ASTR 210 HW 10

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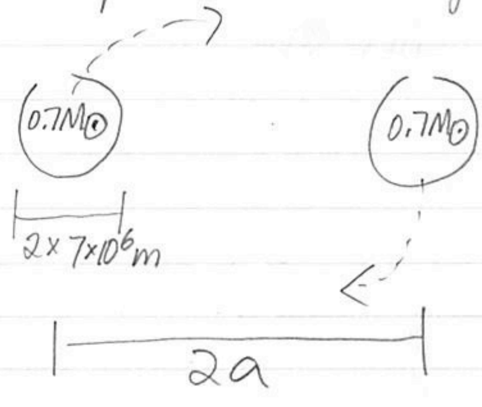
Q10.1 Pulsars

5 pts a) Suppose a pulsar is a binary WD.



What is a if orbital $P = 1.337$ s?
Use Kepler III for binary stars: $P^2 = \frac{4\pi^2 a^3}{G(M_A+M_B)}$

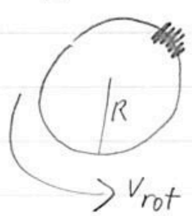
or $a^3 = \frac{P^2 \cdot G(M_A+M_B)}{4\pi^2}$

$= \frac{(1.337\,s)^2\; 6.67\times10^{-11}\,kg^{-1}m^3s^{-2} \cdot 2 \cdot 0.7 \cdot 1.99\times10^{30}\,kg}{4\pi^2}$

gives $a^3 = 8.4\times10^{18}\,m^3$
and $a = 2.0\times10^{6}$ m

2 pts b) Oops. That's not going to work, as the required separation of the stars ($2a$) is smaller than the radius of a white dwarf star.

5 pts c) OK, suppose it's one rotating star.



Period is related to velocity v_{rot} and radius R:

$P = 2\pi R / v_{rot} \Rightarrow v_{rot} = \frac{2\pi R}{P}$

and WD follow a mass-radius relationship: $R_{wd} \approx 0.01 R_\odot \left(\frac{M}{0.7M_\odot}\right)^{-1/3}$ eq 18.17

plug that: $v_{rot} \approx 2\pi \cdot \frac{0.01 R_\odot}{P} \left(\frac{M}{0.7M_\odot}\right)^{-1/3}$

let us also write $P = \left(\frac{P}{1s}\right) \cdot 1s$

$$V_{rot} \approx \frac{2\pi \times 0.01 \times 6.96 \times 10^5 \text{ Km}}{1s \ (P/1s)} \left(M/0.7M_\odot\right)^{-1/3}$$

$$\approx \underbrace{2\pi \times 6.96 \times 10^3 \frac{km}{s}} \left(\frac{P}{1s}\right)^{-1} \left(\frac{M}{0.7M_\odot}\right)^{-1/3}$$

$$\approx 14\,000\pi \text{ km/s} = 4.4 \times 10^4 \text{ km/s} = 0.15c$$

$c = 3 \times 10^5$ km/s

Thus, as suggested, the required rotation speed at the equator of a WD would be

$$V_{rot} \approx 0.15\,c \left(\frac{P}{1s}\right)^{-1} \left(\frac{M}{0.7M_\odot}\right)^{-1/3}$$

It is ok to call this $0.15c \approx 0.1c$ in this context since high accuracy is not necessary.

4 pts

d) breakup speed balances centrifugal accel with gravity

in uniform circular motion, centrip. accel is

$a = V^2/R$ or $\Omega^2 R$

$\frac{v^2}{R} = \frac{GM}{R^2}$ at breakup

Local gravity always $g = \frac{GM}{R^2}$

$\Rightarrow$ breakup speed $V_{crit} = \sqrt{GM/R}$.

Same speed as the circular speed for a Keplerian orbit.

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Q 10.1 cont'd

5 pts e) $V_{crit} = \sqrt{\frac{GM}{R}}$ breakup speed. Let's again use the mass-radius relationship to eliminate $R \approx 0.01 R_\odot \left(\frac{M}{0.7 M_\odot}\right)^{-1/3}$

$$V_{crit} = \left(\frac{GM}{0.01 R_\odot}\left(\frac{M}{0.7M_\odot}\right)^{1/3}\right)^{1/2} = \left(\frac{G \cdot 0.7 M_\odot}{0.01 R_\odot}\left(\frac{M}{0.7M_\odot}\right)^{4/3}\right)^{1/2}$$

$$= \left(\frac{6.67\times10^{-11}\, kg^{-1} m^3 s^{-2} \cdot 0.7 \cdot 1.99\times10^{30}\, kg}{0.01\times 7\times10^{8}\, m}\right)^{1/2}\left(\frac{M}{0.7M_\odot}\right)^{2/3}$$

$$= 3.6\times10^{6}\,\frac{m}{s}\left(\frac{M}{0.7M_\odot}\right)^{2/3} \approx 0.01c\left(\frac{M}{0.7M_\odot}\right)^{2/3}$$

2 pts f) A rotating WD with $M \sim 0.7 M_\odot$ is not a plausible model for a pulsar.
The speed that it would be rotating (at its equator) is about 10 times larger than its breakup speed.
WD are not dense enough to explain pulsars, basically.

