Genuine Quantum Non-Locality on Triangle Network and its Experimental Considerations

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Abstract

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Introduction

In 1964, Bell [1] proved that the predictions of quantum theory are incompatible with those of any physical theory satisfying a natural notion of locality. Bell's theorem has deeply influenced our perception and understanding of physics and is one of the most profound scientific discoveries ever made. It marks a remarkable junction of physics and philosophy that had previously been outside of our access and grew into the field of quantum foundations. On the other hand, interest in non-locality from an information theoretic perspective grew in the novel form of shared randomness that appears in quantum theory, entanglement. Information theorists are interested in the non-local certified shared randomness and device independent schemes for protocols like cryptography and private voting [2].

When we set up multiple individual sources and measurement nodes in an interconnected webbed structure, we have a network. Network non-locality are behaviours on a network that cannot be described by a hidden variable theory. In Fritz [5], it was shown that there exist non-local behaviours in network structure without the need to have measurement input (choice). He provides a more general theory of corelations in networks [6]. Avoiding the free will argument by having no inputs comes with a new assumption of multiple independent sources which makes the problem non-convex, a much more difficult and richer problem. Fritz initially proved non-locality on straight networks by showing how one can construct the experiment such that one can embed the standard bell scenarios in these networks. He then considered a closed triangle network and provided an entropic inequality for the structure followed by an explicit non-local distribution on the triangle network. He again, embedded the standard CHSH scenario in the triangle network and considered the behaviour with the maximal CHSH violation. This behaviour is known as the Fritz distribution.

Although the Fritz distribution is non-local, it is highly asymmetrical to permutations of nodes and it can be implemented using just one source of entanglement, rather than three. In fact, even the measurement basis used to obtain this distribution is completely separable. Also, since it is just a different way to setup the standard CHSH scenario, it offers no additional insight into the types of non-classical resources made available by quantum mechanics. Recognising this, Fritz presented the following problem. *Find an example of non-classical quantum corelations in the triangle structure with a proof that does not rely on Bell's Theorem.*

Motivated by the question above, there is a crude categorisation of non-local behaviour. We define non-local behaviours that are symmetric with respect to permutations of nodes and sources, all sources sharing entanglement and all measurement parties measuring in an entangled basis as "fully network non-local". Similarly, "genuine network non-locality" is the term for non-local behaviours that cannot be derived from/reduced to standard Bell scenarios. However, these are qualitatively definitions and the point of this classification is to look at forms of non-locality that arise strictly from experimental topology, i.e., the network structure, itself. We know that although for some networks these two sets are the same, however we cannot show this in general. We may also observe that one is simply not a special case of the other and that the two are genuine classifications. Furthermore, it only seems intuitive that further classifications of network non-local behaviours will arise in future study.

This led to effort in researching forms on genuine and full network non-locality in multiple directions [7]. Being a relatively new research field there aren't clear connections between the methods or understand them very well. Since the independence of sources makes the behaviours non-convex, we are no longer dealing with polytopes in probability space but much more complicated faces. Given that we have entropic characterisations that do not depend on cardinality, we may also speculate that some of the faces generalise to all measurements for a particular network structure. This makes theoretical characterisation much harder. The different studies on network non-locality might be working on different faces of this volume. The motivation behind that sentence is the crude observation that all of these approaches are completely unrelated to each other.

Types of Non-Locality in the triangle network

Closed network structures show rich forms on non-locality not seen in straight line networks. Furthermore, they represent a closer picture of networks in our everyday life, which motivates the study behind these networks even more. It has also been proved that the range for non-local behaviours in an n-ring structure reduces with increasing n. Thus, the C3 or triangle network serves as the perfect testbed for our theory as it holds the highest capacity to display non-local behaviour. Also, from an experimental point of view it is the most robust to noise and thus ideal for experiments. Below is a picture of the C3 network that will simply be referred to as the triangle network in the rest of this report.

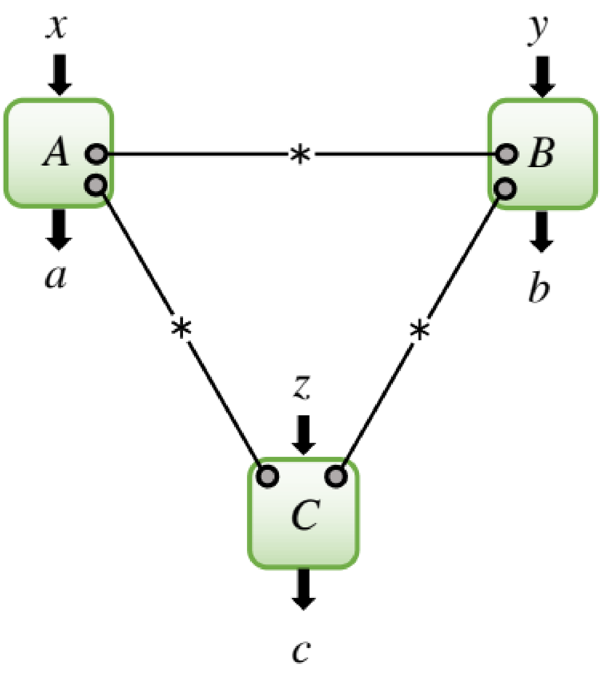

FIG. 1. The asterisks represent sources of information, the measurement nodes are given by A, B and C. Their choice of measurement is given by the variable x, y and z (although for most of this report we will consider no inputs, i.e., fixed x, y and z) and a, b and c represent the measurement outcome.

1. Inflation of networks

Inflation is a powerful method that allows us to set theory-independent constraints on the correlations that can arise in a network structure. The idea behind inflation is to not analyse the original system of interest, but an extended system where we have access to copies of the network elements and we can arrange them arbitrarily. If a particular distribution is proven to not have an implementation in a particular inflation of a network, then it cannot be implemented in the original one either.

Classical Inflation

Classical inflation is a general method for finding causal compatibility. If a particular probability distribution can be generated on a specific causal structure, they are said to be compatible. In our case, this is the act of determining if a distribution on the triangle network admits a local model or not.
The general method is then to identify a distribution we wish to scrutinise P_{obs}, and assume that it can be generated in the network i.e., there exist a set of random variables and a deterministic function of those variables that give rise to the target distribution. Then one considers inflations of the network such

that each copy of the party is connected to only the copies of sources that it was connected to in the original network.

