CPSC 421 - Homework 1

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A.1(3,4,6)

A.1.3

Fibonacci:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f(n)	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

$F_nF_{n+3} - F_{n+1}F_{n+2}$ 1. $1*3-1*2=1$ 2. $1*5-2*3=-1$ 3. $2*8-3*5=1$ 4. $3*13-5*8=-1$ 5. $5*21-8*13=1$

Theorem: The output of $f(n) = F_nF_{n+3} - F_{n+1}F_{n+2}$ is 1 when n is even and -1 when n is odd.

Proof by induction.

Case: n is odd.

Base cases: - $n=1$: $1*3-1*2=1$ - $n=2$: $1*5-2*3=-1$

Induction step ($n \geq 3$):

$f(n) = f(n+2)$

$F_nF_{n+3} - F_{n+1}F_{n+2} = F_{n+2}F_{n+5} - F_{n+3}F_{n+4}$

$= F_{n+2}(F_{n+1} + F_{n+5})$

$= (F_{n+1} + F_n)(F_{n+3} + F_{n+2} + F_{n+3}) - (F_{n+2} + F_{n+1})(F_{n+3} + F_{n+2})$

$$= 2*F_{n+1}F_{n+3} + F_{n+1}F_{n+2} + 2*F_nF_{n+3} + F_nF_{n+2} - F_{n+2}F_{n+3} - F_{n+2}F_{n+2} - F_{n+1}F_{n+3} - F_{n+1}F_{n+2}$$

$$= F_{n+1}F_{n+3} + 2*F_nF_{n+3} + F_nF_{n+2} - F_{n+2}F_{n+3} - F_{n+2}F_{n+2}$$

$$= F_{n+1}F_{n+3} + 2*F_nF_{n+3} + F_nF_{n+2} - F_{n+1}F_{n+3} - F_nF_{n+3} - F_{n+1}F_{n+2} - F_nF_{n+2}$$
$$= F_nF_{n+3} - F_{n+1}F_{n+2}$$

Thus, since $f(n) = f(n+2)$ we prove via induction that the output will be 1 for all odd values of n and -1 for all even values of n.

A.1.4

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In [15]: def F(n):
    ...:         return int(0.5+((1+sqrt(5))**n-(1-sqrt(5))**n)/(2**n*sqrt(5)))
In [16]: [(n, F(n) * F(n+8) - F(n+1)*F(n+7)) for n in range(1,6)]
Out[16]: [(1, 13), (2, -13), (3, 13), (4, -13), (5, 13)]
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For all even values of n, $f(n) = -13$. For all odd values of n, $f(n) = 13$.

Proof by induction.

Base cases: * $n=1$: $f(1) = 13$ * $n=2$: $f(2) = -13$

Induction step:

We must show that for all $n > 2$, $f(n) = f(n+2)$.

$$F_nF_{n+8} - F_{n+1}F_{n+7} = F_{n+2}F_{n+10} - F_{n+3}F_{n+9}$$

$$= F_{n+1}F_{n+10} + F_nF_{n+10} - F_{n+2}F_{n+9} - F_{n+1}F_{n+9}$$

$$= F_{n+1}F_{n+9} + F_{n+1}F_{n+8} + F_nF_{n+9} + F_nF_{n+8} - F_{n+0}F_{n+9} - F_{n+1}F_{n+9} - F_{n+1}F_{n+9}$$

$$= F_{n+1}F_{n+8} + F_nF_{n+8} - F_{n+1}F_{n+9}$$

$$= F_{n+1}F_{n+8} + F_nF_{n+8} - F_{n+1}F_{n+8} - F_{n+1}F_{n+7}$$

$$= F_nF_{n+8} - F_{n+1}F_{n+7}$$

Thus, by the property of induction it holds for all $n \geq 1$, $f(n) = 13$ for odd numbers and $f(n) = -13$ for even numbers.

A.1.6

$$x^2 = x + 1$$

$$x = \{\frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 + \sqrt{5})\}$$

We will show that $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{)}^n}{\sqrt{5}}$

Proof via induction.

Base case (n=1)

$$F_1 = \frac{(\frac{1+\sqrt{5}}{2})^1 - (\frac{1-\sqrt{5}}{2})^1}{\sqrt{5}}$$
$$1 = 1$$

Induction step.

$$\frac{(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^{n} - (\frac{1-\sqrt{5}}{2})^{n}}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}}{\sqrt{5}}$$

$$(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1} = (\frac{1+\sqrt{5}}{2})^{n} - (\frac{1-\sqrt{5}}{2})^{n} + (\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}$$

$$(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1} = ((\frac{1+\sqrt{5}}{2}) + 1)(\frac{1+\sqrt{5}}{2})^{n-1} - ((\frac{1-\sqrt{5}}{2}) + 1)(\frac{1-\sqrt{5}}{2})^{n-1}$$

Using $x^2 = x + 1$:

$$(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1} = (\frac{1+\sqrt{5}}{2})^2(\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^2(\frac{1-\sqrt{5}}{2})^{n-1}$$

$$(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1} = (\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}$$

Identity.

