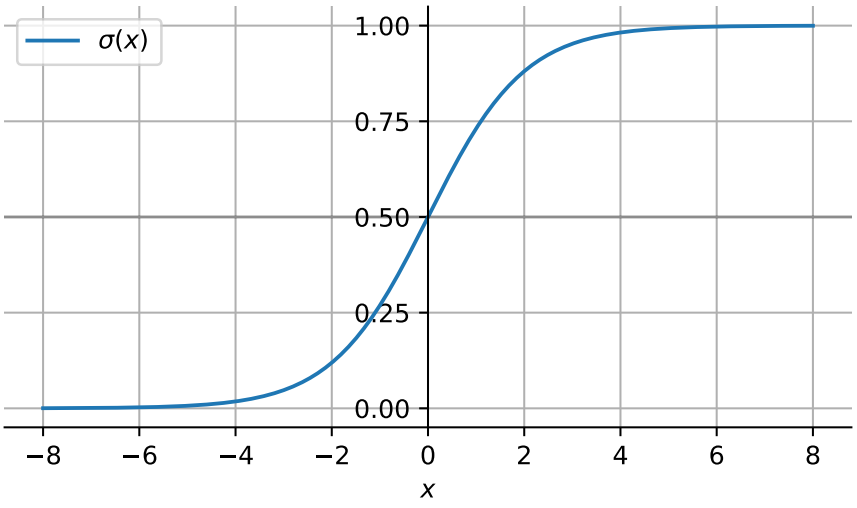
Single-Layer Network

1 Activation functions

Activation functions can be divided into discrete/continuous and unipolar/bipolar. Some common activation functions include (for $net = \sum_i w_i x_i - \theta$):

- **sign function:**
$$f(net) = \text{sgn}(net)$$
- **'step' function:**
$$f(net) = \begin{cases} 1 & \text{for } net \geq 0 \\ 0 \text{ (or } -1) & \text{for } net < 0 \end{cases}$$
- **sigmoid (unipolar):**
$$f(net) = \frac{1}{1 + e^{-net}}$$
- **sigmoid (bipolar):**
$$f(net) = \frac{2}{1 + e^{-net}} - 1$$
- **linear:**
$$f(net) = net$$



Sigmoid (logistic) function

2 Gradient Descent

For a perceptron with a continuous activation function, let's define the following error function:

$$E = \frac{1}{2}(d-y)^2 = \frac{1}{2}(d - f(net)) = \frac{1}{2}(d - f(\mathbf{w}^{\mathrm{T}}\mathbf{x}))$$

We can use gradient descent to minimize the error function. The method involves moving the weights in the direction opposite to the gradient of the error function in small increments:

$$\mathbf{w}' = \mathbf{w} - \alpha \nabla E(\mathbf{w}),$$
$$\theta' = \theta - \alpha \frac{\partial E}{\partial \theta},$$

where

$$\nabla E(\mathbf{w}) = -(d-y) f'(net)\mathbf{x},$$
$$\frac{\partial E}{\partial \theta} = (d-y) f'(net).$$

The derivative of the sigmoid function is

$$\sigma'(x) = \sigma(x)(1-\sigma(x)).$$

The weights of a perceptron using this activation function will be modified as follows:

$$\mathbf{w}' = \mathbf{w} + \alpha(d-y)\sigma'(net)\mathbf{x} = \mathbf{w} + \alpha(d-y)y(1-y)\mathbf{x},$$
$$\theta' = \theta - \alpha(d-y)\sigma'(net) = \theta - \alpha(d-y)y(1-y).$$

3 Single-layer network

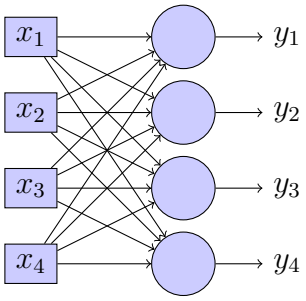


Figure 1: Example of a one-layer network with four inputs and four outputs.

