Quadrature Measurement for parameter estimation in SU(1,1) interferometry

Conventional interferometers utilize beam splitters for wave splitting and recombination. In this study we consider SU(1,1) interferometers, where the beam splitters are replaced by active four wave mixers, namely optical parametric amiplifiers(OPA). In this report, we study the theoretical sensitivities of SU(1,1) interferometers with a variety of input states with quadrature measurements. We show that the SU(1,1) interferometer squeezes one pair of output quadrature modes, while it un-squeezes another pair. Additionally, we show that one may also squeeze either input states to further squeeze one of these quadratures. Lastly, we present a crude Heisenberg limit analysis and a comparison to MZI with squeezed input.

Introduction— Interferometry is the technique of extracting information from the interfernce of waves. It is a powerful technique, having powered some of the biggest advancements in science and technology. It has found historical use in precision sensing and metrology, allowing scientists to calculate various quantities like distance and force fields.

In traditional interferometry, an input field is split into two by a beam splitter. One of the output fields is encoded with a phase modulation and the two fields are made to interfere at a second beam splitter. The change in phase will correspond to a change in output observables which may be analyzed to determine the phase shift. It was shown in the seminal work by Caves [1] that one can reduce the uncertainty in output modes by reducing the uncertainty of the input modes, using squeezed vacuum states in the second input of the beam splitter to achieve sub shot noise sensitivity [2, 3].

SU(1,1) interferometers have recently been of interest in quantum interferometry, where the passive beam splitters in the traditional MZI are replaced by active four wave mixers, namely optical parametric amplifiers (OPA). The name SU(1,1) arises from the nature of the transformation of field modes in the OPA. In this regard, MZI may be called SU(2) interferometry [4].

Unlike the MZI, SU(1,1) interferometry can be done with no inputs. However, it is much more experimentally feasible to boost the signal with coherent state injection [5–8]. There has been extensive theoretical study of SU(1,1) interferometry, considering a variety of input states and measurement schemes, along with the characterization of fundamental sensitivity bounds in the form the Quantum Fisher Information (QFI) and Quantum Crammer-Rao Bound (QCRB) [9–11].

In this work we investigate the effectiveness of quadrature measurement for single phase parameter estimation in SU(1,1) interferometry. In section I, we introduce a mathematical formalism, where input states are always considered to be vacuum and different probe states are described by evolving operators. Correspondingly, any experiment can be constructed by repeated matrix multiplication. In section II we present some of the results of well studied cases as benchmarks to compare our to results. In section III we present the sensitivities for quadrature measurement with different input states. We also consider a joint quadrature measurement, for coherent state input which we show has better sensitivity than the number and quadrature measurements. In section IV we present a discussion on the Heisenberg limit and a comparison with MZI with squeezed state input. Finally, in section V we discuss our findings and further questions for future study.

I. MATHEMATICAL FORMALISM

A. Heisenberg picture of probe states

Inspired by the fact that the squeezing operator and OPA both, share the same transition, albeit on different modes, we introduce a fully Heisenberg approach for interferometry. Rather than considering different probe states, we take our input states to be vacuum and evolve our operators accordingly. For example, if we consider the case of a coherent state input, in our approach the input state remains as vacuum, while the annihilation operator transforms as follows $a \to a + \alpha$. We will now write all the matrices explicitly. Note that we consider the phase parameter ϕ to be zero for the BS, OPA and squeezing matrices.

$$D_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha^* \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad S_1(r) = \begin{pmatrix} R & 0 & r & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ r & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$BS = \begin{pmatrix} c & s & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 \\ 0 & 0 & c & s & 0 \\ 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} OPA = \begin{pmatrix} G & 0 & 0 & g & 0 \\ 0 & G & g & 0 & 0 \\ 0 & g & G & 0 & 0 \\ g & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\phi_1 = \begin{pmatrix} e^{i\phi_1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\phi_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_1^\dagger \\ a_2^\dagger \\ 1 \end{pmatrix} \tag{1}$$

