CIS 575. Introduction to Algorithm Analysis
Material for April 22, 2024

The Ford-Fulkerson Method

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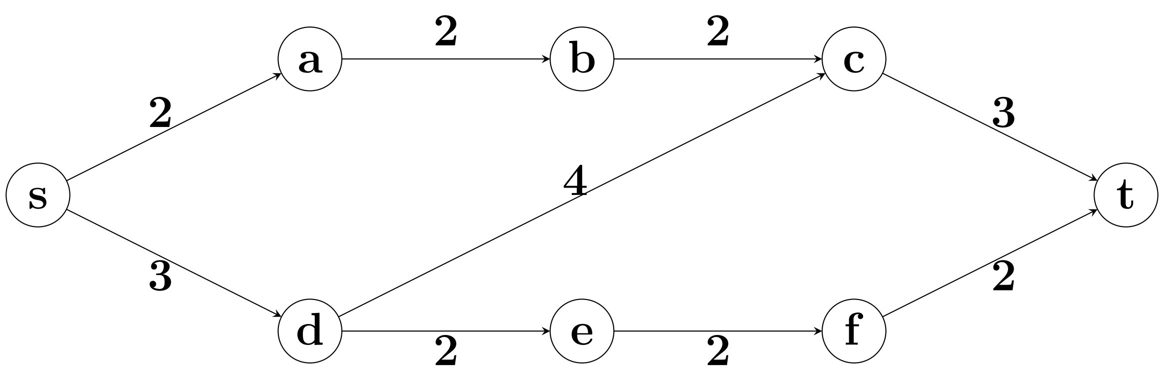
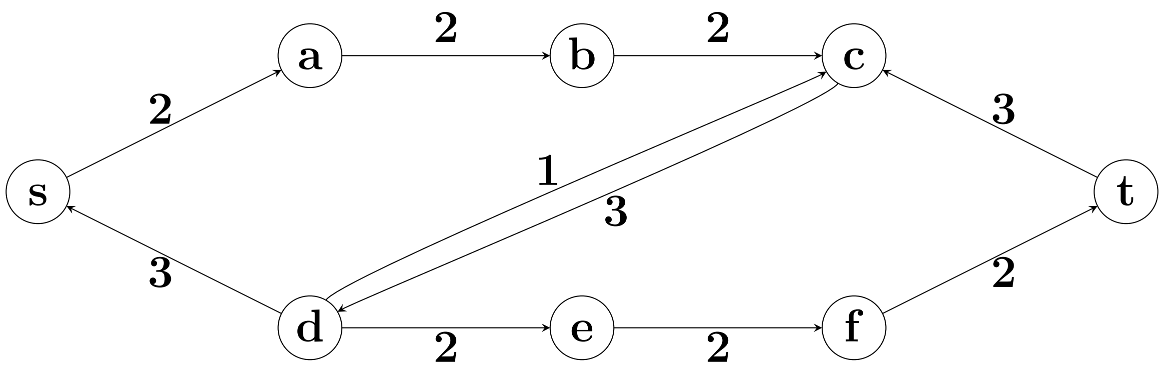
The topic of this note is covered in *Cormen*'s Section 24.2.

1 Residual Networks

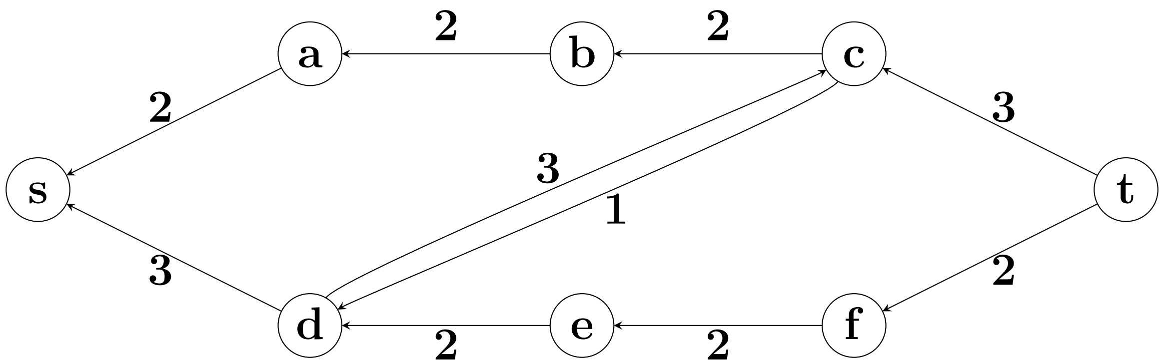
Recall that for a given network, an augmenting path may suggest a flow that is not optimal, as for



