Stochastic Uncoupled Dynamics and Nash Equilibrium

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May 2017

Introduction

Definition (Uncoupledness)

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Hart and Mas-Colell '03

There are no deterministic uncoupled stationary dynamics that guarantee almost sure convergence to pure Nash equilibria in all games where such equilibria exist.

Introduction

	α	β	γ
α	$1, 0$	$0, 1$	$1, 0$
β	$0, 1$	$1, 0$	$1, 0$
γ	$0, 1$	$0, 1$	$1, 1$

Figure: A simple two-player game U

Observation

In each action combination a at least one of the two players is best-replying.

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Lemma

Under uncoupled dynamics f, if player i is best-replying in state a he will play the same move again.

	α	β	γ
α	1, **2**	0, 1	1, 0
β	0, 1	1, 0	1, 0
γ	0, 1	0, 1	1, **0**

Figure: Another two-player game U'

Proof 1/2.

- Pick some a where player 1 is best-replying
- Create a new game $U' = (u^1, \bar{u}^2)$ such that a becomes the unique Nash equilibria

Introduction

The Bad News

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α	1, **2**	0, 1	1, 0
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Figure: Another two-player game U'

Proof 2/2.

▶ The dynamics f converge and thus neither player will move away from a

▶ Yet by uncoupledness $f^1(U) = f^1(U')$

$\square$

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Observation 2

In any action combination a in which only player i plays γ, player i is not best-replying and thus player j is.

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Conclusion.

It follows that the state (γ, γ) can never be reached when starting from any other state. $\qquad \square$

What if players could **remember** previous plays?
What if they had **memories**?

Recall
Influence from the Past

Definition

A strategy has R-**recall** if only the last R action combinations matter, i.e. f^i is of the form $f^i(u^i; a(t-R), ..., a(t-1))$.

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Definition

A strategy has R-**recall** if only the last R action combinations matter, i.e. f^i is of the form $f^i(u^i; a(t - R), ..., a(t - 1))$.

Theorem

There exist uncoupled, 2-recall, stationary strategy mappings that guarantee almost sure convergence to pure Nash equilibria in every game where such equilibria exist.

Proof

Definition (State)

A state is identified as the play of the two previous periods
$(a', a) := (a(t-1), a(t)) \in A \times A$.

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Strategy f^i

- if $a' = a$ and a^i is a best reply of player i to a^{-i} then player i plays the same action a^i;
- otherwise player i picks an action $\bar{a}^i$ uniformly at random from A^i

Proof

	$a' = a$	$a' \neq a$
$a \in$ PNE	$\mathbf{S}_1$	$\mathbf{S}_2$
$a \notin$ PNE	$\mathbf{S}_4$	$\mathbf{S}_3$

Figure: State space S partition

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Each state in S_1 is absorbing.

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Lemma

For all states $s \in S_2 \cup S_3 \cup S_4$ there is a strictly positive probability $p > 0$ to reach a state $s' \in S_1$ in finitely many periods.

Proof

$$
\begin{array}{c|c|c|}
 & a' = a & a' \neq a \\
\hline
a \in \text{PNE} & \mathbf{S}_1 & \mathbf{S}_2 \\
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S_2.

All players randomly pick a^i again $\Rightarrow (a, a) \in S_1$

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S_3.

All players randomly pick some $\bar{a}^i$ s.t. $\bar{a} \in$ PNE $\Rightarrow (a, \bar{a}) \in S_2$

Proof

	$a' = a$	$a' \neq a$
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S_4.

Some player randomly picks $\bar{a}^i \Rightarrow (a, \bar{a}) \in S_2 \cup S_3$

Proof

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$a \in \mathsf{PNE}$	$\mathbf{S}_1$	$\mathbf{S}_2$
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Conclusion.

f induces an absorbing Markov Chain over S. $\qquad\square$

Can We Go Further?

What about **Mixed** Nash Equilibria?

Mixed Equilibira

Theorem

*For every small enough $\epsilon > 0$, there are no uncoupled, **finite recall**, stationary strategy mappings f that guarantee in every game, the almost sure convergence of the behavior probabilities to Nash ϵ-equilibria.*

Proof

	α	β
α	1, 0	0, 1
β	0, 1	1, 0

(a) $U = (u^1, u^2)$

	α	β
α	1, 1	0, 0
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(b) $U' = (u^1, \bar{u}^2)$

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Observation

- Unique equilibria in U is $\underline{x}$ with $\underline{x}^i = (0.5, 0.5)$
- Unique equilibiria in U' is $\underline{a} = (\alpha, \alpha)$

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Suppose f exists.

- f assigns $x^i(\alpha) > 0$ in both U and U'

$\square$

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- We eventually reach the state $s = (\underline{a}, ..., \underline{a})$ in both games

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- $f^1(s)$ should then be close to the unique Nash Equilibrium

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Suppose f exists.

- f assigns $x^i(\alpha) > 0$ in both U and U'
- We eventually reach the state $s = (\underline{a}, ..., \underline{a})$ in both games
- $f^1(s)$ should then be close to the unique Nash Equilibrium
- Contradicts uncoupledness since $f^1(U; s) = f^1(U'; s)$

$\square$

But...

What about if players had arbitrary **memories**?

Definition

A player's strategy f^i has finite R-memory if it can be implemented by a finite-state automaton in $|A|^R$ states.

Memory
No Continuity Restriction

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A player's strategy f^i has finite R-memory if it can be implemented by a finite-state automaton in $|A|^R$ states.

Theorem

For every $\epsilon > 0$ there exists an uncoupled, R-memory, stationary strategy mapping that guarantees the almost sure convergence of the behavior probabilities to a Nash ϵ-equilibrium $\underline{x}$.

Proof

Construction 1/2

- Find some K such that there is a Nash ϵ-Equilibrium $y = (y^1, ..., y^N)$ with all probabilities being multiples of $1/K$.

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- The state $\tilde{s}^i$ of player i are his R memories $(a_0, a_1, ...a_{2K})$

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- s^i denotes his $2K$ most recent memories $(a_1, ..., a_{2K})$

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- The state $\tilde{s}^i$ of player i are his R memories $(a_0, a_1, ...a_{2K})$
- s^i denotes his $2K$ most recent memories $(a_1, ..., a_{2K})$
- Here all players will have the same memories $\tilde{s}$!

Proof

Definition

We denote by z^i the frequency distribution of actions from player i over his K most recent memories $(a_{K+1}, ..., a_{2K})$.

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Example

For the $K = 5$ memories $((\alpha, \alpha), (\alpha, \beta), (\beta, \gamma), (\gamma, \gamma), (\gamma, \gamma))$:

$$z^1(\alpha) = 2/5, \quad z^1(\beta) = 1/5, \quad z^1(\gamma) = 2/5$$
$$z^2(\alpha) = 1/5, \quad z^2(\beta) = 1/5, \quad z^2(\gamma) = 3/5$$

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  - otherwise, **he breaks his periodic play**

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The new state of f^i becomes $\tilde{s}' = (a_1, ..., a_{2K}, \bar{a})$.

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if $\tilde{s} = (a_0, s)$ is K-periodic (Mode II)

- player i plays the mixed action z^i

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if $\tilde{s} = (a_0, s)$ is K-periodic (Mode II)

- player i plays the mixed action z^i

The state of f^i remains the same $\tilde{s}' = \tilde{s}$.