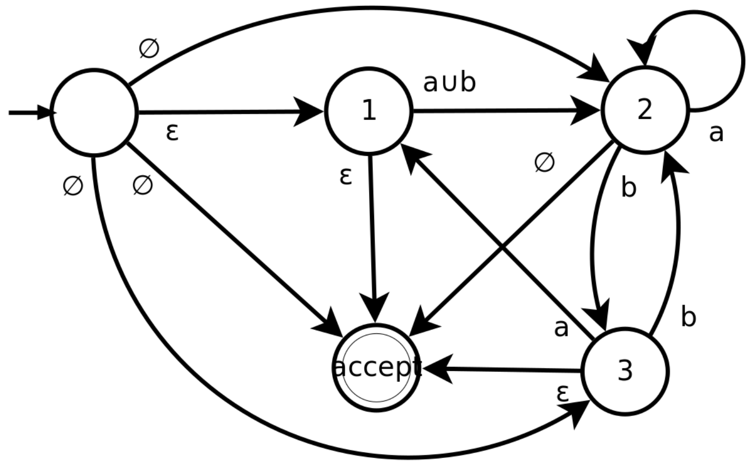
Possible states:

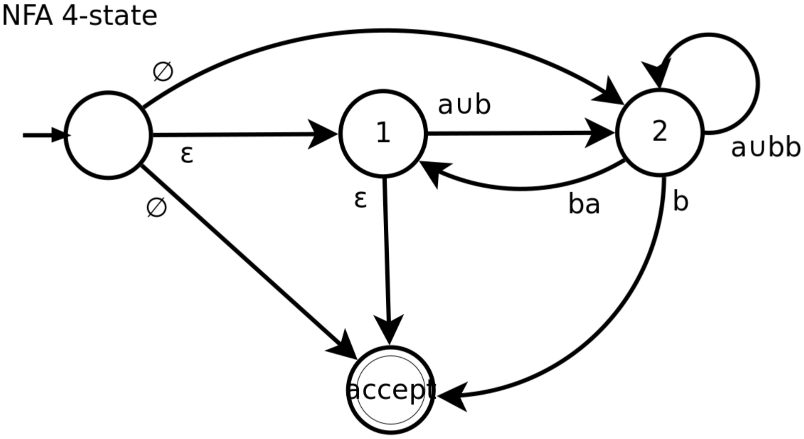
- empty {1}
- a {1, 2}
- b {2}
- aa {1}
- ab {1, 2}
- ba {}
- bb {1}



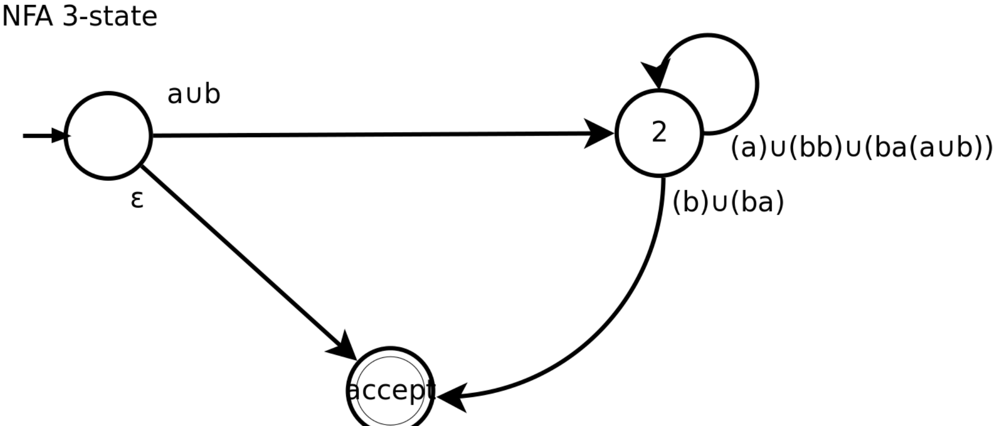
GNFA 5-state



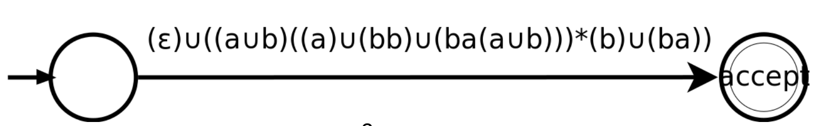
GNFA 4-state



GNFA 3-state



GNFA 2-state



3

3.a

Claim: $\binom{2n}{n} \sim \frac{2^{2n}\gamma}{\sqrt{n}}$.