Adopting this method, we have to use $5 * 5$ matrices rather than $4 * 4$ because displacement adds a scalar (can be seen as a scaled identity operator). This can be seen clearly as the other matrices are all block diagonalizable except the displacement matrix. Correspondingly, our vector also has five entries. We abuse notation slightly

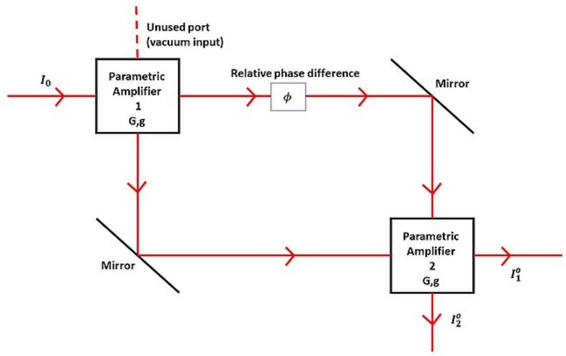

FIG. 1. A schematic of SU(1,1) interferometers. The two OPAs create and destroy interference respectively. Measurement is done at the two outputs of the second OPA. Here the interferometer is operated in the balanced condition ($G_1 = G_2 = G$).

here as, $R = cosh(r), r = sinh(r), G = cosh(g), g = sinh(g), c = cos(\theta), s = sin(\theta)$.

B. Constructing desired experiment

To construct the desired experiment, one must simply now multiply the matrices above in the correct order. The composite transformation acts on the input vector to produce the output vector. For example, we construct the matrix for an SU(1,1) interferometer shown above, with a phase shift in it's top arm, with coherent state input in the first mode.

$$E = (OPA2).\phi_1.(OPA1).D_1(\alpha) \tag{2}$$

$$\vec{a^{(o)}} = E\vec{a^{(i)}} \tag{3}$$

Now we may directly compute output mode observables. Notice how we skip a step of inverting our matrix that we would have to do in traditional analysis in order to express $D_1(\alpha)$ in terms of output mode operators.

C. Single Parameter Estimation

We estimate single phase parameter make observations of output mode observables, that in general, depend on the phase. The error in our estimation will be proportional to the error in the measurement of the observable. If we choose to estimate error in the estimation by measuring an observable $X(\phi)$, we obtain the expression for sensitivity as

$$\Delta\phi = \left(\frac{\partial\langle X(\phi)\rangle}{\partial\phi}\right)^{-1} \Delta(X(\phi)). \tag{4}$$

II. BENCHMARKS

Before we present our results with the quadrature measurement, we will look two benchmark cases that are already well studied in the literature.

A. Number difference measurement with coherent and squeezed state input in MZI

Squeezing typically requires nonlinear optical interactions, such as those found in optical parametric amplifiers or other nonlinear media, where an external pump field is used to generate squeezed states, so it represents a similar type of active resource as an OPA. MZI with squeezed state input is then comparable with SU(1,1) interferometry. Indeed, the minimum sensitivity reported in the literature for MZI [12] with input $|\alpha, r\rangle_{1,2}$, performing a number difference measurement is

$$\Delta\phi_{min} = \frac{\sqrt{r^2 + |\alpha|^2(R-r)^2}}{||\alpha|^2 - r^2|} \approx \frac{1}{(R+r)|\alpha|}. \tag{5}$$

This approximation holds when $|\alpha|^2 \gg r^2 = n_r$. With this approximation we also have $n = n_\alpha + n_r \approx n_\alpha = |\alpha|^2$. Thus with squeezed state we achieve sub shot noise sensitivity as

$$\Delta\phi_{min} = \frac{1}{(R+r)\sqrt{n}}. \tag{6}$$

B. Number measurement with coherent and vacuum state input in SU(1,1) interferometer

Single number measurement for input state $|\alpha, 0\rangle_{1,2}$ in SU(1,1) interferometry is another well studied scenario. For high $|\alpha|^2$, we obtain the following expression.

$$\Delta\phi = \frac{\sqrt{2(1+cos(\phi)((G^2+g^2)^2+4G^2g^2))}}{2|\alpha|Gg}|cosec(\phi)| \tag{7}$$

