Fluid Mechanics and Elasticity

Mini-Project

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Note: The language being used in this report is inspired heavily by the paper written by John Guckenheimer and language used in other studies will be substituted by the above.

A quick Heads-Up

On researching for this material, one would notice that even in the published research papers on this topic there is not much mathematics. Most of the papers on the matter are in the format of articles that discuss quantitatively the nature of chaotic analysis in turbulent flow. I have thus chosen to omit all mathematics from this report in favour of a compilation of known results in the topic and the some basic mathematical definitions have been substituted with a written explanation of the above scenario. Thus, I have chosen to write the report in the format of a comprehensive collection of data and inferences from the influential works on the matter.

The Goal

The final goal in any fluid mechanics problem is the computation of the velocity field constrained by the Navier-Stokes equation and the boundary conditions. The motivation behind using the strange attractor theory to analyse fluid flow is the form of the differential equations governing fluid flow. The right-hand side of the equation is the proper time derivative of the velocity, aka the acceleration and the right-hand side is the tally of the forces applied on the fluid, all of which are functions of velocity. The governing equations may then be written in the form

$$\dot{v} = f(v)$$

Where, v represents the collection of velocities v_i.

The above form of the equation looks similar to a problem that may be analysed using chaos theory and the strange attractor.

The intuitive reason behind the chaotic theory approach is that it would seem very obvious that turbulence is a chaotic behaviour. Turbulence is very initial condition dependant and geometrically very complex. It is often impossible (using current mathematical models and tools) to write solutions of turbulent flows analytically unless the scenario is narrowed quite a bit and many assumptions/approximations are applied. This is the hallmark of a chaotic system. Thus, it would seem natural to try and apply the framework and extensive knowledge of the chaotic theory to the problem of turbulent fluid flow.

Approaches of Analysis using Strange Attractor Theory

There are three paths through which one may understand the behaviour of a turbulent system using strange attractor theory

1. The theory may be used to predict new aspects of turbulent behaviour which would then suggest the parameters values to be used and dynamical variables to be measured in an experimental setup to obtain the desired predicted observations.
   An example would be the period doubling bifurcation for the route to transition into chaotic behaviour (Feigenbaum transitions).
2. The analysis of the density of certain behaviours on the strange attractor can be done using the framework of chaos theory. This then allows us to classify behaviours as either what Guckenheimer referred to as typical and atypical behaviour. Atypical behaviour is classified as the behaviours exhibited by a fluid strictly under the constraints of very narrow intersection of the loci of parameters. Essentially what this means is that the above behaviour is extremely specific, so specific in fact, that it is considered an experimental anomaly and disregarded in most analysis due to its essential "non-existence" in real world fluid flows and in experimental setups. This allows us to classify and also narrow down the set of solutions we are interested in and also provides insight onto what sort of flows are physically possible but implausible in the real world.
3. A direct numerical computation by feeding in data from experiments to a computer and a comparison of the generated and experimental data.

   The issues regarding analysis of turbulent fluid systems is not only the mathematical complexity of the equations, but also the fact that chaos theory furnishes data that may not be easily interpreted and the experimental measurement of dynamical quantities of interest is a task that's easier said than done.

   Thus, the analysis is mostly restricted to
   1. Exploring "scenarios" that describe the route to chaos.
      This method has been applied to study turbulent flow in Taylor-Couette flow to a good amount of success.
   2. Understanding the geometric structure of the strange attractor of the system to determine its properties

   Only the first point will be covered in this report.

Power Series and Definitional Chaos

The power series is essentially the discreet fourier series of the dynamic variable. It produces a function of frequencies that spikes up (appropriate to the magnitude of the behaviour described by the frequency) at specific frequencies.

If the power series of a fluid flow has broadband components, it is said that the attractor is chaotic in nature while narrower spikes in the power series correspond to typical periodic behaviour.

What is a strange attractor?

An attractor is an object to which all but a small subset of points in the phase space converge to at large timescales. Attractors with complex geometrical structure are furnished the name strange attractor in academia.

Bifurcations

Bifurcations describe, mathematically, the phenomenon of qualitative change in velocity fields. It is a “function” of parameters involved and the goal is the find the loci of the bifurcation in parameter space. This, however, becomes a task that becomes tougher as the dimension of the dynamical system increases. Clearly, this would mean that the computations of a continuum mechanics problems are essentially impossible. Thus, most of the study is a qualitative look at the mathematical setup and the experimental results. Bifurcations are the main focus of our analysis in fluid problems as they describe changes in qualitative behaviour of the flow, or in other words describe the onset and nature of turbulence.