Q 10.2 MW mass models

Rotation curve data

R (kpc)	V (km/s)	$M(<r)$ ($M_\odot$)
0.4	255	6.0×10^9
2.8	192	2.4×10^{10}
8.0	220	9.0×10^{10}
17.0	235	2.1×10^{11}

Sample calculation : $M(<r) = V^2 R / G$

$$@ R = 0.4 \text{ kpc}, \; M(<r) = \frac{(255 \times 10^3 \text{ m/s})^2 \, 0.4 \times 10^3 \times 3.09 \times 10^{16} \text{ m}}{6.67 \times 10^{-11} \text{ kg}^{-1} \text{m}^3 \text{s}^{-2}}$$

$$= 1.2 \times 10^{40} \text{ kg} = 6.0 \times 10^9 M_\odot$$

8 pts total; 2 for each radius value.

Q10.3 · GC BH

SO-2 orbits with a semimajor axis $a = 960$ AU
period $P = 15.6$ yr, orbital eccentricity $e = 0.867$

This is a rehash of Kepler's laws, yet again!

4 pts a) Since the star's mass is so much smaller than the BH mass we can do $P^2 = \frac{4\pi^2 a^3}{GM_\bullet}$ or $M_\bullet = \frac{4\pi^2 a^3}{GP^2}$

$$M_\bullet = \frac{4\pi^2 (960 \text{ AU} \times 1.496\times10^{11} \text{ m/AU})^3}{6.67\times10^{-11} \text{ kg}^{-1} \text{m}^3 \text{s}^{-2} (15.6 \times \pi \times 10^7 \text{ s})^2}$$
$$= 3.6\times10^6 M_\odot$$

3 pts b) pericenter distance is given by $r_{pe} = a(1-e)$ and speeds are given by $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$.

$$r_{pe} = a(1-e) = 960 \text{ AU}(1-0.867) = 128 \text{ AU} = \boxed{1.9\times10^{13} \text{ m}}$$

@ perihelion $v^2 = GM\left(\frac{2}{r_{pe}} - \frac{1}{a}\right) = GM\left(\frac{2}{a(1-e)} - \frac{1}{a}\right) = GM\left(\frac{2-(1-e)}{a(1-e)}\right)$

$$= \frac{GM}{a}\left(\frac{1+e}{1-e}\right)$$

$\rightarrow v_{pe} = 6.9\times10^3$ km/s which is about 2% of the speed of light!

5 pts c) Where is the Roche limit for this system?

eq 19.68 $r_{rp} \approx \left(\frac{M_{bh}}{M_\star}\right)^{1/3} \cdot r_\star$ here $r_\star$ is the radius of the star and $M_\star$ is its mass. Spectral classification suggests SO-2 is B1 V but for simplicity we will just read parameters for B0 V from the table — a B0 V star should have $17.5 M_\odot$ and $R = 7.4 R_\odot$

so that means $r_{rip} \approx \left(\frac{3.6\times10^{6} M_{\odot}}{17.5 M_{\odot}}\right)^{1/3} \cdot 7.4 \times 6.96\times10^{5}$ Km

$r_{rip} \approx 3.0\times10^{8}$ Km $= 3.0\times10^{11}$ m

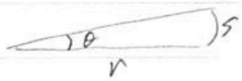
SO-2 does not get close enough to be in danger of disruption; it approaches to about $100 \times r_{rip}$.

Q10.4

Mass estimate for the BH at the center of M87 based on ionized gas kinematics in the vicinity of the BH.

2 pts

a)



$s = r\theta$ with θ in radians

For a radius of 0.1″ @ 16.8 Mpc, the linear radius is

$$s = 16.8\times10^{6} \times 3.09\times10^{13}\ \text{km} \times 0.1'' \times \frac{\pi\ \text{rad}}{180\times3600''}$$
$$= 2.5\times10^{14}\ \text{Km}$$

or 8.2 pc

2 pts

b)

read values off the figure. Looks to me like the velocities are around 1780 km/s @ −0.1″ and maybe 650 km/s @ +0.1″

$$V_{circ} = \frac{1780-650}{2}\ \text{km/s} = 565\ \text{km/s}$$

3 pts

c)

So for a circular velocity of 565 km/s @ radius of 2.5×10^{14} km

$$M_{bh} = V^2R/G \quad \text{again} = \frac{(565\times10^{3}\ \text{m/s})^2\,(2.5\times10^{17}\ \text{m})}{6.67\times10^{-11}\ \text{kg}^{-1}\text{m}^3\text{s}^{-2}}$$
$$= 1.2\times10^{39}\ \text{kg}$$
$$= 6.0\times10^{8}\ M_\odot$$

More careful estimates are a bit higher than that, but this exercise gives some idea of where the estimates come from.