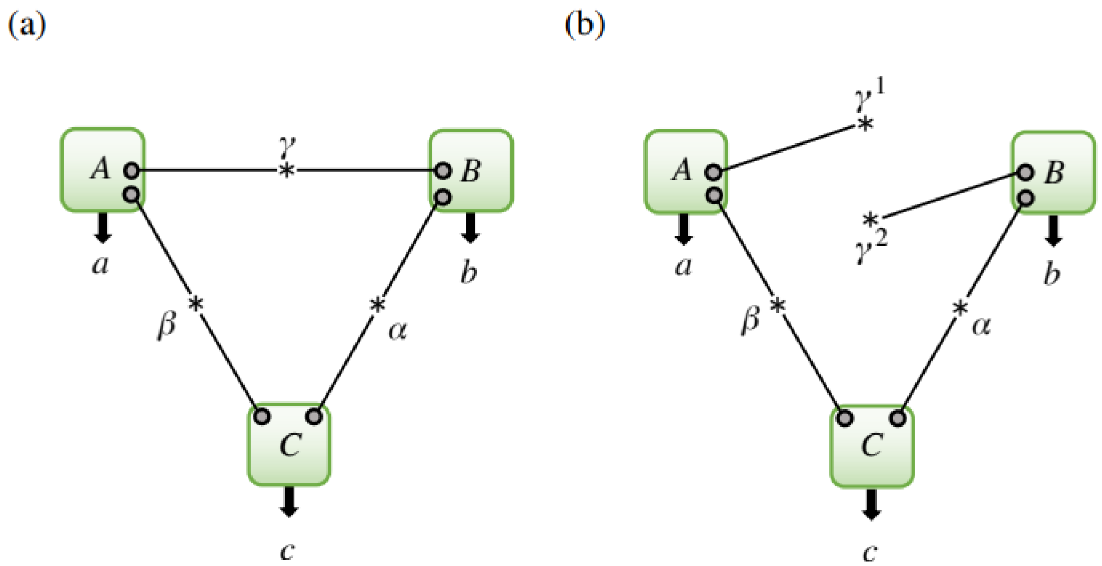

FIG. 2. (a) C3 network (b) Inflated C3 network

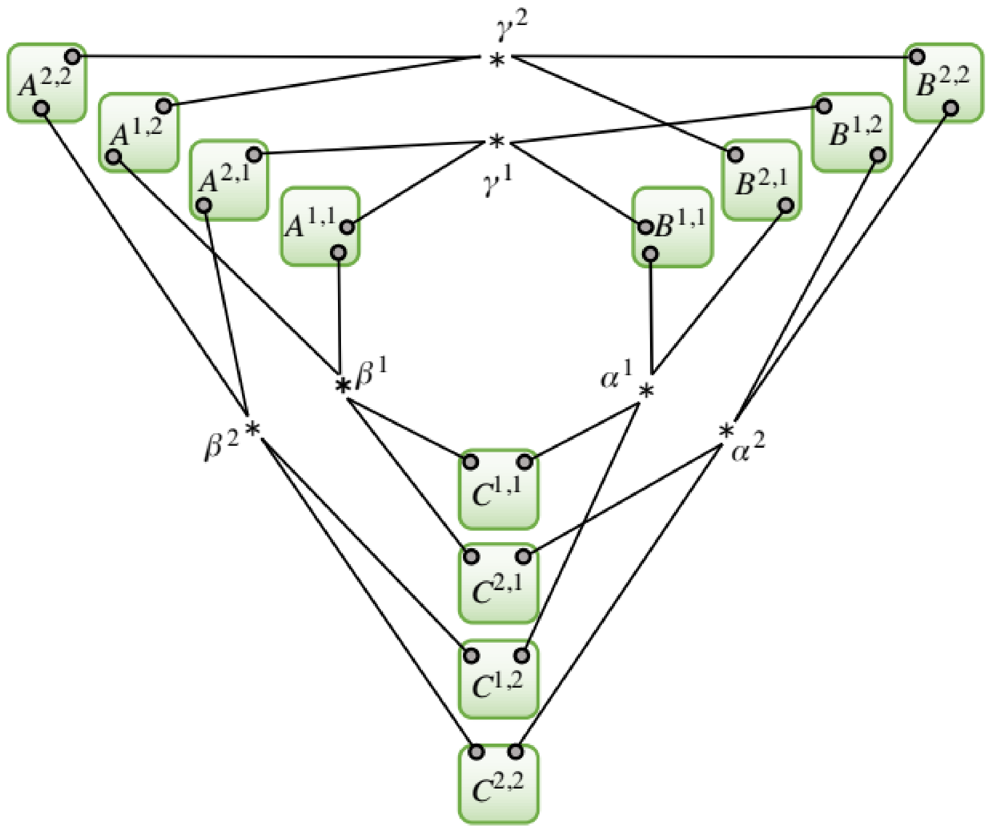

FIG. 3. The web inflation of the triangle network. It is regarded as the second level in the hierarchy of inflations as defined by E. Wolfe (2020).

The inflated system is then analysed and necessary constraints on compatible probability distributions are derived. We call the behaviour of the inflated system P_{inf}. This can be "translated" into constraints for the original system. This is because of they way we have inflated the system, considering the marginals of the inflated network that contain only the original network, are all connected to the same sources. Thus, anything that can be said about the marginal also applies to the original network itself. Alternatively, we can check each individual distribution. Being able to prove a certain distribution can or cannot be obtained on an inflation on a network means that it cannot be obtained in the original network. However, to find the right inflation for the job is a non-trivial question. One may even ask why we consider larger, more complicated networks to understand smaller networks in the first place? Consider the web network in FIG. 3. Observe that the setup is very symmetrical and a lot of those properties are abused to figure out constraints on the behaviours that one cannot in the original network. [8] proved the existence of a hierarchy of inflation tests that converge to a conclusive answer, if all the levels are passed then there is a local model in the network.

Quantum Inflation

Quantum Inflation is a bit tricky because what we have discussed above requires use to clone information which we know is not allowed in quantum mechanics as per the no cloning theorem. We can circumvent this problem by considering copies of sources rather than information. This means that rather than receiving multiple copies of the original source and performing measurements of the whole system and marginalising, in the quantum case the parties receive multiple copies and they choose to perform measurement on only some of them. A behaviour is realisable in the original network using quantum resources if there is a global inflated state whose measurements and statistics (i) are invariant with respect the permutation of the Hilbert spaces corresponding to different copies of the same source and (ii) reproduces the original distribution when the choice of copies reproduces the original network. Using hierarchies of semidefinite programs one can solve the above problem. The non-existence of a solution on a level of the hierarchy constitutes a proof that the distribution cannot be implemented in the original distribution.

The discussion above highlights why inflation is the most versatile and arguably most powerful tool for non-locality analysis. However, there are issues with inflation. For one, there is no analytical proof that the distributions obtained here are not connected to the original bell scenario in some way. However, the multiple inequalities obtained in [8] relied on the Fritz distribution for their derivation and the distributions that maximally violated it shared possibilistic structure with the Fritz distribution.

2. Entropic Inequalities

Network corelations can also be analysed in terms of entropies of those variables that are jointly measurable in the network. This approach has a couple of advantages. Firstly, the variable entropies are insensitive to the cardinality of the measurement being made. This implies that the conclusions derived from this approach hold independently of the number of measurement outcomes. Secondly, as will be shown later, entropic approaches allow us to transform non-linear probability inequalities into linear equations in entropy terms.