At $\phi = \pi$, we obtain the minimum sensitivity as

$$\Delta\phi_{min} = \frac{1}{2|\alpha|Gg} \tag{8}$$

III. QUADRATURE MEASUREMENT FOR PARAMETER ESTIMATION IN SU(1,1) INTERFEROMETER

We now look at quadrature measurement scheme for SU(1,1) interferometry using a variety of input states. We will then also investigate a joint quadrature measurement. In this report we have set the interferometer in the balanced condition, ie $g_1 = g_2$ and $\phi_1 = \phi_2 = 0$. Phase of input beam is also usually taken as a reference, so α will be taken be real, yet to stay consistent with notation, it will often be written as $|\alpha|$.

A. Displaced and Vacuum state inputs

The construction of the composite experiment matrix is given in (2). We calculate the first and second expectations to obtain $\Delta X^{(o)}$ and $\langle X^{(o)}\rangle$. We find that the first order expectation of the different quadratures have the same amplitude (for high gain) but are shifted by phase. They variance is same for each measurement. The variance is a function of G, g, ϕ_1, which is in general proportional to g but at $\phi_1 = \pm\pi$, it drops to 1. Since, the pairs of quadratures are out of phase, the sensitivity of one quadrature increases at the expense of the other quadratures. In this case the sensitivity of $Y_{1/2}^{(o)}$ reduces at the expense of $X_{1/2}^{(o)}$. This is a behavior we will notice in SU(1,1) interferometers, with any choice of inputs. In general, the optimal measurement is always in the set $Y_{1/2}^{(o)}$. The sensitivities of $Y_{1/2}^{(o)}$ for the case of one displaced input is

$$\Delta\phi^{Y_1^{(o)}} = \frac{\sqrt{(G^2+g^2)^2 - 4G^2g^2cos(\phi)}}{2|\alpha|G^2}|sec(\phi)| \quad (9)$$

$$\Delta\phi^{Y_2^{(o)}} = \frac{\sqrt{(G^2+g^2)^2 - 4G^2g^2cos(\phi)}}{2|\alpha|Gg}|sec(\phi)|. \quad (10)$$

At $\phi = \pi$ and at high enough g

$$\Delta\phi_{min} = \frac{1}{2|\alpha|G^2} \approx \frac{1}{2|\alpha|Gg}. \quad (11)$$

This is also the minimum sensitivity achievable using direct measurement for this choice of probe states.

B. Displaced and Squeezed state inputs

Now we consider the input probe state $D_1(\alpha) \otimes S_2(r)|0,0\rangle_{1,2}$. Again, we observe that sensitivity of measurements $Y_{1/2}^{(o)}$ is increased at the expense of $X_{1/2}^{(o)}$.

$$\Delta\phi^{Y_1^{(o)}} = \frac{\sqrt{k_1}}{2|\alpha|G^2}|sec(\phi)| \quad (12)$$

$$\Delta\phi^{Y_2^{(o)}} = \frac{\sqrt{k_2}}{2|\alpha|Gg}|sec(\phi)| \quad (13)$$

The two constants are given as

$$k_1 = G^4 + g^4 + [(R^2 - Rr + r^2) + (1 + (R-r)^2)cos(\phi) - Rrcos(2\phi)]2G^2g^2 \quad (14)$$

$$k_2 = [R-r]^2G^4 + [R^2 + r^2 - 2Rr]g^4 + \left[1 + (1 + (R-r)^2)cos(\phi))\right]2G^2g^2. \quad (15)$$

They both take on their minimum value at $\phi = \pi$ as $k_1 = 1$ and $k_2 = (R-r)^2$. Correspondingly we obtain minimum sensitivities as,

$$\Delta\phi_{min}^{Y_1^{(o)}} = \frac{1}{2|\alpha|G^2} \quad (16)$$

$$\Delta\phi_{min}^{Y_2^{(o)}} = \frac{1}{2|\alpha|Gg(R+r)}. \quad (17)$$

In this case there is an asymmetry between the $Y_{1/2}^{(o)}$. We observe that since input mode 2 was squeezed, thus the sensitivity of output mode is enhanced.

