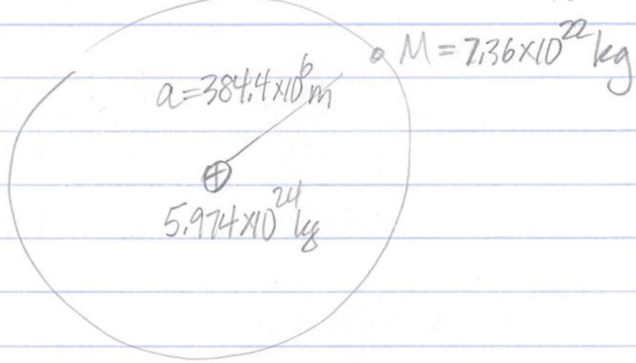
4.1 Tidal braking of Earth-moon system



3 pts

a) Assuming we're supposed to use one of Kepler's laws here

$$P^2 = \frac{4\pi^2}{GM_\oplus} a^3$$

and orbital period relates to ang. vel ω as $\omega = 2\pi/P$. So $\left(\frac{\cancel{2\pi}}{\omega}\right)^2 = \frac{\cancel{4\pi^2} a^3}{GM_\oplus}$.

$$\omega^2 = GM_\oplus/a^3 \quad \rightarrow \quad \omega_{orb} = \left(GM_\oplus/a^3\right)^{1/2}$$

3 pts

b) orbital Ang. Mom $\vec{L} = m\vec{r}\times\vec{V}$. In nice circular orbits this becomes

$$L = mr^2\omega \quad \text{or in this case} \quad L_{orb} = M_M\, a^2 \frac{\sqrt{GM_\oplus}}{a^{3/2}}$$

$$L_{orb} = M_M\sqrt{GM_\oplus a}$$

3 pts

c) Earth as a sphere with $M_\oplus$ and $R_\oplus$ and ω_{sid}

spin ang. mom is $L_{rot} = I\omega_{sid} = \frac{2}{5}M_\oplus R_\oplus^2\,\omega_{sid}$

5 pts

d) $L_{tot} = L_{rot} + L_{orb} = \text{const}$ so $dL_{tot}/dt = 0$.

$$\frac{dL_{rot}}{dt} = -\frac{dL_{orb}}{dt}.$$

$$\left.\begin{aligned}\frac{dL_{rot}}{dt} &= \frac{2}{5}M_\oplus R_\oplus^2 \frac{d\omega_{sid}}{dt}\\ \frac{dL_{orb}}{dt} &= M_M\sqrt{GM_\oplus}\,\frac{1}{2}a^{-1/2}\frac{da}{dt}\end{aligned}\right\} \quad \frac{2}{5}M_\oplus R_\oplus^2\frac{d\omega_{sid}}{dt} = -\frac{M_M}{2}\frac{\sqrt{GM_\oplus}}{a^{1/2}}\,\dot{a}$$

continued →

d) from prev. side $\frac{2}{5} M_\oplus R_\oplus^2 \frac{d\omega_{sid}}{dt} = \frac{-M_m}{2} \frac{\sqrt{GM_\oplus}}{a^{1/2}} \dot{a}$

want $\frac{\dot{a}}{a}$ = something

$$\frac{\dot{a}}{a^{1/2}} = \frac{-4}{5} \frac{M_\oplus}{M_m} R_\oplus^2 \frac{d\omega_{sid}}{dt} \cdot \frac{1}{\sqrt{GM_\oplus}}$$

From (a): $\omega_{orb} = \sqrt{GM_\oplus} \cdot a^{-3/2}$ so $\sqrt{GM_\oplus} = \omega_{orb} \cdot a^{3/2}$

$$\left[\frac{\dot{a}}{a^{1/2}} = \frac{-4}{5} \frac{M_\oplus}{M_M} \left(\frac{d\omega_{sid}}{dt}\right) \frac{R_\oplus^2}{\omega_{orb} a^{3/2}}\right] \times \frac{1}{a^{1/2}}$$

$$\frac{\dot{a}}{a} = \frac{-4}{5} \frac{M_\oplus}{M_m} \left(\frac{d\omega_{sid}}{dt}\right) \frac{R_\oplus^2}{a^2} \cdot \frac{1}{\omega_{orb}}$$

5 pts e) $dP_{rot}/dt = 5.2 \times 10^{-13}$ s/s $\Rightarrow$ use for $d\omega_{sid}/dt$

here we are using P_{rot} to refer to $\oplus$ sidereal day

so $\omega_{sid} = 2\pi / P_{rot}$

$\frac{d\omega_{sid}}{dt} = 2\pi(-1) P_{rot}^{-2} \frac{dP_{rot}}{dt}$. $P_{rot} = 0.997\,d$ (Tab A3)

