CIS 575. Introduction to Algorithm Analysis
Material for March 25, 2024

Reflections on Optimality of Solutions to Subproblems

The topic of this note is mentioned in *Cormen*'s Section 14.3.

1 Optimality of SubSolutions

We shall now show that not all problems can be solved by combining solutions to subproblems.

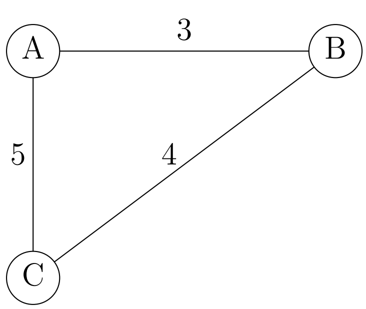
When presenting Floyd's algorithm (and later when presenting Dijkstra's algorithm), we talk about finding the **shortest paths** in a directed graph where edges have positive distances. To keep things simple, in the following we shall consider **un**directed graphs. Let $SP(X, Y)$ denote the length of a shortest path from X to Y. For all A, B, C, we would certainly expect that if a shortest path from A to C goes thru B then

$$SP(A, C) = SP(A, B) + SP(B, C)$$

Now let us look at **longest paths**, where we better restrict ourselves to **simple paths**, that is paths where each node is visited once only, since otherwise we can construct arbitrarily long paths. Let $LP(X, Y)$ denote the length of a longest simple path from X to Y. Then we *may* expect that if a longest simple path from A to C goes thru B then

$$LP(A, C) = LP(A, B) + LP(B, C)$$

But this is obviously not the case, as can be seen from the graph



where we have

$$\begin{aligned} LP(A, B) &= 9 \\ LP(A, C) &= 7 \end{aligned}$$

$$
\begin{aligned}
LP(B,C) &= 8 \\
LP(A,C) &\neq LP(A,B) + LP(B,C)
\end{aligned}
$$

Why is the *longest simple* path different from the *shortest* path? Observe that

- Assume that π is a shortest path from A to C, and that we can write $\pi = \pi_1\pi_2$ where π_1 is a path from A to B, and π_2 is a path from B to C. Then $\boldsymbol{\pi_1}$ *must be* a **shortest path** from A to B, for if π_1' were a shorter path from A to B, then $\pi_1'\pi_2$ would be a path from A to C which is shorter than π, which is a contradiction. Similarly, $\boldsymbol{\pi_2}$ *must be* a **shortest path** from B to C.

- Assume that π is a longest simple path from A to C, and that we can write $\pi = \pi_1\pi_2$ where π_1 is a path from A to B, and π_2 is a path from B to C. Then we can **not** infer that $\boldsymbol{\pi_1}$ is a longest simple path from A to B, as there could easily be a longer simple path π_1' from A to B — for even though $\pi_1'\pi_2$ would be a path from A to C longer than π, it would not necessarily be simple, and hence not yield a contradiction.

For many problems, we have made a tacit assumption:

> the optimal solution for a given problem
> can be compiled from optimal solutions for certain subproblems

This assumption does indeed hold for many problems; not just *shortest path* but also for say *binary knapsack*:

- if $x_1 \ldots x_{i-1}1$ is the best way to select among items $1 \ldots i$ such that the total weight is at most w

- then the restriction $x_1 \ldots x_{i-1}$ is the the best way to select among items $1 \ldots i-1$ such that the total weight is at most $w - w_i$

But we have seen that the assumption does *not* hold for *longest simple path* which hence can not be *immediately* computed by dynamic programming.