Relación de ejercicios 3.1

(1) $\int \frac{1}{x}\,dx = \log|x| + C$; $\int \frac{2}{2x}\,dx = \log|2x| + C$

¿Son distintos?

Lógicamente no son distintos y se puede demostrar de la siguiente forma:

$$\int \frac{1}{x}\,dx = \log|x| \pm C = \log|2x| \pm C = \int \frac{2}{2x}\,dx$$

$$\log|x| = C$$

$$\log|2x| = \log|2| + \log|x| \pm C$$

$$\log|2x| = \log|x| = C'$$

donde C y C' son constantes.

2) a) $\int (4x^3 - 3x^2 + \frac{1}{2}x - \pi)\,dx =$

$= \int 4x^3 dx - \int 3x^2 dx + \int \frac{1}{2} x dx - \pi \int dx$

$= 4\int x^3 dx - 3\int x^2 dx + \frac{1}{2}\int x dx - \pi \int dx =$

$= 4\frac{x^4}{4} - 3\frac{x^3}{3} + \frac{1}{2}\cdot\frac{x^2}{2} - \pi x = \underline{x^4 - x^3 + \frac{1}{4}x^2 - \pi x + C}$

b) $\int \sqrt[3]{x^2}\,dx = \int x^{2/3} dx = \frac{x^{5/3}}{5/3} + C = \underline{\frac{3\sqrt[3]{x^5}}{5} + C}$

c) $\int \cos x \operatorname{sen}^3 x\,dx = \int u^3 du = \frac{u^4}{4} + C = \frac{\operatorname{sen}^4(x)}{4} + C$

$(\operatorname{sen}^3 x)' = 3\operatorname{sen}^2 x$

$u = \operatorname{sen}(x)$ $\qquad \frac{du}{dx} = \cos(x)$

$u^3 = \operatorname{sen}^3(x)$

$dx = \frac{1}{\cos(x)} du$

d) $\int x \operatorname{sen}(x^2)\,dx = \frac{1}{2}\int \operatorname{sen}(u)\,du = \frac{-1}{2}\cos(u) + C = \frac{-1}{2}\cos(x^2) + C$

$u = x^2$ $\qquad \frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

(3) INTEGRACIÓN POR PARTES

a) $\int e^{2x}\cos(x)\,dx = e^{2x}\operatorname{sen}(x) - \int \operatorname{sen}(x)\,2e^{2x}\,dx =$

$u = e^{2x} \rightarrow du = 2e^{2x}$ | $u = e^{2x}$ $du = 2e^{2x}$

$dv = \cos(x)$ $v = \operatorname{sen}(x)$ | $dv = \operatorname{sen}(x)$ $v = -\cos(x)$

$= e^{2x}\operatorname{sen}(x) - 2\int \operatorname{sen}(x)\,e^{2x}\,dx = e^{2x}\operatorname{sen}(x) - 2\left[-e^{2x}\cos(x) + 2\int e^{2x}\cos(x)\,dx\right]$

$= e^{2x}\operatorname{sen}(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,dx = \int e^{2x}\cos(x)\,dx$

$5\int e^{2x}\cos(x)\,dx = e^{2x}\operatorname{sen}(x) + 2e^{2x}\cos(x) + C$

$$\boxed{\int e^{2x}\cos(x)\,dx = \frac{1}{5}e^{2x}\left(\operatorname{sen}(x) + 2\cos(x)\right) + C}$$

b) $\int x^5 \operatorname{sen}(x^3)\,dx = \int x^3 \cdot x^2 \operatorname{sen}(x^3)\,dx =$

$u = x^3$ $\qquad du = 3x^2$

$dv = x^2 \operatorname{sen}(x^3)\,dx$ $\qquad v = \int x^2 \operatorname{sen}(x^3)\,dx$

$= \frac{1}{3} - \cos(x^3) + c$

$= x^3 \cdot \frac{-1}{3} \cos(x^3) - \int 3x^2 \cdot \frac{-1}{3} \cos(x^3)\,dx =$

