수학으로부터 **인류**를 **자유**롭게 하라

Free Humankind from Mathematics

Basic Algebra

Chap.2 Sets

Data Structures in Math

(1) Sets

(2) Sequences

(3) Vectors

(4) Matrices

(3) Trees

⋮

Characteristics of Data Structures

(1) Sets: unordered objects

(2) Sequences: ordered objects with specific patterns

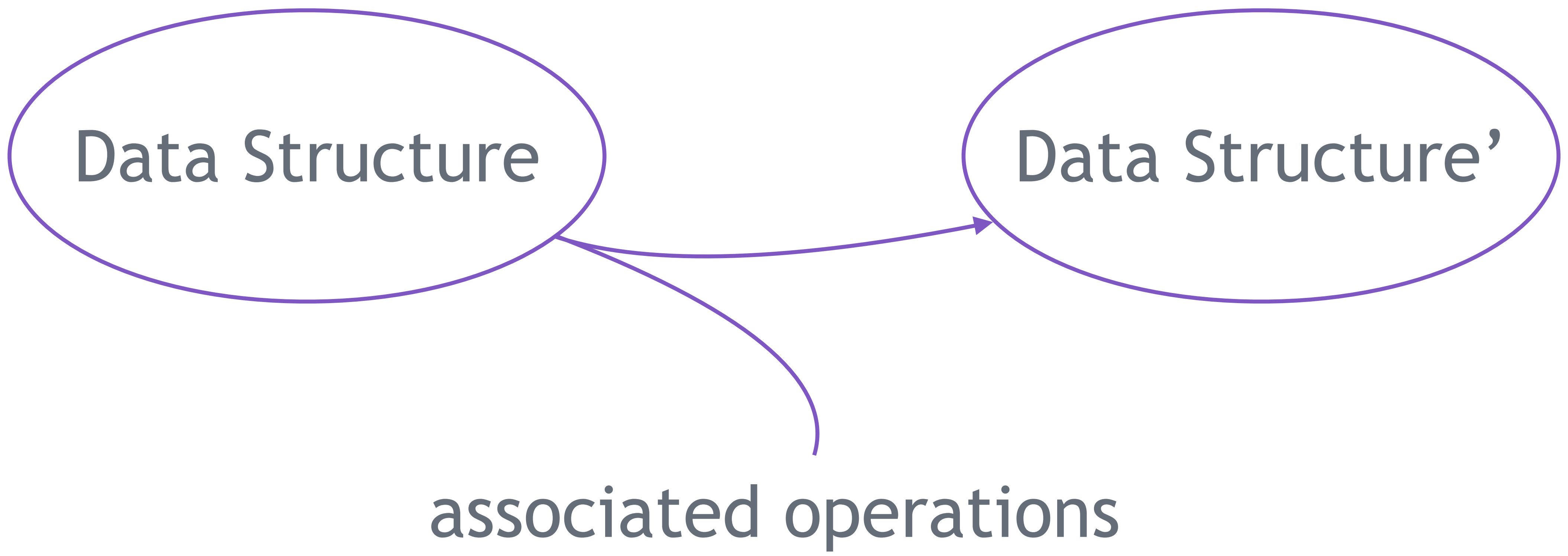
(3) Vectors: ordered numbers

(4) Matrices: numbers arranged in rectangular grids

(3) Trees: nodes connected by edges

Data Structures in Math

Operations on Data Structures



What's Sets?

a **collection** of **distinct** and **well-defined** things(or elements)

distinct: 서로 같지 않은

well-defined: doesn't change from person to person

things: 서로 같은 종류의 object들

don't have to be numbers
e.g. natural numbers, letters, rectangulars, images, persons

Notations

Enumerating Elements(Roster Form)

Set = {element$_1$, element$_2$, ..., element$_n$}

$$A = \{1, 2, 3, 4\}$$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$C = \{red, green, blue\}$$

$$D = \{free, guarantee, help, prices, winner, chance\}$$

Set Builder

Set = {element |element’s condition}

$$A = \{x \mid 1 \le x \in \mathbb{N} \le 4\}$$

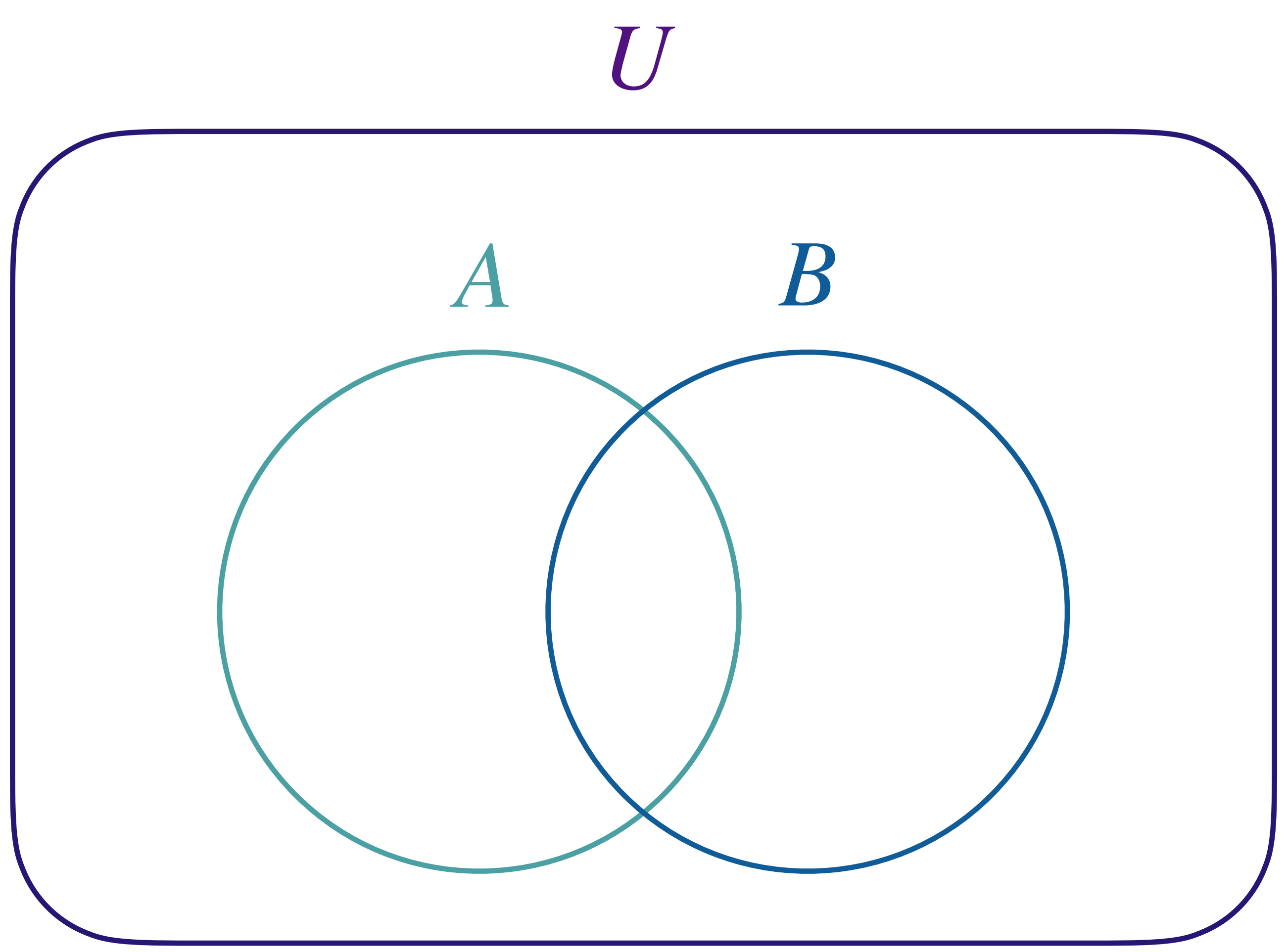
$$B = \{x \mid x\text{는 } \mathbb{R}^{2\times 2}\text{의 standard basis vector}\}$$

$$C = \{x \mid x\text{는 빛의 삼원색}\}$$

$$D = \{x \mid x\text{는 스팸메일에 자주 등장하는 단어}\}$$

Notations

Venn Diagram



Universal/Empty Sets

Empty Sets

$\varnothing = \{\}$

Universal Sets

U : 가능한 모든 원소들의 집합

ex.1) $U = \{x \mid (x \text{ is a 200x200 images})\}$

$A = \{x \mid (x \text{ is a 200x200 images}) \wedge (x \text{ contains humans in it})\}$

$B = \{x \mid (x \text{ is a 200x200 images}) \wedge (x \text{ contains dogs in it})\}$

$C = \{x \mid (x \text{ is a 200x200 images}) \wedge (x \text{ contains humans and dogs in it})\}$

ex.2) $U = \{x \mid (x \text{ is an English word})\}$

$A = \{x \mid (x \text{ is a frequenctly occurred word in spams})\}$

Algebra

Common Number Sets

Natural Numbers(자연수) $\mathbb{N} = \{1, 2, 3, \ \dots\} = \{x \mid (x\text{는 자연수})\}$

Whole Number $\mathbb{W} = \{0, 1, 2, \ \dots\} = \{x \mid (x\text{는 } 0 \vee x\text{는 자연수})\}$

Integers(정수) $\mathbb{Z} = \{\dots \ , -2, \ -1, 0, 1, 2, \ \dots\} = \{x \mid (x\text{는 정수})\}$

Rational Numbers(유리수) $\mathbb{Q} = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{101}{100}, \ \dots\} = \{x \mid (x\text{는 유리수})\}$

$= \{x \mid x = \frac{p}{q}, \ (p, q \text{ are integers } \wedge q \neq 0)\}$

Irrational Numbers(무리수) $\mathbb{I} = \{\pi, e, \sqrt{2}, \ \dots\} = \{x \mid (x\text{는 무리수})\} = \{x \mid \neg(x\text{는 유리수})\}$

Real Numbers(실수) $\mathbb{R} = \{x \mid (x\text{는 실수})\} = \{x \mid (x\text{는 유리수} \vee x\text{는 무리수})\}$

Complex Numbers(복소수) $\mathbb{C} = \{x \mid (x\text{는 복소수})\} = \{a + j \cdot b \mid (a, b\text{는 실수})\}$

Algebra

Coordinate Spaces

Coord. Plane, Space

$$\mathbb{R}^2 = \{(x, y) \mid x, y\text{는 실수}\}$$

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z\text{는 실수}\}$$

Higher Dimensional Spaces

$$\mathbb{R}^n = \{(x_1, x_2, \ \dots \ , x_n) \mid \forall x_i\text{는 실수}\}$$

Algebra

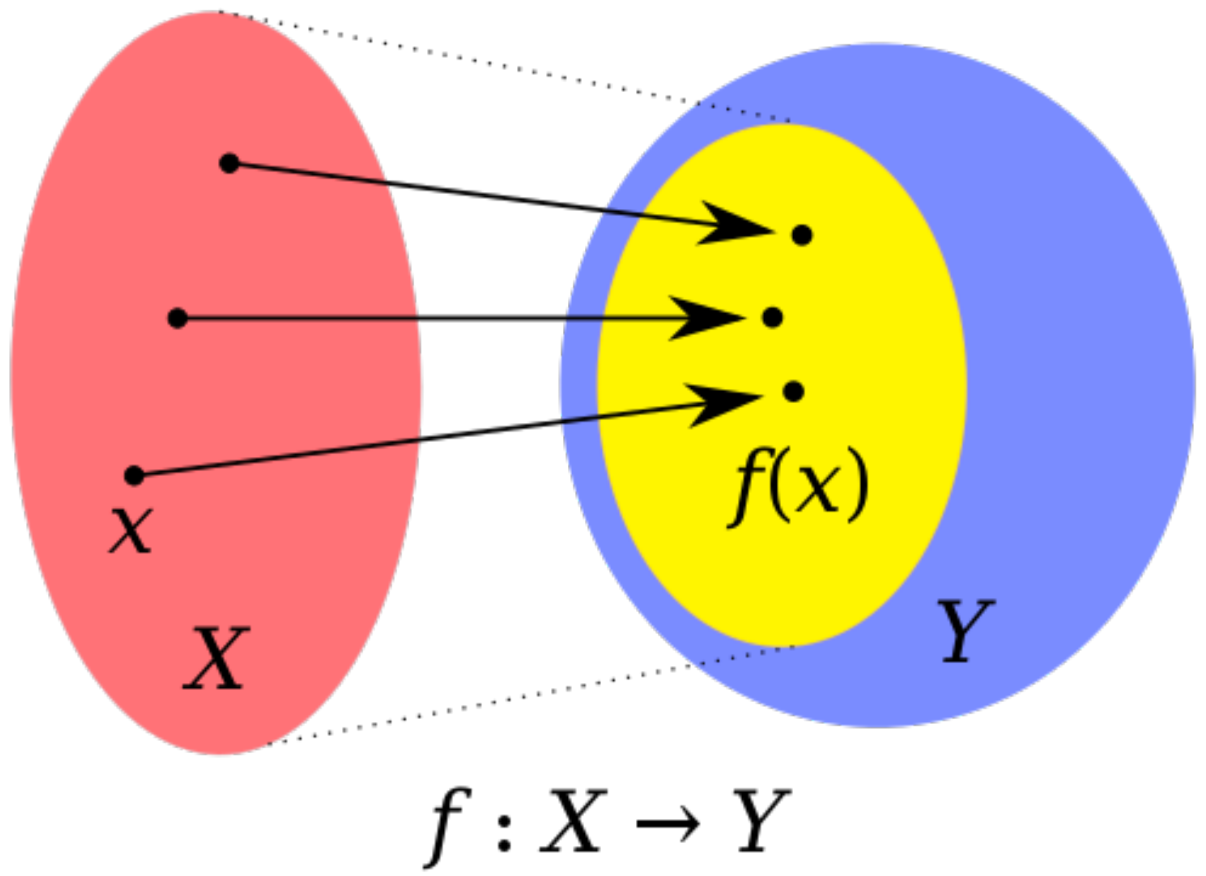
Functions

Domain: **a set of** departure of a function

Codomain: **a set of** destination of a function

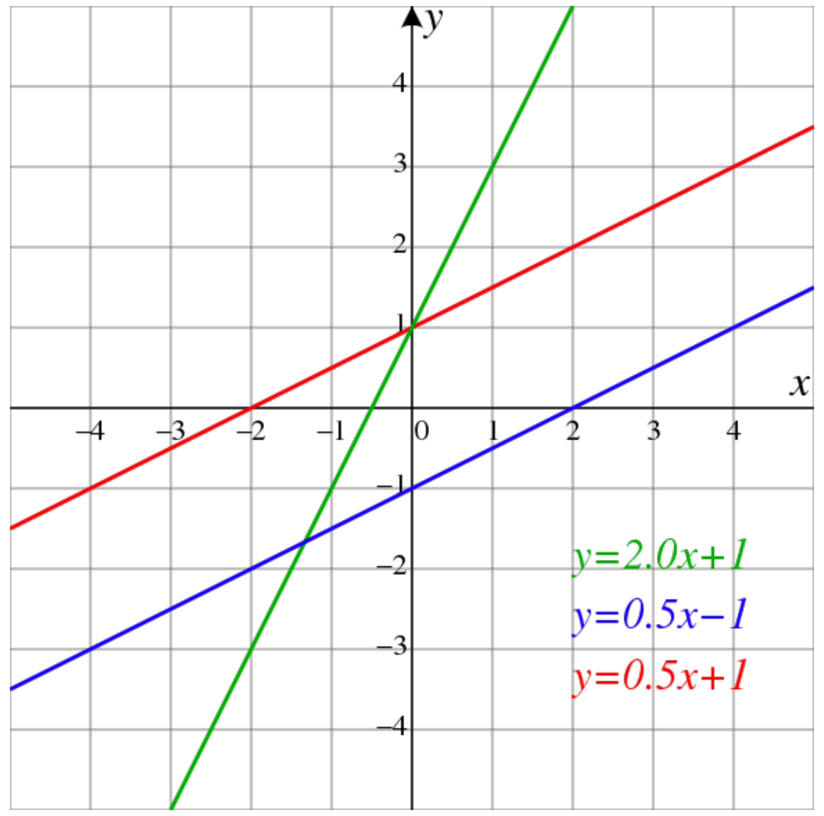
Range(Image): The image of f is then **the subset of** Y consisting of only those elements y of Y such that there is at least one x in X with f(x) = y.

Function: a binary relation between two sets that associates to each element of the first set exactly one element of the second set.



Algebra

Lines and Planes

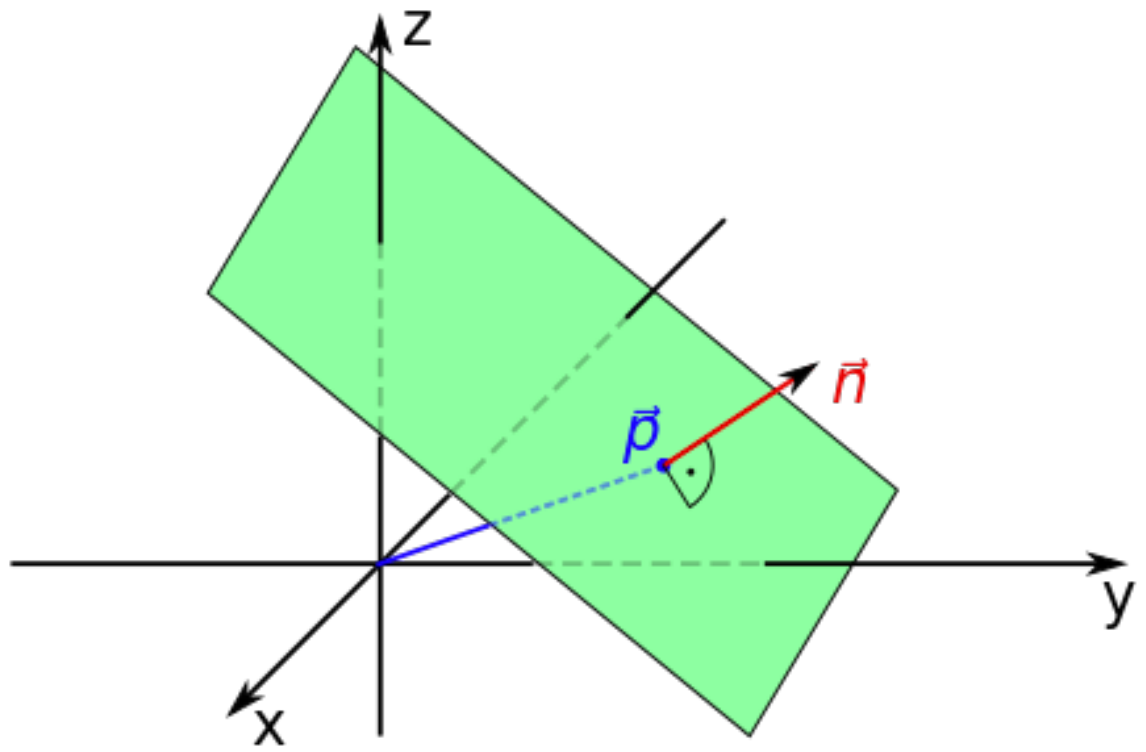


Line: a set of points whose coordinates satisfy a given linear equation

$$y = ax + b$$
$$L = \{(x, y) \mid y = ax + b\}$$

Plane: a set of all points $\mathbf{r}$ such that, $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

$$P = \{(x, y, z) \mid \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\}$$

