# Drivers in the Case of Two Assets

# Future Returns Are Random In Nature

**Optimizing Portfolio requires expectations:**

- about average portfolio return (mean)
- about how far off it may be (variance)

**Why?**

**Portfolio Return Is A Random Variable**

# Past Performance to Predictions

| | Mean Portfolio Return |
|---|---|
| Computed on a sample of T Historical Returns | $\hat{\mu} = \frac{R_1 + R_2 + \ldots + R_T}{T}$ |
| When the return is a random variable | $\mu = E[R]$ |

| | Portfolio Return Variance |
|---|---|
| Computed on a sample of T Historical Returns | $\hat{\sigma}^2 = \frac{(R_1 - \hat{\mu})^2 + (R_2 - \hat{\mu})^2 + \ldots + (R_T - \hat{\mu})^2}{T - 1}$ |
| When the return is a random variable | $\sigma^2 = E[(R - \mu)^2]$ |

# Drivers of Mean & Variance

- Assume two assets:

| Asset 1 | Asset 2 |
|---|---|
| Weight: $w_1$ | Weight: $w_2$ |
| Return: $R_1$ | Return: $R_2$ |

- **Portfolio Return** $P = w_1 * R_1 + w_2 * R_2$
- Thus: $\boxed{E[P] = w_1 * E[R_1] + w_2 * E[R_2]}$

