Lecture 2: Protoplanetary discs I

1 Disc formation and the angular momentum problem

Obervations of both the solar system and exo-planetary systems suggest that planets form in discs around young stars, and in order to understand how planets form it is first necessary to consider how stars form. Naively we assume that stars are able to form by simple gravitational collapse of gas clouds, but we must also consider the effects of rotation. "Cores" in star-forming molecular clouds are observed to have angular velocities of order $\Omega_c \sim 10^{-14}$–$10^{-13}\mathrm{s}^{-1}$, and we can thus compute the angular momentum of a core by appealing to the Jeans length

$$R_J \simeq \frac{c_s}{\sqrt{G\rho}} \tag{1}$$

and Jeans mass

$$M_J \simeq \frac{c_s^3}{G^{3/2}\rho^{1/2}} . \tag{2}$$

(Here we have neglected order-of-unity constants for clarity.) Molecular clouds have temperatures $T \simeq 10\mathrm{K}$, yielding sound speeds of order $c_s \simeq 0.2\mathrm{km/s}$. We therefore see that forming a star of solar mass by Jeans collapse requires densities that exceed $\rho \simeq 10^{-19}\mathrm{g/cm}^3$, and requires that material fall inwards from distances of order $R_J \simeq 0.1\mathrm{pc}$. The specific angular momentum of the collapasing core is thus

$$j_c \simeq \Omega_c R_J^2 \simeq 10^{21} - 10^{22}\mathrm{cm}^2/\mathrm{s} . \tag{3}$$

By contrast, the break-up velocity of a star (the maximum velocity at which it can rotate) can be computed by equating centrifugal acceleration with gravity thus

$$\Omega_b^2 r_* = \frac{GM_*}{r_*^2} \tag{4}$$

$$\Omega_b = \sqrt{\frac{GM_*}{r_*^3}} . \tag{5}$$

A star like the Sun therefore has a break-up velocity $\Omega_b \sim 10^{-3}\mathrm{s}^{-1}$ (corresponding to a few hundred km/s), and a break-up specific angular momentum (assuming solid-body rotation) of

$$j_b \simeq \Omega_b r_*^2 \simeq 10^{18} - 10^{19}\mathrm{cm}^2/\mathrm{s} . \tag{6}$$

Thus $j_b \ll j_c$, and most stars in fact rotate well below break-up. We see therefore that young stars have much lower angular momenta than the gas clouds from which they form, and how this angular momentum is lost is the so-called "angular momentum problem" of star formation.

In Lecture 1 we discussed various reasons for believing that the Solar System formed from a single rotating disc, and we can similarly appeal to discs as a solution to the angular momentum problem of star formation. If we assume that discs around young stars are in Keplerian rotation, we can estimate their typical size from the angular momentum of the system. In a Keplerian orbit the specific angular momentum of a mass orbiting a star of mass M_* at radius R is

$$j_K = \sqrt{GM_* R} \tag{7}$$

and if we set $j_K = j_c$ we find that protostellar discs around solar-mass stars should have typical sizes of

$$R = \frac{j_c^2}{GM_*} \sim 10^3 - 10^4\mathrm{AU} . \tag{8}$$

Star formation is in fact much more dynamic than the process we have described here, and interactions between protostars and their surroundings can redistribute some of the excess angular momentum. Nevertheless, when we observe young stars we see resolved discs with sizes of 100–1000AU, and numerical simulations of star formation typically produce discs of similar sizes. Thus, disc accretion is crucial to the star formation process, and planet-forming discs are an inevitable consequence of star formation.

2 Observations of protoplanetary discs

The first protoplanetary discs were observed in the 1980s, and since then we have amassed a huge body of research into their structure and evolution. The subject of protoplanetary disc observations is large enough to form an entire lecture course by itself, so here we merely summarise the most relevant points. The bulk of the disc mass is gas (mostly molecular hydrogen), but the trace dust component dominates the opacity. The dust is therefore crucial for disc thermodynamics (and for planet formation), and despite representing only around 1% of the total mass the dust is also much easier to observe than the gas.