Proof:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} \\
&= \frac{(2n)!}{n!n!} \\
&\sim \frac{\sqrt{2\pi 2n}(2n/e)^{2n}}{\sqrt{2\pi n}(n/e)^n\sqrt{2\pi n}(n/e)^n} \\
&= \frac{(2n/e)^{2n}}{(n/e)^{2n}\sqrt{\pi n}} \\
&= \frac{2^{2n}}{\sqrt{\pi n}}
\end{aligned}$$

This is clearly equal to $\frac{2^{2n}\gamma}{\sqrt{n}}$ where $\gamma = \frac{1}{\sqrt{pi}}$.

Thus, $\binom{2n}{n} \sim \frac{2^{2n}\gamma}{\sqrt{n}}$.

3.b

Claim: $\binom{2n+1}{n} \sim \frac{2^{2n+1}\gamma}{\sqrt{n}}$.

Proof:

We know that $\binom{2n}{n} \sim \frac{2^{2n}\gamma}{\sqrt{n}}$. Thus, if we can show that the asymptotic ratio is the same for $\frac{\binom{2n+1}{n}}{\binom{2n}{n}}$ and $\frac{2^{2n+1}\gamma/\sqrt{n}}{2^{2n}\gamma/\sqrt{n}}$, we know will that $\binom{2n+1}{n} \sim \frac{2^{2n+1}\gamma}{\sqrt{n}}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{\binom{2n+1}{n}}{\binom{2n}{n}} &= \lim_{n\to\infty} \frac{(2n+1)!n!n!}{(2n)!n!(2n+1-n)!} \\
&= \lim_{n\to\infty} \frac{(2n+1)!n!}{(2n)!(n+1)!} \\
&= \lim_{n\to\infty} \frac{2n+1}{n+1} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^{2n+1}\gamma/\sqrt{n}}{2^{2n}\gamma/\sqrt{n}} &= \lim_{n\to\infty} \frac{2^{2n+1}}{2^{2n}} \\
&= \lim_{n\to\infty} 2 \\
&= 2
\end{aligned}$$

Thus, since the asymptotic ratios are the same, we know that $\binom{2n+1}{n} \sim \frac{2^{2n+1}\gamma}{\sqrt{n}}$ must be proportional.

3.c

$$L = \{s \in \{0,1\}^* | s \text{ has } n \text{ zeros and } m \text{ ones, and } m = n \text{ or } m = n+1\}$$

If a DFA were to exist, it would have a walk counting function $f(l)$ equal to the number of words of length l.

When length l is even, $m = n$ and there are $\binom{2n}{n}$ words. When length l is odd, $m = n+1$ and there are $\binom{2n+1}{n}$ words.

For the even case, we know $\binom{2n}{n}$ is proportional to $\sqrt{2^{2n}}\gamma n^{-\frac{1}{2}}$. This means that $f(l) = \Theta(2^l n^r), r = -\frac{1}{2}$. Since r is not an integer, $f(l)$ must not be a walk count due to Theorem 7.1 from handout 1.

For the odd case, we know $\binom{2n+1}{n}$ is proportional to $\sqrt{2^{2n+1}}\gamma n^{-\frac{1}{2}}$. This means that $f(l) = \Theta(2^l n^r), r = -\frac{1}{2}$. Since r is not an integer, $f(l)$ must not be a walk count due to Theorem 7.1 from handout 1.

Thus, since the function $f(l)$ is not a valid walk count and no DFA can be constructed for the language. Since all regular languages have a corresponding DFA, L must not be a regular language.

4

4.a

Claim: $L = \{1^m | 7 \text{ divides } m\}$ is not recognized by a DFA with fewer than 7 states.

