# CIS 575. Introduction to Algorithm Analysis
# Material for February 9, 2024

# The Master Theorem: Gaps Between the Cases



The topic of this note is mentioned in *Cormen*'s Section 4.5.

## 1 When $q$ is Slightly Greater Than $r$

Let us consider the recurrence

$$T(n) = 2T(\frac{n}{2}) + n\lg(n) \tag{1}$$

and investigate, using various versions of the Master Theorem, what we can say about the asymptotic behavior of $T(n)$. For all versions, we have $a = b = 2$ and thus $r = \log_b(a) = 1$. Our version of the Master Theorem states, for $X \in \{O, \Omega, \Theta\}$, that

1. if $\mathbf{n}\lg(\mathbf{n}) \in \mathbf{X}(\mathbf{n^q})$ with $\mathbf{q} > \mathbf{1}$ then $\mathbf{T}(\mathbf{n}) \in \mathbf{X}(\mathbf{n^q})$
2. if $\mathbf{n}\lg(\mathbf{n}) \in \mathbf{X}(\mathbf{n})$ then $\mathbf{T}(\mathbf{n}) \in \mathbf{X}(\mathbf{n}\lg\mathbf{n})$

By applying 1 with $X = O$ (possible since any root grows asymptotically faster than any logarithm) we get $T(n) \in O(n^q)$ for all $q > 1$, and by applying 2 with $X = \Omega$ we get $T(n) \in \Omega(n\lg(n))$ (which by the way is obvious from the recurrence). Thus

> for all real $q > 1$, $T(n)$ is sandwiched between $n\lg(n)$ and $n^q$

which is a relatively precise description of the asymptotic behavior of $T(n)$, but we would like an *exact* description (using $\Theta$ notation). For that purpose, let us examine some other versions of the Master Theorem.

**Cormen**'s version (Theorem 4.1) and **Wikipedia**'s version (which uses $c_{\text{crit}}$ for $\log_b(a)$) both have a case 2 that states that

$$\text{if } \mathbf{f}(\mathbf{n}) \in \boldsymbol{\Theta}(\mathbf{n^r}\lg^{\mathbf{k}}(\mathbf{n})) \text{ then } \mathbf{T}(\mathbf{n}) \in \boldsymbol{\Theta}(\mathbf{n^r}\lg^{\mathbf{k+1}}\mathbf{n})$$

which (with $r = 1$ and $k = 1$) shows that recurrence (1) has solution

$$T(n) \in \Theta(n\lg(n)\lg(n)).$$

## 2 When $q$ is Slightly Smaller Than $r$

Next let us consider the recurrence

$$T(n) = 2T(\frac{n}{2}) + \frac{n}{\lg(n)} \tag{2}$$

and investigate, using various versions of the Master Theorem, what we can say about the asymptotic behavior of $T(n)$. For all versions, we have $a = b = 2$ and thus $r = \log_b(a) = 1$. Our version of the Master Theorem states, for $X \in \{O, \Omega, \Theta\}$, that

1. if $\mathbf{\frac{n}{\lg n} \in X(n^q)}$ with $\mathbf{q > 1}$ then $\mathbf{T(n) \in X(n^q)}$
2. if $\mathbf{\frac{n}{\lg n} \in X(n)}$ then $\mathbf{T(n) \in X(n \lg n)}$
3. if $\mathbf{\frac{n}{\lg n} \in O(n^q)}$ with $\mathbf{q < 1}$ then $\mathbf{T(n) \in \Theta(n)}$.

While applying 3 is impossible, we can apply 2 with $X = O$ to get $T(n) \in O(n \lg(n))$; as obviously $T(n) \in \Omega(\frac{n}{\lg(n)})$ we see that

$$T(n) \text{ is sandwiched between } \frac{n}{\lg(n)} \text{ and } n \lg(n)$$

which is not a very precise description of the asymptotic behavior of $T(n)$. Unfortunately, it appears that **Cormen**'s version is not even applicable to this recurrence (as stated on p. 105 of that book). On the other hand, the *extended version* (2b) of Wikipedia's version (as of the time of writing these notes) tells us that recurrence (2) has solution

$$T(n) \in \Theta(n \lg(\lg(n))).$$