CIS 575. Introduction to Algorithm Analysis
Material for January 26, 2024

Upper and Lower Estimates of Sums

1 Estimates for Sums

In the recent note, we proved that

$$\sum_{i=1}^{n^2} i^3 \in \Theta(n^8)$$

Since $n^8 = n^2 \cdot (n^2)^3$, this suggests a more general result (from which our recent result would follow by letting $g(n) = n^2$ and $f(i) = i^3$):

$$\sum_{i=1}^{g(n)} f(i) \in \Theta(g(n) \cdot f(g(n))) \tag{1}$$

That is, we can approximate a sum by the result of multiplying

- the number of terms, by
- the last term.

We shall show that (1) does indeed hold, provided that f satisfies certain conditions.

1.1 Upper Bound

If f is non-decreasing, and thus the last term of the sum is the largest, we obviously have

$$\sum_{i=1}^{g(n)} f(i) \in O(g(n) \cdot f(g(n)))$$

as can be verified by the calculation

$$\sum_{i=1}^{g(n)} f(i) \leq \sum_{i=1}^{g(n)} f(g(n)) = g(n) \cdot f(g(n))$$

where the inequality follows from $i \leq g(n)$ and f being non-decreasing.

1.2 Lower Bound

We first observe that in general we can **not** claim

$$\sum_{i=1}^{g(n)} f(i) \in \Omega(g(n) \cdot f(g(n)))$$

To see this, let $f(i) = 2^i$ and $g(n) = n$; then the claim would be

$$\sum_{i=1}^{n} 2^i \in \Omega(n \cdot 2^n)$$

which amounts to

$$2 \cdot 2^n - 2 \in \Omega(n\ 2^n)$$

which is obviously not the case. The problem is that in this case, the last term is bigger than all the other terms combined, and hence multiplying the last term by the number of terms would vastly overapproximate the sum of the terms. For (1) to hold, we cannot allow f to grow "too fast" but must demand f to be "smooth" as to be defined in the next paragraph.

Smoothness Let f be a function from positive integers to non-negative reals. We say that f is *smooth* if it is non-decreasing, and if there exists $c > 0$ and $n_0 \geq 0$ such that for all $n \geq 0$ it holds that

$$f(2n) \leq cf(n).$$

It is easy to see that the identify function is smooth, that all positive constant functions are smooth, and that the sum and product of two smooth functions are also smooth. Hence all polynomials with positive coefficients are smooth.

The notion of smoothness was designed to exclude exponentials, and the function $f(n) = 2^n$ is indeed not smooth as we shall now prove by contradiction: assume that c is such that $2^{2n} \leq c2^n$ for all $n \geq n_0$; this amounts to $(2^n)^2 \leq c2^n$ and thus $2^n \leq c$ for all $n \geq n_0$ which is impossible.

(Even a function that is bounded by a polynomial may not be smooth, for example $2^{2^{\lfloor \lg \lg n \rfloor}}$.)

Smoothness Suffices for Lower Bound Assume that f is *smooth* (and hence non-decreasing), in that there exists $c > 0$ and $m_0 \geq 0$ such that $f(2m) \leq cf(m)$ for all $m \geq m_0$. Then

$$\sum_{i=1}^{g(n)} f(i) \in \Omega(g(n) \cdot f(g(n)))$$

provided that g satisfies (as will almost always be the case) that there exists $n_0 \geq 0$ such that $g(n) \geq 2m_0$ for all $n \geq n_0$.

To see this, first observe that for all $x \geq 2m_0$ we have

$$f(x) \leq f(2 \cdot \lceil \frac{x}{2} \rceil) \leq c \cdot f(\lceil \frac{x}{2} \rceil)$$

where the first inequality comes from f being non-decreasing, and the second from f being smooth. As a consequence, for all $x \geq 2m_0$ we have

$$f(\lceil \frac{x}{2} \rceil) \geq \frac{f(x)}{c} \tag{2}$$

For $n \geq n_0$, and thus $g(n) \geq 2m_0$, we then have the calculation

$$
\begin{aligned}
& \sum_{i=1}^{g(n)} f(i) \\
\text{(fewer terms)} \quad \geq \quad & \sum_{i=\lceil \frac{g(n)}{2} \rceil}^{g(n)} f(i) \\
(i \geq \lceil \tfrac{g(n)}{2} \rceil \text{ and } f \text{ non-decreasing}) \quad \geq \quad & \sum_{i=\lceil \frac{g(n)}{2} \rceil}^{g(n)} f(\lceil \frac{g(n)}{2} \rceil) \\
(\text{at least } \tfrac{g(n)}{2} \text{ terms}) \quad \geq \quad & \frac{g(n)}{2} \cdot f(\lceil \frac{g(n)}{2} \rceil) \\
(\text{Inequality (2) as } g(n) \geq 2m_0) \quad \geq \quad & \frac{g(n)}{2} \cdot \frac{f(g(n))}{c} \\
= \quad & \frac{1}{2c}\, g(n) \cdot f(g(n))
\end{aligned}
$$

which generalizes the last calculation in the previous note and shows the desired result

$$\sum_{i=1}^{g(n)} f(i) \in \Omega(g(n) \cdot f(g(n)))$$