$= -x^3/3 \cos(x^3) + \frac{1}{3}\int 3x^2 \cos(x^3)\,dx = \frac{-x^3}{3}\cos(x^3) + \frac{1}{3}\operatorname{sen}(x^3) + C =$

$= \frac{1}{3}\left(\operatorname{sen}(x^3) - x^3\cos(x^3)\right) + C$

c) $\int \operatorname{sen}^2 x\,dx = \int \operatorname{sen}(x) \cdot \operatorname{sen}(x)\,dx =$

$u = \operatorname{sen}(x)$ $\qquad du = \cos(x)$

$dv = \operatorname{sen}(x)\,dx$ $\qquad v = -\cos(x)$

$= -\operatorname{sen}(x)\cos(x) + \int \cos(x)\cos(x) = -\operatorname{sen}(x)\cos(x) + \int (1 - \operatorname{sen}^2(x))\,d$

$u = \cos(x)$ $\qquad du = -\operatorname{sen}(x)$

$dv = \cos(x)$ $\qquad v = \operatorname{sen}(x)$

$\Rightarrow \int \operatorname{sen}^2(x)\,dx = -\operatorname{sen}(x)\cos(x) + x - \int \operatorname{sen}^2(x)\,dx$

$2\int \operatorname{sen}^2(x)\,dx = -\operatorname{sen}(x)\cos(x) + x + C$

$\int \operatorname{sen}^2(x)\,dx = \frac{1}{2}\left(-\operatorname{sen}(x)\cos(x) + x\right) + C$

(4) a) $\int tg(x)dx = \int \frac{sen(x)}{cos(x)} dx =$

$u = cos(x) \quad \frac{du}{dx} = -sen(x)$

$du = -sen(x)dx$

$= -\int \frac{-sen(x)}{cos(x)} dx = -\int \frac{1}{u} du = -ln|u| + C = -ln|cos(x)| + C$

b) $\int (ln(x))^2 dx = ln^2(x) \cdot x - 2\int \frac{ln(x)}{x} \cdot x dx = ln^2(x)x - 2\int ln(x)dx =$

$u = (ln(x))^2 \quad du = 2\frac{ln(x)}{x}dx$

$dv = dx \quad v = x$

$u = ln(x) \longrightarrow du = \frac{1}{x}dx$

$dv = dx \longrightarrow v = x$

$= ln^2(x) \cdot x - 2\left[ln(x) \cdot x - \int \frac{1}{x} \cdot x \, dx \right] = ln^2(x) \cdot x - 2\,ln(x) \cdot x + 2x + C$

c) $\int e^x \, tg \, e^x dx \approx \int u' \tan(u) du = \int \tan(u) du$

??

$\int e^x \, tg \, e^x dx = -ln|cos(e^x)| + C$

d) $\int \frac{\ln^2(\operatorname{arctg}(x))}{1+x^2}\,dx = \int \ln^2(u)\,du =$

cambio de variable

$u = \operatorname{arctg}(x) \quad - \quad du = \frac{1}{x^2+1}\,dx$

$= \ln^2(u)u + 2u(1-\ln(u)) + C = \ln^2(\operatorname{arctg}(x))\operatorname{arctg}(x) + 2\operatorname{arctg}(x)(1-\ln(\operatorname{arctg}(x))) + C$

Integrales Racionales

(5)

a) $\int \frac{x^2}{x+1}\,dx = \int x - 1 + \frac{1}{x+1}\,dx =$

$$\begin{array}{l|l} x^2 & x+1 \\ \hline -(x^2+x) & x-1 \\ \quad -(x) & \\ \quad -(-x-1) & \\ \qquad +1 & \end{array}$$

$u = x+1 \rightarrow du = 1\,dx$

$= \int x\,dx - \int 1\,dx + \int \frac{1}{x+1}\,dx = \frac{x^2}{2} - x + \ln|x+1| + C$

b) $\int \frac{x^2}{(x+1)^2}\,dx = \int 1 - \frac{2x+1}{(x+1)^2}\,dx =$

$$\begin{array}{l|l} x^2 & x^2+2x+1 \\ \hline & 1 \\ -2x-1 & \end{array}$$

$\frac{2x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$ $\qquad u = x+1 \quad du = 1\,dx$

