ASTR 210 HW 5 ©LYoung 2023

Q5.1

5 pts a) radiative transfer $I(x) = I_0 e^{-n\sigma x}$

$x_m$ is the dist. over which $I_0 \rightarrow I_0/e$

$I(x_m) = I_0 e^{-n\sigma x_m} = I_0/e = I_0 e^{-1}$

$\therefore \; n\sigma x_m = 1$

$x_m = 1/n\sigma.$

3 pts b) $\tau \equiv x/x_m \quad \Rightarrow \quad x = \tau x_m$

$I(\tau) = I_0 e^{-n\sigma x_m \tau} = I_0 e^{-(1)\cdot\tau} = I_0 e^{-\tau}$

6 pts c) slab of thickness 1 cm absorbs 15% of the light.

ie $I(1\,\text{cm}) = 0.85 I_0$.

since $I(\tau) = I_0 e^{-\tau}$, this means $e^{-\tau} = 0.85$

$\ln(0.85) = -\tau$

$\tau = 0.16$

And $x_m = x/\tau$, so $x_m = \dfrac{1\,\text{cm}}{-\ln(0.85)} = 6.15\,\text{cm}$

Now how thick would the slab be to absorb 99% of the light?

That occurs when $e^{-\tau} = 0.01$

$\tau = \ln(100) = 4.61$

and since $x = \tau x_m$, that thickness required is

$x = (4.61)(6.15\,\text{cm}) = 28.3\,\text{cm}.$

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Q5.2

5 pts a) $\langle E\rangle = \int_0^\infty E F(E)\,dE = \int_0^\infty E \cdot \frac{2}{\sqrt{\pi}\,kT}\left(\frac{E}{kT}\right)^{1/2} \exp\left(-\frac{E}{kT}\right) dE$

sub $x \equiv E/kT$

$dx = \frac{1}{kT}\,dE$

$$\langle E\rangle = \frac{2}{\sqrt{\pi}\,\cancel{kT}} \int_0^\infty \cancel{kT}\, x \; x^{1/2} e^{-x}\, kT\, dx$$

$$= \frac{2kT}{\sqrt{\pi}} \int_0^\infty x^{3/2} e^{-x}\, dx = \frac{2kT}{\cancel{\sqrt{\pi}}} \frac{3\cancel{\sqrt{\pi}}}{4} = \frac{3}{2} kT$$

The only tricky part is keeping track of all the $kT$'s in the substitution.

6 pts b) $\langle E\rangle = \frac{3}{2} kT$

A particle whose energy is 4 times the mean has energy $4 \times \frac{3}{2} kT = 6kT$.

Suppose that $6kT$ is enough to ionize an H atom, i.e.

$$6kT = 13.6\,\text{eV} \times \frac{1.602\times10^{-19}\,\text{J}}{\text{eV}} = 2.18\times10^{-18}\,\text{J}$$

$$T = 13.6\,\text{eV} \times \frac{1.602\times10^{-19}\,\text{J}}{\text{eV}} \cdot \frac{1}{6\times1.38\times10^{-23}\,\text{J/K}}$$

$$= 2.6\times10^{4}\,\text{K}$$

and if this is a proton its KE $(\frac{1}{2}mv^2)$ $\rightarrow$ speed $v$ given by

$$\frac{1}{2}mv^2 = E$$

$$v = \left(\frac{2E}{m}\right)^{1/2} = \left(\frac{2\times 2.18\times10^{-18}\,\text{J}}{1.67\times10^{-27}\,\text{kg}}\right) = 5.1\times10^{4}\,\text{m/s}$$

51 km/s

c) Similar computation but now instead of requiring 13.6 eV, we need less to go $n=1 \rightarrow n=2$

5 pts c) Continued

Bohr model energies are $13.6\,\text{eV}\left(\frac{1}{n^2}\right)$

so $\Delta E$ for $n=1 \rightarrow n=2$ is $\Delta E = 13.6\,\text{eV}\left(1-\frac{1}{4}\right)$
$= 13.6\,\text{eV}\,(3/4)$

by analogy with previous part, the necessary temp is

$$T = (2.6\times10^4\,\text{K}) \times 3/4 = 2.0\times10^4\,\text{K}$$

speed will be $51\,\text{km/s}\,(3/4)^{1/2} = 44\,\text{km/s}$

Both of these temperatures are much higher than the surface of the Sun, which is only about 6000K. H atoms on the Sun's surface probably exist in a variety of ionization and excitation states but most of them will probably be neutral atoms in the $n=1$ state. (More info in later chapters.)