# Portfolio Return Variance

Again, for a portfolio with 2 assets

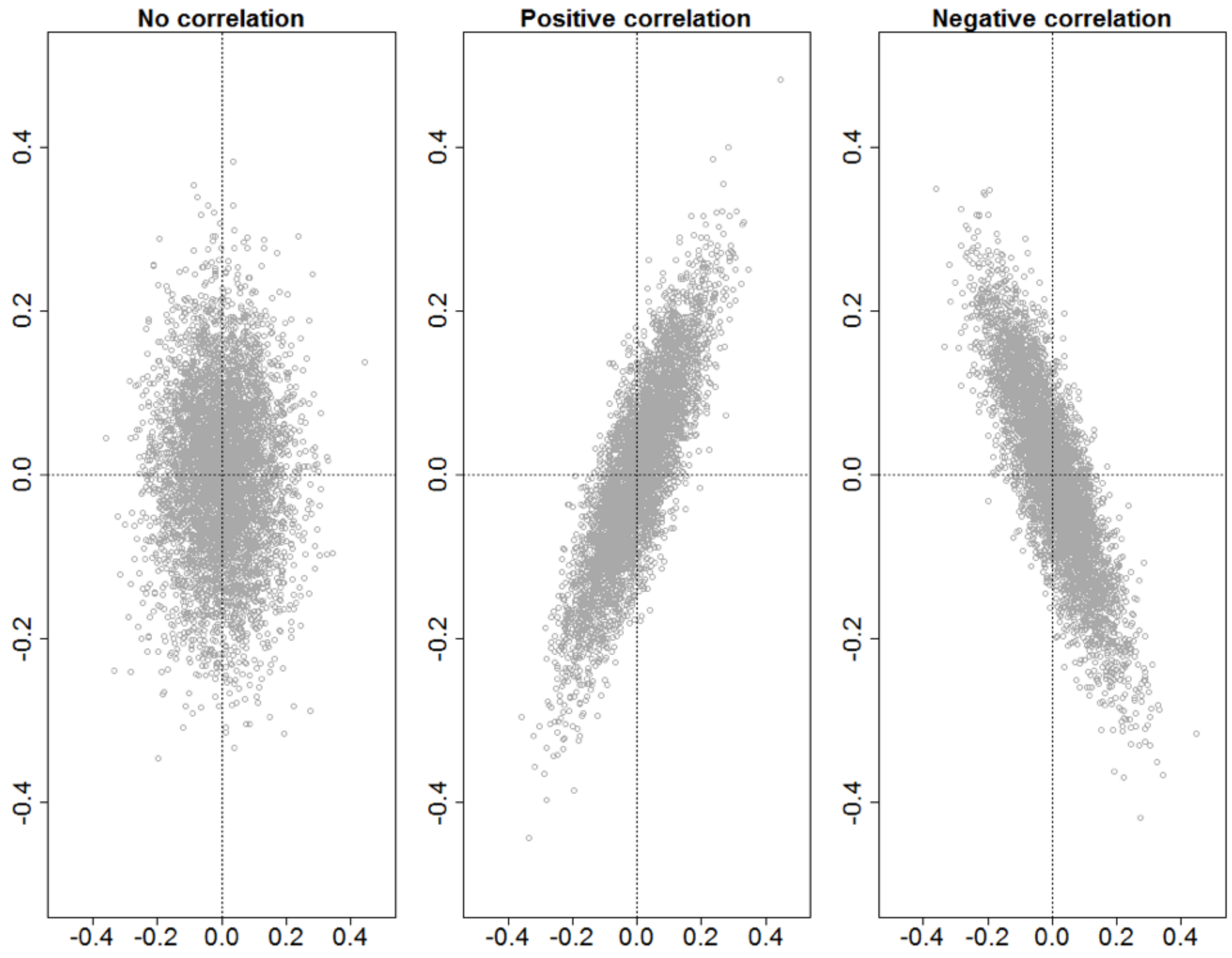
## Variance of Portfolio Return

$$var(P) = \boxed{E[(P - E[P])^2]} = w_1^2 * var(R_1) + w_2^2 * var(R_1) + 2 * w_1 * w_2 * \boxed{cov(R_1, R_2)}$$

## Covariance between return 1 and 2

$$Cov(R_1, R_2) = \boxed{E[(R_1 - E[R_1])(R_2 - E(R_2))]} = StdDev(R_1) * StdDev(R_1) * corr(R_1, R_2)$$

# Correlations



# Take Away Formulas

- E[Portfolio Return] = $E(P) = w_1 * E[R_1] + w_2 * E[R_2]$
- var(Portfolio Return) = $$\begin{aligned} var(P) = \ & w_1^2 * var(R_1) \\ & + w_2^2 * var(R_1) \\ & + 2 * w_1 * w_2 * cov(R_1, R_2) \end{aligned}$$

# Let's practice!

# Using Matrix Notation

# Variables at Stake for N Assets

- w: the N x 1 column-matrix of portfolio weights

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

- R: the N x 1 column-matrix of asset returns

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix}$$

- μ: the N x 1 column-matrix of expected returns

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

# Variables at Stake for N Assets

- $\Sigma$: The N x N covariance matrix of the N asset returns:

$$\begin{array}{cc} & \sigma_i^2 \qquad \sigma_{ij} \\ \Sigma = & \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{bmatrix} \end{array}$$

Covariance: Outside Diagonal
Variance: On Diagonal

# Generalizing from 2 to N Assets

## Portfolio Return

$$\boxed{w_1 * R_1 + w_2 * R_2} \Rightarrow \boxed{w_1 * R_1 + \ldots + w_N * R_N}$$

## Portfolio Expected Return

$$\boxed{w_1 * \mu_1 + w_2 * \mu_2} \Rightarrow \boxed{w_1 * \mu_1 + \ldots + w_N * \mu_N}$$

## Portfolio Variance

$$\boxed{\begin{array}{c} w_1^2 * var(R_1) + w_2^2 * var(R_1) \\ +2 * w_1 * w_2 * cov(R_1, R_2) \end{array}} \Rightarrow \boxed{\begin{array}{l} w_1^2 + var(R_1) + \ldots + w_N^2 * var(R_N) \\ +2 * w_1 * w_2 * cov(R_1, R_2) + \ldots \\ +2 * w_{N-1} * w_N * cov(R_{N-1}, R_N) \end{array}}$$

# Matrices Simplify the Notation

- Avoid large number of terms by using matrix notation
- We have 4 matrices:
    - weights (w), returns (R), expected returns (μ), and covariance matrix (Σ)

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \qquad w' = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix}$$

# Simplifying the Notation

## Portfolio Return

$$w_1 * R_1 + \ldots + w_N * R_N \Rightarrow w'R$$

## Portfolio Expected Return

$$w_1 * \mu_1 + \ldots + w_N * \mu_N \Rightarrow w'\mu$$

## Portfolio Variance

$$\begin{aligned} &w_1^2 + var(R_1) + \ldots + w_N^2 * var(R_N) \\ &+2 * w_1 * w_2 * cov(R_1, R_2) + \ldots \\ &+2 * w_{N-1} * w_N * cov(R_{N-1}, R_N) \end{aligned} \Rightarrow w'\Sigma w$$

