Stochastic Uncoupled Dynamics and Nash Equilibrium

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May 2017

Table of Contents

Introduction

Definition (Uncoupledness)

A dynamic process in a game is called **uncoupled** if the strategy of each player does not depend on the utility/payoff function of other players.

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Definition (Uncoupledness)

A dynamic process in a game is called **uncoupled** if the strategy of each player does not depend on the utility/payoff function of other players.

Hart and Mas-Colell '03

There are no deterministic uncoupled stationary dynamics that guarantee almost sure convergence to pure Nash equilibria in all games where such equilibria exist.

Introduction

What if players could remember previous plays?
What if they had memories?

Table of Contents

Static Setup

you already know this

Definition (Static game)

- $N \geq 2$ players denoted by $i \in \{1, 2, ..., N\}$
- A finite set of actions A^i for each player i
- The set of action combinations $A := A^1 \times A^2 \times ... \times A^N$
- A payoff (or utility) function $u^i : A \to \mathbb{R}$ for each player i

We then identify a game by its payoff functions $U := (u^1, ..., u^N)$.

Static Setup

you already know this

Definition (randomized or mixed actions)

Player i assigns a probability $x^i(a)$ to each action $a \in A^i$.

- The set of all mixed actions for player i is $\Delta(A^i)$
- We denote by $\Delta := \Delta(A^1) \times ... \times \Delta(A^N)$ the set of randomized action combinations or N-tuples.
- The payoff function is multi-linearly extended to $u^i : \Delta \to \mathbb{R}$

Nash Equilibria

should also look familiar

Definition (best replying)

We say that the randomized actions $x^i \in \Delta(A^i)$ is an ϵ-*best reply* to $x^{-i} := (x^1, ..., x^{i-1}, x^{i+1}, ..., x^N)$ if for all $y^i \in \Delta(A^i)$:

$$u^i(x) \geq u^i(y^i, x^{-i}) - \epsilon$$

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Definition (Nash ϵ-equilibrium)

A *Nash ϵ-equilibrium* is a randomized action combination $\underline{x} = (\underline{x}^1, ..., \underline{x}^N) \in \Delta$ such that each $\underline{x}^i$ is an ϵ-best reply to $\underline{x}^{-i}$.

Dynamic Setup

Definition (History of Play)

For repeated play of U at discrete time periods $t = 1, 2, ...$

- $a^i(t) \in A^i$ the action of player i at time t
- $a(t) = (a^1(t), ..., a^N(t)) \in A$ the action combination at t
- $(a(1), ..., a(t-1)) \in H$ the history of play

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Definition (Strategies)

- $f^i : H \to \Delta(A^i)$ the **stationary** strategy of player i
- $f(U) = (f^1(u^1), ..., f^N(u^n))$ the **uncoupled** strategy mapping for U

Recall

Influence from the past

Definition

A strategy has R-**recall** if only the last R action combinations matter, i.e. f^i is of the form $f^i(a(t-R), ..., a(t-1))$ for all $t > R$.

Table of Contents

The Bad News

Theorem

There are no uncoupled, 1-recall, stationary strategy mappings that guarantee almost sure convergence to pure Nash equilibria in all games where such equilibria exist.

The Bad News
Proof

	α	β	γ
α	$1,0$	$0,1$	$1,0$
β	$0,1$	$1,0$	$1,0$
γ	$0,1$	$0,1$	$1,1$

Figure: A simple two-player game U

Observation

In each action combination $a(t)$ at least one of the two players is best-replying.

The Bad News

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In each action combination $a(t)$ at least one of the two players is best-replying.

Lemma

If player i is best-replying in state $a(t)$ he will play the same move at $t+1$

The Bad News
Proof

	α	β	γ
α	$1, \mathbf{2}$	$0, 1$	$1, 0$
β	$0, 1$	$1, 0$	$1, 0$
γ	$0, 1$	$0, 1$	$1, \mathbf{0}$

Figure: Another two-player game U'

Proof 1/2.

- Pick $a(t)$ where player 1 is best-replying
- Create a new game $U' = (u^1, \bar{u}^2)$ such that $a(t)$ becomes the unique Nash equilibria

The Bad News

Proof

	α	β	γ
α	$\mathbf{1, 2}$	$0, 1$	$1, 0$
β	$0, 1$	$1, 0$	$1, 0$
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Figure: Another two-player game U'

Proof 2/2.

- The strategy mapping f converges and is 1-recall and thus neither player will move from $a(t)$ at $t+1$
- Yet by uncoupledness $f^1(U) = f^1(U')$

$\square$

The Bad News

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Figure: The initial two-player game U

Observation 2

In any action combination $a(t)$ in which only player i plays γ, player i is not best-replying and thus player j is.

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Proof

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Observation 2

In any action combination $a(t)$ in which only player i plays γ, player i is not best-replying and thus player j is.

Conclusion.

It follows that the state (γ, γ) can never be reached when starting from any other state. $\square$

Recall to the Rescue

Theorem

There exist uncoupled, 2-recall, stationary strategy mappings that guarantee almost sure convergence to pure Nash equilibria in every game where such equilibria exist.

Table of Contents

Table of Contents

Table of Contents

Table of Contents

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