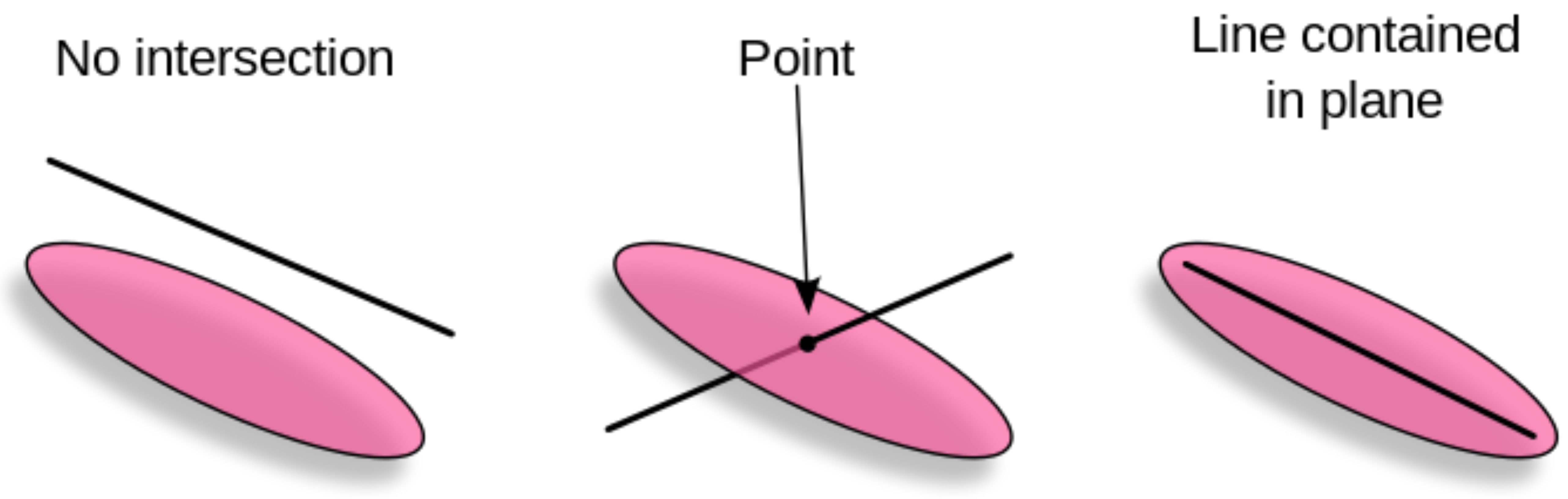


Algebra

Intersections

Intersection of L and P = the set of all points that lie on both L and P

$$\begin{aligned} \text{No intersection} \quad &\Longrightarrow \quad S = \varnothing \\ \text{Point intersection} \quad &\Longrightarrow \quad S = \{(x_s, y_s, z_s)\} \\ \text{Lines intersection} \quad &\Longrightarrow \quad S = \{\mathbf{r} \mid (\mathbf{r} \text{ is on } L)\} \end{aligned}$$



Algebra

Solution Sets

Solution Set of Equations $f(x) = 0$

solution set of the equation = the set of all x's that satisfy the equation

$$(x-2)(x+3) = 0 \rightarrow S = \{2, \ -3\}$$

$$sin(x) = 0 \rightarrow S = \{\dots\ ,\ -2\pi,\ -\pi, 0, \pi, 2\pi,\ \dots\} = \{x \mid x = k\pi, k \text{ is an integer}\}$$

Solution Set of Inequalities $f(x) < 0$

solution set of the inequality = the set of all x's that satisfy the inequality

$$(x-2)(x+3) < 0 \rightarrow S = \{x \mid \ -3 < x < 2\}$$

Linear Algebra

Vector Space	**a set of** objects called vectors
Linear Subspace	W is a **subset** of V, then W is a linear subspace of V if under the operations of V, W is a vector space over K.
Linear Span	**the set of** linear combinations of elements of S.
Basis	**a set B** is a basis if its elements are linearly independent and every element of V is a linear combination of elements of B.
Spectrum of a Matrix	**the set of** its eigenvalues.
Eigenspace	**The set of** all eigenvectors of T corresponding to the same eigenvalue, together with the zero vector, is called an eigenspace

Probability and Statistics

Sample Space	**the set of** all possible outcomes or results of that experiment.
Event	A subset of the sample space is an event
Random Process	A stochastic or random process can be defined as **a collection of random variables** that is indexed by some mathematical set
Statistical Model	A statistical model is a mathematical model that embodies **a set of** statistical assumptions concerning the generation of sample data

Cardinality of Sets

or cardinal number

$$|A| = (\#\ \text{elements})$$

ex.1) $A = \{0, 1\} \longrightarrow |A| = 2$

ex.2) $B = \{a, b, c\} \longrightarrow |B| = 3$

ex.3) $C = \{x \mid (x \text{ is an 1-digit integer})\} \longrightarrow |C| = 10$

ex.4) $D = \{x \mid (x \text{ is an alphabet})\} \longrightarrow |D| = 26$

Cardinality of Empty Set $|\varnothing| = 0$

Singleton Set $|A| = 1$

Equivalent Sets $|A| = |B|$

Finite/Infinite Sets

Finite Sets

원소의 개수가 한정되어 있는 집합

$|A| = 0$ or n

Encoding of Elements

주로 컴퓨터의 연산을 위해, 원소들을 index에 대응시키는 과정

$$A = \{a, b, c, \ \dots \ , x, y, z\}$$

$$\Big\downarrow \quad \begin{array}{l} a:1, \ b:2, \ c:3 \\ \dots \ x:24, \ y:25, \ z:26 \end{array}$$

$$A_E = \{1, 2, 3, \ \dots \ , 24, 25, 26\}$$

Finite/Infinite Sets

Infinite Sets

원소의 개수가 무한한 집합

$|A| = \infty$

ex) $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}$

Countably Infinite Sets

원소들을 index에 대응시킬 수 있는 무한집합

ex) $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}$

encoding이 가능함

Uncountably Infinite Sets

원소들을 index에 대응시킬 수 없는 무한집합

ex) $\mathbb{R}, \mathbb{C}$

Inclusion/Exclusion of Elements

원소들은 어떤 집합에 포함될 수도, 포함되지 않을 수 있다.

$$(\text{원소 } a\text{가 집합 } A\text{에 포함됨}) = \left(a \in A\right)$$

$$(\text{원소 } a\text{가 집합 } A\text{에 포함되지 않은}) = \left(a \notin A\right)$$

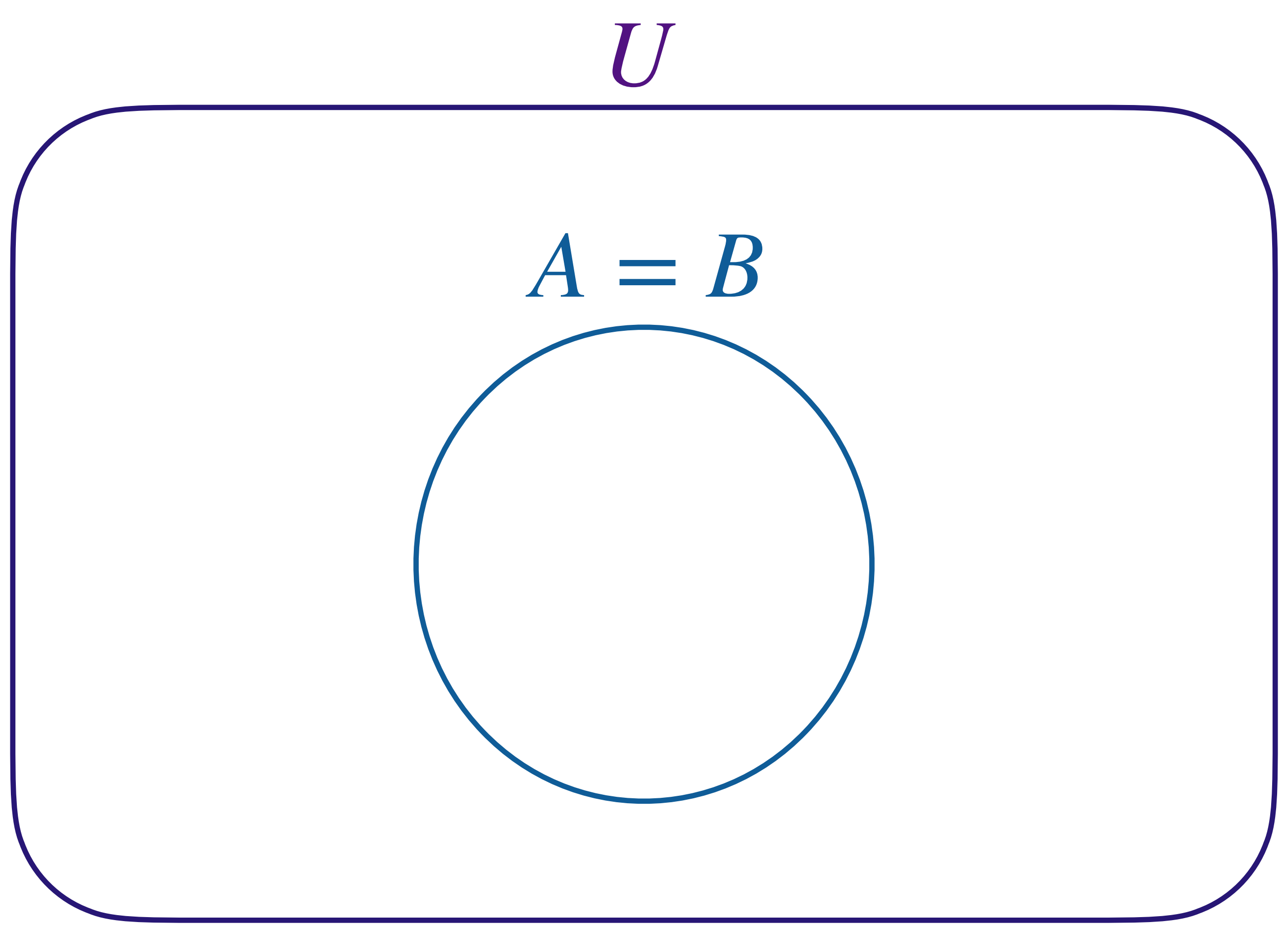
ex) $A = \{a, b, c, d\}$ $\qquad a \in A,\ b \in A,\ c \in A,\ d \in A$

$e \notin A,\ f \notin A,\ g \notin A,\ h \notin A$

Equal Sets

집합 A의 모든 원소가 집합 B에 포함되고 반대도 성립할 때, A, B는 서로 같은 집합이다.

$$A = B \longleftrightarrow [(\forall a \in A) \in B] \wedge [(\forall b \in B) \in A]$$



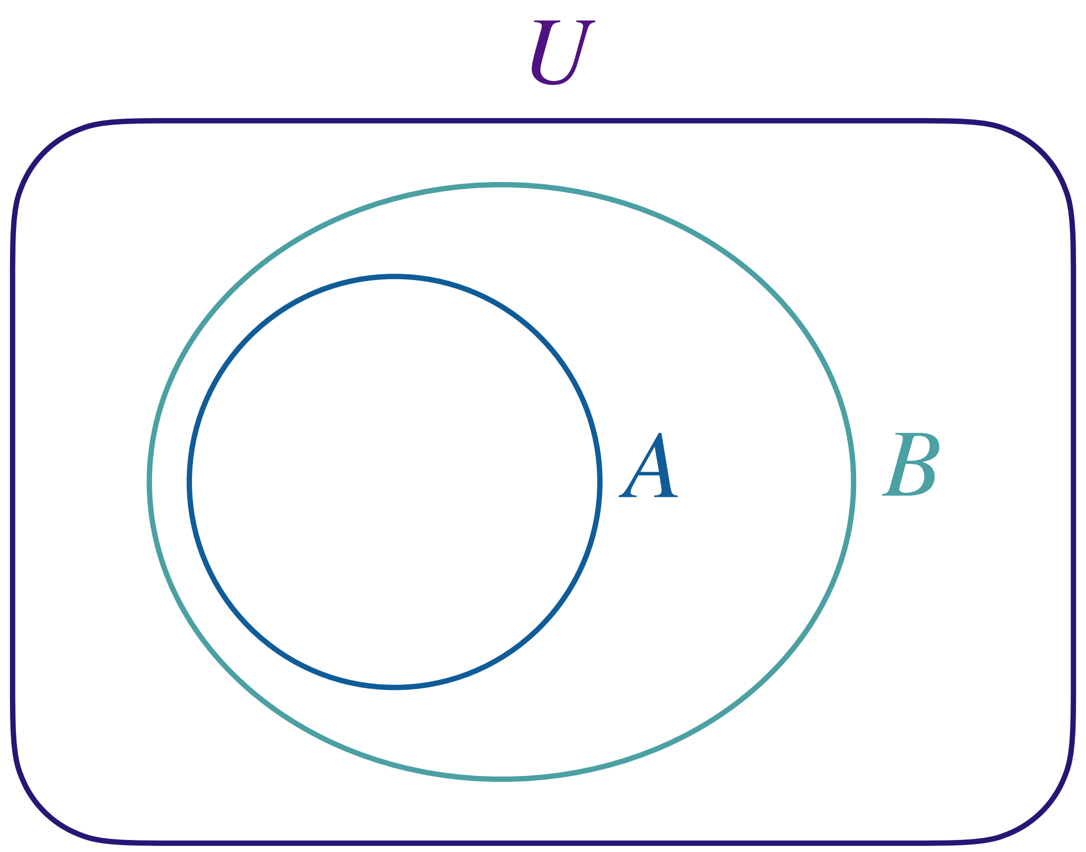
Inclusion/Exclusion of Sets

Subset

집합 A의 모든 원소가 집합 B에 포함될 때, A는 B의 subset이라 한다.

$$A \subseteq B \longleftrightarrow (\forall a \in A) \in B$$

$$A \subseteq B \longrightarrow |A| \leq |B|$$

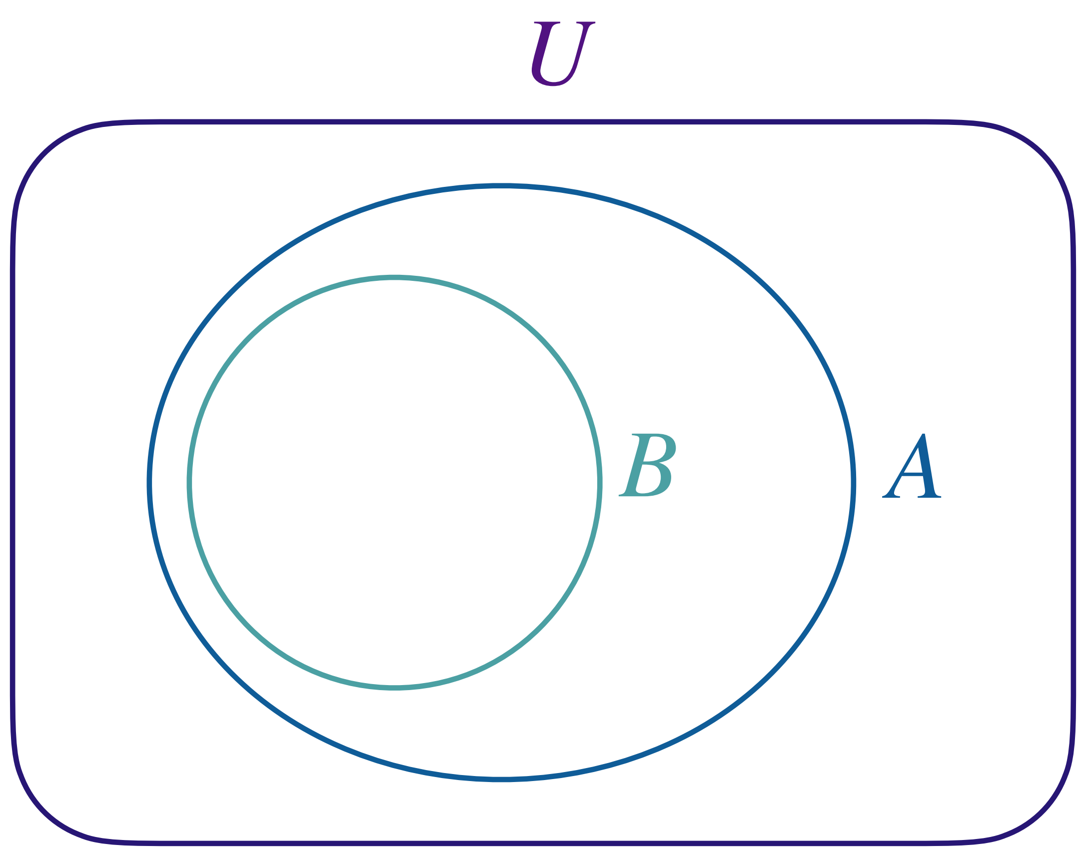


Superset

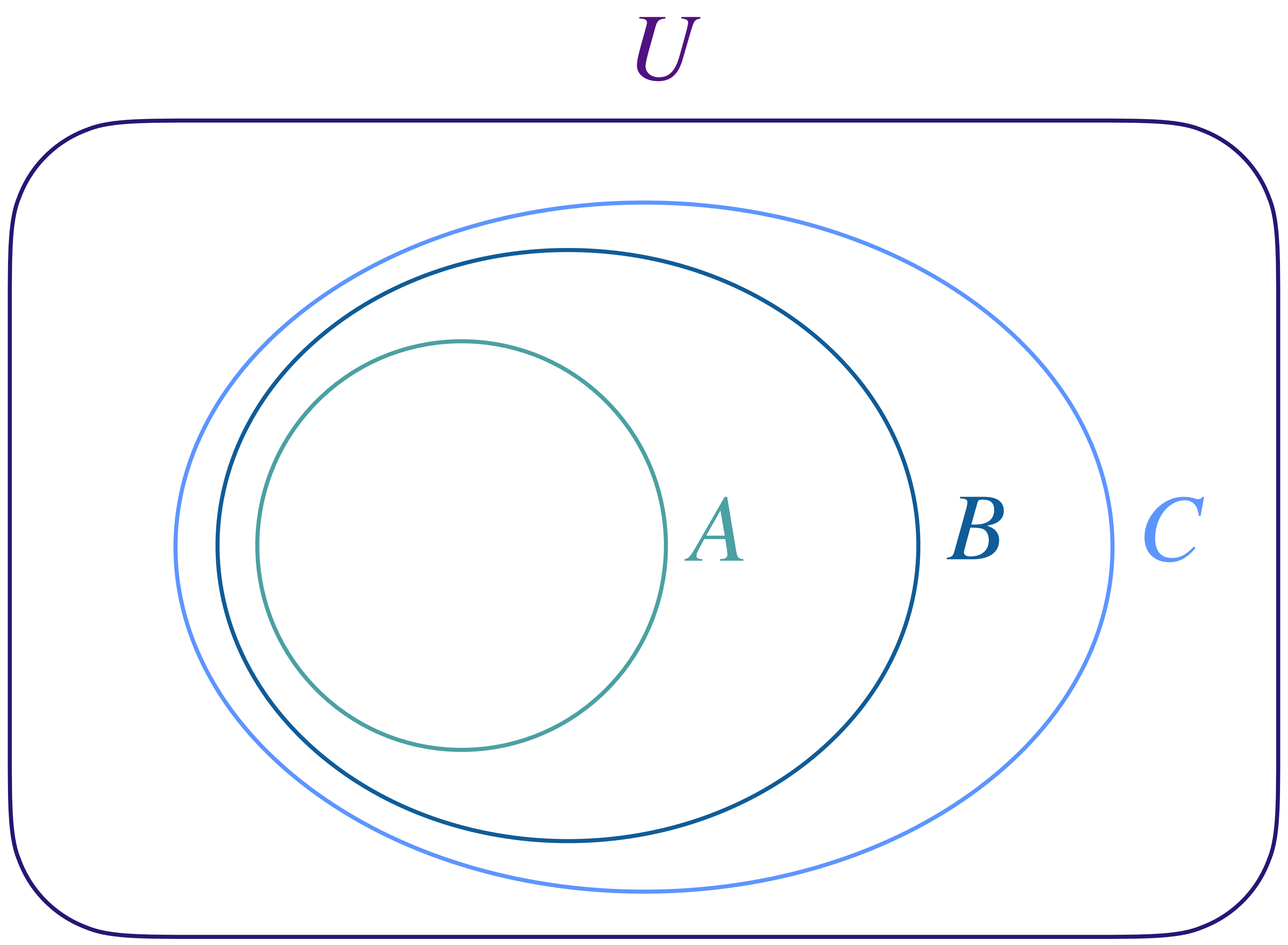
집합 B의 모든 원소가 집합 A에 포함될 때, A는 B의 superset이라 한다.

$$A \supseteq B \longleftrightarrow (\forall b \in B) \in A$$

$$A \supseteq B \longrightarrow |A| \geq |B|$$



Inclusion/Exclusion of Sets



$$A \subseteq B,\ B \subseteq C,\ A \subseteq C$$
$$B \supseteq A,\ C \supseteq B,\ C \supseteq A$$

Inclusion/Exclusion of Sets

Example

$$A = \{a, b, c, d\}$$

$\varnothing \subseteq A$

$\{a\} \subseteq A,\ \{b\} \subseteq A,\ \{c\} \subseteq A,\ \{d\} \subseteq A$

$\{a, b\} \subseteq A,\ \{a, c\} \subseteq A,\ \{a, d\} \subseteq A,$
$\{b, c\} \subseteq A,\ \{b, d\} \subseteq A,\ \{c, d\} \subseteq A$

$\{a, b, c\} \subseteq A,\ \{a, b, d\} \subseteq A,$
$\{a, c, d\} \subseteq A,\ \{b, c, d\} \subseteq A$

$\{a, b, c, d\} \subseteq A \longrightarrow A \subseteq A$

A is a subset of itself.