$\omega_{orb} = \frac{2\pi}{P_{orb}}$ for which we'll use sidereal month 27.32 d

$$\dot{a} = +\frac{4}{5}\left(\frac{5.974 \times 10^{24}\,\cancel{kg}}{7.36 \times 10^{22}\,\cancel{kg}}\right) \frac{\cancel{2\pi}}{(0.997\,d)^2} \left(5.2 \times 10^{-13} \frac{\cancel{s}}{\cancel{s}}\right) \left(\frac{6.378 \times 10^6\,m}{384.4 \times 10^6\,m}\right)^2 \cdot \frac{27.32\,d}{\cancel{2\pi}}$$

= units will be $\frac{m^2}{m} \cdot \frac{d}{d^2} = m/day$ ok

$$\dot{a} = 9.82 \times 10^{-5} \frac{m}{d} \times \frac{100\,cm}{m} \times \frac{365.256\,d}{yr} = 3.6\ cm/yr!$$

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Q4.2 Radiative Transfer

3 pts

a) radiative transfer $I(x) = I_0 e^{-n\sigma x}$

x_m is the dist. over which $I_0 \to I_0/e$

$I(x_m) = I_0 e^{-n\sigma x_m} = I_0/e = I_0 e^{-1}$

$\therefore \quad n\sigma x_m = 1$

$x_m = 1/n\sigma.$

3 pts

b) $\tau \equiv x/x_m \quad \Rightarrow \quad x = \tau x_m$

$I(\tau) = I_0 e^{-n\sigma x_m \tau} = I_0 e^{-(1)\cdot\tau} = I_0 e^{-\tau}$

5 pts

c) slab of thickness 1 cm absorbs 15% of the light.

ie $I(1\,cm) = 0.85 I_0$.

since $I(\tau) = I_0 e^{-\tau}$, this means $e^{-\tau} = 0.85$

$\ln(0.85) = -\tau$

$\tau = 0.16$

And $x_m = x/\tau$, so $x_m = \dfrac{1\,cm}{-\ln(0.85)} = 6.15\,cm$

Now how thick would the slab be to absorb 99% of the light?

That occurs when $e^{-\tau} = 0.01$

$\tau = \ln(100) = 4.61$

and since $x = \tau x_m$, that thickness required is

$x = (4.61)(6.15\,cm) = 28.3\,cm.$

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Q43 Thermodynamics & collisional excitation

4 pts

a) $\langle E \rangle = \int_0^\infty E F(E) dE = \int_0^\infty E \cdot \frac{2}{\sqrt{\pi} kT} \left(\frac{E}{kT}\right)^{1/2} \exp\left(-\frac{E}{kT}\right) dE$

sub $x \equiv E/kT$

$dx = \frac{1}{kT} dE$

$$\langle E \rangle = \frac{2}{\sqrt{\pi}\,\cancel{kT}} \int_0^\infty \cancel{kT}\, x\, x^{1/2} e^{-x}\, kT\, dx$$

$$= \frac{2kT}{\sqrt{\pi}} \int_0^\infty x^{3/2} e^{-x} dx = \frac{2kT}{\cancel{\sqrt{\pi}}} \frac{3\cancel{\sqrt{\pi}}}{4} = \frac{3}{2} kT$$

The only tricky part is keeping track of all the kT's in the substitution.

4 pts

b) $\langle E \rangle = \frac{3}{2} kT$

A particle whose energy is 4 times the mean has energy $4 \times \frac{3}{2} kT = 6kT$.

Suppose that $6kT$ is enough to ionize an H atom, i.e.

$$6kT = 13.6\,eV \times \frac{1.602 \times 10^{-19} J}{eV} = 2.18 \times 10^{-18} J$$

$$T = 13.6\,eV \times \frac{1.602 \times 10^{-19} J}{eV} \cdot \frac{1}{6 \times 1.38 \times 10^{-23} J/k}$$

$$= 2.6 \times 10^4 K$$

and if this is a proton its KE $(\frac{1}{2} m v^2) \rightarrow$ speed v given by

$$\frac{1}{2} m v^2 = E$$

$$v = \left(\frac{2E}{m}\right)^{1/2} = \left(\frac{2 \times 2.18 \times 10^{-18} J}{1.67 \times 10^{-27} kg}\right) = 5.1 \times 10^4\ m/s$$

