CIS 575. Introduction to Algorithm Analysis
Material for February 5, 2024

The Substitution Method: Case 1

The topic of this note is covered in *Cormen*'s Section 4.3.

1 The Substitution Method

For a given recurrence

$$T(n) = \ldots$$

one may **guess** a g such that one **believes** that $T(n) \in O(g(n))$. To **verify** that belief, we may use the **substitution method** which if successful will construct an inductive proof that $T(n) \leq c \cdot g(n)$ for a certain constant $c > 0$ (and for big enough n). Observe that we do *not* have to guess the constant c as it will emerge from the substitution method.

Similarly, one may use the substitution method to prove $T(n) \in \Omega(h(n))$ with h a given guess.

We shall illustrate the substitution method by presenting 3 examples, each with a different flavor.

2 Applying the Substitution Method, Example 1

We shall consider the recurrence

$$T(n) = 2 \cdot T(\lfloor \frac{n}{2} \rfloor) + n^2$$

and shall look for a g (as small as possible) such that $T(n) \in O(g(n))$. (As our current interest is in big-O, we have chosen to wrap the floor rather than the ceiling around $\frac{n}{2}$, so as to avoid unnecessary complications.)

Let us *guess* that we can use $g(n) = n^2$ (obviously we cannot use a polynomial of lower degree). Our goal is thus to prove that for all $n \geq 1$ we have

$$T(n) \leq cn^2$$

for some $c > 0$ to be found along the way.

Let us first consider the **inductive step**, where $n \geq 2$. Then

$$1 \leq \lfloor \frac{n}{2} \rfloor < n$$

and thus we can apply the *induction hypothesis* to $\lfloor \frac{n}{2} \rfloor$. Hence we have the calculation

$$\begin{array}{rcl} & & T(n) \\ \text{by recurrence} & = & 2 \cdot T(\lfloor \dfrac{n}{2} \rfloor) + n^2 \\ \text{by } \textbf{induction hypothesis} & \leq & 2 \cdot c(\lfloor \dfrac{n}{2} \rfloor)^2 + n^2 \\ \text{by properties of floor operator} & \leq & 2 \cdot c(\dfrac{n}{2})^2 + n^2 \\ & = & (\dfrac{c}{2} + 1) \cdot n^2 \\ & \leq & c \cdot n^2 \end{array}$$

which will show the desired $T(n) \leq cn^2$, *provided* we choose $c > 0$ such that the last inequality does indeed hold. That will be the case if $\frac{c}{2} + 1 \leq c$, so we need to choose $\mathbf{c \geq 2}$ for the inductive step to go through.

For the base case, with $n = 1$, we need $T(1) \leq c \cdot 1^2$, that is $c \geq T(1)$. By choosing $c \geq \max(T(1), 2)$, we thus allow the inductive proof to go through. This confirms that $T(n) \leq cn^2$ for all $n \geq 1$, and hence $T(n) \in O(n^2)$.