Young, low-mass stars are traditionally classified by the shape of their infrared (IR) spectral energy distribution (SED), which is typically measured through broad-band photometry (allowing large numbers of objects to be observed simultaneously). The SED classification scheme, originally proposed by Lada (1987) and subsequently updated several times, is based on the IR spectral index

$$\alpha_{\rm IR} = \frac{d\log(\lambda F_\lambda)}{d\log\lambda} \, . \tag{9}$$

In practice $\alpha_{\rm IR}$ is usually measured between two fixed wavelengths: early work used 2.2μm (K-band) and 10μm, but more recent studies generally use one of the *Spitzer* bands for the long-wavelength point (usually 24μm). Light from the central (proto-)star is absorbed by circumstellar dust and re-emitted at longer wavelengths, so a "redder" SED, resulting from more material in the circumstellar environment, is thought to be indicative of an earlier evolutionary phase. The modern classification scheme is as follows:

- **Class 0:** SED peaks in the far-IR or sub-mm, with no measurable flux being emitted in the near- or mid-IR. These objects are typically interpreted as proto-stars which are still in the collapse phase.
- **Class I:** SED peaks in the mid- or far-IR, with a rising SED slope ($\alpha_{\rm IR} \gtrsim 0$) in the near-IR. These objects are inferred to be embedded protostellar discs with substantial circumstellar envelopes.
- **Class II:** SED is declining in the near- and mid-IR, but shows significant excess emission over the stellar photosphere ($-1.5 \lesssim \alpha_{\rm IR} \lesssim 0$). These are optically-visible pre-main-sequence stars with a surrounding disc, but little or no remaining envelope.
- **Class III:** SED is effectively that of a stellar photosphere, with $\alpha_{\rm IR} \sim -1.5$. These are pre-main-sequence stars which have lost their discs.

Additional sub-classes exist (notably "flat spectrum" sources between Classes I & II, and "transitional discs" between Classes II & III), but few such objects are known. The relationship between the SED classification scheme and the physical state of the YSO is now supported by a large body of evidence, and although there is not a perfect one-to-one correspondence between SED class and evolution the term "Class X" is commonly used to refer to the physical state of the object. For the purposes of this course we are most interested in Class II & III objects, but we will also discuss

Class I objects. Note also that Classes II & III correspond almost perfectly with an older classification scheme, based on the strength of optical emission lines: objects with Hα equivalent widths $\gtrsim 10$Å are referred to as Classical T Tauri stars (CTTs), while objects with Hα equivalent widths $\lesssim 10$Å are referred to as Weak-lined T Tauri stars (WTTs). We now understand that the observed emission lines are primarily due to accretion on to the stellar surface: almost all disc-bearing Class II sources are CTTs, while disc-less Class III are usually WTTs.

Observations of dust continuum emission are the most straightforward means of observing protoplanetary discs, and we now have large statistical samples across a broad range of wavelengths. Near- and mid-IR emission comes from the inner disc, at radii $\lesssim 1$AU, where the disc is optically thick, while longer wavelength emission primarily comes from the colder outer disc. At mm wavelengths the emission is optically thin, and (sub-)mm observations allow us to measure the total (dust) mass in the disc. Estimating disc masses in this manner is subject to significant uncertainties (particularly in the dust-to-gas ratio), but standard assumptions derive disc masses which range from $\gtrsim 0.1\mathrm{M}_\odot$to $\lesssim 0.001\mathrm{M}_\odot$. The median disc mass (for Class I & II sources) is approximately 1% of the stellar mass (i.e., $\sim$ 5–10$\mathrm{M}_{\mathrm{Jup}}$).

Observing the gas component of the disc is more difficult, due to the fact that H_2 has no bright emission lines (because the hydrogen molecule has no permanent dipole). In general we are limited to detecting emission from the hot disc surface, and/or detecting emission from (trace) heavier elements and molecules (notably CO). Our only means of detecting the bulk of the disc mass lies in observing accretion on to the stellar surface, and typical accretion rates[1] for Class II sources lie in the range 10^{-7}–$10^{-9}\mathrm{M}_\odot\mathrm{yr}^{-1}$. The fact that discs are observed to accrete tells us that they must evolve: we will discuss accretion and disc evolution in more detail in the next lecture.

These observations also tell us that essentially all young stars form with discs. The disc fraction in the youngest ($\lesssim$ 1Myr) star clusters is close to 100%, but the number of discs drops rapidly with time and in $\sim$ 10Myr-old clusters almost no discs remain. This result holds across the full range of disc signatures, and implies that typical disc lifetimes are a few Myr. This in turn sets a strict limit on the process(es) of planet formation: by 10Myr T Tauri stars have insufficient gas to form even Neptune mass planets, so giant planets must be able to form within the few Myr lifetime of protoplanetary discs.