Proof:

Assume that $p < 7$. The walk counting function for a graph with p vertices must satisfy the recurrence equation

$$f(n) = c_1 f(n-1) + \ldots + c_p f(n-p)$$

Thus, the walk counting function can only take into account the last p outputs of the walk counting functions.

The walk counting function $f(n) = 0$ if 7 doesn't divide m since there is no valid walk that satisfies having a length that is a multiple of 7. Thus, $f(n)$ is only non-zero when n is a multiple of 7. Since $p < 7$, $\sum_{i=1}^{p} f(n-i) = 0$ when n is a multiple of 7. Since the recurrence relation must be based on the previous p $f(n-i)$ values, there are no possible constants that will result in a non-zero $f(n)$ value since each of the previous values are 0. This is a contradiction and thus there can't be a graph with fewer than 7 vertices for L. Since there's no graph, there must not be a DFA with fewer than 7 states that recognizes L.

4.b

Claim: $L = \{1^m | m \text{ is a perfect square}\}$ is not regular.

Proof:

Say L is regular and there is some constant p such that it is possible to construct a walk counting function for a graph of L that satisfies a recurrence equation

$$f(n) = c_1 f(n-1) + \ldots + c_p f(n-p)$$

for all $n \geq p+1$ where c_i are constants (and integers).

The walk counting function $f(n)$ is zero when m is not a perfect square since there isn't a valid word of that length. Thus, the recurrence relation must be include the previous valid non-zero walk, otherwise it can only return zero which we know is incorrect. Thus, the non-zero values must separated by at most $p-1$ zero values. Since p is a constant, that must mean the nth perfect square is $= O(n)$.

This is a contradiction since the nth perfect square is $= n^2 = O(n^2)$.

Thus, there must not be a walk-counting function on a graph with p vertices and thus no DFA satisfying L can be created. Since no valid DFA can be constructed, that means that L must not be a regular language.

4 (alternate proofs using Myhill-Nerode + Pumping Lemma) (not applicable to this assignment)

4.a

Claim: $L = \{1^m | 7 \text{ divides } m\}$ is not recognized by a DFA with fewer than 7 states.

Proof: We can apply the Myhill-Nerode Theorem.

Say that for some $w \in \sum^*, xw \in L, yw \notin L$, then x, y are in different states.

$$L \in \sum^*, x \in \sum^*$$

$$AcceptingFuture(L, x) = \{w \in \sum^*\}$$

$$L = \{1^m | 7 \text{ divides } m\}$$

$$AcceptingFuture(L,\epsilon) = (1^7)^*$$
$$AcceptingFuture(L,1) = 1^6(1^7)^*$$
$$AcceptingFuture(L,1^2) = 1^5(1^7)^*$$
$$AcceptingFuture(L,1^3) = 1^4(1^7)^*$$
$$AcceptingFuture(L,1^4) = 1^3(1^7)^*$$
$$AcceptingFuture(L,1^5) = 1^2(1^7)^*$$
$$AcceptingFuture(L,1^6) = 1(1^7)^*$$

Thus, since there are at least 7 distinct accepting futures, by the Myhill-Nerode Theorem, the DFA representing this language must have at least 7 states.

4.b

Pumping Lemma: Say that L is regular and accepted by a DFA of p states or fewer. Then if $s \in L$ and $|S| \geq p$ we can write $S = xyz$ such that

1. $xz, xyz, xy^2z, xy^3z, \ldots \in L$
2. $y \neq \epsilon$
3. $|xy| \leq p$

Claim: $L = \{1^m | m$ is a perfect square$\}$ is not regular.

Proof: Say L is regular and accepted by a DFA of p states.

Now consider $S = 1^p \in L$. Using the pumping lemma, let $s = xyz$ such that 1-3 above hold. Since $S = 1^{2p} = xyz$ and $|xy| \leq p$, we have $x = 1^a, y = 1^b, z = 1^{2p-a-b}$

Thus, $\forall i \in \mathbb{N}$, such that $1^{a+bi}1^{2p-a-b} \in L$.

$1^{2p+b(i-1)}$ cannot be in L since there is no constant b such that $\forall i \in \mathbb{N}, 2p+b(i-1)$ is a perfect square. That would imply perfect squares are separated by a constant factor. Proof by contradiction. L must not be regular.