In a neural network, classes can be represented in two ways:

- **Locally**: each neuron represents one class. For a given class only one neuron should have the value 1 and the others should have value 0. For a given example we choose the class represented by the neuron with the highest activation.
- **Globally**: each class is encoded as a combination of outputs. K output layer neurons can represent 2^K classes.

For the single-layer network, we introduce the following notation:

- input vector: $\mathbf{x} = \begin{pmatrix} x_1 & \dots & x_J \end{pmatrix}^{\mathrm{T}}$
- output vector: $\mathbf{y} = \begin{pmatrix} y_1 & \dots & y_K \end{pmatrix}^{\mathrm{T}}$
- weight matrix:

$$\mathbf{W} = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1J} \\ w_{21} & w_{22} & \dots & w_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ w_{K1} & w_{K2} & \dots & w_{KJ} \end{pmatrix}$$

  where w_{kj} is the weight of the j-th input of the k-th neuron.
- bias vector: $\boldsymbol{\theta} = \begin{pmatrix} \theta_1 & \dots & \theta_K \end{pmatrix}^{\mathrm{T}}$, where θ_k is the bias of the k-th neuron.
- activation function matrix: $\boldsymbol{\Gamma} = \mathrm{diag}[f(\cdot)]$. Multiplying this matrix by a vector determines the value of f for each element.

The output is calculated as:

$$\mathbf{y} = \boldsymbol{\Gamma}[\mathbf{W}\mathbf{x} - \boldsymbol{\theta}]$$

Delta rule for one layer:

$$\mathbf{W}' = \mathbf{W} + \alpha(\mathbf{d} - \mathbf{y})\mathbf{x}^{\mathrm{T}}$$

$$\boldsymbol{\theta}' = \boldsymbol{\theta} - \alpha(\mathbf{d} - \mathbf{y})$$

Questions

Question 1.

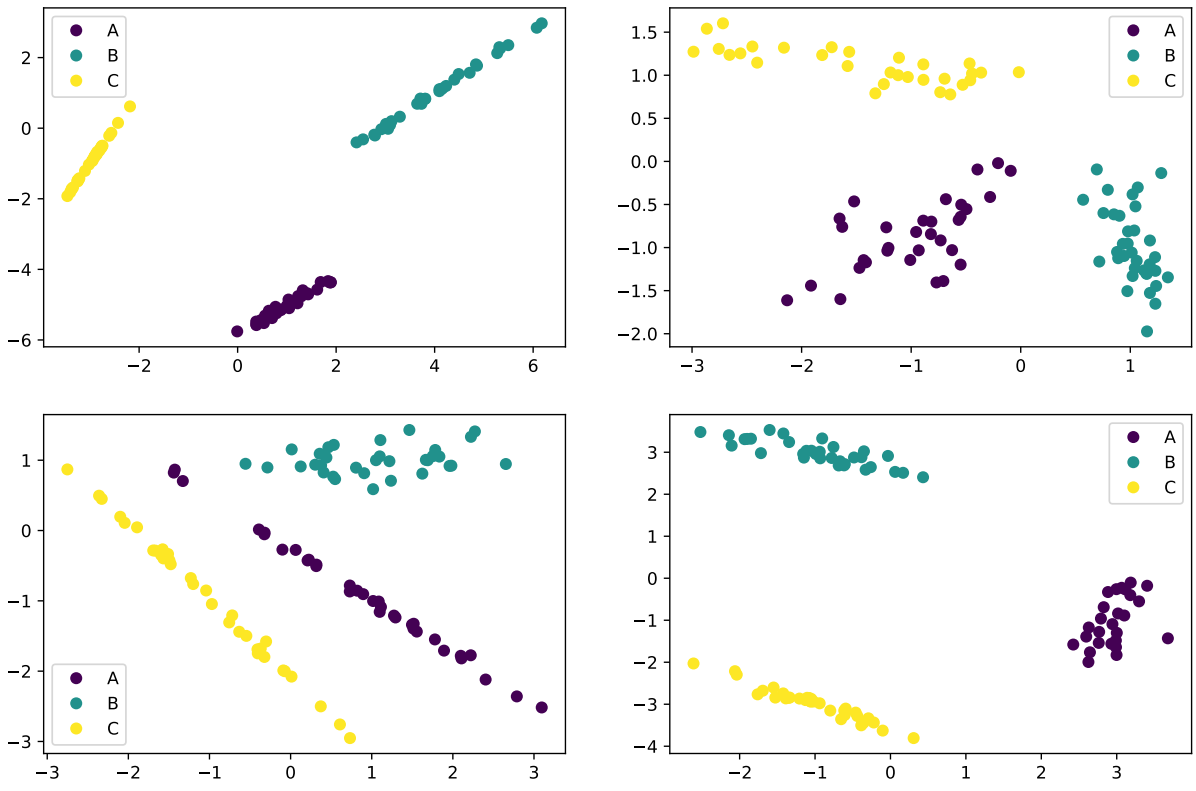
Given a perceptron with the weight vector $\mathbf{w} = (2, -1, 4, 1)$, bias $\theta = 3$, which uses the **sigmoid** activation function, calculate the output and modify the weights and bias for the given input vectors ($\alpha = 1$).