Thus, $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{)}^n}{\sqrt{5}}$.

Asymptotic ratio of F_n:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

$$= \lim_{n\to\infty} \frac{(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{\sqrt{5}} * \frac{\sqrt{5}}{(\frac{1+\sqrt{5}}{2})^{n} - (\frac{1-\sqrt{5}}{2})^{n}}$$

$$= \lim_{n\to\infty} \frac{(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{(\frac{1+\sqrt{5}}{2})^{n} - (\frac{1-\sqrt{5}}{2})^{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2} * \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2}(\frac{(1+\sqrt{5})^{n+1}}{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}} - \frac{(1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}})$$

$$= \lim_{n\to\infty} \frac{1}{2}(\frac{1+\sqrt{5}}{1 - \frac{(1-\sqrt{5})^n}{(1+\sqrt{5})^n}} - \frac{1-\sqrt{5}}{\frac{(1+\sqrt{5})^n}{(1-\sqrt{5})^n} - 1})$$

$$\lim_{n\to\infty} (\frac{1-\sqrt{5}}{1+\sqrt{5}})^n = 0$$

$$\lim_{n\to\infty} (\frac{1+\sqrt{5}}{1-\sqrt{5}})^n = \infty$$

$$= \frac{1}{2}(\frac{1+\sqrt{5}}{1-0} - \frac{1-\sqrt{5}}{\infty - 1})$$

$$= \frac{1}{2}(1+\sqrt{5})$$

A.2(1,2)

A.2.1

Prove $\sum_{m=1}^{n} m = \binom{n+1}{2}$ for any $n \in \mathbb{N}$.

Proof via induction.

Base case $n = 1$.

$\sum_{m=1}^{1} m = \binom{2}{2}$

$1 = 1$

Identity.

Induction step.

Must prove that:

$$\binom{n+1+1}{k+1} = \binom{n+1}{k+1} + \binom{n+1}{k}$$

$$\frac{(n+2)!}{(k+1)!(n+2-k-1)!} = \frac{(n+1)!}{(k+1)!(n+1-k-1)!} + \frac{(n+1)!}{k!(n+1-k)!}$$

$$\frac{(n+2)!}{(k+1)!(n-k+1)!} = \frac{(n+1)!}{(k+1)!(n-k)!} + \frac{(n+1)!}{k!(n-k+1)!}$$

$$\frac{(n+2)(n+1)!}{(k+1)k!(n-k+1)!} = \frac{(n+1)!}{(k+1)k!(n-k)!} + \frac{(n+1)!}{k!(n-k+1)!}$$

$$\frac{(n+2)}{(k+1)(n-k+1)(n-k)!} = \frac{1}{(k+1)(n-k)!} + \frac{1}{(n-k+1)(n-k)!}$$

$$\frac{(n+2)}{(k+1)(n-k+1)} = \frac{1}{(k+1)} + \frac{1}{(n-k+1)}$$

$$(n+2) = (n-k+1) + k + 1$$

$$n+2 = n+2$$

Identity.

Thus, $\sum_{m=1}^{n} m = \binom{n+1}{2}$ for any $n \in \mathbb{N}$. QED.

A.2.2

Prove $\sum_{m=1}^{n} \binom{m}{k} = \binom{n+1}{k+1}$.

Proof by induction.

Let $k \in \mathbb{N}$ be fixed and arbitrary.

Base case $n = 1$:

$$\binom{1}{k} = \binom{1+1}{k+1}$$

If $k = 1$, $\binom{1}{1} = \binom{2}{2} = 1$. If $k > 1$, $\binom{1}{k} = \binom{2}{1+k} = 0$.

Inductive step:

We must show that $\binom{n+1}{k} = \binom{n+2}{k+1} - \binom{n+1}{k+1}$.

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+2)!}{(k+1)!(n+2-k-1)!} - \frac{(n+1)!}{(k+1)!(n+1-k+1)!}$$

$$\frac{(n+1)!}{k!(n-k+1)!} = \frac{(n+2)!}{(k+1)!(n-k+1)!} - \frac{(n+1)!}{(k+1)!(n-k)!}$$

$$\frac{(n+1)!(k+1)!}{k!} = \frac{(n+2)!(n-k+1)!}{(n-k+1)!} - \frac{(n+1)!(n-k+1)!}{(n-k)!}$$

$$(n+1)!(k+1) = (n+2)! - (n+1)!(n-k+1)$$

$$(k+1) = \frac{(n+2)!}{(n+1)!} - (n-k+1)$$

$$k+1 = n+2-n+k-1$$

$$k+1 = k+1$$

Identity.

Thus, $\sum_{m=1}^{n} \binom{m}{k} = \binom{n+1}{k+1}$ holds for all $n \geq 1$.

