Final Exam Solution, CPSC 302, FALL 2007

1. (a) Newton is faster, bisection is more robust. Newton is extendable to systems, bisection requires less smoothness of nonlinear function.

   (b) SOR can be much faster than Jacobi with the right choice of parameter ω. For the model problem it can be $O(J)$ vs $O(J^2)$ iterations.

   On the other hand Jacobi is more easily and directly parallelizable.

   (c) The inverse power method can be applied also when $|\lambda_1| \approx |\lambda_2|$, where λ_i are the eigenvalues sorted in descending magnitude order.

   The power method is simpler and the iteration can be much cheaper when solving $A\mathbf{x} = \mathbf{b}$ is costly.

   (d) Iterative methods are preferable for very large very sparse symmetric positive definite matrices.

   Direct methods are preferable when the matrix A is general: it is not unbearably large and does not have a special structure such as sparsity.

   (e) Secant does not require knowing $f'(x)$.

   But secant is also slower than Newton.

2. (a) For this matrix in general L and U inherit their nonzero structure directly from A. The matrix L will therefore be **bidiagonal**: $l_{ij} = 0$ if $i > j + 1$ or $i < j$.

   (b) U is in general **full upper triangular**, because there is no special corresponding structure known for A.

   (c) Of the i, j, k-loops the ith one (zeroing below the pivot) is shortened to have only one operation. The other two (on the diagonals and updating rows) are as usual. Hence the flop count is $O(n^2)$.

   (d) Partial pivoting here only potentially exchanges rows with a similar sparsity structure. The answer is therefore the same as in (a), (b) above.

3. (a) For $k > 1$ use $Q^TQ = I$ to obtain

$$A^k = A \cdot A \cdots A = Q\Lambda Q^T Q\Lambda Q^T \cdots Q\Lambda Q^T = Q\Lambda Q^T.$$

For $k = -1$ by definition $A^{-1} = Q\Lambda^{-1}Q^T$, where $Q^{-1} = Q^T$ is used.

(b) We have

$$e^A = Q\left(\sum_{j=0}^{\infty} \Lambda^j/j!\right)Q^T.$$

An obvious way is therefore to define $e^\Lambda = diag\{e^{\lambda_1}, \ldots, e^{\lambda_n}\}$ and

$$e^A = Qe^\Lambda Q^T.$$

(c) When z tends to 0 both numerator and denominator in f tend to 0. Hence it is better to use the Taylor expansion

$$f(z) = (z + z^2/2 + \ldots)/z = 1 + z/2 + z^2/3! + \ldots.$$

This expansion can be cut very quickly if $|z|$ is very small.

(d) We have

$$f(A) = (Qe^\Lambda Q^T - QQ^T)(Q\Lambda^{-1}Q^T) = Q(e^\Lambda - I)\Lambda^{-1}Q^T = Qf(\Lambda)Q^T.$$

Hence each eigenvalue can be dealt with separately. For very small eigenvalues use the expression from (c) for $z = \lambda_i$, and for other eigenvalues use the original definition to evaluate $f(\lambda_i)$. This gives $f(\Lambda)$, from which we get $f(A)$ stably.

4. (a)

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

(b) The above can be written as $A^T\mathbf{b} - A^T A\mathbf{x} = \mathbf{0}$, or

$$A^T(\mathbf{b} - A\mathbf{x}) = \mathbf{0}.$$

(c) By the definition of the new right hand side and since $A^T\mathbf{r} = \mathbf{0}$ we have

$$A^T\hat{\mathbf{b}} = A^T(\mathbf{b} + \alpha\mathbf{r}) = A^T\mathbf{b}.$$

Hence the right hand side of the normal equations does not change and neither does their solution.

(d) SVD is significantly slower the QR decomposition for non-small problems. Also, sparsity is lost: if A is large and sparse then SVD is much worse than the alternatives both in terms of storage and in terms of CPU time.

1. (a) Here $M = \beta^{-1} I$ and $T = I - \beta A$.

   (b) Order $\lambda_1 \geq \cdots \geq \lambda_n > 0$. (Since A is symmetric positive definite we can do this). The eigenvalues of T are obviously

   $$\mu_i = 1 - \beta \lambda_i,$$

   and convergence holds if $\rho(T) < 1$, i.e. if $|\mu_i| < 1$, all i. Since $\lambda_n > 0$ we must have $\beta > 0$. The only other implied constraint is that $\mu_i > -1$, which happens iff $1 - \beta\lambda_1 > -1$. This gives the range

   $$0 < \beta < \frac{2}{\lambda_1}.$$

   (c) If $\lambda_1 > \lambda_2$ then the power method may be applied to approximately find only λ_1 and substitute in the bound for β. The other eigenvalues do not matter and need not be found.

2. (a) We have

   $$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 - 1 \\ x_1 x_2 - 1 \end{pmatrix}, \quad \Rightarrow \; J(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ x_2 & x_1 \end{pmatrix}.$$

   (b) Newton's method will fail if it starts from an initial guess such that the Jacobian matrix is singular. This happens here if $x_1 = 0$.