$\{a, b, c, d\} \supseteq A \longrightarrow A \supseteq A$

$\{a, b, c, d, e\} \supseteq A$

$\{a, b, c, d, f\} \supseteq A$

$\{a, b, c, d, e, f\} \supseteq A$

Proper Subsets/Supersets

Proper Subsets

집합 A, B에 대해 A가 B의 subset이지만 완전히 같지는 않을 때, A는 B의 proper subset이라 한다.

적어도 B의 원소 중 하나는 A에 포함되지 않아야 한다.

$$A \subset B \longleftrightarrow [(\forall a \in A) \in B] \wedge [A \neq B]$$

$A \subset B \longrightarrow |A| < |B|$

Proper Supersets

집합 A, B에 대해 A가 B의 superset이지만 완전히 같지는 않을 때, A는 B의 proper superset이라 한다.

적어도 A의 원소 중 하나는 B에 포함되지 않아야 한다.

$$A \supset B \longleftrightarrow [(\forall b \in B) \in A] \wedge [A \neq B]$$

$A \subseteq B \longrightarrow |A| \leq |B|$

Proper Subsets/Supersets

Example.1

$$A = \{a, b, c, d\}$$

$\varnothing \subset A$

$\{a\} \subset A, \{b\} \subset A, \{c\} \subset A, \{d\} \subset A$

$\{a, b\} \subset A, \{a, c\} \subset A, \{a, d\} \subset A,$
$\{b, c\} \subset A, \{b, d\} \subset A, \{c, d\} \subset A$

$\{a, b, c\} \subset A, \{a, b, d\} \subset A,$
$\{a, c, d\} \subset A, \{b, c, d\} \subset A$

A is not a proper subset of A.

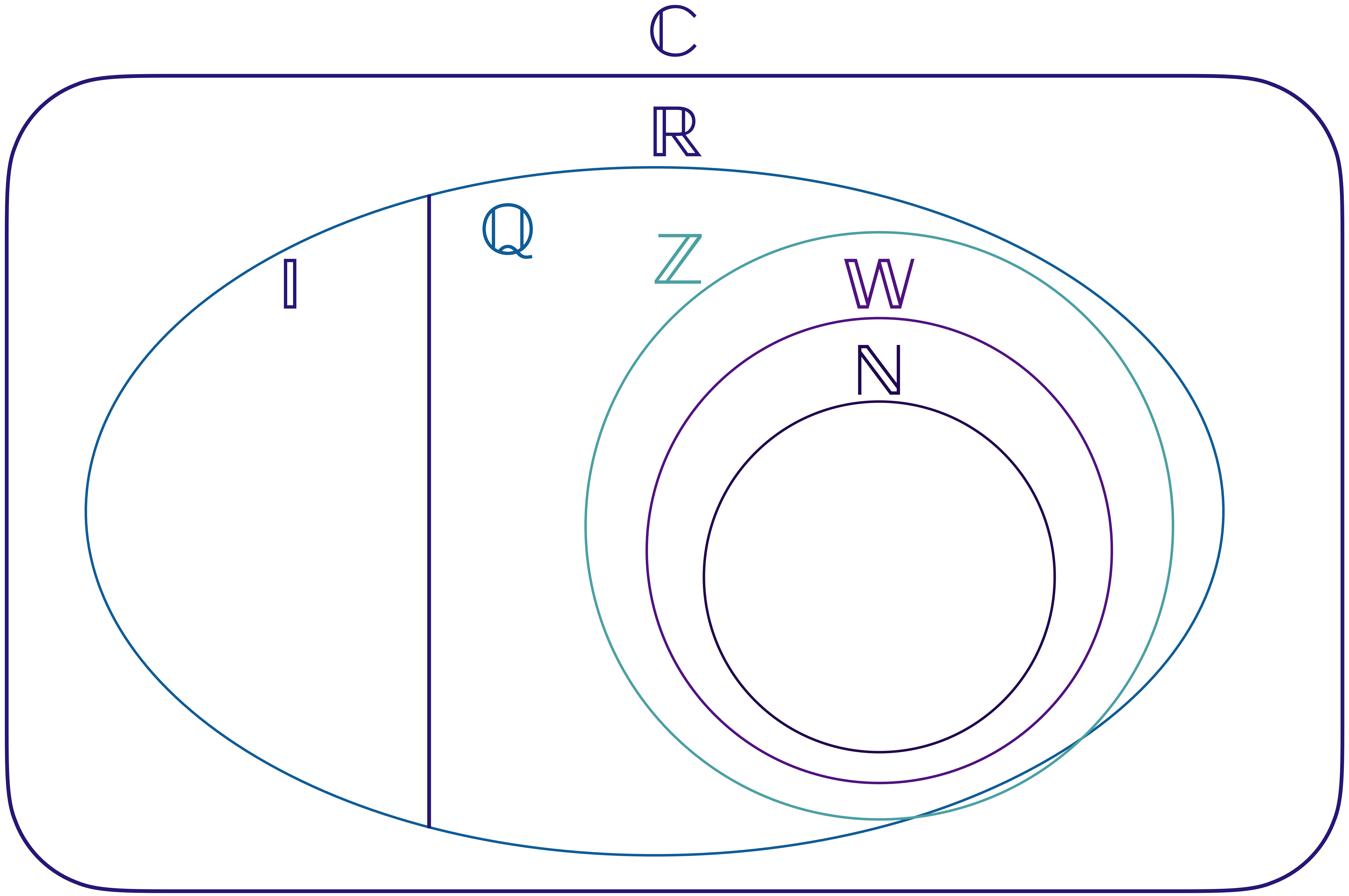
A is not a proper superset of A.

$\{a, b, c, d, e\} \supset A$
$\{a, b, c, d, f\} \supset A$
$\{a, b, c, d, e, f\} \supset A$

Proper Subsets/Supersets

Example.2

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$
$$\mathbb{I} \subset \mathbb{R}$$



Proper Subsets/Supersets

ex.3)

$X = \{x \mid (x$는 4의 배수$)\}$ $\qquad$ $Y = \{x \mid (x$는 8의 배수$)\}$

$= \{4, 8, 12, 16, 20, \ \dots\}$ $\qquad$ $= \{8, 16, 24, 32, 40, \ \dots\}$

$$X \supset Y, Y \subset X$$

ex.4)

$A = \{2, 4\}$ $\qquad$ $B = \{x \mid (x$는 12의 약수$)\}$

$= \{1, 2, 3, 4, 6, 12\}$

$$A \subset B, B \supset A$$

ex.5)

$A = \{3, 5, 7, 9\}$ $\qquad$ $B = \{x \mid \alpha \leq x \in \mathbb{N} \leq \beta\}$

$A \subseteq B$이기 위해, $\alpha \leq 3, \beta \geq 9$

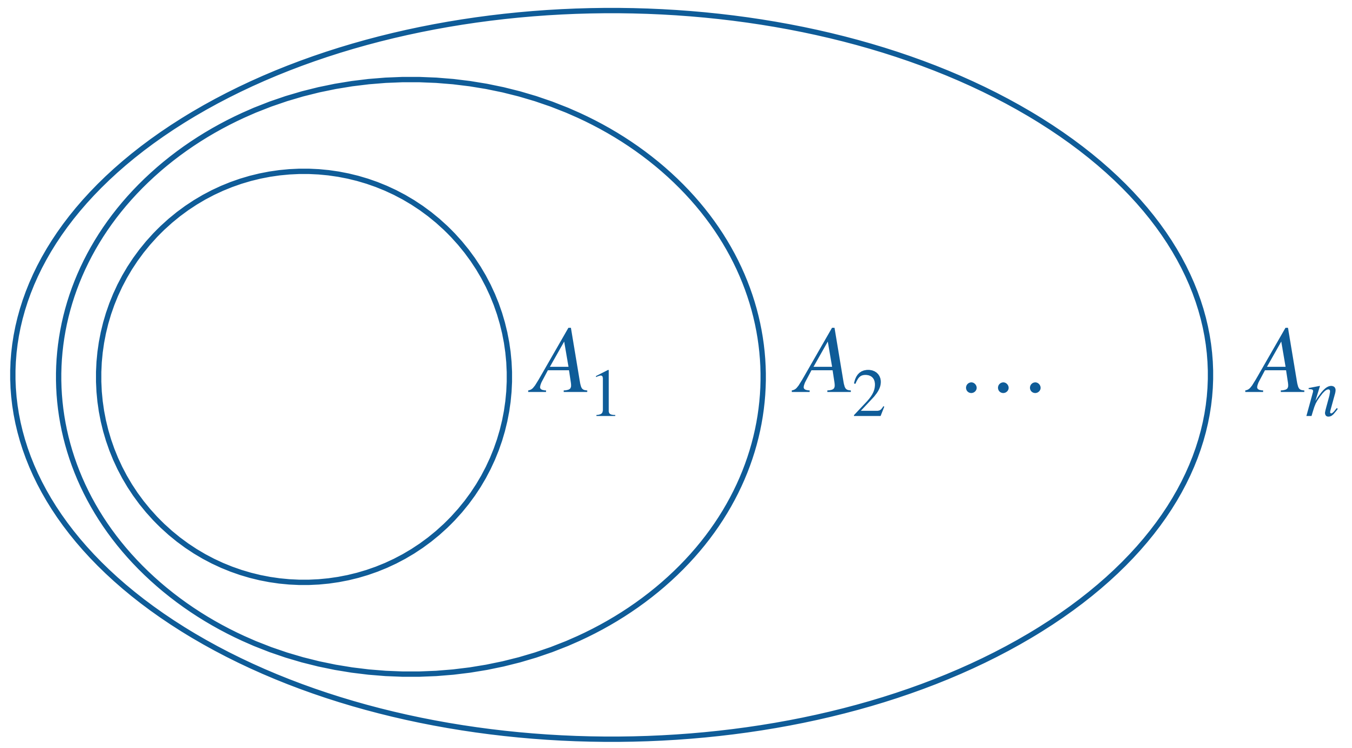
Increasing/Decreasing Sequences of Sets

Increasing Sequences of Sets

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$$

$$A_k \subseteq A_{k+1}, 1 \leq k \in \mathbb{N} \leq n-1$$

$$A_k \subseteq A_{k+1}, 1 \leq k \in \mathbb{N} \leq n-1 \longrightarrow |A_k| \leq |A_{k+1}|$$

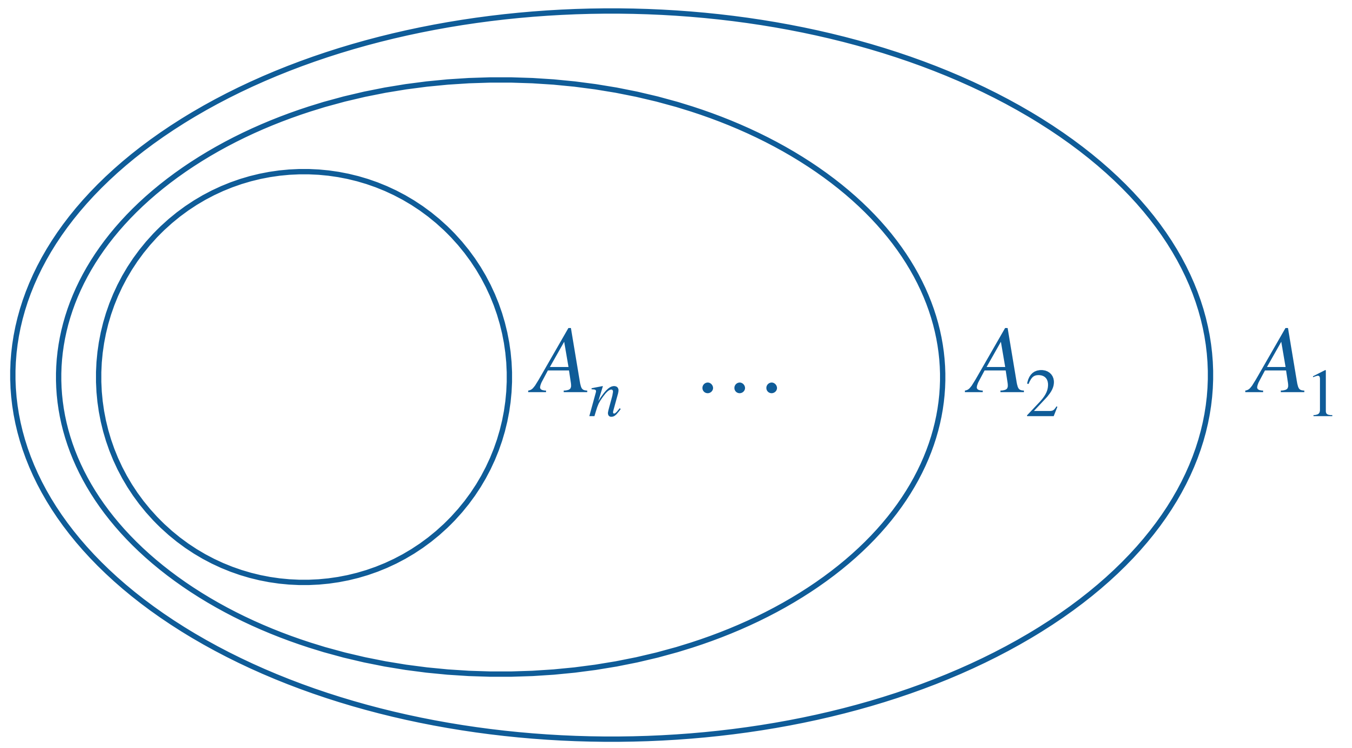


Decreasing Sequences of Sets

$$A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n$$

$$A_k \supseteq A_{k+1}, 1 \leq k \in \mathbb{N} \leq n-1$$

$$A_k \supseteq A_{k+1}, 1 \leq k \in \mathbb{N} \leq n-1 \longrightarrow |A_k| \geq |A_{k+1}|$$



Increasing/Decreasing Sequences of Sets

Example.1

$A_i = \left\{x \mid (x\text{는 } 2^i\text{의 배수})\right\}$

$A_1 = \left\{x \mid (x\text{는 } 2\text{의 배수})\right\} = \{2, 4, 6, 8, 10, \ldots\}$

$A_2 = \left\{x \mid (x\text{는 } 4\text{의 배수})\right\} = \{4, 8, 12, 16, 20, \ldots\}$

$A_3 = \left\{x \mid (x\text{는 } 8\text{의 배수})\right\} = \{8, 16, 24, 32, 40, \ldots\}$

$\Longrightarrow A_1 \supset A_2 \supset A_3 \supset \ldots$

$\Longrightarrow A_1, A_2, A_3, \ldots$:decreasing sequnce

Increasing/Decreasing Sequences of Sets

Example.2 The key idea of residual networks

Unary/Binary Operations

Operations on Sets

일정한 규칙을 통해 **새로운 집합**을 만들어내는 과정

Unary Operations $f : A \longrightarrow B$

- power set of sets
- complement of sets

Binary Operations $f : A \times B \longrightarrow C$

- Intersection of sets
- union of sets
- set difference
- symmetric difference
- Cartesian product of sets

Unary Operations - Power Sets

Power Sets

집합 A의 모든 subset들의 집합 $\mathscr{P}(A)$

모든 원소들은 "집합"

$$\mathscr{P}(A) = \{X \mid X \subseteq A\}$$

Power Set and Cardinality

$$|\mathscr{P}(A)| = 2^{|A|}$$

ex.1)

$A = \{0, 1\}$

Subsets => $\varnothing, \{0\}, \{1\}, \{0, 1\}$

$$\varnothing \in P(A)$$
$$\{0\} \in P(A)$$
$$\{1\} \in P(A)$$
$$\{0, 1\} \in P(A)$$

$$\mathscr{P}(A) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$$

ex.2)

$B = \{a, b, c\}$

Subsets => $\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

$$\varnothing \in P(B) \qquad \{a, b\} \in P(B)$$
$$\{a\} \in P(B) \qquad \{b, c\} \in P(B)$$
$$\{b\} \in P(B) \qquad \{b, c\} \in P(B)$$
$$\{c\} \in P(B) \qquad \{a, b, c\} \in P(B)$$

$$\mathscr{P}(B) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Unary Operations - Power Sets

Power Sets and Binary Numbers

$$B = \{a, b, c\}$$

Decimal	Bin. Num.	Subset
0	000	$\varnothing$
1	001	$\{c\}$
2	010	$\{b\}$
3	011	$\{b, c\}$
4	100	$\{a\}$
5	101	$\{a, c\}$
6	110	$\{a, b\}$
7	111	$\{a, b, c\}$

*Symmetric Table

Unary Operations - Power Sets

Note!

1. Power set의 원소들은 “집합”

2. $\varnothing$과 A는 $\mathscr{P}(A)$의 원소

3. $\mathscr{P}(A)$의 원소들은 binary number로 인코딩이 가능하다.

