Stochastic Uncoupled Dynamics and Nash Equilibrium

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Introduction

Definition (Uncoupledness)

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Hart and Mas-Colell '03

There are no deterministic uncoupled stationary dynamics that guarantee almost sure convergence to pure Nash equilibria in all games where such equilibria exist.

Introduction

The Bad News

	α	β	γ
α	$1, 0$	$0, 1$	$1, 0$
β	$0, 1$	$1, 0$	$1, 0$
γ	$0, 1$	$0, 1$	$1, 1$

Figure: A simple two-player game U

Observation

In each action combination a at least one of the two players is best-replying.

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Lemma

Under uncoupled dynamics f, if player i is best-replying in state a he will play the same move again.

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	α	β	γ
α	$\mathbf{1, 2}$	$0, 1$	$1, 0$
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γ	$0, 1$	$0, 1$	$\mathbf{1, 0}$

Figure: Another two-player game U'

Proof 1/2.

- Pick some a where player 1 is best-replying
- Create a new game $U' = (u^1, \bar{u}^2)$ such that a becomes the unique Nash equilibria

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Proof 2/2.

- The dynamics f converge and thus neither player will move away from a
- Yet by uncoupledness $f^1(U) = f^1(U')$

$\square$

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Observation 2

In any action combination a in which only player i plays γ, player i is not best-replying and thus player j is.

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Conclusion.

It follows that the state (γ, γ) can never be reached when starting from any other state. $\square$

Introduction

But ...

What if players could **remember** previous plays?
What if they had **memories**?

Recall

Influence from the Past

Definition

A strategy has R-**recall** if only the last R action combinations matter, i.e. f^i is of the form $f^i(u^i; a(t-R), ..., a(t-1))$.

Recall

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A strategy has R-**recall** if only the last R action combinations matter, i.e. f^i is of the form $f^i(u^i; a(t-R), ..., a(t-1))$.

Theorem

There exist uncoupled, 2-recall, stationary strategy mappings that guarantee almost sure convergence to pure Nash equilibria in every game where such equilibria exist.

Proof

Definition (State)

A state is identified as the play of the two previous periods $(a', a) := (a(t-1), a(t)) \in A \times A$.

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Strategy f^i

- if $a' = a$ and a^i is a best reply of player i to a^{-i} then player i plays the same action a^i;
- otherwise player i picks an action $\bar{a}^i$ uniformly at random from A^i

Proof

	$a' = a$	$a' \neq a$
$a \in$ PNE	$\mathbf{S}_1$	$\mathbf{S}_2$
$a \notin$ PNE	$\mathbf{S}_4$	$\mathbf{S}_3$

Figure: State space S partition

Proof

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Each state in S_1 is absorbing.

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- if $a' = a$ and a^i is a best reply of player i to a^{-i} then player i plays the same action a^i;

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Lemma

For all states $s \in S_2 \cup S_3 \cup S_4$ there is a strictly positive probability $p > 0$ to reach a state $s' \in S_1$ in finitely many periods.

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S_2.

All players randomly pick a^i again $\Rightarrow (a, a) \in S_1$

Proof

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Lemma

For all states $s \in S_2 \cup S_3 \cup S_4$ there is a strictly positive probability $p > 0$ to reach a state $s' \in S_1$ in finitely many periods.

S_3.

All players randomly pick some $\bar{a}^i$ s.t. $\bar{a} \in$ PNE $\Rightarrow (a, \bar{a}) \in S_2$

Proof

	$a' = a$	$a' \neq a$
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Lemma

For all states $s \in S_2 \cup S_3 \cup S_4$ there is a strictly positive probability $p > 0$ to reach a state $s' \in S_1$ in finitely many periods.

S_4.

Some player randomly picks $\bar{a}^i \Rightarrow (a, \bar{a}) \in S_2 \cup S_3$

Proof

	$a' = a$	$a' \neq a$
$a \in$ PNE	$\mathbf{S}_1$	$\mathbf{S}_2$
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Conclusion.

f induces an absorbing Markov Chain over S. $\square$

Can We Go Further?

What about **Mixed** Nash Equilibria?

Mixed Equilibira

Theorem

*For every small enough $\epsilon > 0$, there are no uncoupled, **finite recall**, stationary strategy mappings f that guarantee in every game, the almost sure convergence of the behavior probabilities to Nash ϵ-equilibria.*

Proof

	α	β
α	$1, 0$	$0, 1$
β	$0, 1$	$1, 0$

(a) $U = (u^1, u^2)$

	α	β
α	$1, 1$	$0, 0$
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(b) $U' = (u^1, \bar{u}^2)$

Proof

$$\begin{array}{c|c|c|} & \alpha & \beta \\ \hline \alpha & 1,0 & 0,1 \\ \hline \beta & 0,1 & 1,0 \\ \hline \end{array}$$

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(b) $U' = (u^1, \bar{u}^2)$

Observation

- Unique equilibria in U is $\underline{x}$ with $\underline{x}^i = (0.5, 0.5)$
- Unique equlibiria in U' is $\underline{a} = (\alpha, \alpha)$

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(b) $U' = (u^1, \bar{u}^2)$

Suppose f exists.

- f assigns $x^i(\alpha) > 0$ in both U and U'

$\square$

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Suppose f exists.

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- We eventually reach the state $s = (\underline{a}, ..., \underline{a})$ in both games

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- $f^1(s)(\alpha)$ should then be close to the unique Nash Equilibrium

□

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Suppose f exists.

- f assigns $x^i(\alpha) > 0$ in both U and U'
- We eventually reach the state $s = (\underline{a}, ..., \underline{a})$ in both games
- $f^1(s)(\alpha)$ should then be close to the unique Nash Equilibrium
- Contradicts uncoupledness!

$\square$

But...

What about if players had arbitrary **memories**?

Memory

No Continuity Restriction

Definition

A player's strategy f^i has finite R-memory if it can be implemented by a finite-state automaton in $|A|^R$ states.

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No Continuity Restriction

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A player's strategy f^i has finite R-memory if it can be implemented by a finite-state automaton in $|A|^R$ states.

Theorem

For every $\epsilon > 0$ there exists an uncoupled, R-memory, stationary strategy mapping that guarantees the almost sure convergence of the behavior probabilities to a Nash ϵ-equilibrium $\underline{x}$.

Proof

Buckle in

Definition