The “scenarios” that describe the route to chaos (Theory)

The method of analysis of the route to chaos is to vary a parameter through a significant portion of its range, allowing steady/non-turbulent motion to begin with and eventually leading to full blown chaos. The qualitative changes that occur on the route to chaos are often observed to be discreet and threshold/cut-off values of parameters and regularities in the sequence of the type of bifurcation occurring can be observed in different systems. This leads to some level of predictability in the qualitative nature of the flow. The eigenvalues of dynamical variables in fluids commonly form a set of 1, -1 and arbitrary complex values. These correspond to tangent, period doubling and quasi-periodic scenarios respectively. It can then be seen that on the progression of these parameters into the regime of quasi periodic scenarios leads to chaotic behaviour.

The eigen values of 1 and -1 are associated with the creation and destruction of periodic orbits. The eigen value -1 has the more interesting property that orbits formed by the bifurcation result in a doubling of period. This is a well-studied phenomenon and is sometimes even described as the Feigenbaum transition.

The complex eigen values lead to formation of 2-d invariant flow on tori. The dyanmics of the tori are dominated by power series functions with 2 distinct spikes with rational independence of frequencies that describe phase locked periodic solutions.

Symmetries, forces or any other factor that constraints some of the variables of motion are known to have effects on the appearance of bifurcations. Constraints usually kill of certain bifurcation behaviours.

Above is the general framework of analysis of a continuum mechanics problem using the framework of strange attractor theory.

The “scenarios” that describe the transition to chaos (Experiment)

There are 2 scenarios in which the attractor theory of turbulence has been applied and tested experimentally.

This is where the chaos theory of turbulence hits a brick wall. The major issues with experiments to verify the attractor theory of turbulence

1. The measurement of dynamic variables in fluid flow is highly flawed. In the research papers referred to for the making of this report have all referred to Laser-Doppler velocimetry. This is said to be a fine balance of precision and influence on the flow. Other methods of higher precision have the effect of significantly influencing the flow, to the point of changing its qualitative qualities in the most extreme/delicate cases. This, is an issue that is being resolved to a certain extent as new technologies allow us to map dynamic variables as spatial functions.
2. The theoretical study, in the words of Guckenheimer, “issue that is lacking in these studies is a thorough appreciation of the role played by aspect ratio and boundaries”. In general, the theoretical study of fluids allows the fluid to extend infinitely through all of space and often imposes unrealistic boundary conditions. This is, obviously, not reproducible in real experimental setups. This leads to qualitatively different flows from what is predicted by theory. In fact, the nature of flow is also a function of the aspect ratio in which the setup is based as realistic boundary conditions and finite aspect ratios can be seen as new constraints imposed on the flow.

However, the regimes in which the experimental verification has been done for are the cases of Taylor-Couette flow and Rayleigh-Benard convection.

Conclusions

The application of strange attractor theory in turbulent liquid flow is an interpretation of fluid flow as a vector field in a finite dimensional phase space. The intention is to apply years of development and progress in the field of chaotic theory to the problem of fluids mechanics to make qualitative predictions of the fluid flow. So, do the predictions made by chaotic theory hold in experimental tests? Well…… kind of. There are more nuances and context to the answers that the theory does furnish that render it to be very limited in its applications. The study of turbulence under the framework of chaotic theory however faces its biggest challenges in the nature of the analytical and numerical complexity of the governing differential equations. An application of the study of the simplest generic bifurcations that are often studied in chaotic systems (like the ones described above) are not applicable to the highly complex situations of fluid flow. Currently our most powerful tool to analyse complex systems is the power series. However, the power series furnishes very limited data that simply does not tell us enough about the system we are interested in.

Most of the theoretical analysis is based in numerical solutions and little is known of the precise structure of attractors and the methods to obtain said structure rigorously and virtually nothing is known about the dependence of flow on dimensions. Even the numerical solutions get out of hand very soon for most cases and very advanced algorithms and hardware to compute the numerical values accurately.

The main issue lies in the complexity of analysis of structures in higher dimensional analysis. The next development in the chaotic theory of turbulence would be a generalised method of analysis of N dimensional systems.

With all the above being said, it would seem to me personally, that the chaotic theory of turbulence strikes a dead end given the current mathematical tools available to us. Given the sheer complexity of the systems involved and the fact that most research papers on the topic also do not contain much of the math, analytical or numerical. In fact, given that most of the papers on this topic are from the 1970-1980s seems like a clear admission of the failure of the model in the academic community. Newer methods for numerical computation of velocity fields based on vorticity transport theorem and other relevant methods seem to have a great deal of success. Not all hope is lost though as a breakthrough in the tools of mathematical analysis of chaotic systems that extends to phase spaces of higher dimensions.

References

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