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Q5.3

4 pts a) Check blackbody flux per unit area for a temp of 310 K.

Flux from a blackbody surface $F = \sigma_{SB} T^4$

for 310 K this is $5.67 \times 10^{-8} \frac{W}{m^2 \cdot K^4} \times (310\,K)^4 = 523\ W/m^2$, ok!

4 pts b) $523 \frac{W}{m^2} = 523 \frac{J}{s \cdot m^2} \times 24 \times 3600 s \times 2.0\,m^2 = 9.0 \times 10^7\ J$

Assuming a surface area of maybe $2.0\ m^2$ ?

This value is hard to guess but $1\ m^2$ seems too little

food calories: $9.0 \times 10^7\ J \times \frac{1\ Cal}{4200 J} = 2.1 \times 10^4\ Cal$

Q5.4

6 pts a) Want to resolve 2 stars separated by $50\times10^{-6}$ arcsec.

$\theta_{min} \approx 1.22\,\lambda/D \Rightarrow D \approx 1.22\,\lambda/\theta_{min}$

remember for this you have to convert $\theta_{min}$ to radians

$$D = \frac{1.22\times0.5\,mm}{50\times10^{-6}\,''} \times \frac{180^\circ \times 3600''}{\pi\,rad} \times \frac{1\,m}{10^3\,mm} = 2.5\times10^6\,m = 2500\,Km$$

$R_\oplus = 6378$ km, $D_\oplus = 12756$ km

So a telescope 2500 Km in diameter is a hefty fraction of the size of Earth.

Same calculation for $\lambda = 1\,\mu m = 10^{-6}\,m$ gives $D = 5.0$ km – still pretty big compared to JWST (6.5m)

b) time estimates for observing based on eq 6.27 in text

$$S/N \propto \frac{F_\lambda D\,\varphi_a}{\theta}\left(\frac{\Delta\lambda\,\varphi_t\,q}{S_\lambda}\right)^{1/2} \cdot t^{1/2}$$

Parameters are such that $t = 1$ hr $\rightarrow S/N = 2$

3 pts i) Now keep everything the same, including $\theta$ and $t$, but $D$ increases by a factor of 2.

$S/N \propto D$ so $S/N$ now becomes $2\times2=4$.

3 pts ii) The previous case was appropriate when $\theta$ is fixed by Earth's atmosphere. If instead $\theta$ scales with $D$ according to $\theta_{min} \propto \lambda/D$, then larger $D$ also makes smaller $\theta_{min}$.

In this case the new $S/N$ is $2\times2\times2 = 8$.

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Q5.5

5 pts

$$I_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT}-1} \quad \text{eq } 5.86$$

Black Body Flux is $F = \int_0^\infty d\nu \int_{\Omega_f} I_\nu \cos\theta \, d\Omega$

solid angle $d\Omega = \sin\theta \, d\theta \, d\varphi$

$$F = \int_0^\infty I_\nu \, d\nu \cdot \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \cos\theta \sin\theta$$

$$= \int_0^\infty I_\nu \, d\nu \cdot 2\pi \cdot \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= \int_0^\infty I_\nu \, d\nu \cdot 2\pi \cdot \frac{1}{2} = \pi \int_0^\infty \frac{2h\nu^3/c^2}{e^{h\nu/kT}-1} d\nu$$

sub $x \equiv h\nu/kT$. $\quad dx = \left(\frac{h}{kT}\right) d\nu$

$$x^3 = \frac{h^3\nu^3}{(kT)^3} \quad \text{so} \quad \frac{2h\nu^3}{c^2} = \frac{h^3\nu^3}{(kT)^3} \cdot \frac{(kT)^3}{h^2} \cdot \frac{2}{c^2} = \frac{2x^3}{c^2} \frac{(kT)^3}{h^2}$$

$$F = \pi \int_0^\infty \frac{2x^3}{c^2} \frac{(kT)^3}{h^2} \cdot \frac{1}{e^x - 1} \left(\frac{kT}{h}\right) dx$$

$$= \frac{2}{c^2} \pi \frac{(kT)^3}{h^2} \left(\frac{kT}{h}\right) \int_0^\infty \frac{x^3}{e^x - 1} dx \cong \frac{2\pi}{c^2} \frac{(kT)^4}{h^3} \frac{\pi^4}{15}$$

$$\cong \frac{2\pi^5}{15} \frac{k^4}{c^2 h^3} \cdot T^4 \approx \sigma_{SB} \cdot T^4$$