In the same way that the primary object in the probability picture is the set of distributions compatible with a network structure, in the entropic picture it is the entropy cone. For a fixed number of random variables n, the entropy cone is the closure of the set of all vectors $v \in \mathbb{R}^{2^n-1}$ of the form

$$v = \left(H(X_1), H(X_2), \dots, H(X_1X_2), H(X_2X_3), \dots, H(X_1X_2 \dots X_n)\right)$$

where H is an entropy function. Traditionally, in the analysis of classical models we chose to entropy to be the Shannon entropy, given by

$$H(\mathfrak{X}) = -\sum_{x \in X} p(x) log_2 p(x)$$

where p(x) is the distribution of events for a subset of variables $\mathfrak{X} \subset \{X_1, X_2, \ldots, X_n\}$ whose range is X. In quantum experiments we use the von Neumann entropy

$$H(\mathfrak{X}) = Tr\big(\rho_{\mathfrak{X}} log(\rho_{\mathfrak{X}})\big)$$

The boundary of the entropy cone may be potentially characterised by infinite linear inequalities, and thus it is a difficult task in general. For this reason, outer approximations of the cone are often used. The most common approximation is the Shannon cone, which is the polyhedral cone bound by three constraints, positivity (the uncertainty about a variable should be non-negative), monotonicity (the uncertainty about a set of variables is not lower than that of its subsets), and sub modularity (the conditional positive information is positive). The Shannon cone coincides with the entropy cone for $n \leq 3$. For $n > 3$ it is an outer approximation of the entropy cone. Inequalities that describe the entropy cone but cannot be derived from these conditions are called non-Shannon type inequalities. Additional constraints need to be applied to the Shannon cone to account for the network structure.

Local Corelations

Locality implies that all variables are jointly measurable, and thus one can define a probability distribution over all events. The corresponding Shannon cone can be built by imposing the three conditions to all possible subsets of the variables. The constraints applied on probability elements due to network structure can be written as constraints on the entropies. These constraints are

$$I\big(\lambda_j : \lambda_k\big) = 0$$
$$I\big(A_j : A_k \big| \lambda_{i\to j,k}\big) = 0$$

Here $\lambda_{i\to j,k}$ is the source that is connected to parties j and k. The first sentence captures the independence of sources, while the second establishes that all correlations between the two parties are related to only the sources connected to both of them. In the case of networks, the locality condition gives rise to polynomial constraints, however one may note here that the above entropic constraints are linear. The problem of if an entropy vector lies within the set defined by the Shannon cone can be cast as a linear program, which can be efficiently solved using numerical methods.

The fact that for $n > 3$ in the entropic approach we are working with an approximation means that we can never find all solutions with it. Besides, having access to just the entropies of variables tells us less than having access to the probability distributions. So, while it is possible to witness non-locality using entropic approaches, it is not clear as to if it is a good approach for further characterisation of these sorts of behaviours. For example, none of the entropic approaches have identified the non-classicality of the W distribution, while inflation methods have.

Quantum Corelations

In contrast with corelations of local models, quantum models are characterised by the fact that particular variables cannot be jointly measured. Therefore, when one imposes the Shannon condition on the quantum probabilities, we have to restrict ourselves to only the subsets of variables that may be measured jointly. Furthermore, here we use von Neumann entropies, these new functions do not necessarily follow monotonicity, so a weaker condition in imposed, which instead ensures the non-negativity of the sum of conditional entropies. Finally, to relate the entropies of different jointly measured variables, one makes use of data processing inequalities that capture the notion that the information content of a quantum system cannot be increased by performing local operations, An

example of this type of inequality is the upper bound of mutual information of two outcome variables is the mutual information of their parent quantum states.
Another constraint that has been employed for the triangle and its extensions is:

$$I(XYZ) = 0$$

This immediately proves that GHZ and W distribution are impossible on the triangle. However, with no general definition for higher order multi-party mutual information it is hard to apply this to models beyond the triangle [9].

3. RGB4 distribution

In [10-12], it is shown that network non-locality can be proven by using the rigidity property of some classical strategies. In a first step, this property is used to restrict the possible explanation of the quantum probabilities to a tiny family of local strategies. In a second step, this family is ruled out using extremely general consistency properties of probability distributions.

Token Counting Strategies

Given a fixed network structure, a classical strategy associated to the corelation P_R is said to be rigid if it is the only classical strategy which obtains it. It can be thought of as a self-testing of classical strategies. Two types of classical strategies have been identified to be rigid on the triangle network, the token-counting and colour-matching strategies.
Here is the proof that quantum triangle P_{RGB4} distribution does not admit a local model. Before we define the distribution. We consider the triangle network where each source distributes the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Each party performs measurement in the same basis given by $\{\ \bar{0} = |00\rangle\langle 00|,\ \bar{1} = |01\rangle\langle 01| + |10\rangle\langle 10|,\ \bar{2} = |11\rangle\langle 11|\}$, where $\bar{n}$ counts the number of tokens received, obtaining the corelation P_{count}. Whenever the outcome $\bar{1}$ is obtained the party who obtained it performs a second, dichotomic measurement given by $\{|\ \bar{1}_0\ \rangle = u|01\rangle + v|10\rangle,\ \{|\bar{1}_1\rangle = -v|01\rangle + u|10\rangle\}$. One may observe that the P_{count} may easily be obtained by a classical source $\{(|01\rangle\langle 01| + |10\rangle\langle 10|)/2\}$ as well since the projectors are diagonal in the number basis. Since these strategies are rigid, this immediately fixes the local variables and response function. The proof is based on the fact that the total number of tokens detected stays fixed, at three in this case. This means if there is a change in α such that B receives a different number of tokens, then corresponding to that value C must also receive one more token as the total number of tokens must remain the same (and it is not connected to A). Thus, for any P_{RGB4} there is a P_{count} that is generated that corresponds to a rigid classical strategy. To accommodate the statistics of the second measurement we introduce another hidden variable t, which contains information about the direction in which a source sends its token (it is easy to recognise that when each party receives only one token then each source must have sent out the token in the same direction). Thus, t is a binary variable that takes on values 0 and 1 and represents each direction. We can then define

$$q(a,b,c,t) = P(a,b,c,t|all\ parties\ measure\ \bar{1}\ token)$$

where a, b and c are values obtained on the second measurement. It can be easily checked that this is a valid probability distribution. It can then be shown, by computing these marginals and using their positivity constraint, that this distribution cannot be explained using an incoherent classical description for values of u (v is simply $\sqrt{1-u^2}$). The key difference is that in the quantum wave function there is a coherent superposition of the two directions, which when measured over the distance entangled basis, has statistical properties that cannot be explained by the incoherent classical theory. This style of proof has been employed for larger systems. In general, it applies to any network where any two nodes share at most one source, like larger ring network (Cn).

Very recently, in [13], the surprising result that the RGB4 distribution is actually non-local in the vast majority of the range of the parameter u was obtained. Due to symmetry, we only consider $1/\sqrt{2} < u < 1$. Initially, in [10], it was proved that the behaviour is non-local for $0.8860 < u < 1$. Motivated by the numerical analysis done using deep learning, it was conjectured in N. Brunner et al. (2020) that the behaviour is non-local all points but $u = 1/\sqrt{2}$ & 0.8860. Following this, as stated above, it was proved using inflation that behaviours were non-local in the additional range $0.7504 < u < 0.8101$. This leaves a relatively time range of local behaviour. Furthermore, we observe that the answer we obtain is from a finite level of inflation which is an approximation. The approximated polytope facet is qualitatively the same in the endpoints found, thus it is further conjectured that the current range is a reflection of limited computational power and that the entire range is non-local. Also quite specially, this is, to the best of my knowledge, the first study that applies inflation to behaviours that are genuinely non-local.

