CIS 575. Introduction to Algorithm Analysis
Material for April 22, 2024

Flow Networks

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The topic introduced in this note is covered in *Cormen*'s Chapter 24.

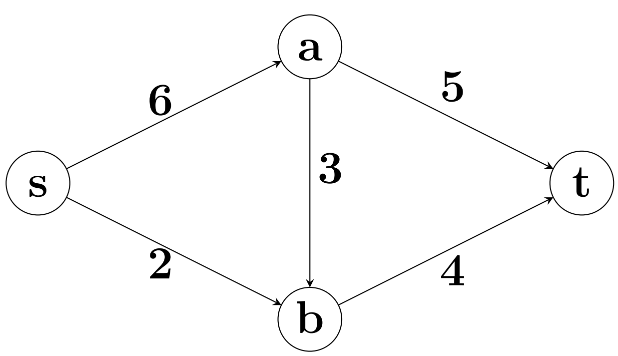
1 Flow Networks

In this set of notes we shall consider the problem of leading flow through a network where each edge has a finite capacity; to find the maximum possible flow we shall develop techniques which will also reveal "bottlenecks".

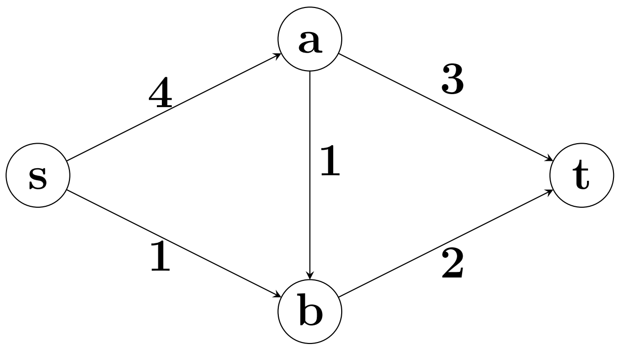
The setting is that we have a *directed* graph (V, E) where

- one node is the **source** (we shall often call that node s)
- one node is the **sink** (we shall often call that node t)
- each edge e has a **capacity**, denoted $C(e)$, which is a positive real number (often an integer).

As an example, consider the network



Such a network can be used to create a flow (of unspecified goods) from the source to the sink. For example, in the given network we could have the flow



Formally, a **flow** F assigns to each edge e a number $F(e)$ that satisfies the **capacity constraint**

$$0 \leq F(e) \leq C(e)$$

and also satisfies **flow conservation**:

> for all nodes except the source and sink, the sum of incoming flow equals the sum of outgoing flow.

For the example, flow conservation holds for a since $4 = 1 + 3$, and for b since $1 + 1 = 2$.

Flow Value The *value* of a flow is given as

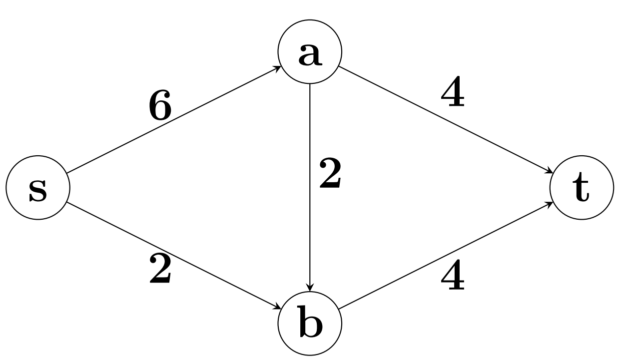
> the **sum** of the flow **out from the source**
> (minus the sum of the flow into the source which is usually zero).

Equivalently, the value is given as

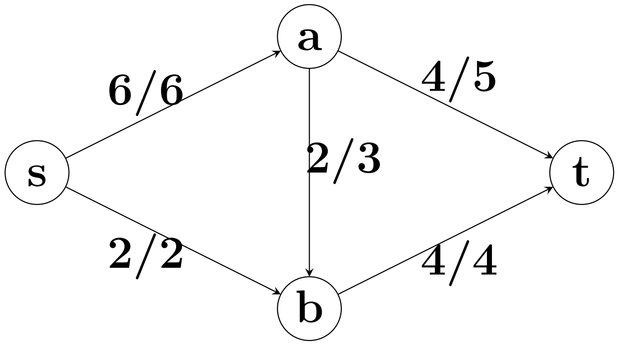
> the **sum** of the flow **into the sink**
> (minus the sum of flow out from the sink which is usually zero)

For our example, the given flow has value 5, since the flow out of the source is $4 + 1 = 5$ (and the flow into the sink is $3 + 2 = 5$).

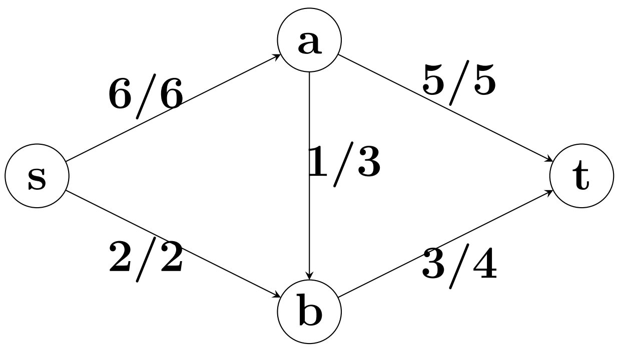
Maximum Flow Our goal is to find a flow with maximum value. For the given example, since the total outgoing capacity from the source is $6 + 2 = 8$, we cannot hope for a flow with value greater than 8. But there does exist a flow with value 8:



which we may depict together with the capacity constraints:



For the given network, there are *two* flows with maximum value; the other is



In the subsequent notes, we shall develop techniques for finding a maximum flow.