CIS 575. Introduction to Algorithm Analysis
Material for February 9, 2024

The Master Theorem: One Version

This note corresponds to material covered in *Cormen*'s Section 4.5.

1 A Master Theorem

In the previous note, we considered a general recurrence (with q a non-negative real)

$$T(n) = aT(\frac{n}{b}) + n^q$$

and through calculations based on repeated unfoldings we claimed that

1. if $b^q > a$ then $T(n) \in \Theta(n^q)$
2. if $b^q = a$ then $T(n) \in \Theta(n^q \ \log_b(n))$
3. if $b^q < a$ then $T(n) \in \Theta(n^{\log_b(a)})$.

With $\mathbf{r} = \log_{\mathbf{b}}(\mathbf{a})$, we can (applying $\log_b$ to each side in the comparison) rephrase that into

1. if $q > r$ then $T(n) \in \Theta(n^q)$
2. if $q = r$ then $T(n) \in \Theta(n^q \ \log_b(n))$
3. if $q < r$ then $T(n) \in \Theta(n^r)$.

We see that the solution is a polynomial whose degree is the maximum of q and r, except that if $q = r$ we multiply by $\lg(n)$ (observe that the base of the logarithm does not matter since $\log_b(n) \in \Theta(\lg(n))$ for all $b > 1$).
We have motivated a key result, commonly known (in various versions) as the "Master Theorem".

Master Theorem Consider a recurrence of the form

$$T(n) = aT(\frac{n}{b}) + f(n) \text{ for } n \geq b$$

where

- b is an integer with $b \geq 2$

- either $\lfloor\ \rfloor$ ("floor") or $\lceil\ \rceil$ ("ceiling") (the choice doesn't matter) is wrapped around $\frac{n}{b}$
- $T(n)$ is positive ($T(n) > 0$ for $n \geq 1$), and eventually non-decreasing (that is, there exists m such that $T(n) \leq T(n+1)$ for all $n \geq m$).

Let $\mathbf{r} = \log_\mathbf{b} \mathbf{a}$. Then, for $X \in \{O, \Omega, \Theta\}$, we have (where q is a non-negative real number)

1. if $\mathbf{f(n) \in X(n^q)}$ with $\mathbf{q > r}$ then $\mathbf{T(n) \in X(n^q)}$
2. if $\mathbf{f(n) \in X(n^q)}$ with $\mathbf{q = r}$ then $\mathbf{T(n) \in X(n^r \lg n)}$
3. if $\mathbf{f(n) \in O(n^q)}$ with $\mathbf{q < r}$ then $\mathbf{T(n) \in \Theta(n^r)}$.

Observe that to get an upper bound for $T(n)$ we need an upper bound for $f(n)$, and that to get a lower bound for $T(n)$ we need a lower bound for $f(n)$ except in the last case where the lower bound is given by the recursive call structure.

Applications Let us apply the Master Theorem to the three recurrences for which we guessed a solution which we verified using the substitution method; all of them have $a = b = 2$ and thus $r = 1$.

1. for $T(n) = 2\,T(\lfloor\frac{n}{2}\rfloor) + n^2$ we have $q = 2$; thus case 1 applies, giving us $T(n) \in \Theta(n^2)$ (we verified $T(n) \in O(n^2)$).
2. for $T(n) = 2\,T(\lfloor\frac{n}{2}\rfloor) + n$ we have $q = 1$; thus case 2 applies, giving us $T(n) \in \Theta(n \lg(n))$ (we verified $T(n) \in O(n \lg(n))$)
3. for $T(n) = 2\,T(\lfloor\frac{n}{2}\rfloor) + 1$ we have $q = 0$; thus case 3 applies, giving us $T(n) \in \Theta(n)$ (we verified $T(n) \in O(n)$).

In principle, to apply the Master Theorem we should first verify that $T(n)$ is eventually non-decreasing, but that is quite straightforward (a simple induction in n) and in general we shall not bother about that requirement.

1.1 Relating to Other Versions

The literature contains many versions of the Master Theorem, for example:

- **Cormen**'s Theorem 4.1 (p.102)
- **Howell**'s Theorem 3.32
- `https://en.wikipedia.org/wiki/Master_theorem_(analysis_of_algorithms)`

They are all quite similar, with some minor differences, such as:

- *Cormen* and *Wikipedia* allow b to be a real number > 1
- *Cormen* and *Wikipedia* list the 3 cases in reverse order compared to our version.

In all three versions, the second case (with $q = r$) is stated so as to be more generally applicable (as we shall illustrate in the next note) than the version presented in these notes (which is based on my desire to have a widely applicable theorem with a proof that I fully understand).