5 pts

Q10.5 Oort constants' meanings

$A \equiv -\frac{R_0}{2} \left.\frac{d\omega}{dR}\right|_{R_0}$ and $B \equiv A - \omega_0$. So $A - B = \omega_0$.

let's evaluate. $\frac{A-B}{A+B} = \frac{\omega_0}{2A - \omega_0} = \frac{\omega_0}{-R_0 \left.\frac{d\omega}{dR}\right|_{R_0} - \omega_0}$

OK, Now look at $\frac{d\omega}{dR}$. For a keplerian circular velocity curve, $V_c = \sqrt{\frac{GM}{R}}$ and M is constant, so $V_c = (GM)^{1/2} R^{-1/2}$

Also $\omega = V_c/R \rightarrow \omega = (GM)^{1/2} R^{-3/2}$

Then $d\omega/dR = (GM)^{1/2} \left(\frac{-3}{2}\right) R^{-5/2}$

$$\left.\frac{d\omega}{dR}\right|_{R_0} = -\frac{3}{2} (GM)^{1/2} R_0^{-5/2}$$

$$R_0 \left.\frac{d\omega}{dR}\right|_{R_0} = \frac{-3}{2} (GM)^{1/2} R_0^{-3/2} = \frac{-3}{2} \omega_0$$

$$-R_0 \left.\frac{d\omega}{dR}\right|_{R_0} - \omega_0 = \frac{+3}{2} \omega_0 - \omega_0 = \frac{1}{2} \omega_0$$

$$\frac{A-B}{A+B} = \frac{\omega_0}{-R_0 \left.\frac{d\omega}{dR}\right|_{R_0} - \omega_0} = \frac{\omega_0}{\frac{1}{2}\omega_0} = 2.$$

Measured local values of Oort constants are
$A = 14.8 \pm 0.8 \, \frac{km}{s \cdot kpc}$ and $B = -12.4 \pm 0.6 \, \frac{km}{s \cdot kpc}$

$$\frac{A-B}{A+B} = \frac{14.8 + 12.4}{14.8 - 12.4} = 11.3 \text{, not } 2.$$

We haven't done careful error propagation here but it seems clear that the local values of the Oort constants are inconsistent with keplerian velocities. (We are in the midst of the MW's mass distribution, so $M(<r) \neq$ const.)

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Q10.6 More on pulsars!

redo parts c, e, and f of Q10.1 for neutron stars rather than WD.

2 pts c) $V_{rot} = \frac{2\pi R}{P}$ but this time we'll use a NS mass-radius relationship.

$$R_{NS} \approx 11\text{ km}\left(\frac{M}{1.4M_\odot}\right)^{-1/3} \quad \text{eq } 18.41$$

$$\Rightarrow V_{rot} = 2\pi \cdot \frac{3\text{ km}}{1\text{ s}}\left(\frac{P}{1\text{ s}}\right)^{-1}\left(\frac{M}{1.4M_\odot}\right)^{-1/3}$$

$$\approx 18.8\ \frac{\text{km}}{\text{s}}\left(\frac{P}{1\text{ s}}\right)^{-1}\left(\frac{M}{1.4M_\odot}\right)^{-1/3}$$

Not very high speed.

2 pts e) V_{crit} for breakup $= \left(\frac{GM}{R}\right)^{1/2}$

$$V_{crit} = \left(\frac{6.67\times10^{-11}\text{ kg}^{-1}\text{m}^3\text{s}^{-2}\cdot 1.4\times1.99\times10^{30}\text{ kg}}{11\,000\text{ m}}\right)^{1/2}\left(\frac{M}{1.4M_\odot}\right)^{2/3}$$

$$= 1.3\times10^8\ \frac{\text{m}}{\text{s}}\left(\frac{M}{1.4M_\odot}\right)^{2/3} \approx 0.4c\left(\frac{M}{1.4M_\odot}\right)^{2/3}$$

2 pts f) A rotating NS with $M\sim1.4M_\odot$ is a plausible model for a pulsar, at least in the sense that it wouldn't break up when rotating at the required speed.