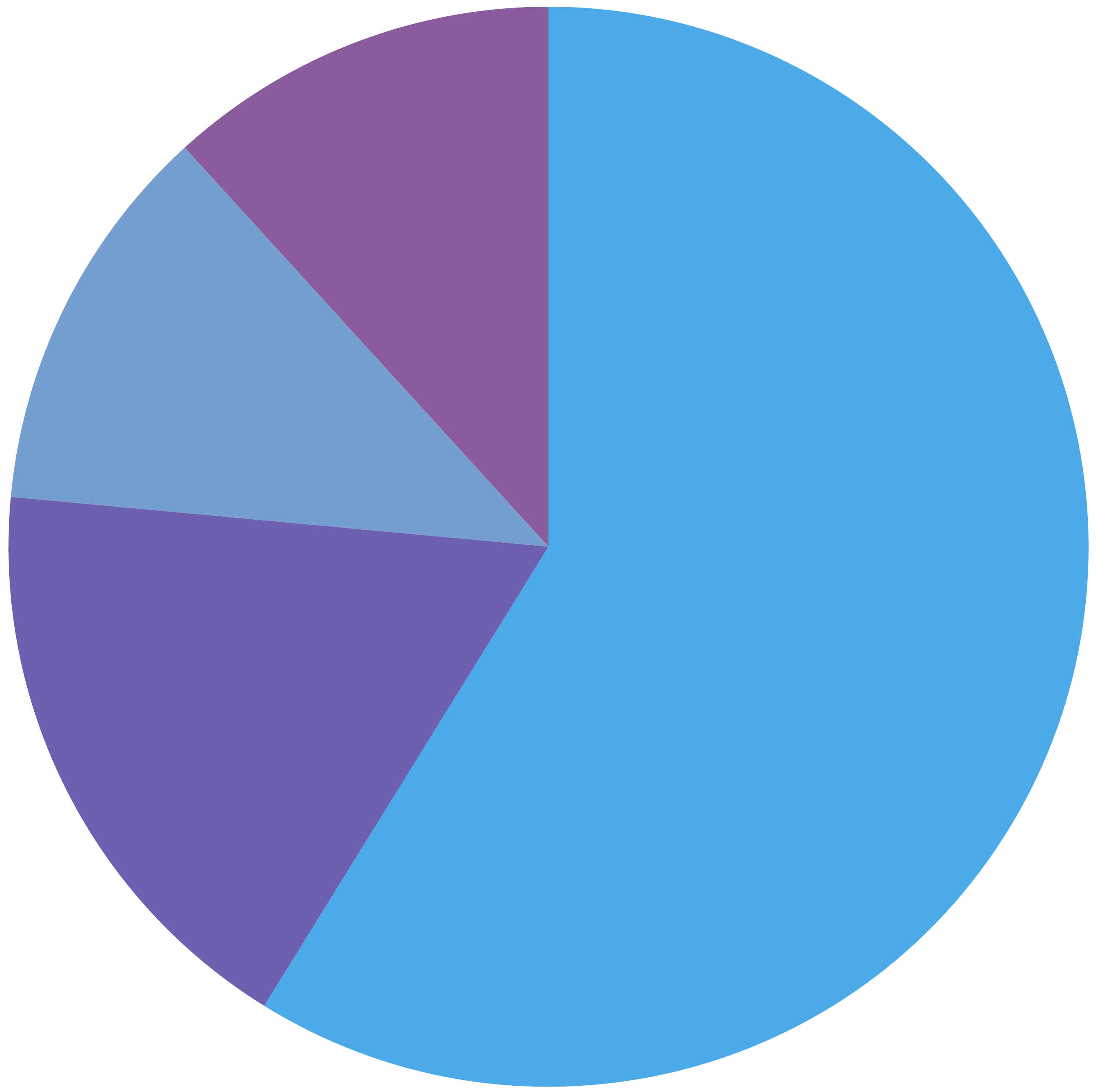
# Let's practice!