Unary Operations - Power Sets

ex.1)

$$A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\longrightarrow \mathscr{P}(A) = \left\{ \varnothing, \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \right\}$$

ex.2)

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\longrightarrow \mathscr{P}(B) = \left\{ \varnothing, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right\}$$

ex.3)

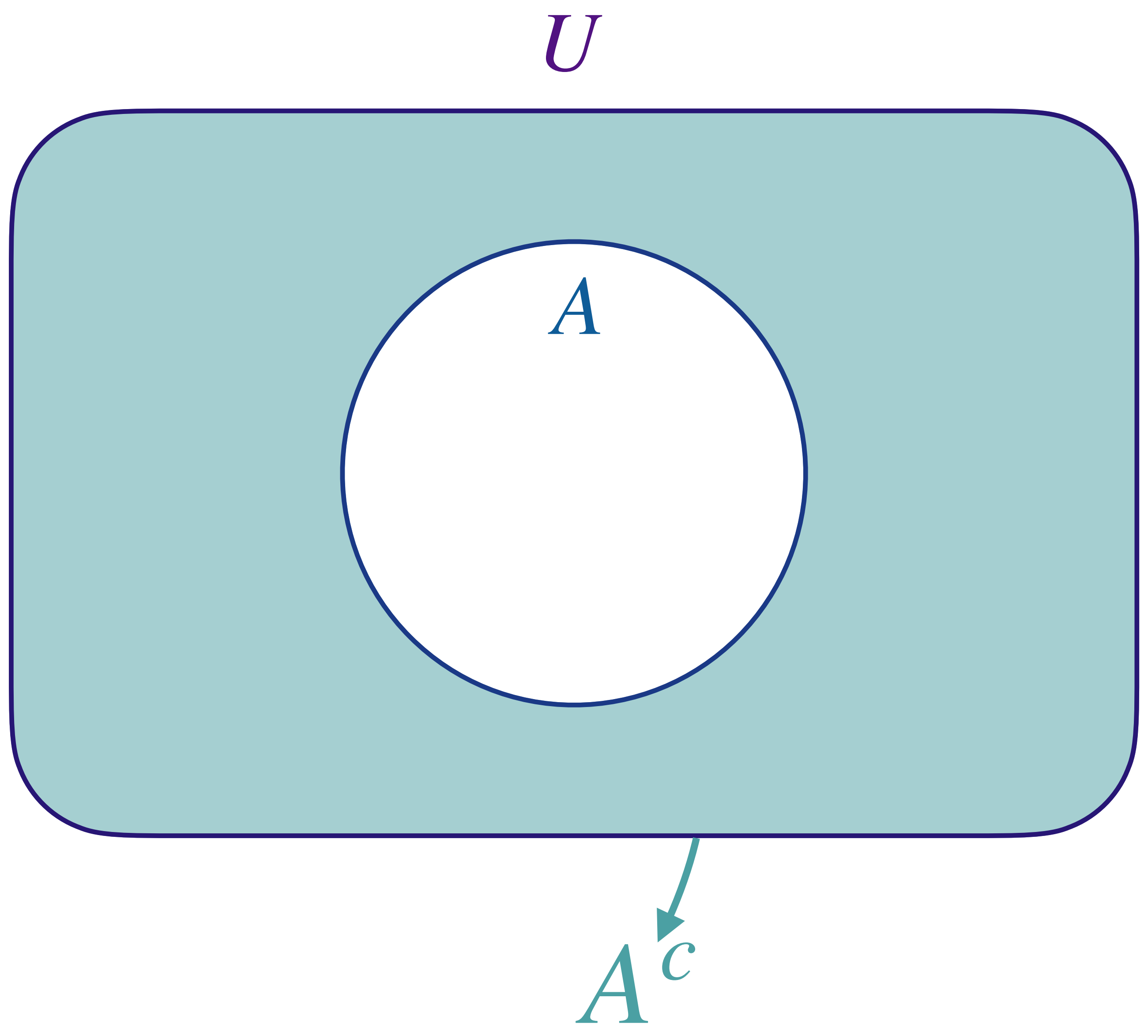
$$C = \{R,\ G,\ B\}$$

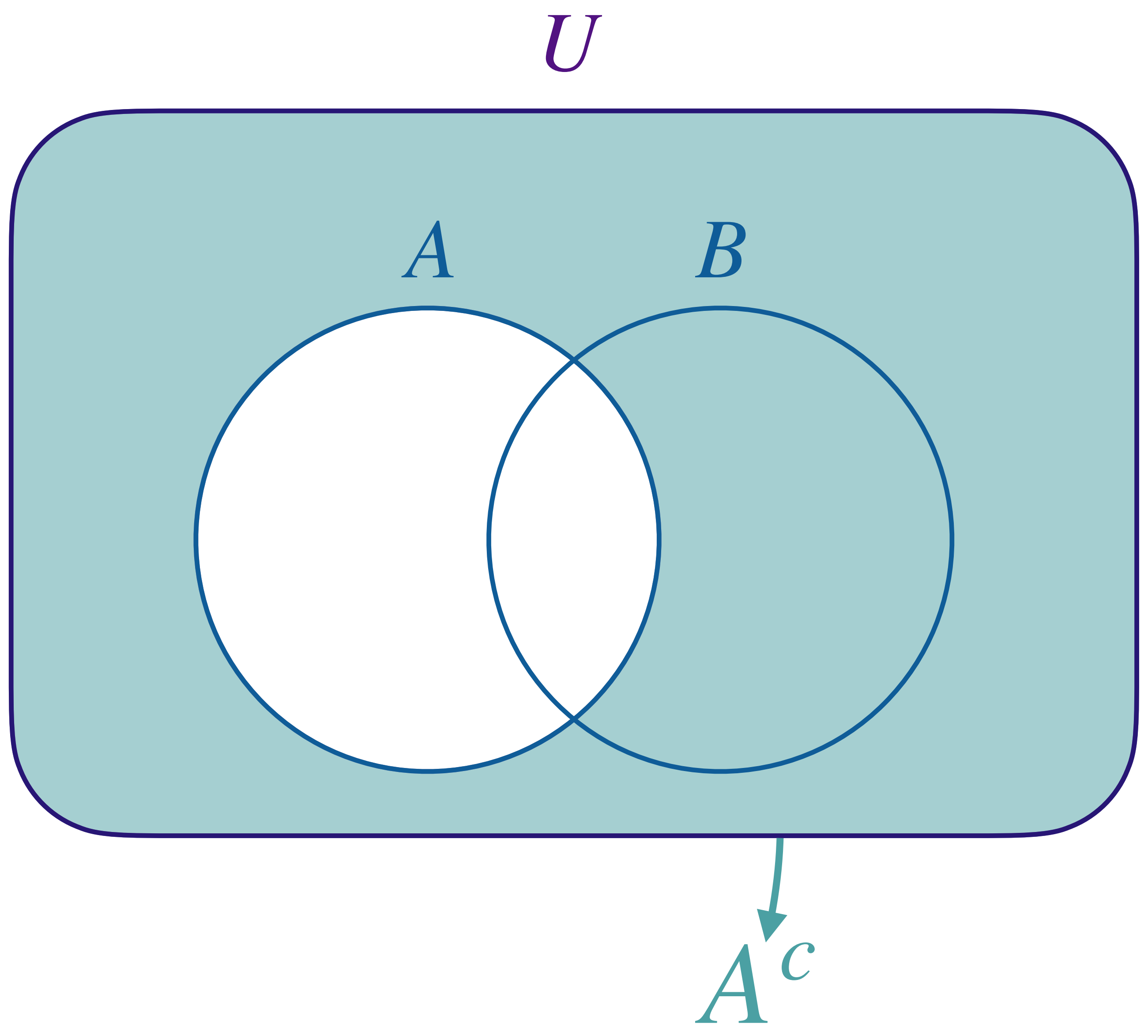
$$\longrightarrow \mathscr{P}(C) = \left\{ \varnothing, \{R\}, \{G\}, \{B\}, \{R, G\}, \{R, B\}, \{G, B\}, \{R, G, B\} \right\}$$

Unary Operations - Complements

A에 **포함되지 않은** 원소들을 모은 집합을 A의 complement이라 하고, A^c로 표현한다.

$$A^c = \{x \mid x \notin A\}$$





Cardinality

$|A^c| = |U| - |A|$

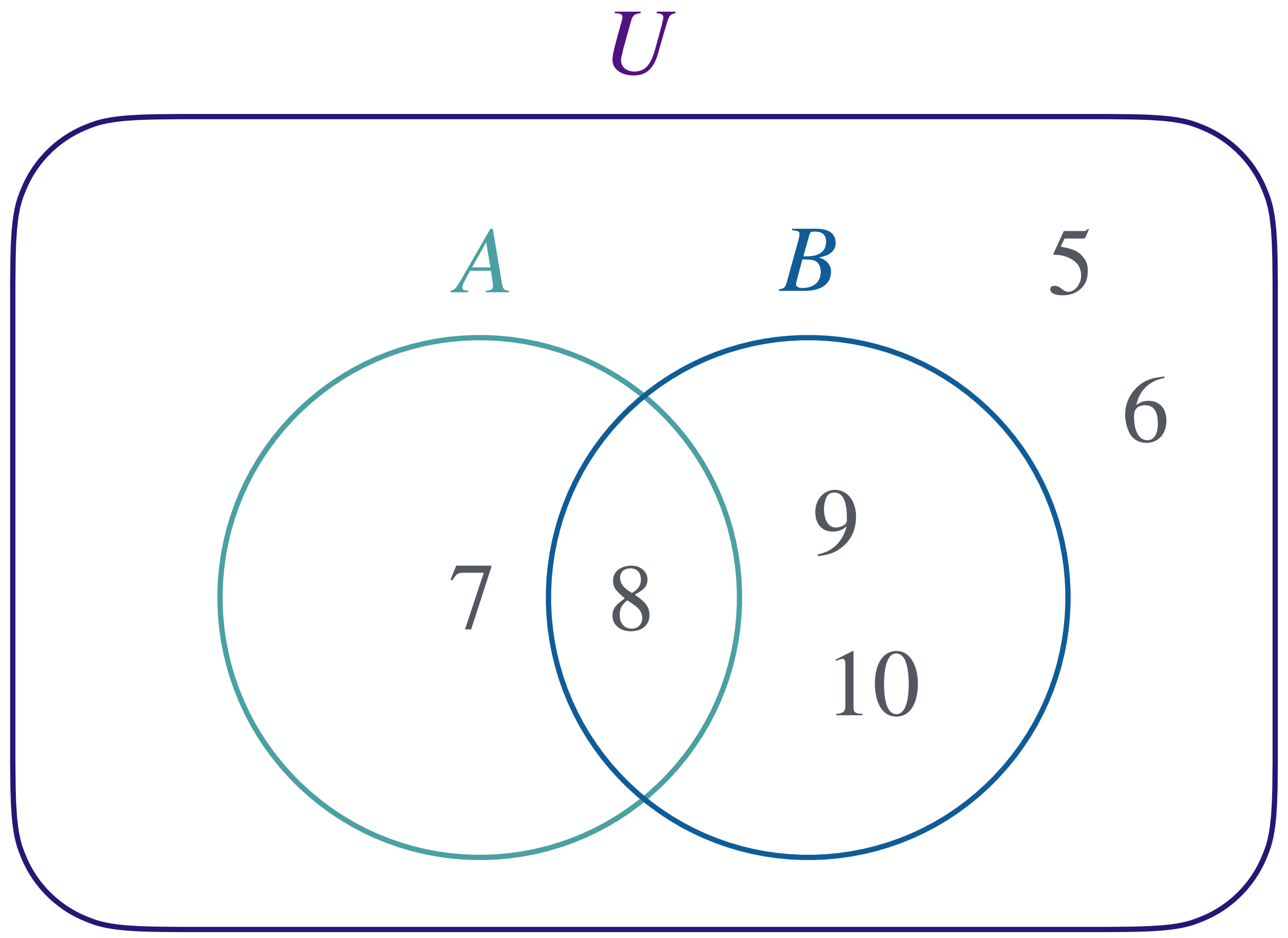
Unary Operations - Complements

ex.1)

$U = \{x \mid 5 \leq (x \in \mathbb{N}) \leq 10\}$

$A = \{7, 8\}$

$B = \{8, 9, 10\}$



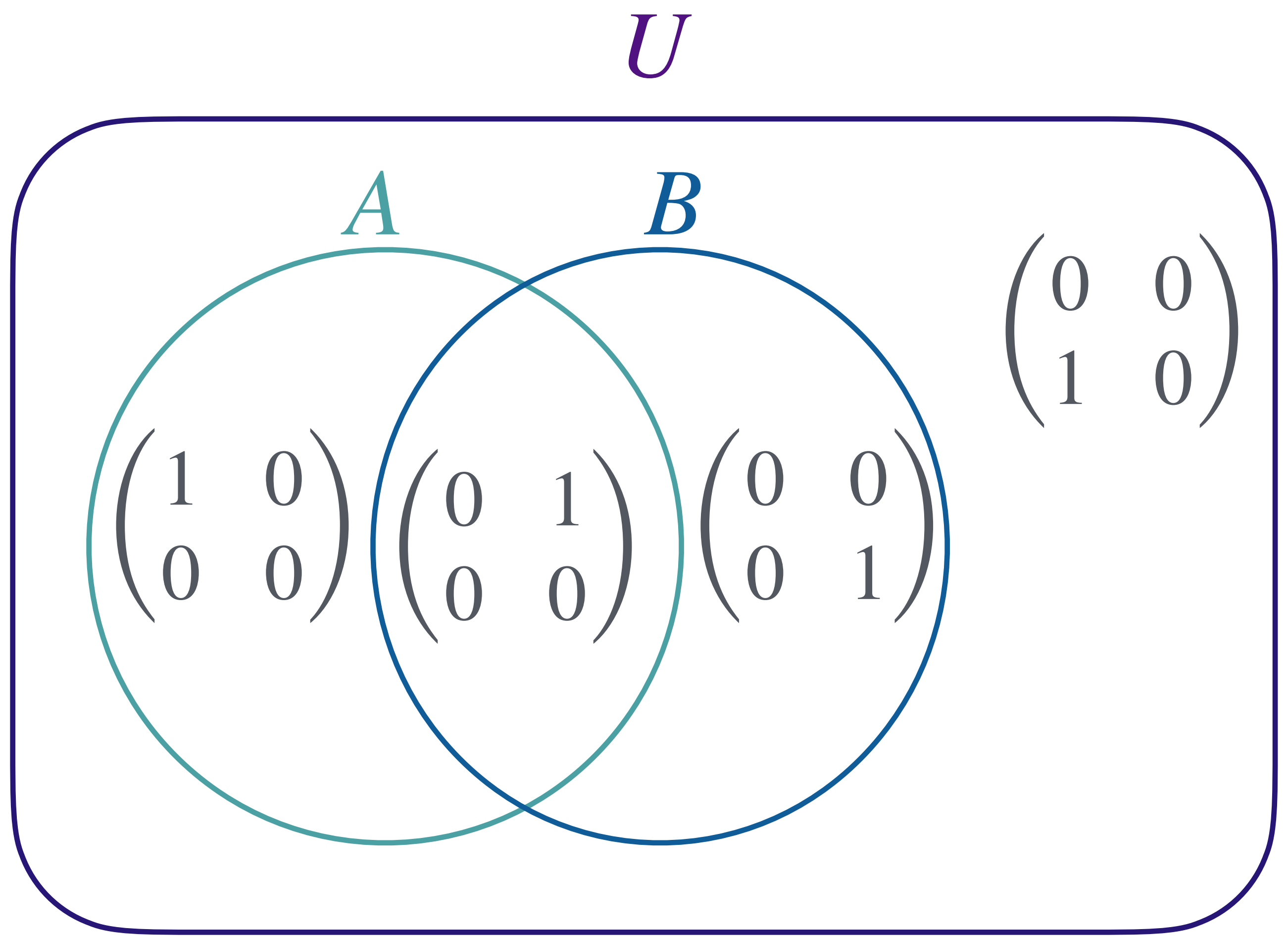
$A^c = \{5, 6, 9, 10\}$

$B^c = \{5, 6, 7\}$

ex.2)

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \qquad B = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$



$$A^c = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$B^c = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

Binary Operations - Intersections and Unions

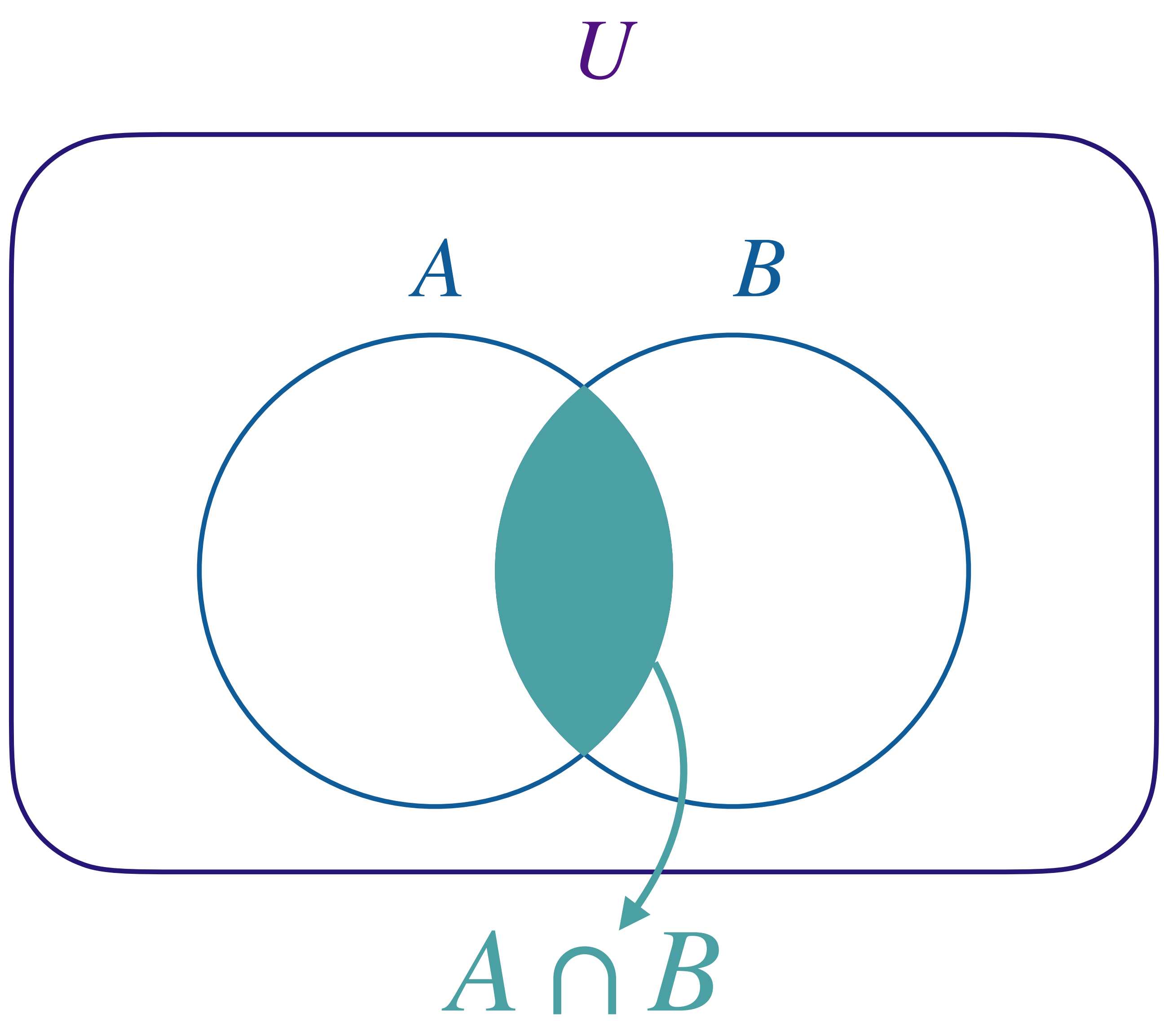
집합 A, B에 모두 포함되는 원소들을 모든 집합을 A와 B의 intersection(교집합)이라고 부르고, $A \cap B$로 나타낸다.