Colour matching strategies

In colour matching strategies, we distribute the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and the parties measure to check if they get the same colour or not from both sources. The measurement basis is $\{\overline{X_0} = |01\rangle\langle 01|, \overline{X_1} = |10\rangle\langle 10|, \overline{M} = |00\rangle\langle 00| + |11\rangle\langle 11|\}$ This generates a distribution like P_{count} that completely defines a classical strategy. We have an additional measurement in the experiment in the entangled colour basis $\{|\overline{M_0}\rangle = u|00\rangle + v|11\rangle, \{|\overline{M_1}\rangle = -v|00\rangle + u|11\rangle\}$. Again, it is easy to observe that in the classical experiment, when the colours have matched, both the source will have sent it incoherent, while when the quantum measurement is performed it remains in a coherent superposition, which when measured with the entangled basis produces statistics, in a particular range of u, that cannot be explained by any local incoherent model. Actually, the quantum colour matching and token counting strategies for the triangle are analogous, they differ simply by global unitary operators. Similar to token counting, colour matching has also been proven for a large number of systems, and in general applies to the n-ring generalisation of the triangle.

Further discussion

The discussion above includes only the most developed methods. The methods that have not been discussed above (like the distribution obtained from the joint elegant measurement) are either too niche or apply to networks other than the triangle. The methods described above all apply different principles. Currently we do not know how these methods are related to each other. The math required to handle these scenarios is underdeveloped right now Even, conceptually, there is a real gap in our understanding of the situation here. This is highlighted by the fact that apart from inflation, the other forms of non-locality are all based on counter example. While these proofs help us identify examples that we can look to for deeper analysis and potentially serve as checks for future tests, a more general proof is desired. Also, they are not robust to noise so are not ideal for experiments, although one can use inflation inequalities to test noisy distributions and determine the visibility threshold.

Experimental Non-Locality

Quantum non-locality is not just a tool for theory, but a phenomenon that is being experimented in laboratories today. Tests of network non-locality are natural extensions of the experiments on standard Bell scenarios that are being performed for multiple decades now. The interest is not only in fundamental tests of the concept but also in the development of entangled measurements and development of quantum information protocols on a network, like a quantum internet, that can leverage the novel features of non-local corelations for a quantum edge. We chose to normally implement these scenarios experimental on a photonic platform. Photonics system bring a level of control and simplicity that is typically not possible with alternatives. Light-atom systems are also used often, especially when we want to use a quantum memory.
Experimental network non-locality presents a big challenge, it demands several, high quality synchronised sources distributing entanglement over large distances and multiple scalable entangled measurements. Experimental network non-locality also suffers from the loophole issues that plague the early tests of the standard bell scenario, which allow the possibility of the explanation of the experimental statistics admitting a local model. Another key consideration for experiments is the justification of the assumption of the independence of sources. Although experimentalists have succeeded in demonstrating loophole free non-locality, elimination of each loophole adds an overhead of sophistication to the system that we do not desire from technology that we wish to scale.

1. Entanglement swapping based experiments

Entanglement swapping is a key primitive in network non-locality tests. The simplest swapping scenario is identical to the bi-local scenario, both parties receive a part of two entangled pairs of particles. By performing entangled measurements on one share of each pair, one can entangle the other two, thus creating entanglement between systems that have never interacted with each other. To achieve this, one must perform a Bell state measurement, i.e., a projection onto the Bell basis. The state of the two-party system after they have been entangled is a maximally entangled state, depending on the outcome of the measurement done on the other two qubits. Thus, we say that the swapped entanglement is event ready. Observe that this procedure has allowed us to generate distributed entanglement which we may now use for non-locality tests. Since standard local models are stronger than quantum network models, a demonstration of violation of local inequalities also witness quantum non-local behaviour.
Over the years, entanglement swapping experiments have increased in sophistication and are more robust. Although these experiments reproduce the statistics of the experimental scenario that we consider in our theory, they come about in the laboratory, in many different ways. When we account for these factors, we often find that in their presence we can simply come up with new classical theories that do indeed satisfy the corelations obtained. Thus, Bell tests have gotten increasingly complex and sophisticated over the decades. Recently, the first loophole free demonstrations of non-locality were done in [14-17].

2. Direct tests on Bi-local and Star Networks

Direct experimental tests of network non-locality that properly reproduce the experimental topology. Most tests have been conducted on bi-local network and star network. Unsurprisingly, most of the technical nuances of the standard non-locality show themselves in the network scenario as well. For experimentalists, this means experimental loopholes that must be patched. Similar to the development of the standard scenario, experiments of increasing sophistication for networks have been performed. One particular experiment will be discussed here. Sun et al, (2019) [18] take a rigorous approach to demonstrate violation in the bi-locality scenario. Independence of source of ensured by using two separate and synchronised SPDC sources, with randomised relative phase, which is achieved by periodically shifting the laser from spontaneous to stimulated emission. Furthermore, the experiment implements fast switching to implement space-like separation, which closes the source independence loophole (also called the locality-loophole) as well as the freedom of choice loophole. It does not address the detection loophole however. This experiment, however, captures the spirit of the network in the true sense, where the parties are actually separated by large spatial dimensions. The experiment demonstrates a violation of a modified BRGP inequality which is tailored to partial Bell state measurement (more on this later).

