CPSC 320 Sample Midterm 2
March, 2016

Name: ____________________ Student ID: ____________________

Signature: ____________________

- You have 70 minutes to write the 5 questions on this examination. A total of 74 marks are available.
- **Justify all of your answers.**
- You are allowed to bring in one hand-written or printed, double-sided 8.5 x 11in sheet of notes, and nothing else.
- Keep your answers short. If you run out of space for a question, you have written too much.
- The number in square brackets to the left of the question number indicates the number of marks allocated for that question. Use these to help you determine how much time you should spend on each question.
- Use the back of the pages for your rough work.
- **Good luck!**

Question	Points	Score
1	20	
2	8	
3	16	
4	16	
5	14	
Total:	74	

UNIVERSITY REGULATIONS:

- Each examination candidate must be prepared to produce, upon the request, his/her UBC card.
- No examination candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
- CAUTION: candidates guilty of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  1. Having at the place of writing, or making use of, any books, papers or memoranda, electronic equipment, or other memory aid or communication devices, other than those authorised by the examiners.
  2. Speaking or communicating with other candidates.
  3. Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

1. [20 points] Short Answers

(a) [4 points] Let T be the MST (Minimum Spanning Tree) of an undirected graph $G = (V, E)$ with cost function $c : E \to \mathbb{R}^+$ produced by the Kruskal's algorithm. Let $c' : E \to \mathbb{R}^+$ is defined as follows: $c'(e) = [c(e)]^2$ for every $e \in E$. **Decide whether the following statement is true or false (justify your answer shortly).**

Statement: *Then T is an MST of G with cost function c'.*

(b) [6 points] Modify the MERGE algorithm so that in addition to merging the sorted subarrays $A[first \ldots mid]$ and $A[mid{+}1 \ldots last]$, it also counts and returns the number of inversions (i, j) such that $first \leq i \leq mid$ and $mid + 1 \leq j \leq last$. You only need to insert a couple of lines.

procedure MERGE($A, first, mid, last$)
  $B \leftarrow$ empty list
  $i \leftarrow first$
  $j \leftarrow mid + 1$
  while $i \leq mid$ and $j \leq last$ **do**
    if $A[i] \leq A[j]$ **then**
      append $A[i]$ to B
      $i \leftarrow i + 1$
    else
      append $A[j]$ to B
      $j \leftarrow j + 1$
    end if
  end while

```
        if i ≤ mid then
            append A[i . . . mid] to B
        else
            append A[j . . . last] to B
        end if
        A[first . . . last] ← B
    end procedure
```

(c) [6 points] In the *merge* part of the algorithm to find the **closest pair of points** among a set of points in the 2D plane, we only considered the subset S_δ of the points whose distance from the line $\ell : x = x_0$ used to separate the two subproblems was smaller than δ.

What does δ represent and why is it sufficient to only consider these points?

(d) [4 points] Why is RANDOMIZEDQUICKSORT better than the regular QUICKSORT?

2. [8 points] Consider an undirected graph $G = (V, E)$ and cost function $c : V \to \mathbb{R}^+$. Assume that all edge costs are distinct. You can assume that the following property holds (without proving it):

   > **Cut property.** Let S be any subset of nodes that is neither empty nor equal to all of V, and let edge $e = (u, v)$ be the minimum cost edge with one end in S and the other in $V \setminus S$. Then every MST of G contains e.

   Prove that there is only one MST for G.

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3. [16 points] Recurrence relations

   (a) [8 points] Write a recurrence relation that describes the running time of the following algorithm MEDIAN as a function of $n = j - i + 1$. Algorithm SORT(A, i, j) runs in time $\Theta(j - i + 1)$.

```
function MEDIAN(A, i, j)
    if j ≤ i + 4 then
        SORT(A, i, j)
        return A[⌈(i + j)/2⌉]
    else
        B ← []
        for k ← 0 to ⌊(j − i + 1)/5⌋ − 1 do
            append MEDIAN(A, i + 5 ∗ k, i + 5 ∗ k + 4) to B
        end for
        return MEDIAN(B, 1, ⌊(j − i + 1)/5⌋)
    end if
end function
```

   Remark: Do not try to figure out what this algorithm does and whether it actually finds a median.

(b) [8 points] Consider the following recurrence: $T(n) = \begin{cases} 3T(n/3) + dn & \text{if } n \geq 3 \\ b & \text{if } n \leq 2 \end{cases}$

You have guessed that $T(n) \in O(n \log n)$. **Verify** that this guess is correct using the substitution method (induction).

4. [16 points] *Recursion Tree method.* Prove tight upper and lower bounds on the function $T(n)$ defined by

$$T(n) = \begin{cases} 3T(n-1) + 4^n & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

Hint. Recall the formula $\sum_{i=0}^{\infty} c^i = \frac{1}{1-c}$ for any $0 < c < 1$.

5. [14 points] Consider the following implementation of the randomized QUICKSORT that waits for a "perfect" pivot. A pivot is **perfect** if the size of the left partition is $\lfloor (n-1)/2 \rfloor$ and the size of the right partition is $\lceil (n-1)/2 \rceil$.

procedure PERFECTPIVOTQUICKSORT($A, first, last$)
    if $first < last$ **then**
        $n \leftarrow last - first + 1$
        repeat
            $mid \leftarrow$ RANDOMIZEDPARTITION($A, first, last$)
            ▷ chooses pivot uniformly at random
        until $mid - first = \lfloor (n-1)/2 \rfloor$
        PERFECTPIVOTQUICKSORT($A, first, mid - 1$)
        PERFECTPIVOTQUICKSORT($A, mid + 1, last$)
    end if
end procedure

(a) [2 points] What's the probability that a **perfect** pivot is selected in one run of RANDOMIZEDPARTITION? Justify shortly your answer.

(b) [6 points] Write down the recurrence for the expected running time $S(n)$ of the PERFECTPIVOTQUICKSORT algorithm. Assume that each call to RANDOMIZEDPARTITION takes time cn. Let $g(n)$ be the number iteration of the **repeat**-**until** loop. If the probability of the perfect pivot is p, then the expected value of $g(n)$, $E[g(n)]$ is $1/p$.

(c) [6 points] Solve the recurrence from part (c).

Hint. If you use the Master Theorem, you can ignore additive constants in the recurrence (in addition to ignoring floor and ceiling parts). For example, you can replace $S(\lfloor (n-2)/3 \rfloor)$ with $S(n/3)$.

End of exam.