C. Squeezed Displaced and Vacuum state inputs

Consider the input probe state $D_1(\alpha)S_1(r)|0,0\rangle_{1,2}$. Looking at the sensitivities of the quadratures squeezed by the OPA we have

$$\Delta\phi^{Y_1^{(o)}} = \frac{\sqrt{k_3}}{2|\alpha|G^2}|sec(\phi)| \quad (18)$$

$$\Delta\phi^{Y_2^{(o)}} = \frac{\sqrt{k_1}}{2|\alpha|Gg}|sec(\phi)|. \quad (19)$$

Where,

$$k_3 = [R^2 + r^2 - 2Rr]G^4 + [R-r]^2g^4 + \left[1 + (1 + (R-r)^2)cos(\phi))\right]2G^2g^2. \quad (20)$$

At $\phi = \pi$, k_3 takes on its minimum value as $(R-r)^2$. Correspondingly we obtain minimum sensitivities as

$$\Delta\phi_{min}^{Y_1^{(o)}} = \frac{1}{2|\alpha|G^2(R+r)} \quad (21)$$

$$\Delta\phi_{min}^{Y_2^{(o)}} = \frac{1}{2|\alpha|Gg}. \quad (22)$$

D. Displaced Squeezed and Vacuum state inputs

Finally we consider the input probe state $S_1(r)D_1(\alpha)|0,0\rangle_{1,2}$. Computing the squeezed quadrature sensitivities we obtain

$$\Delta\phi^{Y_1^{(o)}} = \frac{\sqrt{k_2}}{2|\alpha|G^2((R+r)cos(\phi) + (R-r)sin(\phi))} \approx \frac{\sqrt{k_2}}{2|\alpha|G^2(R+r)}|sec(\phi)| \quad (23)$$

$$\Delta\phi^{Y_2^{(o)}} = \frac{\sqrt{k_1}}{2|\alpha|Gg(R+r)}|sec(\phi)|. \quad (24)$$

Again, they both have minima at $\phi = \pi$.

$$\Delta\phi_{min}^{Y_1^{(o)}} = \frac{1}{2|\alpha|G^2(R+r)^2} \quad (25)$$

$$\Delta\phi_{min}^{Y_2^{(o)}} = \frac{1}{2|\alpha|Gg(R+r)}. \quad (26)$$

Interestingly, the value $\Delta\phi_{min}^{Y_1^{(o)}}$ picks up a $(R+r)^2$ term in the denominator. It is interesting that a change in the relative order of the displacement and squeezing has a rather drastic effect. It is also interesting to note that squeezing followed by displacing produced identical results as separate displaced and squeezed inputs, but displacing followed by squeezing is superior to both.

E. The Joint Quadrature Measurement

When we squeeze either one of the inputs of the OPA we see an assymetry between the phase estimation capabilities of different quadratures. However, for the case of only one displaced input, we observed that for high enough g, the two sensitivities are nearly identical. Thus, we may hope to construct a joint quadrature measurement of sorts, where the two signals add while their noise cancels. Indeed we observe this is the case for joint quadrature measurements. Let us define the joint quadrature measurement $(Y_1^{(o)} \pm Y_2^{(o)})$. The sensitivity of these measurements is given as

$$\Delta\phi^{Y_1^{(o)} \pm Y_2^{(o)}} = \frac{\sqrt{2(G^2+g^2 \pm 2Ggcos(\phi))}}{2|\alpha|G}|sec(\phi)| \quad (27)$$

They both have the same minimum value.

$$\Delta\phi_{min}^{Y_1^{(o)} \pm Y_2^{(o)}} = \frac{\sqrt{2}}{2|\alpha|G(G+g)}|sec(\phi)| \approx \frac{1}{\sqrt{2}}\Delta\phi_{min}^{Y_{1/2}^{(o)}} \quad (28)$$

Thus, the joint quadrature measurement performs better than both the single quadrature and the direct number measurement.

IV. ANALYSIS OF COHERENT STATE INPUT SENSITIVITIES WITH QUADRATURE DETECTION

While the definition of Heisenberg limit and QFI for SU(1,1) interferometry is a delicate matter, here we do some crude analysis based on simpler definitions to develop basic understanding of error scaling.