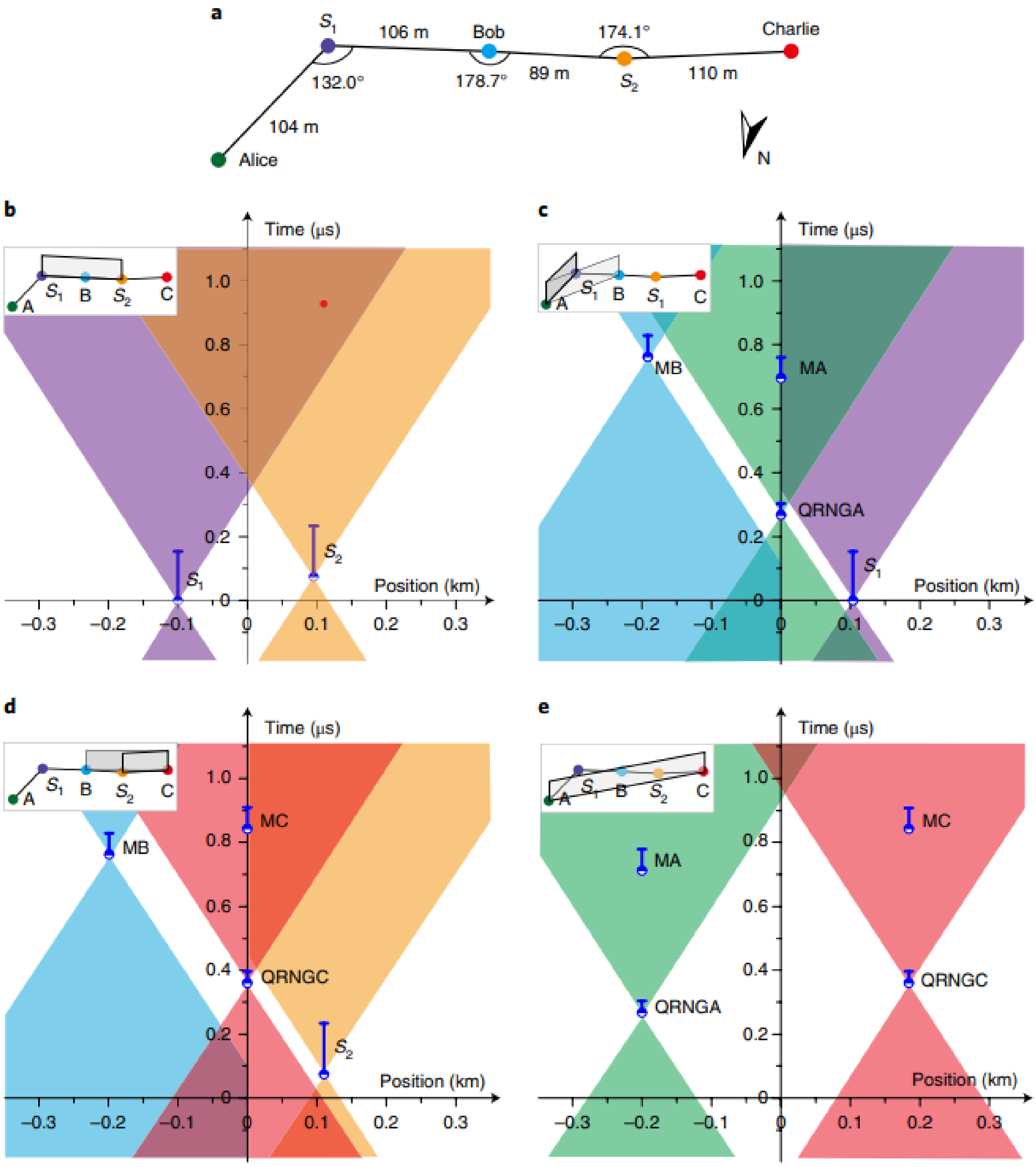

FIG. 4. Space-time configuration of related events in [18]

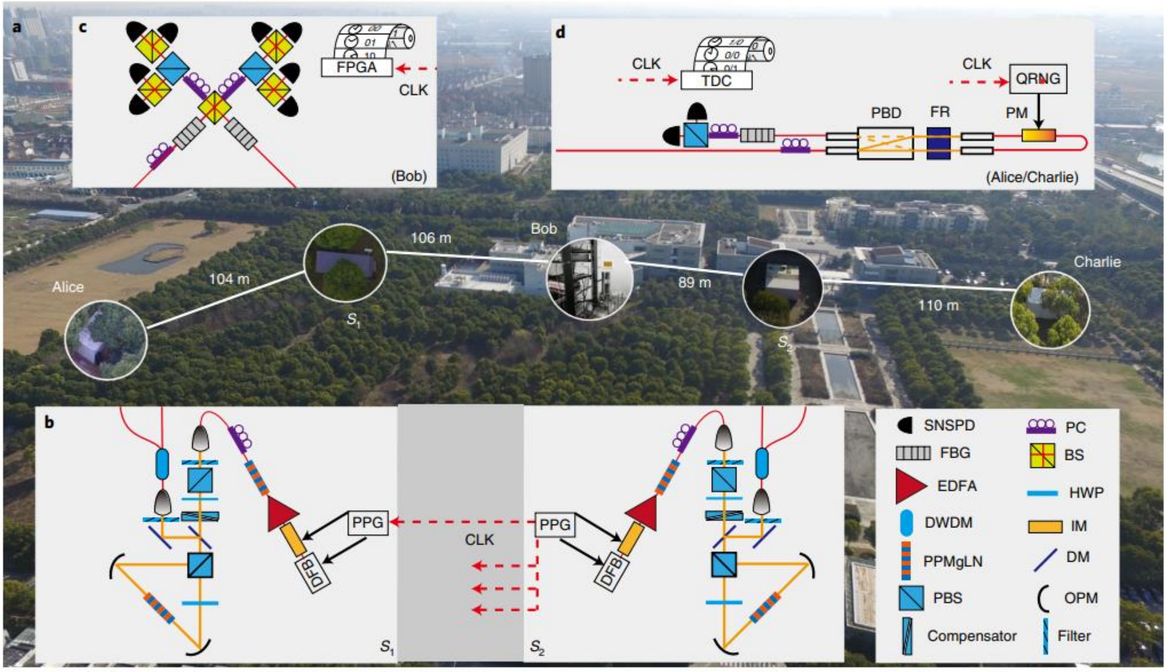

FIG. 5. Schematic and pictures of experimental setup testing nonlocality in bi-local scenario in [18]

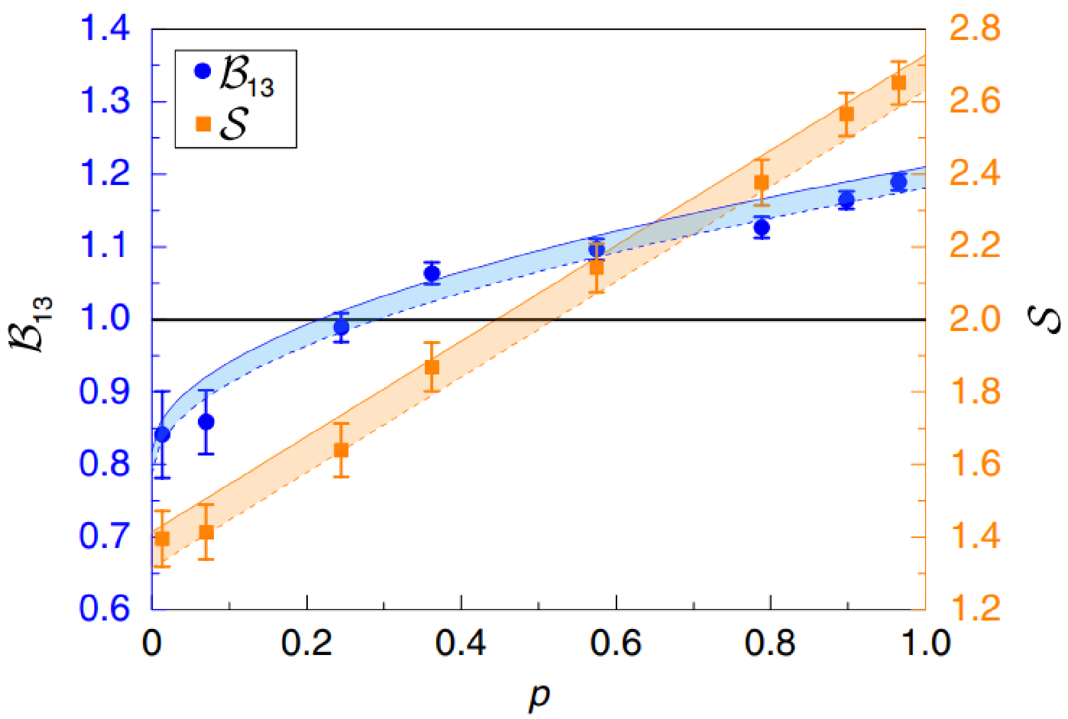

FIG. 6. Violation of locality inequalities obtained in [18]

Violations in star networks, with up to 4 branches have been demonstrated, with four independent sources of entangled qubits, have been demonstrated in [19]. This experiment is based on a separable measurement at the central node, which essentially breaks it down to 4 simultaneous CHSH tests. Similar tests have been employed for bi-local scenario as well. Network Bell tests have also been conducted on transmon quantum computers in N. Gisin et al. [20]. Since all the qubits are located on a single computer, the experiment does not follow the spirit of the network structure of the problem. Of course, another consequence of this setup is that it falls prey to the source independence loophole. It is better viewed as a simulation. Nevertheless, at the level of measurement probabilities, violations in star network scenarios with up to five branches have been demonstrated on public IBM computers. The same platform is used to implement the joint elegant measurement. This allows a demonstration of a bi-local and triangle inequality.