- $\mathbf{x} = (7, -2, -4, 3), d = 1$
- $\mathbf{x} = (1, 0, 1, -2), d = 0$

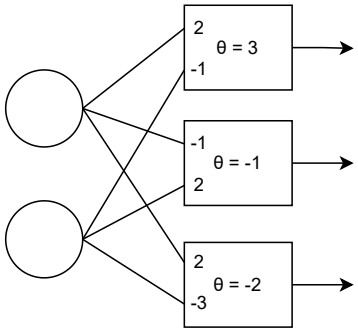
Question 2.

Design a single-layer network to classify the following datasets.

- Which method of output representation (local/global) will be appropriate for each example?
- Draw the decision boundary for each perceptron in the network.
- Write down the mapping of classes to network outputs.



Question 3.



For the single layer network above:

1. Write down the weight matrix.
2. Perform one learning step for the following inputs and expected outputs ($\alpha = 1$).
   - $\mathbf{x} = (3, 1)$, $\mathbf{d} = (0, 0, 1)$
   - $\mathbf{x} = (2, -8)$, $\mathbf{d} = (1, 1, 1)$
   - $\mathbf{x} = (2, 2)$, $\mathbf{d} = (1, 0, 1)$
   - $\mathbf{x} = (0, 1)$, $\mathbf{d} = (1, 1, 0)$

Mini-project: Single-layer network

The aim of the project is to create a single-layer netowrk to identify the language of input texts.

The files `lang.train.csv` and `lang.test.csv` contain a set of texts in four languages – English, German, Polish and Spanish. To classify a given text, count the number of occurences of each letter of the latin alphabet. For the purpose of this task, ignore all other characters (diacritics, puncturation, etc.) – only count the frequencies of the 26 letters of the latin alphabet.

For each text, generate a 26-element input vector containing the number of occurences of each letter. Then normalize it:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

The output of the network should have local representation: one perceptron should be assigned to each language. For a given text only the perceptron corresponding to its language should have value 1 – all others should have value 0.

You can use either a discrete 'step' activation function or a linear activation function ($f(net) = net$) (in both cases the weights are modified using the same formula, because for the linear function $f'(net) = 1$). To classify an input text, select the perceptron with the maximum activation.

Requirements:

- Train the network using data from `lang.train.csv` and output the test accuracy for data in `lang.test.csv`.
- The network should be able to adjust to different datasets, e.g. with a different number of languages.

- Provide an interface for the user to input new texts to classify (e.g. in the terminal).
- (Optionally) display incorrectly classified texts from the test set.

Hints:

- Some of the texts contain commas, so `split(",")` will not work when loading the files (but you can use for example `split(",", 1)` (Python) or `split(",", 2)` (Java)).
- Using a linear activation function decreases the chance of ambiguous network output (when more than one perceptron has output 1).
- You can re-use your implementation of the perceptron from the previous project or implement the network from scratch using matrix operations.