$2x+1 = Ax + (A+B)$

$A = 2 \rightarrow A+B = 1$

$2+B = 1$

$B = -1$

$= \int 1 - \frac{2}{x+1} - \frac{-1}{(x+1)^2}\,dx = \int 1\,dx - 2\int \frac{1}{x+1}dx + \int \frac{1}{(x+1)^2}\,dx =$

$= x - 2\ln|x+1| + \int \frac{1}{u^2}\,du = x - 2\ln(x+1) - \frac{1}{u} = x - 2\ln(x+1) - \frac{1}{x+1} + C$

c) $\int \frac{3}{9x^2-6x+2}\,dx = \int \frac{3}{(3x-1)^2+1}\,dx = \arctg(3x-1) + C$

$9x^2-6x+2 = (3x-1)^2+1$

d) $\int \frac{3x-1}{9x^2-6x+2}\,dx = \frac{1}{6}\ln(9x^2-6x+2) + C$

$9x^2-6x+2 \Rightarrow 18x-6 \div 6 = 3x-1$

⑥ $\displaystyle\int \frac{x^2+6x+5}{x^3-x^2-x-2}\,dx =$

$x^3 \quad -x^2 \quad -x \quad -2 \xrightarrow{\text{Ruffini}}$

$$\begin{array}{c|cccc} & 1 & -1 & -1 & -2 \\ 2 & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$x^2 \quad +x \quad +1$

$$= \int \frac{x^2+6x+5}{(x-2)(x^2+x+1)}\,dx$$

$$\frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+x+1} = \frac{A(x^2+x+1)+(Bx+C)\cdot(x-2)}{(x-2)(x^2+x+1)} =$$

$$= \frac{Ax^2+Ax+A+Bx^2-2Bx+Cx-2C}{(x-2)(x^2+x+1)} = \frac{(A+B)x^2+(A-2B+C)x+(A-2C)}{(x-2)(x^2+x+1)} \Rightarrow$$

$$\Rightarrow \quad x^2+6x+5 = (A+B)x^2+(A-2B+C)x+(A-2C)$$

$$\begin{cases} A+B=1 & B=1-A \\ A-2B+C=6 & \longrightarrow \\ A-2C=5 & C=\frac{-5+A}{2} \end{cases}$$

$A-2(1-A)+\frac{A-5}{2}=6$

$A-2+2A+\frac{A-5}{2}=6$

$2A-4+4A+A-5=12$

$7A=21$

$$\boxed{\begin{aligned} A&=3 \\ B&=-2 \\ C&=-1 \end{aligned}}$$

$$= \int \frac{3}{(x-2)} + \frac{-2x-1}{x^2+x+1}\,dx = 3\int \frac{1}{x-2}\,dx - \int \frac{2x+1}{x^2+x+1}\,dx =$$

$$= \underline{\underline{3\ln|x-2| - \ln|x^2+x+1| + C}} = \ln\left\{\frac{|x-2|^3}{|x^2+x+1|}\right.$$

⑦ a) $\int \frac{x}{4+x^4}\,dx = \frac{1}{4}\int \frac{x}{1+\frac{x^4}{4}}\,dx =$

$= \frac{1}{4}\int \frac{x}{1+\left(\frac{x^2}{2}\right)^2}\,dx = \frac{1}{4}\operatorname{arctg}\left(\frac{x^2}{2}\right)+C$

b) $\int \frac{x^3}{4+x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{4+x^4}\,dx = \frac{1}{4}\cdot \ln|4+x^4|+C$

c) $\int \frac{x^2}{4+x^4}\,dx$

(8) $t=e^x$

a) $\int \frac{e^x}{1-e^x}\,dx = \int \frac{1}{1-t}\,dt = -\int \frac{-1}{1-t}\,dt = -\ln|1-t| + C =$