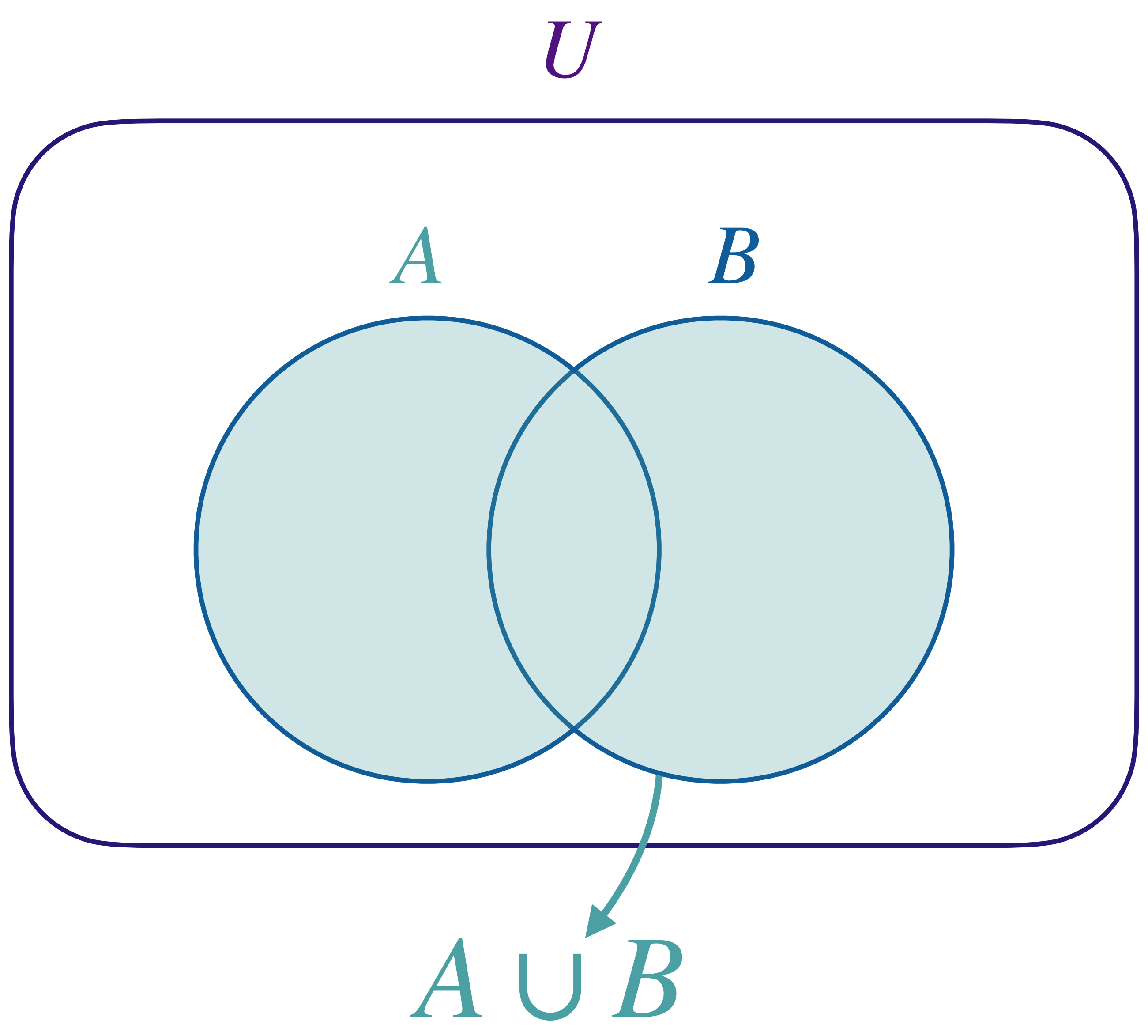
$$A \cap B = \{x \mid (x \in A) \land (x \in B)\}$$

$A \cap B$는 가끔 AB로 표현하기도 한다.

집합 A 또는(or) B에 포함되는 원소들을 모든 집합을 A와 B의 union(합집합)이라고 부르고, $A \cup B$로 나타낸다.

$$A \cup B = \{x \mid (x \in A) \lor (x \in B)\}$$





Binary Operations - Intersections and Unions

Cardinality

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cap B| \leq |A|, \quad |A \cap B| \leq |B|$$
$$|A \cup B| \geq |A|, \quad |A \cup B| \geq |B|$$

Special Cases

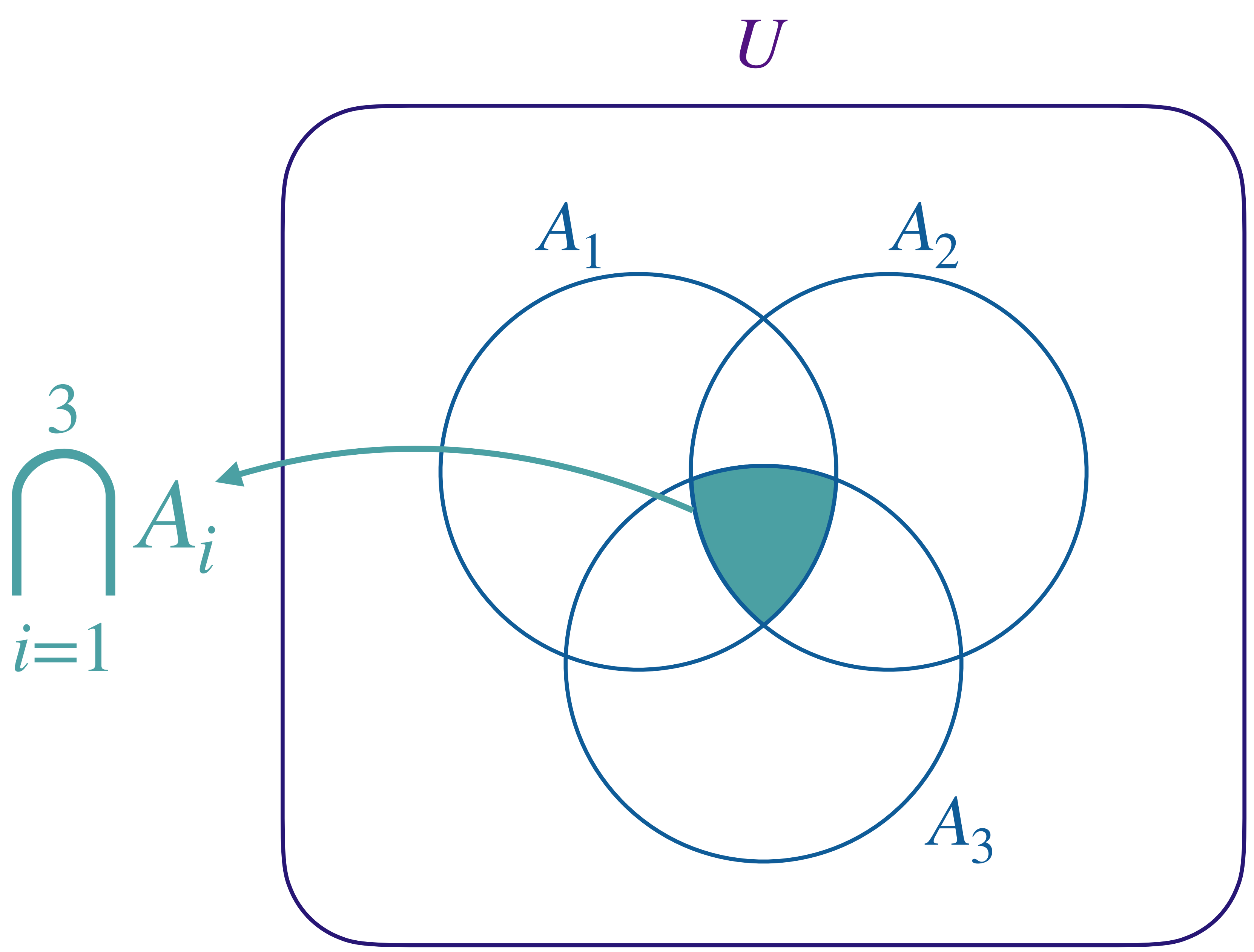
$$A \subseteq B \longrightarrow A \cup B = B, \quad A \cap B = A$$
$$|A \cap B| = |A|, \quad |A \cup B| = |B|$$

$$A \supseteq B \longrightarrow A \cup B = A, \quad A \cap B = B$$
$$|A \cap B| = |B|, \quad |A \cup B| = |A|$$

Binary Operations - Intersections and Unions

General Intersections

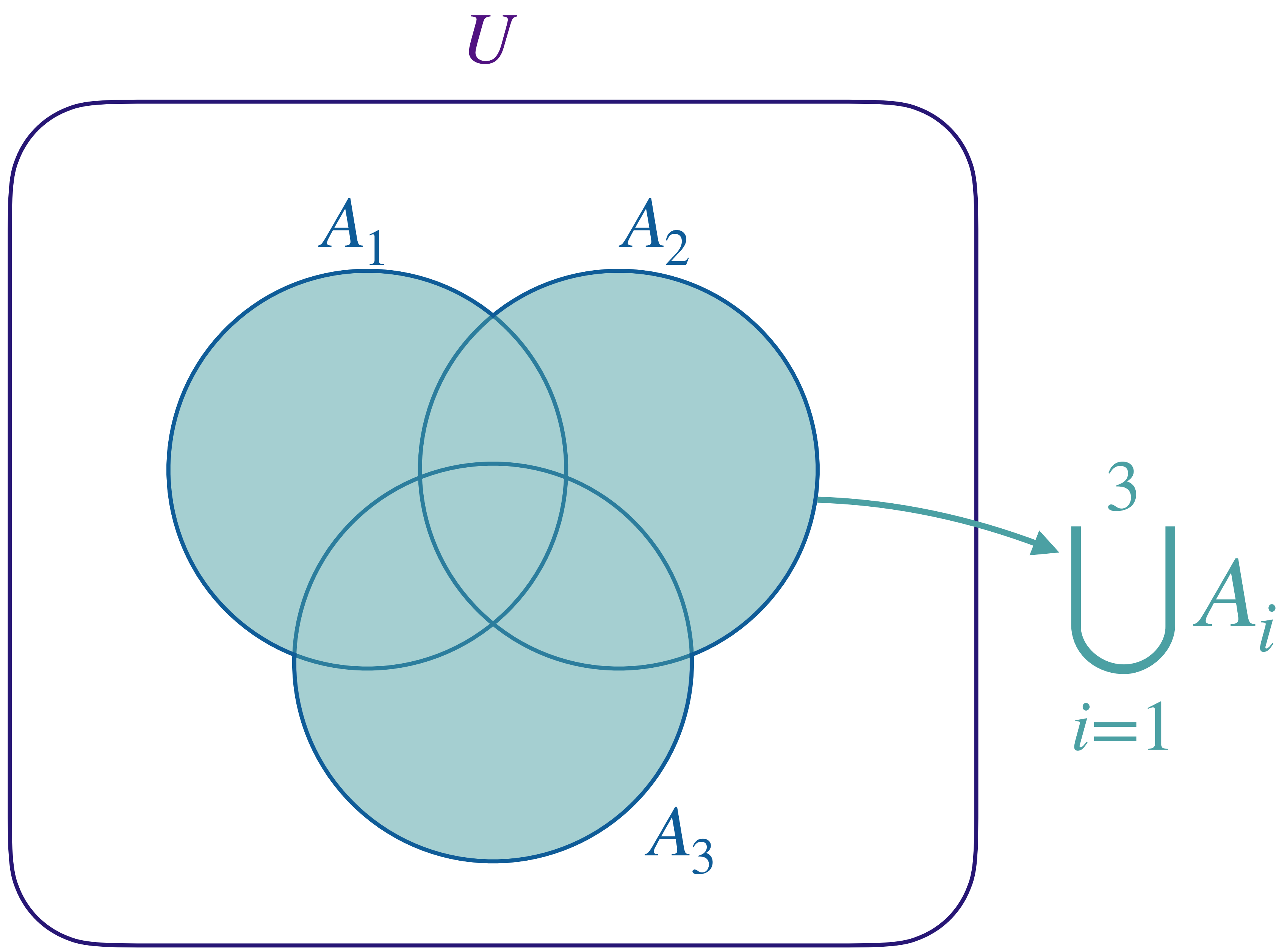
$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_n \qquad \bigcap_{i=1}^{n} A_i = A_1 A_2 \dots A_n$$



Binary Operations - Intersections and Unions

General Unions

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$



Binary Operations - Intersections and Unions

Special Cases(2)

$$A_k \subseteq A_{k+1},\ \ 1 \le k \in \mathbb{N} \le n-1 \quad \longrightarrow \quad \bigcap_{i=1}^{n} A_i = A_1,\ \ \bigcup_{i=1}^{n} A_i = A_n$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = |A_n|,\quad \left|\bigcap_{i=1}^{n} A_i\right| = |A_1|$$

$$\left|\bigcup_{i=1}^{k-1} A_i\right| \le \left|\bigcup_{i=1}^{k} A_i\right|,\quad \left|\bigcap_{i=1}^{k-1} A_i\right| = \left|\bigcap_{i=1}^{k} A_i\right| = |A_1|$$

Binary Operations - Intersections and Unions

Special Cases(2)

$$A_k \supseteq A_{k+1},\ \ 1 \leq k \in \mathbb{N} \leq n-1 \quad \longrightarrow \quad \bigcap_{i=1}^{n} A_i = A_n,\ \ \bigcup_{i=1}^{n} A_i = A_1$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = |A_1|,\quad \left|\bigcap_{i=1}^{n} A_i\right| = |A_n|$$

$$\left|\bigcup_{i=1}^{k-1} A_i\right| = \left|\bigcup_{i=1}^{k} A_i\right| = |A_1|,\quad \left|\bigcap_{i=1}^{k-1} A_i\right| \geq \left|\bigcap_{i=1}^{k} A_i\right|$$

