Final Exam, CPSC 302, FALL 2007
Dec. 7, 2007

Instructions :

- Write down your name and student number in the designated spot in the booklet.
- Make sure this exam has 5 pages.
- Time: $2\frac{1}{2}$ hours.
- **There is choice**: answer only five of the six questions. If you answer more than five, a random choice will be graded.
- Total 100 marks: 20 marks per question.
- Two pages (one sheet) of handwritten notes can be used. No other material is allowed.
- Show your work, but please avoid unnecessarily lengthy answers.
- G O O D L U C K !!

1. Answer briefly the following questions:

   (a) State one advantage and one disadvantage that Newton's method has over bisection for solving nonlinear equations. **[4 marks]**

   (b) State one advantage and one disadvantage that SOR has over Jacobi for solving linear systems. **[4 marks]**

   (c) State one advantage and one disadvantage that the inverse power method has compared to the power method. **[4 marks]**

   (d) State one type of matrices for which iterative methods may be preferable over direct methods, and one type of matrices for which direct methods may be preferable. **[4 marks]**

   (e) State one advantage and one disadvantage that the secant method has compared to Newton's method. **[4 marks]**

2. An $n \times n$ matrix A is said to be in *Hessenberg form* if all its elements below the first sub-diagonal are zero

$$a_{ij} = 0, \quad i > j + 1.$$

   For parts (a), (b) and (c), consider the LU decomposition of such a matrix assuming that no pivoting is needed: $A = LU$.

   (a) What is the nonzero structure of the matrix L (i.e. where are its nonzeros)? **[5 marks]**

   (b) What is the nonzero structure of the matrix U? **[5 marks]**

   (c) How many operations (to a leading order) does it take to solve a linear system $A\mathbf{x} = \mathbf{b}$ where A is upper Hessenberg? **[5 marks]**

   (d) Suppose now that partial pivoting is applied. What are the patterns of the factors of A? **[5 marks]**

3. Let A be a real, square, nonsingular matrix that has an orthogonal spectral decomposition, i.e. there are an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

   (a) Show that for any integer $k > 1$, and also for $k = -1$,

   $$A^k = Q\Lambda^k Q^T.$$

   [5 marks]

   (b) The exponential matrix function is defined by Taylor's expansion

   $$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \ldots.$$

   Assuming we have a good routine for evaluating e^z stably for any scalar z, propose a way to evaluate e^A for our matrix. **[5 marks]**

   (c) In certain applications there is a need to accurately compute the quantity

   $$f(z) = \frac{e^z - 1}{z}.$$

   Explain what numerical difficulties may arise when $|z|$ is very small, and how to perform the computation in a numerically stable fashion. **[5 marks]**

   (d) Consider now our matrix A, assume it has only nonnegative eigenvalues which may however range from very small to large in magnitude, and propose a stable way to compute

   $$f(A) = (e^A - 1)A^{-1}.$$

   [5 marks]

4. Consider the least squares problem

$$\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|_2,$$

where A has a full column rank.

(a) Write down the normal equations for this problem. **[4 marks]**

(b) Show that if $\mathbf{x}$ solves the problem, then $A^T\mathbf{r} = \mathbf{0}$, where $\mathbf{r} = \mathbf{b} - A\mathbf{x}$. (Be brief, please!) **[6 marks]**

(c) With $\mathbf{r}$ defined as the residual at the optimal solution, consider replacing $\mathbf{b}$ by $\widehat{\mathbf{b}} = \mathbf{b} + \alpha\mathbf{r}$ for some scalar α. Show that we will get the same least squares solution to

$$\min_{\mathbf{x}} \|\widehat{\mathbf{b}} - A\mathbf{x}\|_2$$

regardless of the value of α. **[6 marks]**

(d) The SVD is extremely robust, but it is normally not the method of choice for solving least squares problems.
Give two reasons why. **[4 marks]**

5. Let A be a large, sparse, symmetric positive matrix, and consider solving the system $A\mathbf{x} = \mathbf{b}$ using the method

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \beta(\mathbf{b} - A\mathbf{x}_k),$$

where β is a scalar parameter.

(a) If $A = M - N$ is the splitting associated with this method, state what M and the iteration (or amplification) matrix $T = I - M^{-1}A$ are. **[4 marks]**

(b) Suppose the eigenvalues of A are $\lambda_i,\ i = 1, \ldots, n$. Derive a condition on β that guarantees convergence of the scheme to the solution $\mathbf{x}$ for any initial guess. Show all your steps, but be brief. Simplify the condition as much as possible. **[10 marks]**

(c) Suppose the eigenvalues λ_i are not known . Explain how the power method can be used to determine in which range β must be for convergence to occur. State a condition that is required for the power method to work effectively.

[6 marks]

6. Consider the system

$$x_1 - 1 = 0,$$

$$x_1 x_2 - 1 = 0.$$

It is trivial to solve this system without a computer. However, suppose we apply Newton's method.

(a) Write down the Jacobian matrix for this system. [**10 marks**]

(b) For what initial guesses will the method fail? [**10 marks**]