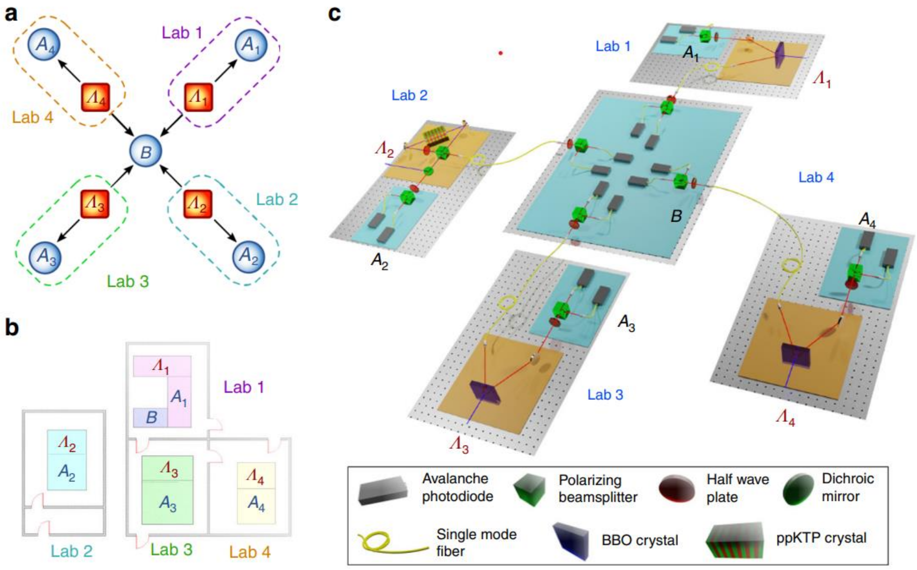

FIG. 7. (a) Schematic of the 4-branch star network considered in [19] (b) physical location of the labs (c) schematic with details about sources and measurement stations

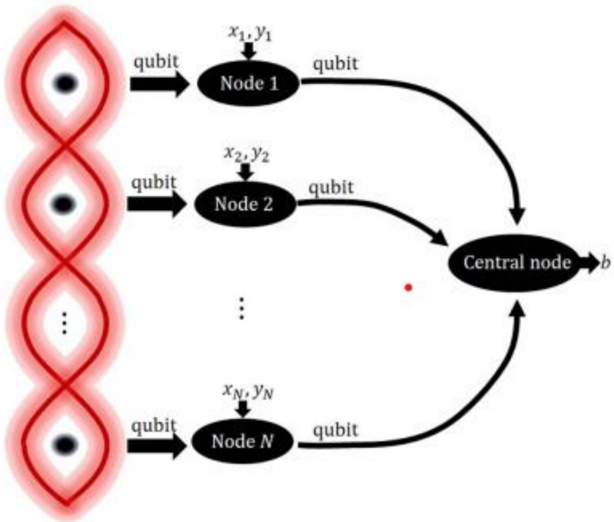

FIG. 8. Schematic of transmon quantum computer simulation of quantum networks in [20]

3. Tests on triangle network

It is evident from the discussion above that experimental network non-locality is an even younger field than its theoretical counterpart. Very few tests have been performed on the triangle network so far. E. Wolfe et al. (2022) [21] demonstrated SGMNL, a popular notion of genuine non-locality, in the triangle network by implementing parallel two-party games on the triangle network. Amongst the myriad of results reported here, perhaps the most important was the proof that the triangle network has the "highest capability to witness nonlocality". Another thing to notice is that so far none of the experiments above demonstrate non-locality without inputs.

A major issue in the above programme is the fact that the rigidity-based proof of TC and CM non-locality is based on a particular distribution and measurement, and is thus not applicable in a noisy system, which constitutes every experimental system. There are two ways around this. One is to recognise that the even though the proof and the distribution arrives from a highly contrived scenario, the witness of non-locality is actually done using an inequality. We do arrive at the inequality using an extremely different route and the inequality is in terms of a measurement parameter u rather than probability terms or expectations of physical quantities. But given that the probability terms are a function of u and that it is possible that the inequality for u can be, upon changing notation, written as an inequality in the probability elements. Also, given the fact that we obtain an inequality that admits a finite violation, it seems reasonable to assume that upon introducing an infinitesimal amount of noise to a behaviour that produced a large violation would decrease the value of violation by an infinitesimal amount. Thus, the slightly noisy behaviour would remain non-local. However, these arguments are not rigorous and the issue remains unsolved. Regardless, this seems to be a promising behaviour to test experimentally. At the time of writing this report, I have discovered that Marc-Olivier Renou et al. (2022) [22] proposed a scheme for the implementation of the RGB4 distribution has been proposed. Single photon sources are used and different modes of a photon are entangled. The measurement is done by beam splitter. However, because of the different modes of a single photon are entangled, the experiment cannot have a truly spatially separated setup, and thus it lacks the spirit of the original problem. The proposal is very similar to the scheme below, however there are a few key differences in the analysis.

4. Bell state measurements

Before we discuss the nuances of non-locality in network structure, it is critical that we discuss Bell state measurements [23]. To properly measure in the Bell basis, one has to use active measurements involving local oscillators creating extra photons. Using passive measurements allow one to measure information at the expense of collapsing the path superposition. Thus, the statistics lose their coherence and can be simulated using classical logic, contrary to the dynamics of a bell states, the final product state of a bell measurement. Thus, it is tempting to think that Bell state measurement using passive elements is impossible. And that would be right. All the experiments that were discussed above relied on abusing features of the experiment which allowed them to discard the results when the measurement returns an inconclusive. Thus, they were not really violating the inequalities that we study in theory, but versions that are modified to account for the fact that only two of the four bases are being measured in.

J Calsamiglia (2001) [24] proved that any linear optical device that does not use auxiliary photons in ancillary modes cannot unambiguously discriminate any state with a Schmidt rank higher than two from any set of two-qudit states spanning the whole two-qudit Hilbert space. It cannot even partially distinguish the two. This has the implication that none of the two-qudit bell state can be distinguished from one another. A. Carollo (2001) [25] also proved that only states that can be distinguished without auxiliary photons can be distinguished using auxiliary photons. This, technically, rules out the possibility of performing a complete bell state measurement using passive optical elements [26]. However, one may observe that for the qubit case, one can indeed distinguish two of the bell states perfectly [27, 28]. Additionally, one may add auxiliary photons in such a way that the probability to distinguish goes up [29]. W. P. Grice (2011) [30] proposed a scheme where the addition of each additional photon would add a success rate of $1/2^N$. We can thus implement bell measurements of arbitrary accuracy as is required for the case at hand. However, it is easy to see the resources required for very precise measurements grow exponentially, which is obviously not a desirable property to have. The scheme for 50% measurement is discussed now.