Binary Operations - Intersections and Unions

The Algebraic Properties

Commutative Law

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

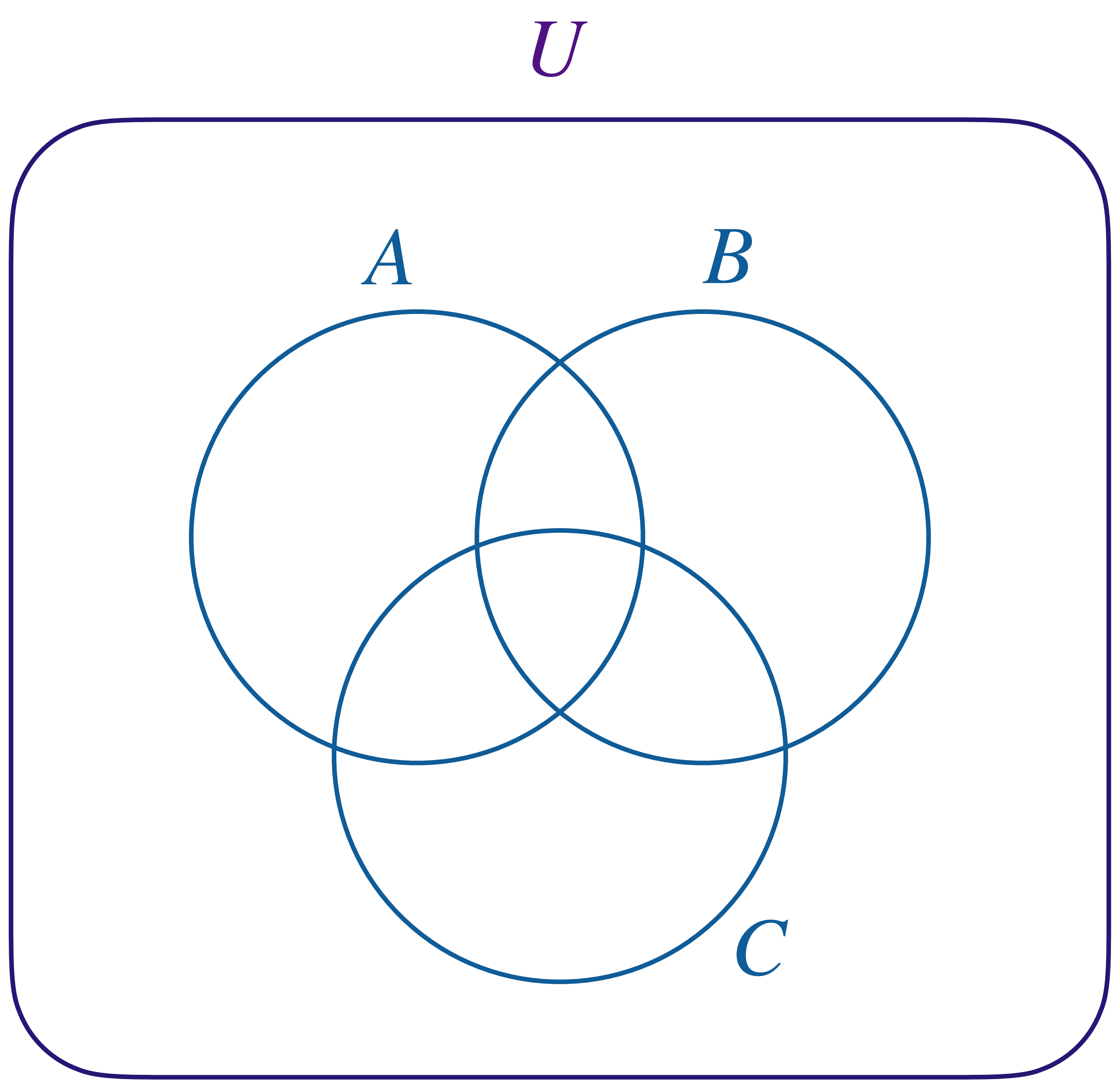
Associative Law

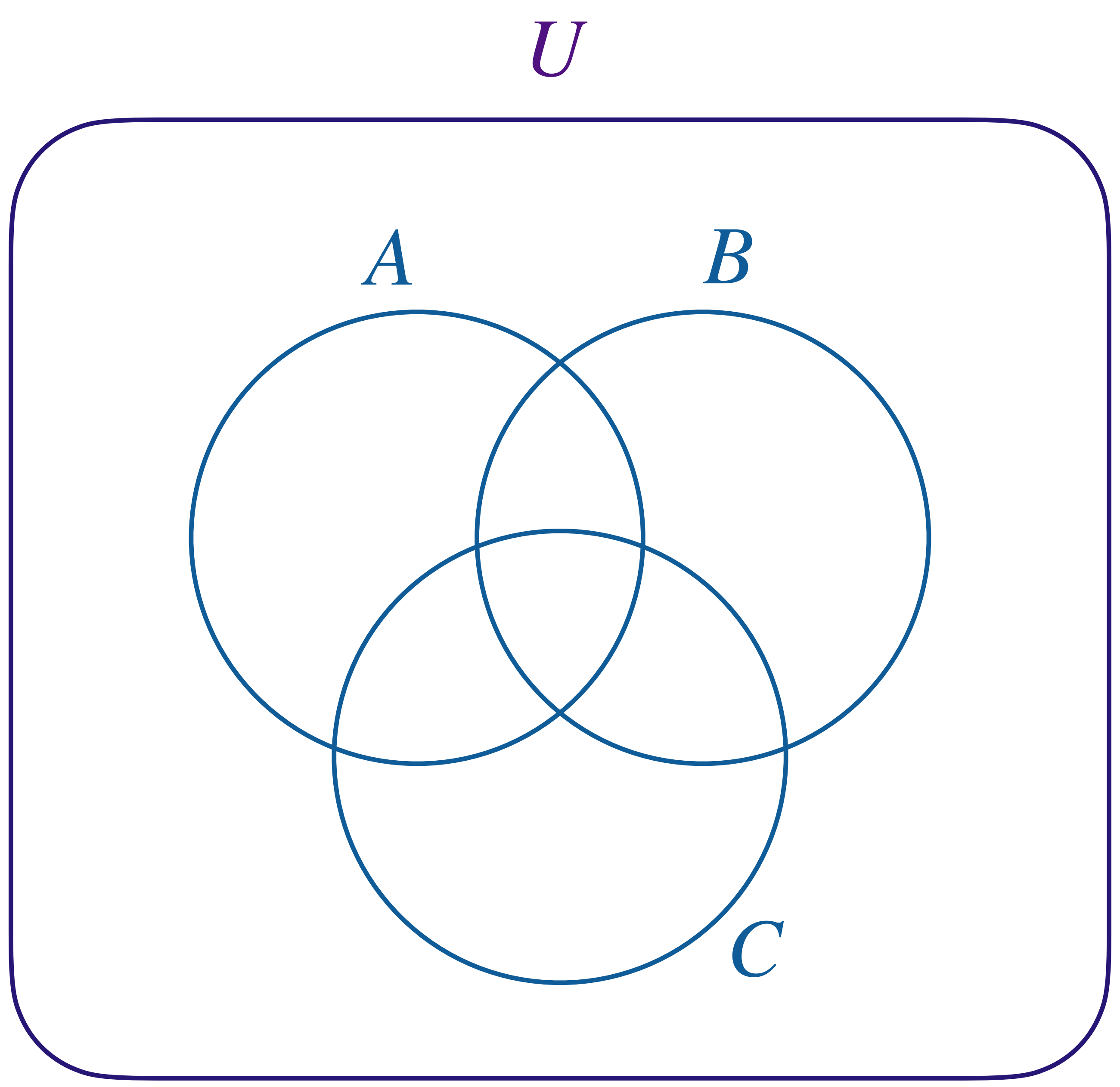
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$$





Binary Operations - Intersections and Unions

Identities

(1) $A \cup \varnothing = A$

(2) $A \cap \varnothing = \varnothing$

(3) $A \cup U = U$

(4) $A \cap U = A$

(5) $A \cup A^c = U$

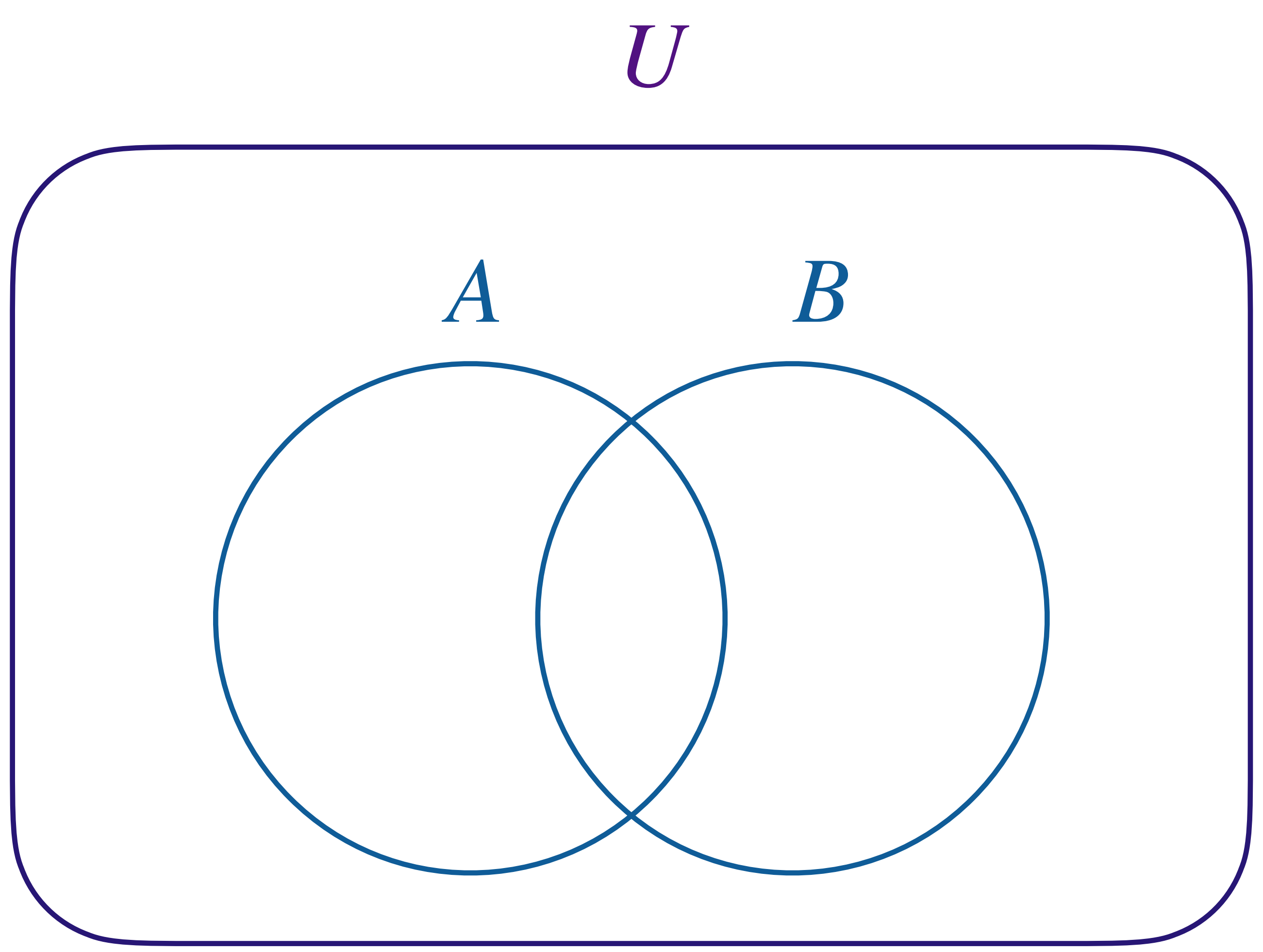
(6) $A \cap A^c = \varnothing$

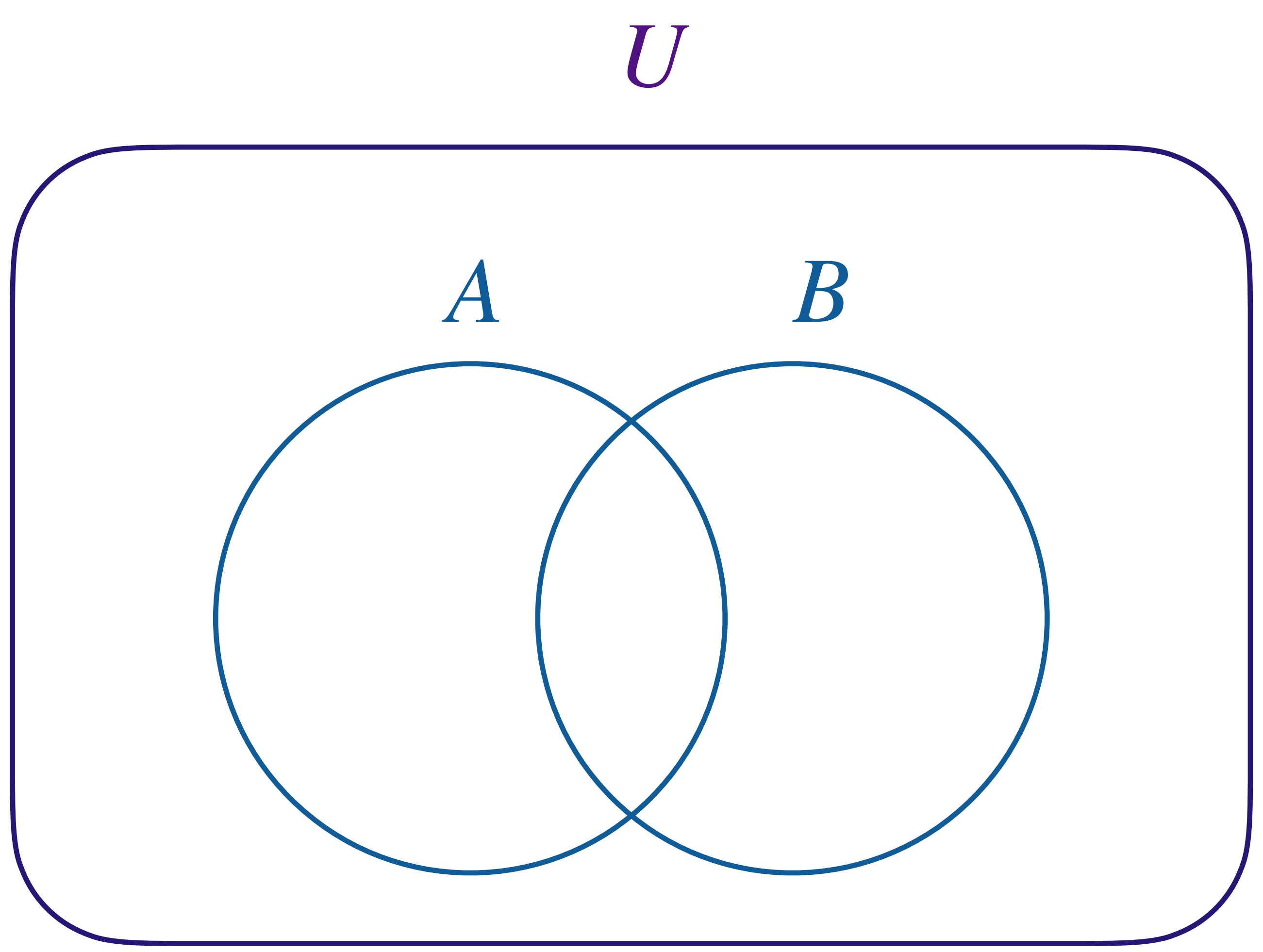
(7) $(A^c)^c = A$

Binary Operations - Intersections and Unions

De Morgan's Law

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$





Examples

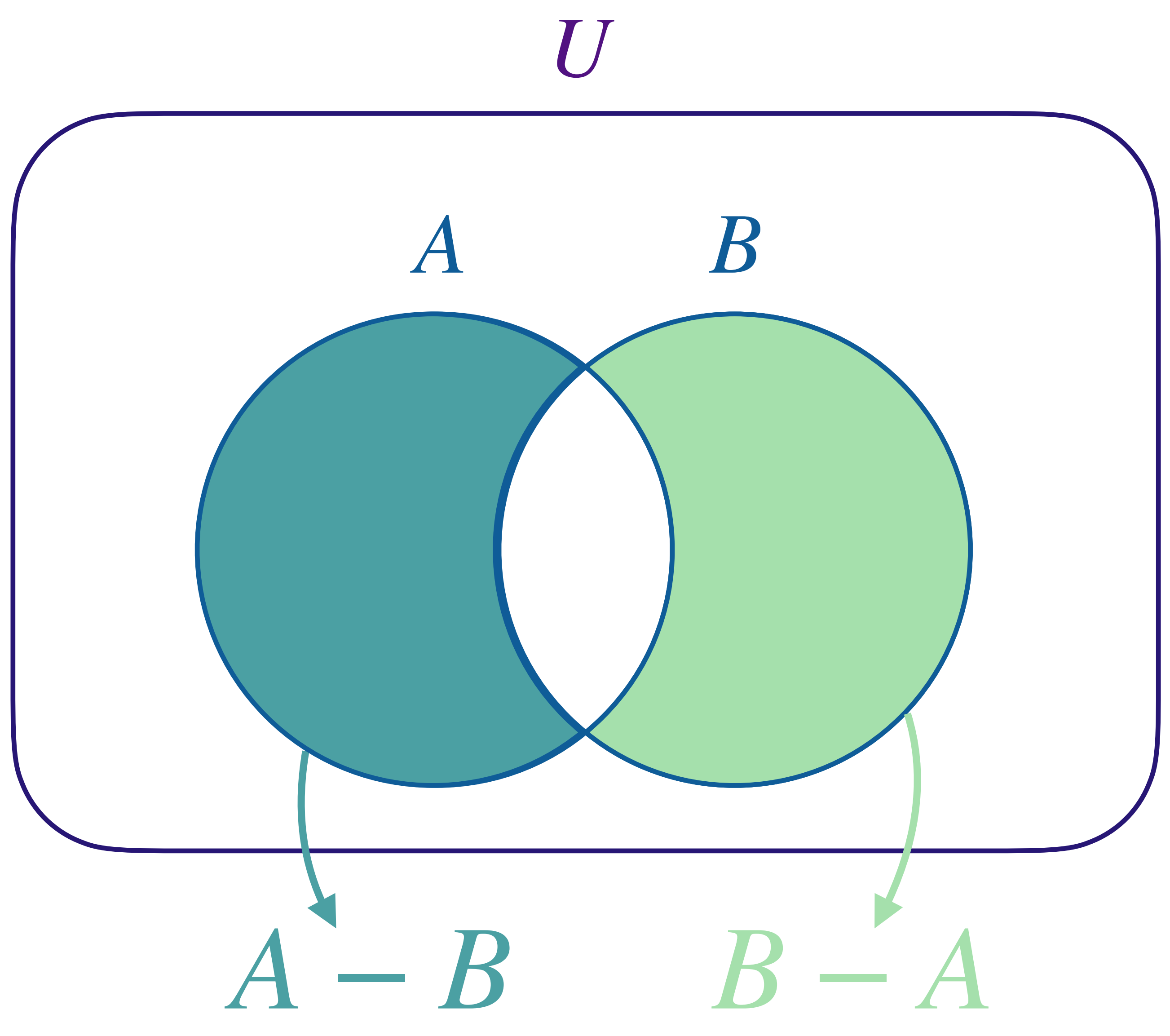
$$(A^c \cap B)^c = \left[(A^c)^c \cup B^c\right] = A \cup B^c$$
$$(A \cup B^c)^c = \left[A^c \cap (B^c)^c\right] = A^c \cap B$$

Binary Operations - Set Differences

집합 A, B에 대해 A에는 포함되고, B에는 포함되지 않은 원소들을 모은 집합을 $A - B$로 나타내고, set difference(차집합)이라고 부른다.

$$A - B = \{x \mid (x \in A) \land (x \notin B)\}$$

$A - B$는 $A \backslash B$로 표현하기도 한다.



Binary Operations - Set Differences

Computation Exercises

(1) $A - B = A \cap B^c$

(2) $B - A = B \cap A^c$

(3) $$A - B = A - (A \cap B) = (A \cup B) - B$$

(4) $$B - A = B - (A \cap B) = (A \cup B) - A$$

Binary Operations - Set Differences

Computation Exercises

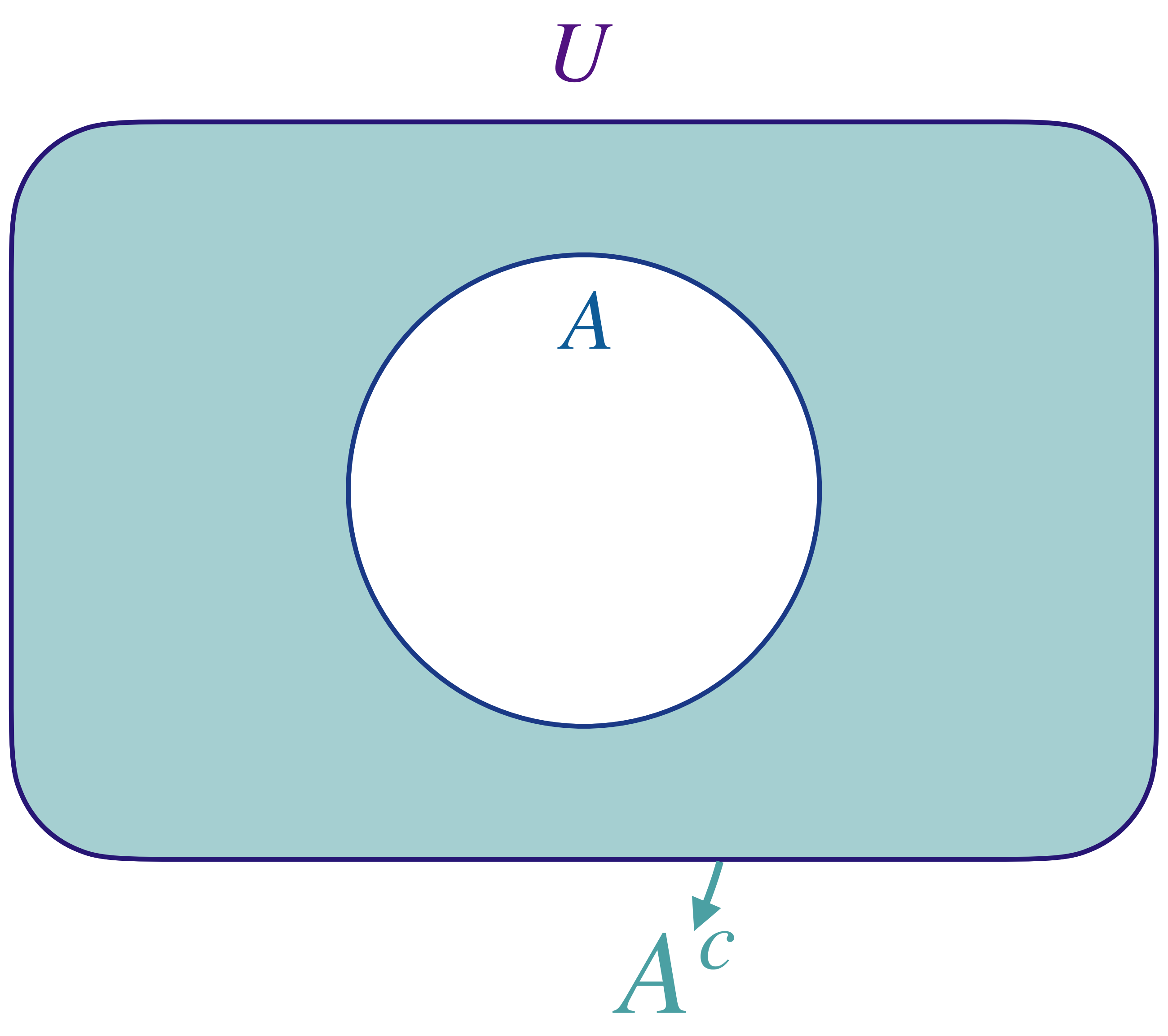
(5) $$\begin{aligned}(A-B)\cup(A\cap B) &= (A\cap B^c)\cup(A\cap B)\\ &= A\cap(B^c\cup B)\\ &= A\cap U\\ &= A\end{aligned}$$