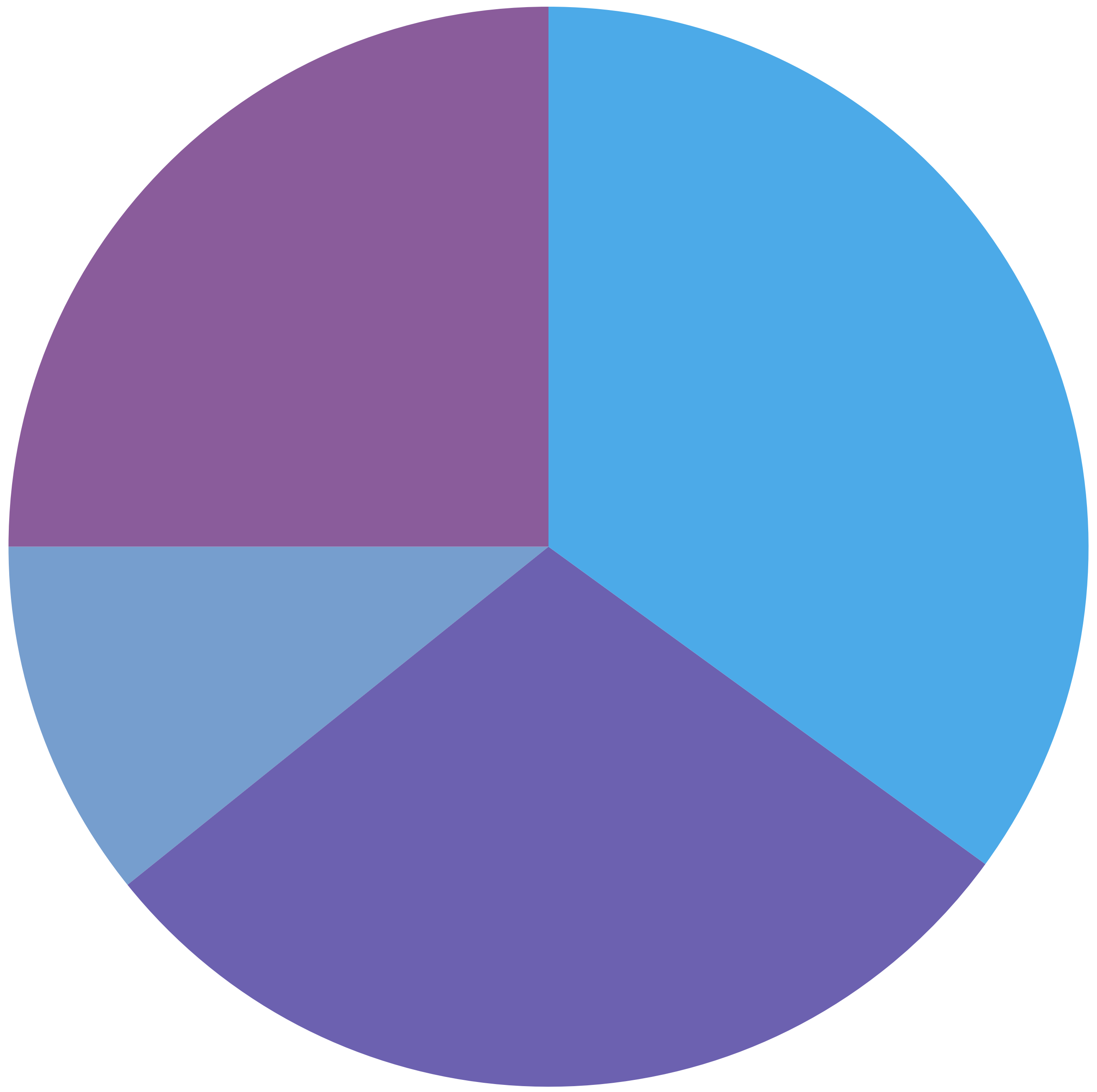
# Portfolio Risk Budget

# Who Did It?

Capital Allocation Budget

Portfolio Volatility Risk





# Portfolio Volatility In Risk Contribution

- Portfolio Volatility = $\sum_{i=1}^{N} RC_i$
  - Where: $RC_i = \frac{w_i(\sum w)_i}{\sqrt{w' \sum w}}$
- risk contribution of asset *i* depends on
  1. the complete matrix of weights $w$
  2. the full covariance matrix $\sum$

# Percent Risk Contribution

$$\boxed{\%RC_i = \frac{RC_i}{\text{portfolio volatility}}}$$

where $\sum_{i=1}^{N} \%RC_i = 1$

Relatively more risky assets: $\%RC_i > w_i$

Relatively less risky assets: $\%RC_i < w_i$

# Let's practice!