CIS 575. Introduction to Algorithm Analysis
Material for February 7, 2024

The Substitution Method: Case 3

The topic of this note is covered in *Cormen*'s Section 4.3.

1 Applying the Substitution Method, Example 3

In previous notes, we considered two recurrences and guessed their solutions; to prove the correctness of our guesses, we had to follow two slightly different approaches. In this note, we shall consider a third recurrence:

$$T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + 1$$

for which we guess that $\mathbf{T(n) \in O(n)}$ but to verify that we must add a twist to our previous approaches.

It seems natural to aim at proving, for suitable $c > 0$ to be found along the way, that for all $n \geq 1$ we have

$$T(n) \leq cn$$

and we may embark on the inductive step:

$$\begin{array}{rrl} & & T(n) \\ & = & 2T(\lfloor \frac{n}{2} \rfloor) + 1 \\ \mathbf{IH} & \leq & 2c\lfloor \frac{n}{2} \rfloor + 1 \\ & \leq & cn + 1 \\ & \leq ?? & cn \end{array}$$

but now we are *stuck* in that it is impossible to choose c such that $cn + 1 \leq cn$.

What to do? It turns out that the trick is to prove a *stronger* property: for suitable $c > 0$ **and $\mathbf{d > 0}$** found along the way, for all $n \geq 1$ it will hold that

$$T(n) \leq cn - d \tag{1}$$

For the inductive step, for $n \geq 2$ we have the calculation

$$
\begin{array}{rcl}
 & & T(n) \\
 & = & 2T(\lfloor \frac{n}{2} \rfloor) + 1 \\
\mathbf{IH} & \leq & 2(c\lfloor \frac{n}{2} \rfloor - d) + 1 \\
 & \leq & cn - 2d + 1 \\
 & \leq & cn - d
\end{array}
$$

which will show the desired $T(n) \leq cn - d$, *provided* we choose $d > 0$ such that the last inequality does indeed hold. That will be the case if $-2d + 1 \leq -d$, so we need to choose $\mathbf{d \geq 1}$ for the inductive step to go through.

For the base case, with $n = 1$, we need $T(1) \leq c - d$, that is $\mathbf{c \geq d + T(1)}$.

We have seen that we can choose $c > 0$ and $d > 0$ such that the inductive proof goes through. This confirms that $T(n) \leq cn - d$ for all $n \geq 1$, implying (since $d > 0$) that also $T(n) \leq cn$ for all $n \geq 1$, and thus the desired $T(n) \in O(n)$.