(6) $$\begin{aligned}(B-A)\cup(A\cap B) &= (B\cap A^c)\cup(A\cap B)\\ &= (B\cap A^c)\cup(B\cap A)\\ &= B\cap(A^c\cup A)\\ &= B\cap U\\ &= B\end{aligned}$$

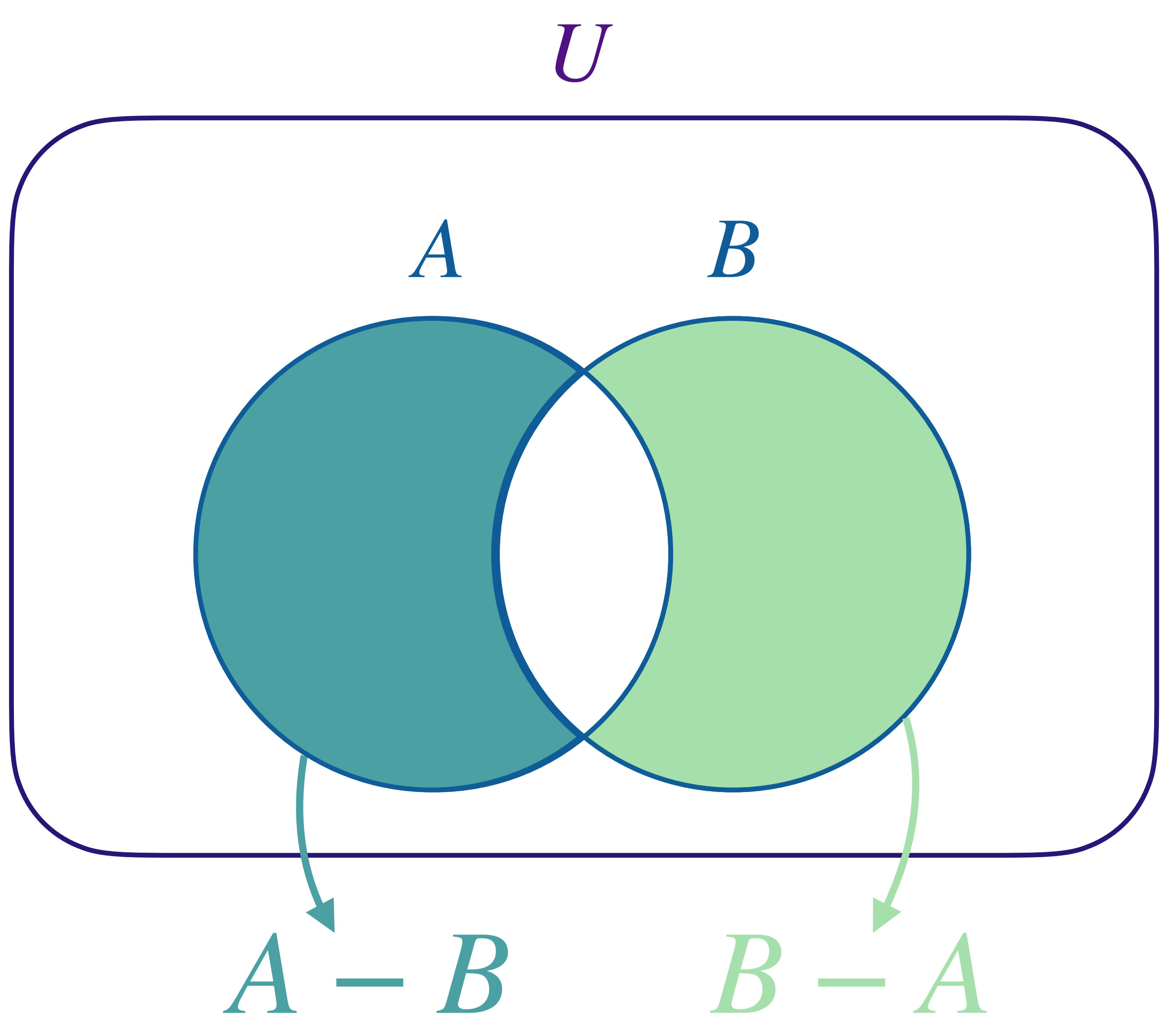
(7) $$\begin{aligned}A-(A-B) &= A-(A\cap B^c)\\ &= A\cap(A\cap B^c)^c\\ &= A\cap(A^c\cup B)\\ &= (A\cap A^c)\cup(A\cap B)\\ &= \varnothing\cup(A\cap B)\\ &= A\cap B\end{aligned}$$

Binary Operations - Set Differences

set difference는 relative complement라고 부르기도 한다.



$$U - A = U \cap A^c$$

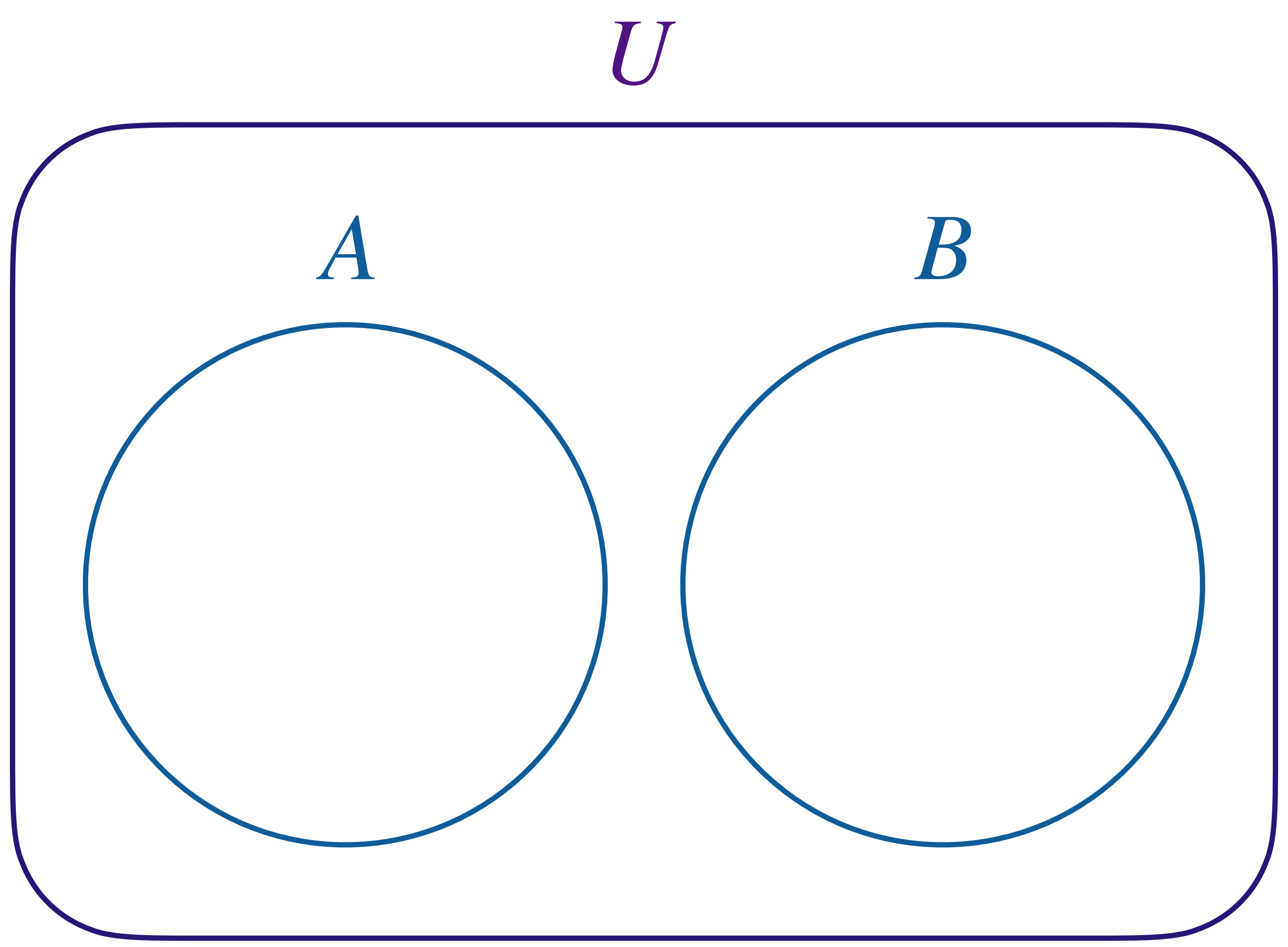


$$A - B = A \cap B^C$$

$$B - A = B \cap A^C$$

Binary Operations - Set Differences

Special Case



$$A \cap B = \varnothing$$

$$A - B = A$$
$$B - A = B$$

Binary Operations - Set Differences

The Algebraic Properties

Anti-commutativity $A - B \neq B - A$

Anti-associativity $A - (B - C) \neq (A - B) - C$

Distributive Law

$$
\begin{aligned}
(1)\ C - (A \cap B) &= C \cap (A \cap B)^c \\
&= C \cap (A^c \cup B^c) \\
&= (C \cap A^c) \cup (C \cap B^c) \\
&= (C - A) \cup (C - B)
\end{aligned}
$$

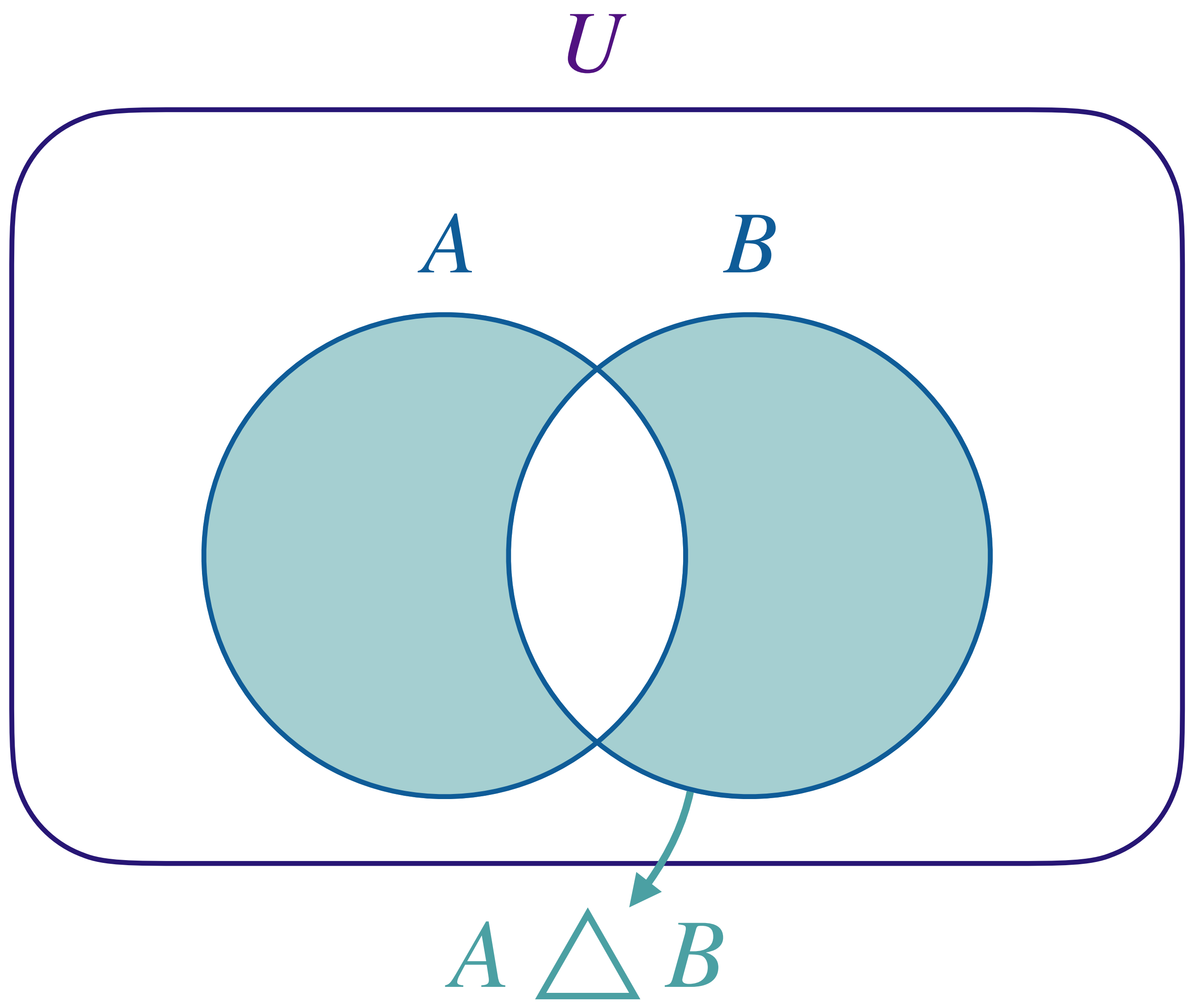
$$
\begin{aligned}
(2)\ C - (A \cup B) &= C \cap (A \cup B)^c \\
&= C \cap (A^c \cap B^c) \\
&= (C \cap A^c) \cap (C \cap B^c) \\
&= (C - A) \cap (C - B)
\end{aligned}
$$

$\cup \Longrightarrow \cap\ ,\ \ \cap \Longrightarrow \cup$

Binary Operations - Symmetric Differences

집합 A, B에 대해 $A - B$와 $B - A$의 union

$$A \triangle B = \{x \mid (A - B) \cup (B - A)\}$$
$$= \{x \mid [(x \in A) \vee (x \in B)] \wedge (x \notin A \cap B)\}$$



$$\begin{aligned} A \triangle B &= (A - B) \cup (B - A) \\ &= (A \cap B^c) \cup (B \cap A^c) \end{aligned}$$

$$A \cap B^c = X$$

$$\begin{aligned} &= X \cup (B \cap A^c) \\ &= (X \cup B) \cap (X \cup A^c) \\ &= [(A \cap B^c) \cup B] \cap [(A \cap B^c) \cup A^c] \\ &= [(A \cup B) \cap (B^c \cup B)] \cap [(A \cup A^c) \cap (B^c \cup A^c)] \\ &= [(A \cup B) \cap U] \cap [U \cap (B \cap A)^c] \\ &= (A \cup B) \cap (B \cap A)^c \\ &= (A \cup B) - (A \cap B) \end{aligned}$$

Binary Operations - Cartesian Product

집합 A, B에서 원소 a, b들을 각각 뽑아 뽑아 (a, b)를 만들 때, 모든 (a, b)들의 집합을 $A \times B$라 한다.

$$A \times B = \{(a, b) \mid (a \in A) \land (b \in B)\}$$

$$A = \{a_1, a_2, \ \dots \ , a_m\}$$
$$B = \{b_1, b_2, \ \dots \ , b_n\}$$

	b_1	b_2	$\cdots$	b_n
a_1	(a_1, b_1)	(a_1, b_2)	$\cdots$	(a_1, b_n)
a_2	(a_2, b_1)	(a_2, b_2)	$\cdots$	(a_2, b_n)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_m	(a_m, b_1)	(a_m, b_2)	$\cdots$	(a_m, b_n)

$$\begin{aligned} A \times B = \{&(a_1, b_1), (a_1, b_2), \ \dots \ , (a_1, b_n) \\ &(a_2, b_1), (a_2, b_2), \ \dots \ , (a_2, b_n) \\ &\vdots \\ &(a_m, b_1), (a_m, b_2), \ \dots \ , (a_m, b_n)\} \end{aligned}$$

Binary Operations - Cartesian Product

ex.1)

$$A = \{0, 1, 2\}, \quad B = \{a, b\} \longrightarrow A \times B$$

	a	b
0	(0,a)	(0,b)
1	(1,a)	(1,b)
2	(2,a)	(2,b)

$$A \times B = \{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\}$$

Binary Operations - Cartesian Product

ex.2)

$$A = \{0, 1, 2\} \longrightarrow A \times A = A^2$$

	0	1	2
0	(0,0)	(0,1)	(0,2)
1	(1,0)	(1,1)	(1,2)
2	(2,0)	(2,1)	(2,2)

$$\begin{aligned} A \times B = \{&(0,0), (0,1), (0,2) \\ &(1,0), (1,1), (1,2) \\ &(2,0), (2,1), (2,2)\} \end{aligned}$$

Binary Operations - Cartesian Product

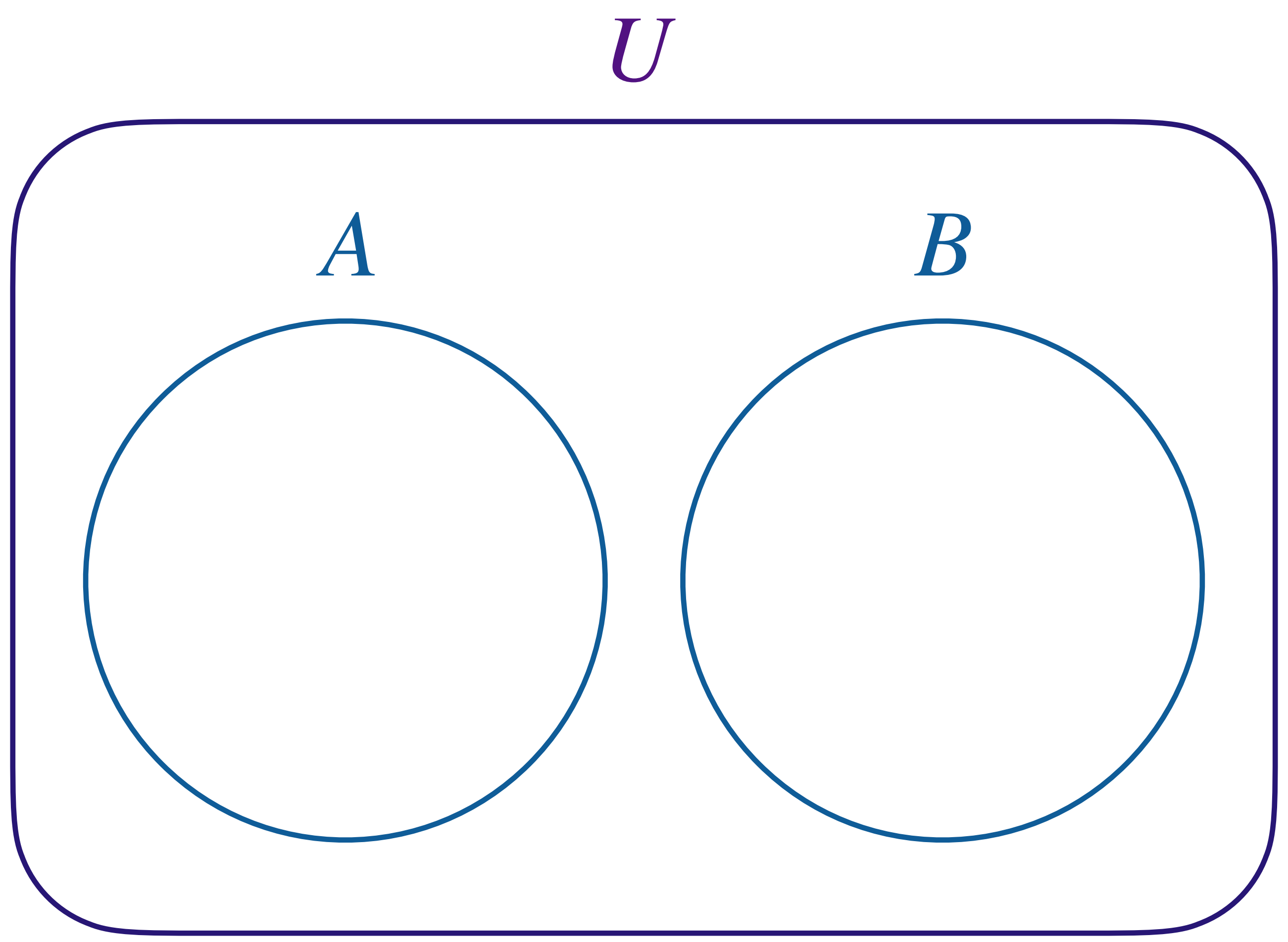
ex.3)

$$\mathbb{R} \longrightarrow \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$
$$\mathbb{R} \longrightarrow \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

Disjoint Sets

집합 A, B에 대해 $A \cap B = \varnothing$일 때 A, B는 disjoint set이다.

disjoint는 mutually exclusive라고도 부른다.