51 km/s

4 pts

c) Similar computation but now instead of requiring 13.6 eV, we need less to go $n=1 \rightarrow n=2$

c) Continued

3 pts

Bohr model energies are $13.6\,eV\left(\frac{1}{n^2}\right)$

so ΔE for $n=1 \rightarrow n=2$ is $\Delta E = 13.6\,eV\left(1-\frac{1}{4}\right)$
$= 13.6\,eV\,(3/4)$

by analogy with previous part, the necessary temp is
$$T = (2.6\times10^4\,K)\times 3/4 = 2.0\times10^4\,K$$

speed will be $51\,km/s\left(3/4\right)^{1/2} = 44\,km/s$

Both of these temperatures are much higher than the Surface of the Sun, which is only about 6000K. H atoms on the Sun's surface probably exist in a variety of ionization and excitation states but most of them will probably be neutral atoms in the $n=1$ state. (More info in later Chapters.)

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Q4A Blackbody radiation

4 pts a) Check blackbody flux per unit area for a temp of 310 K.

Flux from a blackbody surface $F = \sigma_{SB} T^4$

for 310 K This is $5.67 \times 10^{-8} \frac{W}{m^2 K^4} \times (310\,K)^4 = 523\ W/m^2$, ok!

4 pts b) $523 \frac{W}{m^2} = 523 \frac{J}{s \cdot m^2} \times 24 \times 3600\,s \times 2.0\,m^2 = 9.0 \times 10^7$ J per day

Assuming a surface area of maybe $2.0\,m^2$?
This value is hard to guess but $1\,m^2$ seems too little

food calories: $9.0 \times 10^7\,J \times \frac{1\,Cal}{4200\,J} = 2.1 \times 10^4$ Cal

much higher than our typical daily intake (ignores energy absorption from surroundings)

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4.5 Gravity vs electromagnetism

3 pts

a) Do a Bohr model derivation, but use gravity instead of an EM attraction.

Begin with Newton #2, i.e. $\vec{F} = m\vec{a}$ for hypothetical circular orbit:

$$-\frac{G m_e m_p}{r^2} = -\frac{m_e v^2}{r} \qquad (5.1)$$

Thus $m_e v^2 = G m_e m_p / r \Rightarrow KE = \frac{1}{2} m_e v^2 = \frac{G m_e m_p}{2r}$ (5.2)

$$U = \text{Grav. PE} = -\frac{G m_e m_p}{r} \qquad (5.3)$$

$$\text{total } E = K + U = -\frac{G m_e m_p}{2r} \qquad (5.4)$$

$L = n\hbar$ as usual, so $m_e^2 v^2 r^2 = n^2\hbar^2 = (m_e v^2) m_e r^2$

$$n^2\hbar^2 = \frac{G m_e m_p}{r} \cdot m_e r^2 = G m_e^2 m_p r$$

$$\Rightarrow r_n = \frac{n^2\hbar^2}{G m_e^2 m_p} \qquad (5.8)$$

Ground state orbit, $n=1$, has radius $r_1 = \frac{\hbar^2}{G m_e^2 m_p}$

energy is $E_n = -\frac{G m_e m_p}{2 r_n} = -\frac{G m_e m_p \, G m_e^2 m_p}{2\hbar^2 n^2}$

$$E_n = -\frac{G^2 m_e^3 m_p^2}{2\hbar^2} \cdot \frac{1}{n^2}$$

3 pts

b) Plug #s

$$r_1 = \frac{\hbar^2}{G m_e^2 m_p} = \frac{(1.055 \times 10^{-34}\, J\cdot s)^2}{6.67\times10^{-11}\, kg^{-1} m^3 s^{-2} \cdot (9.11\times10^{-31}\, kg)^2 (1.67\times10^{-27}\, kg)}$$

$$= 1.2 \times 10^{29}\ m$$

$$= 8.0 \times 10^{17}\ AU \quad \text{ie large}$$

ground state energy $E_1 = -\dfrac{G^2 m_e^3 m_p^2}{2\hbar^2}$

$$= -\frac{(6.67\times10^{-11}\,kg^{-1}m^3s^{-2})^2(9.11\times10^{-31}\,kg)^3(1.67\times10^{-27}\,kg)^2}{2\,(1.055\times10^{-34}\,J\cdot s)^2}$$

$$= -4.2\times10^{-97}\,J$$
$$= -2.6\times10^{-78}\,eV \quad \text{ie tiny}$$

Gravity would not do a good job at binding atoms together.