CIS 575. Introduction to Algorithm Analysis
Material for February 9, 2024

The Master Theorem: Indirect Applications

1 Twisting Recurrences to Fit the Master Theorem

Consider the recurrence

$$T(n) = 2T(\sqrt{n}) + \lg(n)$$

where it appears that the Master Theorem (no matter the version) is not applicable since that theorem requires that the recursion is with a *fraction* of n, and *not* with the square root. But it turns out, as we shall now demonstrate, that we can "massage" the recurrence so as to fit the required template.

The trick is to make a *change of variables*, and define a new function S by the equation $\mathbf{S(k) = T(2^k)}$. We then have

$$S(k) = T(2^k) = 2T(\sqrt{2^k}) + \lg(2^k) = 2T(2^{\frac{k}{2}}) + k$$

and hence the recurrence

$$S(k) = 2S(\frac{k}{2}) + k$$

to which the Master Theorem is immediately applicable; we have already seen that it gives us ($a = b = 2$, $r = q = 1$) the solution

$$S(k) \in \Theta(k \lg(k))$$

which we can translate into a solution for T:

$$\mathbf{T(n)} = S(\lg(n)) \in \boldsymbol{\Theta}(\lg(\mathbf{n}) \lg(\lg(\mathbf{n}))).$$

You may (should!) feel a bit uneasy about whether this kind of reasoning is really sound. Let us therefore verify, using our old friend the substitution method, that we do indeed have $T(n) \in O(\lg(n) \lg(\lg(n)))$ (the Ω direction is left to the meticulous reader!)

After inserting floor around the recursive call:

$$T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \lg(n)$$

we thus want to prove

$$T(n) \leq c \lg(n) \lg(\lg(n))$$

for a $c > 0$ discovered along the way, and for $\mathbf{n \geq 3}$ so that $\lg(\lg(n))$ is well-defined and > 0.

Since we need to apply the induction hypothesis to $\sqrt{n}$, the inductive step will be for $n \geq 9$. We then get

$$\begin{aligned}
T(n) &= 2T(\lfloor\sqrt{n}\rfloor) + \lg(n) \\
&\leq 2c\lg(\lfloor\sqrt{n}\rfloor)\lg(\lg(\lfloor\sqrt{n}\rfloor)) + \lg(n) \\
&\leq 2c\lg(\sqrt{n})\lg(\lg(\sqrt{n})) + \lg(n) \\
&= 2c\frac{1}{2}\lg(n)\lg(\frac{1}{2}\lg(n)) + \lg(n) \\
&= c\lg(n)(\lg(\lg(n)) - 1) + \lg(n) \\
&= c\lg(n)\lg(\lg(n)) - c\lg(n) + \lg(n) \\
&\leq c\lg(n)\lg(\lg(n))
\end{aligned}$$

where the last inequality will hold if $c \geq 1$. For the base cases, we must consider $n = 3, n = 4, \ldots n = 8$ but they can all be handled by choosing c big enough.