A.3(1)

We aim to show $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

Proof by induction.

Base case:

$$A_1 = A_2 = \varnothing$$

$$|\varnothing \cup \varnothing| = |\varnothing| + |\varnothing| - |\varnothing \cap \varnothing|$$

Inductive step:

Consider adding an element a to A_1.

Assume $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ for A_1, A_2.

Since a only occurs in A_1, only the $|A_1|$ term changes, and the union is maintained. This holds for adding a only to A_2 since set addition, union and intersection are symmetric operations.

Consider adding an element a to A_1 and A_2.

Since a occurs in A_1 and A_2, the right side becomes $|A_1| + a + |A_2| + a - |A_1 \cap A_2| - a$. Since a is in both, it is included in the intersection and cancels out. Thus, it is only added once.

Thus, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ holds for all A_1, A_2.

A.4(1b,1d,2d)

Show that $f(n) = O(g(n))$...

A.4.1b

$f(n) = 100 * 3^n, g(n) = 4^n$.

Pick $C = 100, n_0 = 0$.

Proof by induction.

Base case: $n = 0$

$$100 * 3^0 \leq 100 * 4^0$$
$$100 \leq 100$$

Inductive step:

Must show that

$$100 * 3^{n+1} - 100 * 3^n \leq 100 * 4^{n+1} - 100 * 4^n$$
$$3^{n+1} - 3^n \leq 4^{n+1} - 4^n$$

$$3^n(3-1) \leq 4^n(4-1)$$

$$3^n(2) \leq 4^n(3)$$

$$n * \log(3) + \log(2) \leq n * \log(4) + \log(3)$$

The left side is clearly less than the right side. Thus, via induction we show that $f(n) = O(g(n))$ since $f(n) \leq Cg(n) \forall n \geq n_0$.

A.4.1d

Show that $f(n) = O(g(n)), f(n) = n^2 + 3n + 1, g(n) = n(n-1)$.

$g(n) = n^2 - n$

$C = 4, n_0 = 4$

Proof by induction.

Base case $n = 4$:

$$4^2 + 3*4 + 1 \leq 4*4^2 - 4*4$$

$$16 + 12 + 1 \leq 64 - 16$$

$$29 \leq 48$$

Inductive step:

We must show that $(n+1)^2 + 3(n+1) + 1 - n^2 - 3n - 1 \leq 4(n+1)^2 - 4(n+1) - 4n^2 + 4n$.

$$n^2 + 2n + 1 + 3n + 4 - n^2 - 3n - 1 \leq 4(n^2 + 2n + 1) - 4n + 4 - 4n^2 + 4n$$

$$2n + 4 \leq 4n^2 + 8n + 8 + 4 - 4n^2$$

$$2n + 4 \leq 8n + 12$$

$$n + 2 \leq 8n + 6$$

This holds for $n \geq 0$.

Thus, by induction $f(n) = O(g(n)), f(n) = n^2 + 3n + 1, g(n) = n(n-1)$ since $f(n) \leq Cg(n) \forall n \geq n_0$.

A.4.2d

Compute limit $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ to show that $f(n) = o(g(n))$ for $f(n) = n^2 + 3n + 1, g(n) = n(n-1)(n-2)$.

$$\lim_{n\to\infty} \frac{n^2 + 3n + 1}{n(n-1)(n-2)}$$

$$= \lim_{n\to\infty} \frac{n^2 + 3n + 1}{n(n^2 - 3n + 6)}$$

$$= \lim_{n\to\infty} \frac{n^2 + 3n + 1}{n^3 - 3n^2 + 6n)}$$

$$= \lim_{n\to\infty} \frac{n^2 + 3n + 1}{n^3 - 3n^2 + 6n)}$$

Using L'Hospital's rule.

$$= \lim_{n\to\infty} \frac{2n + 3}{3n^2 - 6n + 6)}$$

$$= \lim_{n\to\infty} \frac{2}{3n - 6)} = 0$$

Since the limit goes to 0 as $n \to \infty$, $f(n) = o(g(n))$.

A.5(2)

Prove that if f, g are functions $\mathbb{Z} \to \mathbb{R}^+$, then $f(n) = o(g(n)) \iff \log g(n) - \log f(n)$ tends to infinity (as $n \to \infty$).

Definition: We know that $f(n) = o(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$$\lim_{n\to\infty} \log g(n) - \log f(n) = \infty$$
$$= \lim_{n\to\infty} \log(\frac{g(n)}{f(n)}) = \infty$$

If $\lim_{x\to\infty} \log x = \infty$, then $\lim_{x\to\infty} x = \infty$, since $x >> \log x$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$$

Taking the reciprocal of this statement gets us to the definition for $f(n) = o(g(n))$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1}{\infty} = 0$$

Thus, if f, g are functions $\mathbb{Z} \to \mathbb{R}^+$, then $f(n) = o(g(n)) \iff \log g(n) - \log f(n)$ tends to infinity (as $n \to \infty$).