If one wants to check if a qubit basis is measurable by beam splitter, we input all the basis states in the two inputs of the beam splitter, and determine if the measurement outcomes for the different states are exclusive. Notice that this is different that the requirement of being orthogonal. For now, we input the four bell states into the beam splitter. We implement the entanglement in different modes of two photons. The modes $|0\rangle$ and $|1\rangle$ are not actually the vacuum and first excited state of a particular field mode but single photon excitations of two different field modes, and are symbolically represented as 0 and 1. We now observe that

$$\frac{|00\rangle_{ab} + |11\rangle_{ab}}{\sqrt{2}} = \frac{1}{2\sqrt{2}}\left(c_0^\dagger c_0^\dagger \; + \; c_1^\dagger c_1^\dagger \; + \; d_0^\dagger d_0^\dagger \; + \; d_1^\dagger d_1^\dagger\right)|vacc\rangle$$

$$\frac{|00\rangle_{ab} - |11\rangle_{ab}}{\sqrt{2}} = \frac{1}{2\sqrt{2}}\left(c_0^\dagger c_0^\dagger \; - \; c_1^\dagger c_1^\dagger \; + \; d_0^\dagger d_0^\dagger \; - d_1^\dagger d_1^\dagger\right)|vacc\rangle$$

$$\frac{|01\rangle_{ab} + |10\rangle_{ab}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(c_0^\dagger d_1^\dagger \; + \; c_1^\dagger d_0^\dagger \right)|vacc\rangle$$

$$\frac{|01\rangle_{ab} - |10\rangle_{ab}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(c_0^\dagger c_1^\dagger \; - \; d_0^\dagger d_1^\dagger\right)|vacc\rangle$$

Where a and b are the creation and annihilation operator for the two different input modes with the subscript 0 or 1 denoting the field mode. The beam splitter considered above is the standard 50-50 beam splitter given by

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

such that

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Here the second bell pair can be distinguished in an experiment which measures in the number basis (as we do in the experiment). Observe that all four states are orthogonal after passing through the beam splitter, however, this measurement scheme can only distinguish between two since the other two still share the same possibilistic structure. Thus, the outputs need to have a stronger condition than orthogonality to be observed by the experiment.

Proposal for experimental genuine network non-locality

Presented below is a proposal for an operational non-local setup. We first start by identifying, from an operational point of view, the resources required to implement genuine quantum non-locality in the triangle network without inputs. Ideally, one looks to propose a scheme in which the resources required is feasible to obtain. One looks to introduce simple apparatus that can produce desired behaviour at high efficiencies. Simplicity of the apparatus is also desired. In this case, let's look at entangled measurement schemes that can be achieved using passive linear optical elements.

We chose to implement the RGB4 distribution, particularly because of the measurement basis. One observes that the measurement basis required to implement this distribution require partially entangled measurements. Namely, two of the basis elements are entangled while the others are separable. Also, the Schmidt rank of the entangled basis elements is two, which means there is a possibility that they can be distinguished by a beam splitter. This is indeed the case. As a test case we try to implement the basis given by $\{(|01\rangle + |10\rangle)/\sqrt{2},\ (|01\rangle - |10\rangle)/\sqrt{2},\ |00\rangle,\ |11\rangle\}$, which is one case of the parametrised basis of the distribution. When we input the basis states into the same beam splitter as above, we observe

$$\frac{|01\rangle_{ab} + |10\rangle_{ab}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(c_0^\dagger d_1^\dagger \ + \ c_1^\dagger d_0^\dagger\right)|vacc\rangle$$

$$\frac{|01\rangle_{ab} - |10\rangle_{ab}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(c_0^\dagger c_1^\dagger \ - \ d_0^\dagger d_1^\dagger\right)|vacc\rangle$$

$$|00\rangle_{ab} \ = \ \frac{1}{2}\ (c_0^\dagger c_0^\dagger \ - \ d_0^\dagger d_0^\dagger)|vacc\rangle$$

$$|11\rangle_{ab} \ = \ \frac{1}{2}\ (c_1^\dagger c_1^\dagger \ - \ d_1^\dagger d_1^\dagger)|vacc\rangle$$

As we can see the final states have exclusive possibilistic structure in their distributions in when measured in the number basis, thus they are completely distinguishable. When we attempted to measure the bell basis on a 50-50 beam splitter, we observed that for the two distinguishable states there were a bunch of terms that cancel out to give the final expression given above. To measure the general basis for the RGB4 distribution parametrised by u we use a beam splitter that is not balanced. This is to ensure proper weights to the terms in the superposition to cancel those terms. It does add extra terms to the expression for $|00\rangle$ and $|11\rangle$, however it is done in a way such that the two still remain exclusive in the output possibilistic distribution. The appropriate beam splitter is one in which the transmittivity $t = u$ and, of course $r = v$. So, our beam splitter transformation is

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

This describes the beam splitter that we wish to use in this experiment. The two output modes of the beam splitter are then sent into two polarisation beam splitters. Two photon number resolving detectors are placed at the outputs of the polarising beam splitter. This demonstrates a successful implementation of the token counting measurement basis using only beam splitter. On the triangle network, a colour matching strategy corresponding to a particular value of u generates the same distribution as token counting strategy corresponding to that value, up to a relabelling on outputs. This would suggest that the colour matching measurement basis would be achievable using a phase shifted beam splitter. However, on checking this, we find that this is not the case. To work with the most general beam splitter, we represent it in the most general form of a unitary matrix. The beam splitter transformation matrix is

$$B \ = \begin{pmatrix} \alpha & \beta \\ -e^{i\phi}\beta^* & e^{i\phi}\alpha^* \end{pmatrix}$$

Again, for the sake of simplicity of algebra, the following is calculation is done on a particular value of $u = 1/\sqrt{2}$, but it applies to all values. Working with $u = 1/\sqrt{2}$, the 50-50 beam splitter is used. In this case we have $\alpha = 1/\sqrt{2}\mathrm{e}^{-\mathrm{i}\theta_\alpha}$ and $\beta = 1/\sqrt{2}\mathrm{e}^{-\mathrm{i}\theta_\beta}$. It is observed that the phase ϕ ends up showing up as a global phase in the output state and the phases θ_α and θ_β show up as relative phases in the state in the number state representation. Thus, when measurement is performed on the state in the number basis, the phases have absolutely no effect as only the magnitudes define experimental probability, and the experiments have the same result regardless of the values of the phases. The results obtained above apply to every beam splitter.