$\frac{dt}{dx} = e^x$

$= -\ln|1-e^x| + C$

$dt = e^x dx$

b) $\int \frac{e^x}{1-e^{2x}}\,dx = \int \frac{1}{1-t^2}\,dt = \int \frac{1/2}{1+t} + \frac{1/2}{1-t}\,dt$

$1-t^2 = (1+t)(1-t)$

$\frac{1}{(1-t)(1+t)} = \frac{A}{(+1-t)} + \frac{B}{(1+t)} = \frac{A(1+t)+B(1-t)}{(1-t)(1+t)} \Rightarrow$

$1 = A + At + B - Bt = (A-B)t + (A+B)$

$t=0 \to A-B=0 \to A=B$

$A+B=1$

$A=1-B$

$A+A=1$

$2A=1$

$A = 1/2 = B$

$= \frac{1}{2}\int \frac{1}{1+t}\,dt + \frac{1}{2}\int \frac{1}{1-t}\,dt = \frac{1}{2}\ln|1+t| + \frac{1}{2}\cdot\ln|1-t| + C =$

$= \frac{1}{2}\ln|(1+t)(1-t)| + C = \frac{1}{2}\ln|(1+e^x)(1-e^x)| + C$

(9) $\int \cos^5(x)\,dx = \int \cos^4 x \cdot \cos x\,dx = \int (1-\text{sen}^2 x)^2 \cos x\,dx =$

a)

$R(\text{sen}x, \cos x) = \cos^5(x)$

$R(\text{sen}x, -\cos x) = (-\cos(x))^5 = -\cos^5(x) = -R(\text{sen}x, \cos x)$

$\text{sen}(x) \to t$

$\frac{dt}{dx} = \cos(x)$

$dt = \cos(x)dx$

$= \int (1-t^2)^2\,dt = \int 1+t^4-2t\,dt = \int t^4 dt - 2\int t^2 dt + \int 1 dt =$

$= \frac{t^5}{5} - 2\frac{t^3}{3} + t + C = \frac{\text{sen}^5(x)}{5} - \frac{2}{3}\text{sen}^3(x) + \text{sen}(x) + C$

(10)

a) $\int sen^2(x)\cos^3(x)dx = \int t^2(1-t^2)\,dt = \int t^2 - t^4\,dt = \int t^2dt - \int t^4dt =$

$R(senx, cosx) = -R(senx, cosx) \rightarrow senx = t$

$dt = cosx\,dx$

$= t^3/_3 - t^5/_5 + C = \frac{sen^3(x)}{3} - \frac{sen^5(x)}{5} + C$

b) $\int \frac{dx}{1+cosx} = \int \frac{\frac{2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\,dt =$

$R(senx, -cosx) = \frac{1}{1-cosx} \quad -R(senx, cosx) = -\frac{1}{1+cosx}$

$cosx = \frac{1-t^2}{1+t^2} \quad dx = \frac{2}{1+t^2}dt$

$= \int \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}}\,dt = \int 1dt = t + C = tg\left(\frac{x}{2}\right) + C$

(11) $\displaystyle\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx = \int \frac{\sqrt{1-sen^2(t)}}{1-\sqrt{sen^2(t)}}\, 2sen(t)cos(t)\,dt =$

$x = sen^2 t \qquad t = arcsen(\sqrt{x})$

$\frac{dx}{dt} = 2\,sen(t)cos(t)$

$= \displaystyle\int \frac{cos\,t}{1-sen(t)}\, 2sen(t)\,cos(t)\,dt = \int \frac{2\,sen(t)\,cos^2(t)}{1-sen(t)}\,dt =$

$= \displaystyle\int \frac{2sen(t)cos^2(t)\cdot(1+sen(t))}{1^2-sen^2(t)}\,dt = 2\int sen(t)+sen^2(t)\,dt =$

$= 2\cdot(-cos(t)) - sen(t)cos(t) + t + C = -2cos(arcsen\sqrt{x}) - \dots$ $\quad t = arcsen\sqrt{x}$

$\displaystyle\int sen^2(t) = \frac{1}{2}(-sen\,t\,cos\,t + t)$