Midterm Exam Solution, CPSC 302, FALL 2007

1. (a) If x is large, then $\sqrt{x^2-1} \approx x$ and hence the expression on the left, namely $\ln(x - \sqrt{x^2-1})$, is likely to suffer from the damaging effect of cancellation error. Therefore, the expression on the right, $-\ln(x + \sqrt{x^2-1})$, is better for numerical computations.

   (b) Both expressions do not handle well a situation of overflow; the exact result is within a range for which there is no overflow unless x itself causes overflow. Therefore, a better way to go would be to write $\sqrt{x^2-1} = \sqrt{x+1}\sqrt{x-1}$ and use the expression

$$-\ln\left(x + \sqrt{x+1}\sqrt{x-1}\right).$$

   Another way around is to write $\sqrt{x^2-1} = x\sqrt{1-\frac{1}{x^2}}$ and use the expression

$$-\ln\left(x + x\sqrt{1-\frac{1}{x^2}}\right).$$

   This may cause underflow, but the latter is significantly less damaging than overflow. In fact, in most cases it would be sufficient to obtain an effective approximation simply by computing $-\ln(2x)$.

2. (a) For $f(x) = x(x-1)^2$ we have that

$$f'(x) = (x-1)^2 + 2x(x-1) = (x-1)(3x-1).$$

   Thus, given an initial guess x_0, the Newton iteration reads

$$x_{k+1} = x_k - \frac{x_k(x_k-1)}{3x_k-1}.$$

   In the question you were asked not to worry about simplifying, and it was enough to give the formula in its un-simplified form

$$x_{k+1} = x_k - \frac{x_k(x_k-1)^2}{(x_k-1)^2 + 2x_k(x_k-1)}.$$

   (b) The root x^* is a double root (indeed, $f'(1) = 0$) and so convergence to $x^* = 1$ is linear. In contrast, $x^* = 0$ is a simple root, and we have that $f'(0) \neq 0$. In fact, f' does not vanish anywhere near the root. Also, f is smooth. Thus, Newton's method converges quadratically to $x^* = 0$ – much faster than to $x^* = 1$ – when starting sufficiently close to the root; see theorem on page 62 of the notes.

This item has proved to be the killer in our midterm, so here is a short program to better see what happens.

```
function [x,iter] = newt_midterm(guess,tol)
x = guess;
for iter = 1:100
    xo = x;
    x = x - x*(x-1)/(3*x-1);
    if (abs(x*(x-1)^2) < tol)*(abs(x-xo) < tol) , break, end
end
```

Invoking this with tol = 1.e-8 and guess=.1 produces x=-3.3e-21 in 5 iterations. Invoking with guess=1.1 produces x=1+6.5e-9 in 24 iterations.

(c) We have that $f(x) > 0$ in the immediate neighborhood of $x^* = 1$ (both when $x < 1$ and when $x > 1$), and so bisection cannot be successfully applied in this case. On the other hand, $f(x)$ changes sign as it passes through $x^* = 0$, and so bisection can be effectively applied to find the simple root.

3. (a) FALSE. (By definition.)

   (b) FALSE. (For example $f(x) = x^3$.)

   (c) TRUE . (A and $A^{(k)}$ are related by a nonsingular transformation so they both are either singular or nonsingular.)

   (d) TRUE. (Example in notes; Section 5.3.)

   (e) FALSE. (For example $A = .1I$, for a largish dimension n, where $\det(A) = .1^n$ is very small while $\kappa(A) = 1$.)

   (f) FALSE. (See notes on banded matrices, Section 5.4.)

   (g) TRUE. (Just look at $TA\mathbf{x} = T\mathbf{b}$.)