3 Protoplanetary disc structure

The equation of motion for an inviscid, non-magnetised fluid is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla\Phi\,, \tag{10}$$

where $\mathbf{v}$ is the fluid velocity, ρ the density, P the pressure and Φ the gravitational potential. We consider a disc as a stationary, axisymemetric flow around a central gravitating mass, and therefore work in cylindrical co-ordinates (R, z, ϕ)[2]. In the limit of a low-mass disc the potential is simply $\Phi = -GM_*/r$, and the radial component of the equation of motion is an expression of centrifugal balance

$$\frac{v_\phi^2}{R} = \frac{1}{\rho}\frac{dP}{dR} + \frac{GM_*}{R^2}\,. \tag{11}$$

If we neglect the gas pressure, we find that the azimuthal velocity v_ϕ is simply the Keplerian velocity $v_{\mathrm{K}} = \sqrt{GM_*/R}$.

In general, the pressure in a disc decreases outwards (as both the surface density and temperature are typically decreasing functions of radius). The dP/dR term in Equation 11 is therefore

[1]Note, however, that the accretion is highly variable, especially during the earlier evolutionary phases.

[2]Note that I use lower-case r for spherical radius and upper-case R for cylindrical radius.

negative, so the orbital velocity of the gas is sub-Keplerian. How sub-Keplerian the gas is depends on radial temperature and density structure of the disc, but for typical parameters we find that $v_\phi/v_\mathrm{K} \simeq 0.995$. This is negligible for gas dynamics, but has important consequences for solid bodies in the disc (as we will see in Lecture 4).

The vertical component of Equation 10 is a statement of hydrostatic balance

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{d\Phi}{dz} = \frac{d}{dz}\left(\frac{GM_*}{r}\right) = \frac{d}{dz}\left(\frac{GM_*}{(R^2+z^2)^{1/2}}\right) , \tag{12}$$

with pressure supporting the disc against gravity. In the limit of a thin disc ($z \ll R$)[3], this reduces to

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GM_* z}{R^3} = -\Omega_\mathrm{K}^2 z , \tag{13}$$

where $\Omega_\mathrm{K} = v_\mathrm{K}/R$ is the Keplerian orbital frequency. If we then assume that the disc is vertically isothermal, the equation of state is $P = c_s^2\rho$ and we have

$$\frac{1}{\rho}\frac{d\rho}{dz} = \frac{d\log\rho}{dz} = -\frac{z}{H^2} , \tag{14}$$

where we have defined the disc scale-height $H = c_s/\Omega$. We can integrate to find the vertical structure

$$\rho(z) = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right) . \tag{15}$$

The disc therefore has a Gaussian density profile in the vertical direction, with scale-height H. (68% of the disc mass lies within $\pm H$ of the midplane.) The midplane density ρ_0 is related to the local surface density Σ by the normalisation condition

$$\rho_0 = \frac{\Sigma}{\sqrt{2\pi}H} . \tag{16}$$

The disc structure $H(R)$ is primarily determined by the radial temperature profile of the disc $T(R)$. The low accretion rates in protoplanetary discs mean that accretion (viscous) heating is usually negligible, and the disc's heating is instead dominated by irradiation from the central star[4]. For a razor-thin disc, we can compute the radial temperature profile by considering the flux absorbed by a patch of the disc at radius r. The star has radius r_* and effective temperature T_*, so if we assume a constant surface brightness we have $I_* = (1/\pi)\sigma_\mathrm{SB}T_*^4$ (where σ_SB is the Stefan-Boltzmann constant). The flux F absorbed by the disc is simply the integral of the brightness over the fraction of the stellar surface "seen" by the disc, so

$$F = \int I_* \sin\theta\cos\phi d\Omega , \tag{17}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the (infinitesimal) solid angle element. If we consider only one hemisphere of the star (and therefore only the flux absorbed by one surface of the disc), we see that the limits on the integral are $-\pi/2 \le \phi \le \pi/2$ and $0 \le \theta \le \sin^{-1}(r_*/R)$. Substituting, we find that

$$F = I_* \int_{-\pi/2}^{\pi/2} \cos\phi d\phi \int_0^{\sin^{-1}(r_*/R)} \sin^2\theta d\theta . \tag{18}$$

[3]In deriving Equations 11 & 12 we have formally assumed that the radial gas velocity $v_\mathrm{R} \ll c_s \ll v_\phi$. The first inequality requires that any radial gas flow (i.e., accretion) be very sub-sonic, while the second inequality is essentially a re-statement of the thin disc approximation (and also implies that the pressure term in Equation 11 is small.) Observations of disc aspect ratios and accretion rates confirm that both of these approximations are justified for protoplanetary discs.