where the edge from d to c is part of an augmenting path $\pi_1 = sdct$ with mimimum capacity 3, even though a maximum flow will assign only 1 to that edge. An algorithm for finding a maximum flow may still select π_1, but needs to be able to (partially) *undo* the resulting flow assignment. To facilitate that, the algorithm will maintain a *residual network*. After processing π_1, that residual network will still have an edge from d to c, though (as in the previous note) with capacity reduced from 4 to 1, but (unlike the previous note) also an edge from c to d with capacity 3 which represents the capability to undo the flow from d to c. In general, for each edge in π_1 there is a **residual edge** in the reverse direction:



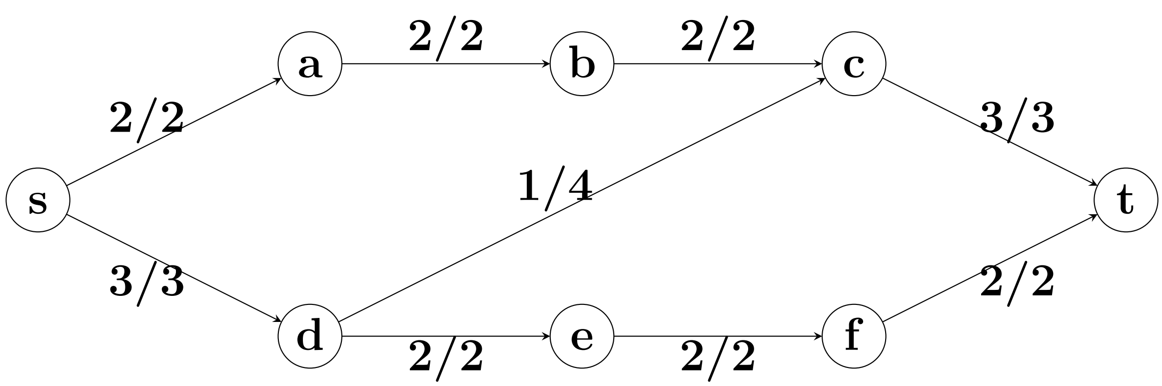
The above network has exactly one augmenting path, $\pi_2 = sabcdeft$ with minimum capacity 2. The resulting residual network is



which has no augmenting path, so we are done. We have constructed a flow F that is the sum of the flow from π_1 and the flow from π_2, that is:

$$\begin{aligned} F(s,a) &= 0+2=2 \\ F(s,d) &= 3+0=3 \\ F(a,b) &= 0+2=2 \\ F(b,c) &= 0+2=2 \\ F(d,e) &= 0+2=2 \\ F(e,f) &= 0+2=2 \\ \mathbf{F(d,c)} &= \mathbf{3-2=1} \\ F(c,t) &= 3+0=3 \\ F(f,t) &= 0+2=2 \end{aligned}$$

which is indeed the maximum flow depicted in the previous note:



We have illustrated a recursive approach, called the **Ford-Fulkerson method**, to build a flow F in a network G:

- if G has no augmenting path, F is 0 everywhere.
- If G has an augmenting path π with minimum capacity m:
  1. construct the residual network G' wrt. π
  2. recursively build a flow F' for G'
  3. construct F from F' by adding m to all edges in π.

Correctness The Ford-Fulkerson method is highly **non**-deterministic, since at each step there may be several augmenting paths to choose from. But one can prove that **when no augmenting paths exist, a maximum flow has been constructed**.

Edges from/to Sources/Sinks Since an augmenting path is acyclic, it can never (re)enter the source s nor exit t. (Hence there is no need to draw residual edges into the source, or out of the sink, and we shall not do that in subsequent examples.) We infer that **the flow of an edge from the source, or into the sink, is never decreased**.

Running time Assume we apply the Ford-Fulkerson method to a network $G = (V, E)$. Retrieving an augmenting path, or determining that no such exists, can obviously be done (perhaps through a depth-first search) in time $O(|E|)$, as can the construction of the residual network.

To analyze the total running time, we need to estimate the number of iterations. It turns out that if capacities may be irrational numbers, it is possible that Ford-Fulkerson's method never terminates! But if all capacities are integers, the maximum flow of the residual network will decrease by at least one for each iteration.

We conclude: with M the maximum flow value of an integer network, the Ford-Fulkerson method will run in time $\mathbf{O}(\mathbf{M}|\mathbf{E}|)$.