A. Operating beyond Heisenberg Uncertainty Limit?

We may define the Heisenberg limit for SU(1,1) as $1/n$ where $n = (G^2+g^2)|\alpha|^2 \approx 2G^2|\alpha|^2$. Now we look at the minimum single quadrature sensitivities.

$$\Delta\phi_{min} = \frac{1}{2|\alpha|G^2} = \frac{|\alpha|}{n} \quad (29)$$

Clearly this implies sensitivity lower than Heisenberg limit for $|\alpha| < 1$, with minima at 0. Clearly, this result cannot be interpret directly and one must define fundamental sensitivity bounds with more careful consideration. However, this might suggest that working with input states that have low intensity, and perhaps amplify the signal with higher OPA gain. While we are not stating a concrete argument, this idea has validity in other forms in existing literature as well.

B. Comparison to squeezed state input in MZI

Let us look at the SNR in the MZI with input $|\alpha, r\rangle_{1,2}$ and number difference detection. Using the sensitivity expression given in (6), for $R = r \gg 1$ and $|\alpha|^2 \gg r^2$ we obtain the following.

$$SNR_{MZI} \approx 4R^2|\alpha|^2 = 4R^2 n \quad (30)$$

On the other for the SU(1,1) interferometer with input $|\alpha, 0\rangle_{1,2}$, performing the single quadrature measurement we have the following SNR.

$$SNR_{OPA} \approx 4G^4|\alpha|^2 = 2G^2 n \quad (31)$$

For the SU(1,1) interferometer, n is taken to be proportional to the strength of the field inside the interferometer, ie $n = 2G^2|\alpha|^2$. Consider the joint quadrature measurement we have the following SNR.

$$SNR_{OPA} \approx 8G^4|\alpha|^2 = 4G^2 n \quad (32)$$

Thus, we observe that for similar gain and squeeze parameters, joint quadrature measurement in SU(1,1) interferometer with input $|\alpha, 0\rangle_{1,2}$ matches SNR of MZI with input $|\alpha, r\rangle_{1,2}$, while the single quadrature SNR is smaller by a factor of 2.

V. CONCLUSIONS

A. Discussions

In this work we study the sensitivity of joint quadrature measurement for parameter estimation in SU(1,1) interferometry. Firstly, we develop a compact formalism that allows one to construct transformation matrices for any choice of experiment. We use mathematica's analytical tools to multiply desired matrices to obtain the final transformation matrices. It is then relatively easy to calculate the expectation values of output mode observables. Using this, we were able to obtain the sensitivity of the quadrature measurement for different input probe states. We observed how the OPA squeezed two of the quadratures, allowing us to get sensitivities lower than MZI. We also observed an interesting behavior, where relative order of squeezing and displacement at the input nodes effects the final sensitivity. Finally, we investigated a joint quadrature measurement for non-squeezed inputs which beats the single quadrature and direct measurement sensitivity.

B. Future Work

In section IV we attempt to look at a crude approach to defining the Heisenberg limit for the SU(1,1) interferometer in the case of input state $|\alpha, 0\rangle_{1,2}$. However, one must consider the QFI with the given input states to define the true lower bound. However, QFI calculations produce erroneous results. In particular, it yields different values when the phase shift is in the upper or lower arm of the interferometer, a situation that is physically equivalent. It is suggested that since the OPA is an active element with a pump, one cannot naively assume the input as a reference. In the future, we plan to study these questions in the context of our work.

In this study, we have only considered the balanced condition of the interferometer ($G_2 = G_1$). It is claimed in the literature [13] that the optimal working condition of the interferometer is unbalanced ($G_2 \gg G_1$). However, the SNR for the SU(1,1) interferometer with input state $|\alpha, 0\rangle_{1,2}$ is reported to be the same for both single and joint quadrature measurements as our study. In the future we plan to investigate this as well. Finally, our analysis so far has been in the ideal case, in the absence of noise. We plan to extend this framework to include noisy apparatus.

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