[4]Accretion heating dominates at small radii and high accretion rates, $\gtrsim 10^{-7}\mathrm{M}_\odot\mathrm{yr}^{-1}$. These conditions are only satisfied in the inner disc ($\lesssim$ 1AU) and in the early stages of disc evolution.

This integral is ugly but basically straightforward [use Pythagoras to evaluate $\cos(\sin^{-1} x)$], and we find that

$$F = I_* \left[\sin^{-1}(r_*/R) - \left(\frac{r_*}{R}\right)\sqrt{1 - \left(\frac{r_*}{R}\right)^2} \right] . \tag{19}$$

In thermodynamic equilibrium this absorbed flux is equal to the flux radiated by the disc. If the disc has local temperature $T(r)$ then $F = \sigma_{\rm SB} T^4$ and

$$\left(\frac{T(R)}{T_*}\right)^4 = \frac{1}{\pi} \left[\sin^{-1}(r_*/R) - \left(\frac{r_*}{R}\right)\sqrt{1 - \left(\frac{r_*}{R}\right)^2} \right] . \tag{20}$$

This expression is not particularly intuitive, but if we expand as a Taylor series[5] in the far-field limit $r_*/R \ll 1$, we find that

$$T(R) \propto R^{-3/4} . \tag{21}$$

This is the radial temperature profile for a flat, optically-thick reprocessing disc[6], by assuming that $c_s^2 \propto T$ we find that $H/R \propto R^{1/8}$. This solution is therefore not self-consistent, as the disc is not flat (the aspect ratio H/R is an increasing function of radius). Self-consistent solutions must take account the relationship between temperature and disc thickness, and also the fact that the disc sub-tends a larger solid angle at larger radii. (This is most readily done by defining a flaring angle $\alpha = d/dR(H/R)$, and modifying Equation 17 accordingly.)

We can use the Planck function and integrate Equation 19 to find the disc SED (we find $\lambda F_\lambda \propto \lambda^{-4/3}$), but it was recognised more than 20 years ago that a flat reprocessing disc produces substantially less IR emission than is typically observed in Class II discs[7]. We now understand that protoplanetary discs are flared, with H/R increasing significantly with radius: the disc is relatively thicker at larger radii, and thus intercepts (and re-emits) a larger fraction of the stellar flux than a thin disc (leading to a larger IR excess). For a vertically isothermal disc the self-consistent solution is $T \propto R^{-1/2}$, which gives $H/R \propto R^{5/4}$: the temperature profile is much flatter than that of a flat disc, and the disc flares substantially with radius. Modern disc models relax the assumption of vertical isothermality (see, e.g., the "two-layer" model of Chiang & Goldreich 1997), and include additional complications such as accretion heating and realistic opacities. Detailed study of disc structure remains an active area of research.

[5]The relevant Taylor series expansion is $\sin^{-1} x = x + x^3/6 + x^5/40\ldots$

[6]Coincidentally, this power-law scaling is the same as that found in a self-luminous accretion disc.

[7]The flat disc solution for λF_λ has $\alpha_{\rm IR} = -4/3$, which is close to the Class II/III boundary and much steeper ("bluer") than is typical for Class II objects.

4 Further Reading

In addition to the main list of references given on the course home-page, the following papers are particularly relevant to this lecture:

Lodato, *Classical disc physics*, 2008, NewAR, 52, 21.
Alexander, *From discs to planetesimals: Evolution of gas and dust discs*, 2008, NewAR, 52, 60.
Williams & Cieza, *Protoplanetary Disks and Their Evolution*, 2011, ARA&A, in press (`arXiv:1103.0556`).
Armitage, *Dynamics of protoplanetary disks*, 2011, ARA&A, in press (`arXiv:1011.1496`).
Dullemond et al., *Models of the Structure and Evolution of Protoplanetary Disks*, 2007, Protostars & Planets V, p555.
Calvet & Natta, *Protoplanetary Disk Structure and Evolution*, 2011, in "Physical Processes in Circumstellar Disks around Young Stars". (This book contains a number of excellent review articles, and should be very useful. A copy is finally coming my way, a mere 4 1/2 years after the conference was held, so let me know if you would like to borrow it.)
Natta et al., *Dust in Protoplanetary Disks: Properties and Evolution*, 2007, Protostars & Planets V, p767.
Chiang & Goldreich, *Spectral Energy Distributions of T Tauri Stars with Passive Circumstellar Disks*, 1997, ApJ, 490, 368.
Lada, *Star formation - From OB associations to protostars*, 1987, IAU Symposium 115, p1.