$A \cap B = \varnothing$

Cardinality

$|A \cap B| = 0$

$|A \cup B| = |A| + |B|$

Partitions of Sets

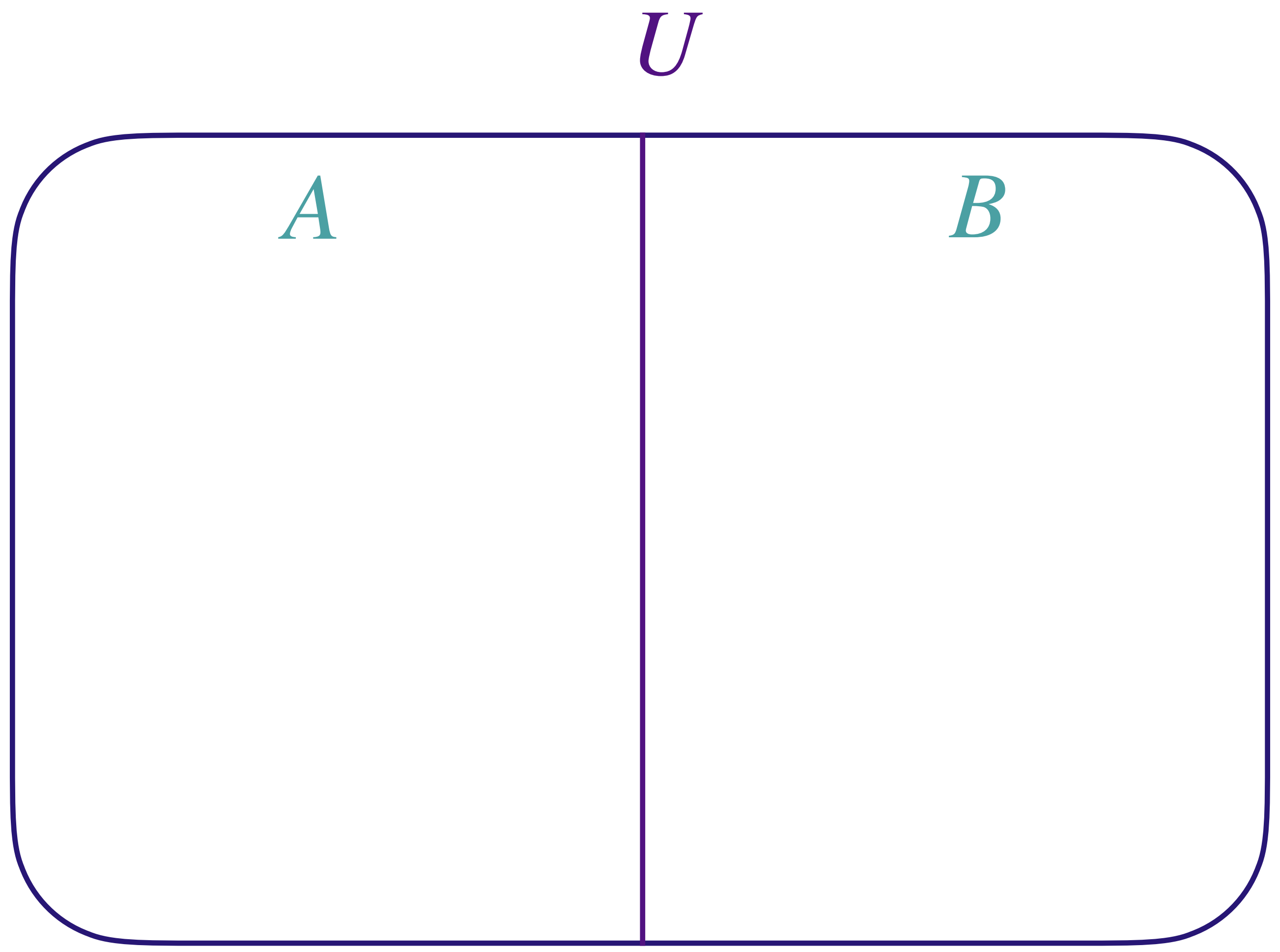
Universal set(U)안에 n개의 집합 $A_1, A_2, \ldots, A_n$이 있고, U의 모든 원소들이 모두 단 하나의 A_i에만 포함될 때, $\{A_1, A_2, \ldots, A_n\}$를 U의 partition이라 부른다.

$$A_i \cap A_j = \varnothing,\ i \neq j$$

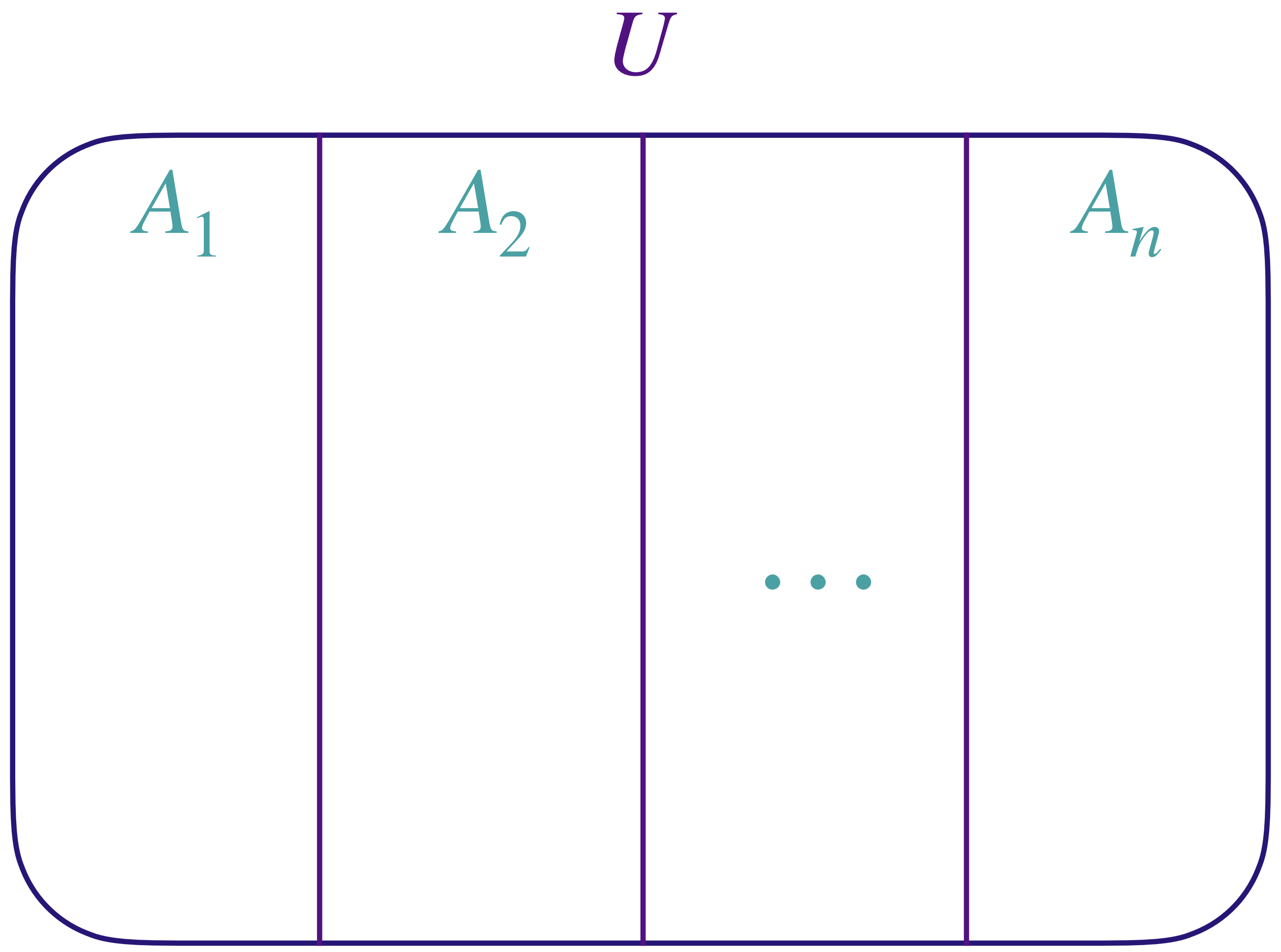
$$\bigcup_{i=1}^{n} A_i = U$$

$A \cap B = \varnothing$

$A \cup B = U$



$|A| + |B| = |U|$



$|A_1| + |A_2| + \ldots + |A_n| = |U|$

Partitions of Sets

Non-uniqueness of Partitions

Partition of Sets is NOT UNIQUE

$$A = \{x \mid 1 \leq x \in \mathbb{N} \leq 20\}$$

$A_1 = \{x \mid 1 \leq x \in \mathbb{N} \leq 10\}$
$A_2 = \{x \mid 10 < x \in \mathbb{N} \leq 20\}$

$A_1 \cap A_2 = \varnothing$
$A_1 \cup A_2 = A$

The $\{A_1, A_2\}$ is a partition of A

$A_3 = \{x \mid (1 \leq x \in \mathbb{N} \leq 20) \wedge (x = 2n)\}$
$A_4 = \{x \mid (1 \leq x \in \mathbb{N} \leq 20) \wedge (x = 2n + 1)\}$

$A_3 \cap A_4 = \varnothing$
$A_3 \cup A_4 = A$

The $\{A_3, A_4\}$ is a partition of A

$A_5 = \{x \mid (1 \leq x \in \mathbb{N} \leq 20) \wedge (x \ mod\ 3 = 0)\}$
$A_6 = \{x \mid (1 \leq x \in \mathbb{N} \leq 20) \wedge (x \ mod\ 3 = 1)\}$
$A_7 = \{x \mid (1 \leq x \in \mathbb{N} \leq 20) \wedge (x \ mod\ 3 = 2)\}$

$A_5 \cap A_6 = \varnothing$
$A_5 \cap A_7 = \varnothing$
$A_6 \cap A_7 = \varnothing$

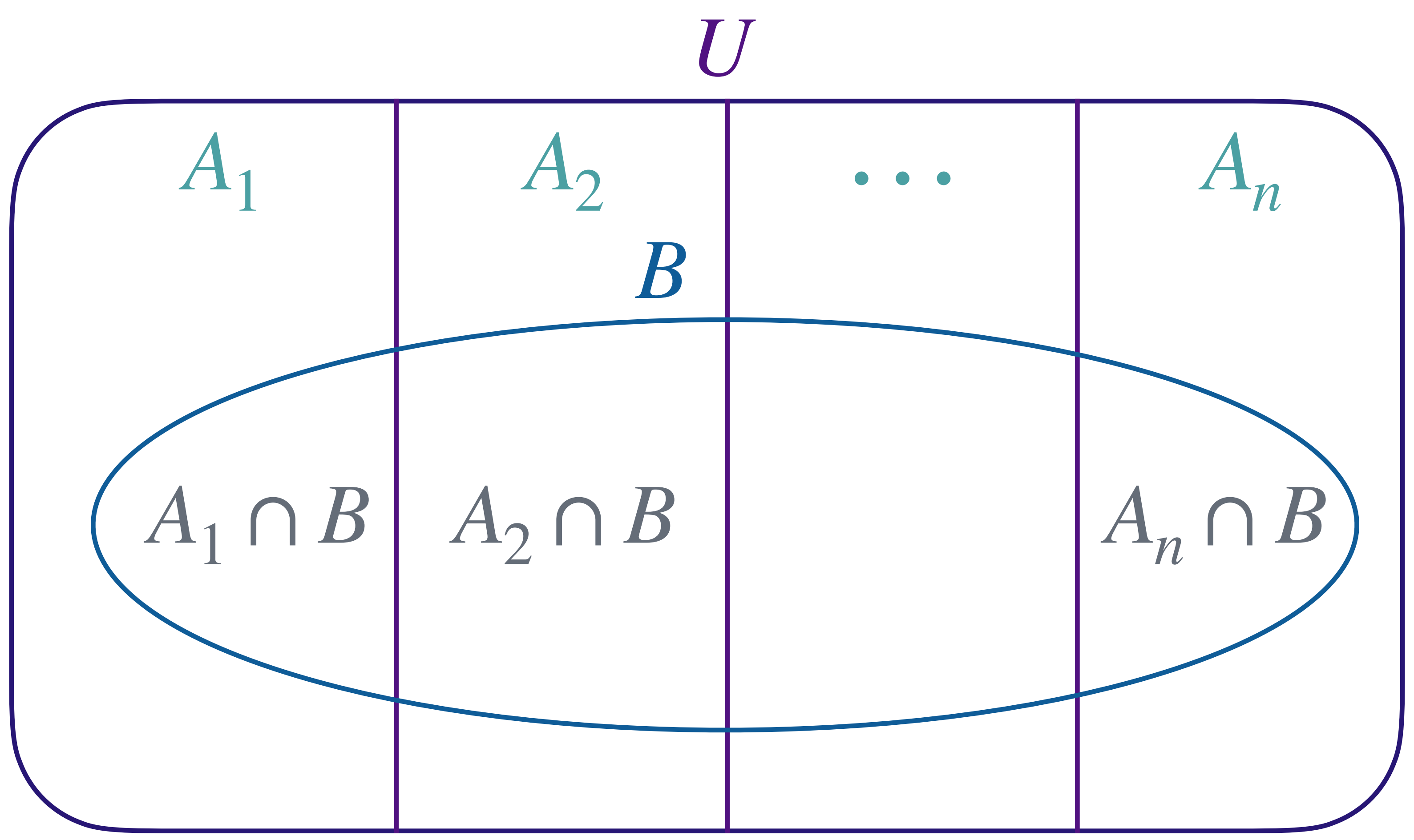
$A_5 \cup A_6 \cup A_7 = A$

The $\{A_5, A_6, A_7\}$ is a partition of A

Partitions of Sets

Making a Complete Set with Partitions

$$B = \bigcup_{i=1}^{n} (A_i \cap B)$$

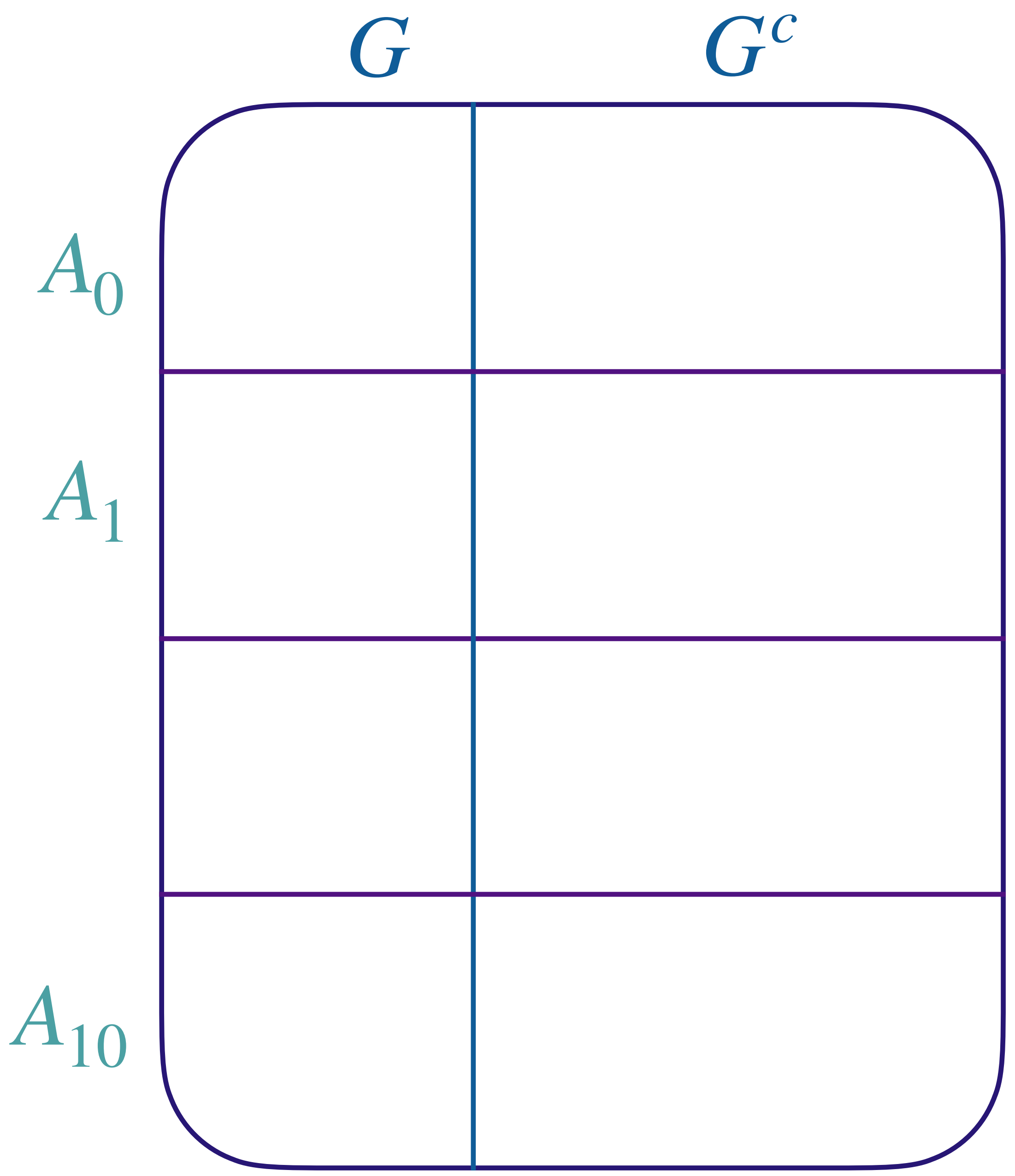


Partitions of Sets

ex.1)

A_i : 나이(x)가 $10 \cdot i \leq x \leq 10 \cdot (i+1)$인 사람들의 집합

G : 안경을 쓴 사람들의 집합



$$G = \bigcup_{i=1}^{10} (A_i \cap G)$$

$$G^c = \bigcup_{i=1}^{10} (A_i \cap G^c)$$

CLOSING

Basic Algebra

Chap.2 Sets