We then observe that a beam splitter can not only not distinguish more than two Bell states but also it can only distinguish the particular states identified above, and not the other two. There is an apparent breaking of equivalence in the linear photonic implementation of the RGB4 distribution. This is quite an odd result. An interesting point to note here is that as stated above, the way that the distribution is achieved experimentally is by symbolically representing excitations in different field modes as representing 0 and 1, so it may be argued that the 0 and 1 were arbitrary labels imposed on our system. It is then just a matter of an alternative definition to obtain the colour matching strategy, both strategies generate identical distributions (up to relabelling). However, they states generated are still optical bell states, and the fact that beam splitters can exclusively only distinguish two specific bell state is still odd and something that should be looked at further. Regardless, the section above shows how quantum number counting strategies measurement scheme can be implemented using just the beam splitter, and these strategies generate the non-local RGB4 distribution.

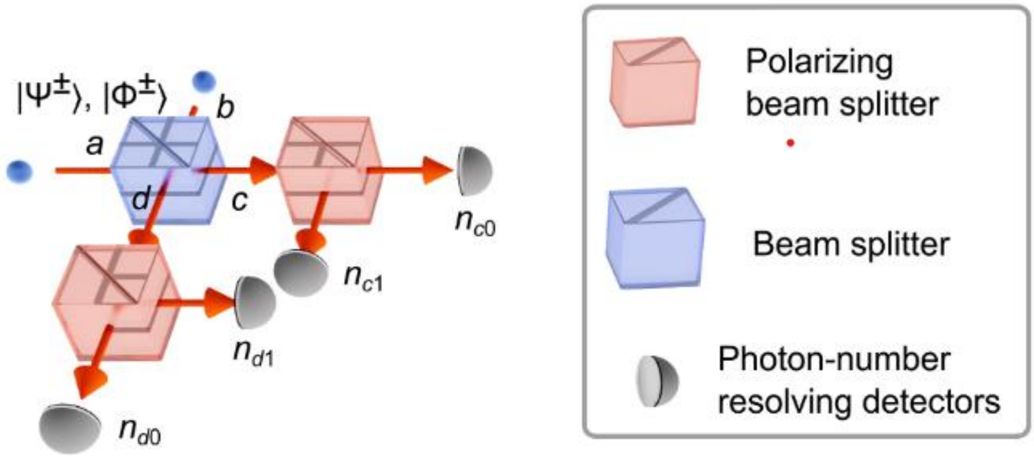

FIG. 9. A schematic of the proposed beam splitter measurement

For the sources, we choose to generate entangled two photon states using a type II degenerate spontaneous parametric down conversion (SPDC) process. To achieve this, we use periodically poled potassium titanyl phosphate (PPKTP) crystals are pumped by a continuous-wave laser. The generated photons are then transported to the measurement nodes by the use of optical fibre cables.

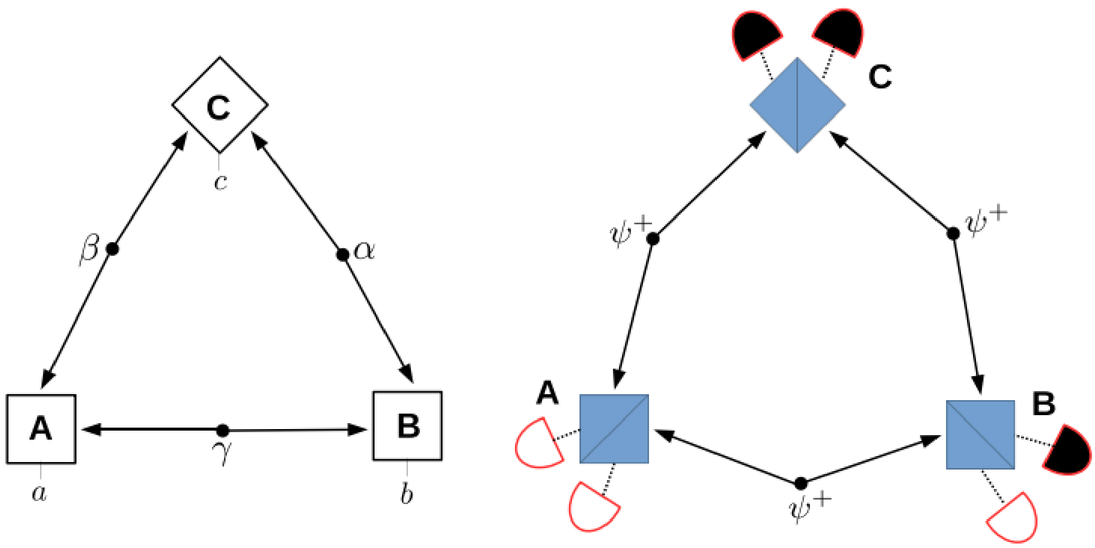

FIG. 10. (left) Causal structure of the tringle network (right) schematics of the proposed experiment

Conclusion

Network scenarios take us into a conceptual landscape far richer than the age-old Bell scenarios. Although, the research on the field currently is young, it is already evident that the behaviours in triangle networks can display quantum non-classical behaviour that simply arrives out of structure topology constraints on the two theories. This is a remarkable observation, and the inflation and RGB4 proofs of quantum network non-locality abuse this property. As we see in [13] especially, that we can analyse these systems very deeply when these two methods when combined. However, the RGB4 scenario is based on a specific distribution and is not ideal for experimental analysis, where one wishes to work with inequalities that are noise resistant. The sort of inflation applied in [13] is very specific and only applies to the specific states we consider in those strategies, so the argument does not improve robustness to noise. The standard inflation method is applied using some reference distribution. This usually results in behaviours that maximize the inequalities obtained that are very similar to the original reference distribution. That does not represent a new non-local resource. On the other entropy-based arguments are strong due to the insensitivity to output cardinality, however they are limited in the sense that our best methods are approximations, which get worse as the number of random variables increases. Additionally, we cannot seem to identify any of the methods as special cases of the others or establish any links. Thus, it is not clear moving forward which approach should be explored, or which combinations. There is likely to be scenarios in which any of the above could be powerful tools and the goal is to identify them. It is evident from the above discussion that the phenomenon is very rich and requires some nuanced analysis.

Experimental network non-locality is an even younger field and extremely constrained by resources and fundamental limits of nature! While massive work has been done to overcome the latter, this only increases the sophistication of the resources required to perform the experiment. Further, one must overcome the locality loophole of the experiment, which adopt the spirit of space separated networks and pave the way for a quantum internet. All of the above demand a level of sophistication required in the experiment setup that very few experiments have been performed on networks, even fewer on the triangle network. The measurement process proposed above requires relatively simple optical elements, with a freedom in choice of relative phase. Thus, this introduces a considerable simplification.

Further work

Further work would be in the exploration of beam splitter for more entangled measurements. The discrimination of the other pair of bell states is an interesting question. I plan to also explore a coarse grained, two step measurement scheme using beam splitter for a bell state measurement. Further, the Elegant Joint Measurement is another interesting symmetric entangled measurement scheme that is used in network non-locality study. Elegant Joint Measurement basis has Schmidt rank 3 and it is therefore not possible to distinguish it according to [24]. However, approximate schemes or alternatives that deliver a level of performance that is acceptable for the task at hand using these simple optical elements could be explored. Novel teleportation behaviours on the triangle network and their connection to standard non-locality analysis is another interesting topic that may be looked at.

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