ASTR 210 HW 5 © 2024 LYoung

5.1

3 pts a) observing at $\nu = 230$ GHz
want angular resolution $\theta \leq 50 \times 10^{-6}$ arcsec
what is the diameter D required?

$\theta = 1.22\, \lambda/D$ for θ in radians

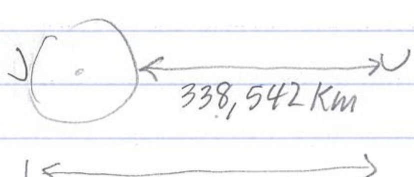
$D = 1.22\, \lambda/\theta$
also $\lambda = c/\nu$.
For $\nu = 230 \times 10^{9}$ Hz, $\lambda = \dfrac{3 \times 10^{5} \text{ km/s}}{230 \times 10^{9} \text{ Hz}} = 1.3 \times 10^{-6}$ Km
aka 1.3 mm

$D = \dfrac{1.22 \times 1.3 \times 10^{-6} \text{ km}}{50 \times 10^{-6} \times (\pi/180 \times 3600) \text{ rad}}$

$= 6500$ Km
slightly larger than the radius of the Earth!

3 pts b) orbiting radio telescope reached apogee of 338,542 km above Earth's surface
operated at $\lambda = 92, 18, 6,$ and 1.3 cm

Maximum theoretical resolution will be when $\lambda = 1.3$ cm and diameter D of interferometer = 338,542 Km



Actually, you might be able to squeeze out a little more if the separation is actually 338,542 km + $2R_\oplus$

338,542 Km $+ 2R_\oplus = 338{,}542 + 2(6371)$ Km $= 351{,}284$ km

$\theta = 1.22\, \lambda/D = \dfrac{1.22 \times 1.3 \times 10^{-5} \text{ Km}}{351{,}284 \text{ Km}} = 4.5 \times 10^{-11}$ rad
$= 9.3 \times 10^{-6}$ arcsec

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5.2

2 pts a) field in sunspot: P_s = gas P + magnetic P

$$P_s = \frac{\rho k_B T_s}{\mu m_p} + \frac{B_s^2}{2\mu_0}$$

2 pts b) photosphere: similar

$$P_p = \frac{\rho k_B T_p}{\mu m_p} + \frac{B_p^2}{2\mu_0}$$

4 pts c) $P_s = P_p$

$$\frac{\rho k_B T_p}{\mu m_p} + \frac{B_p^2}{2\mu_0} = \frac{\rho k_B T_s}{\mu m_p} + \frac{B_s^2}{2\mu_0}$$

solve for B_s

$$\frac{\rho k_B}{\mu m_p}(T_p - T_s) + \frac{B_p^2}{2\mu_0} = \frac{B_s^2}{2\mu_0}$$

$$B_s^2 = B_p^2 + \frac{2\mu_0 \rho k_B}{\mu m_p}(T_p - T_s)$$

5 pts d) $T_s = 4300\,K$, $T_p = 6100\,K$, $\rho = 3.5\times10^{-4}\ kg/m^3$, $\mu = 0.6$

$T_p - T_s = 1800\,K$

First set $B_p = 0$

$$B_s^2 = \frac{2 \cdot 4\pi\times10^{-7}\,W\cdot m \cdot 3.5\times10^{-4}\,kg/m^3 \cdot 1.38\times10^{-23}\,J/K\,(1800\,K)}{0.6 \times 1.67\times10^{-27}\,kg}$$

$B_s = 0.1477\,T$

Now set $B_p = 0.01\,T$. $B_s = \left[B_p^2 + \frac{2\mu_0 \rho k_B (T_p - T_s)}{\mu m_p}\right]^{1/2}$

$B_s = 0.148\,T$. For these values, ignoring B_p is OK.

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5.3

Energy storage in magnetic fields

2 pts a) region with $B = 0.03\,T$ produces a flare with $E = 2\times10^{24}\,J$

What was energy density? $\frac{B^2}{2\mu_0} = \frac{(0.03T)^2}{2\times 4\pi\times10^{-7}\,H/m} = 358\,J/m^3$

2 pts b) Volume required? $E = \varepsilon \cdot V$ so $V = E/\varepsilon$

$$V = \frac{2\times10^{24}\,J}{358\,J/m^3} = 5.6\times10^{21}\,m^3$$

3 pts c) Suppose that's a sphere - what is its radius?

$$V = \frac{4}{3}\pi R^3 \quad \text{so} \quad \left(\frac{3V}{4\pi}\right)^{1/3} = R$$

$$\left(\frac{3\times5.6\times10^{21}\,m^3}{4\pi}\right)^{1/3} = 1.1\times10^{7}\,m = 1.1\times10^{4}\,km$$

Very similar size to that of a sunspot.

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5.4 planetary temperatures

3 pts

a) planet absorbs energy at a rate

$$W_p = \frac{L_\star}{4\pi r^2}(\pi R^2)(1-A).$$

eq 8.6 in text,

$r = 1\text{ AU} = 1.496\times10^8\text{ km}$
$= 1.496\times10^{11}\text{ m}$

blackbody luminosity of planet is

$$L_p = 4\pi R^2 \sigma_{SB} T_p^4$$

in = out;
$$4\cancel{\pi R^2}\sigma_{SB}T_p^4 = \frac{L_\star}{4\pi r^2}(\cancel{\pi R^2})(1-A)$$

$$T_p^4 = \frac{L_\star(1-A)}{16\pi r^2\sigma_{SB}}$$

$$T_p = \frac{1}{2}\left[\frac{L_\star(1-A)}{\pi r^2\sigma_{SB}}\right]^{1/4}$$

3 pts

b)
$$dT_p/dL_\star = \frac{1}{2}\cdot\frac{1}{4}\left[\frac{L_\star(1-A)}{\pi r^2\sigma_{SB}}\right]^{-3/4}\frac{(1-A)}{\pi r^2\sigma_{SB}}$$

$$= \frac{1}{2}\cdot\frac{1}{4}L_\star^{-3/4}\left[\frac{1-A}{\pi r^2\sigma_{SB}}\right]^{1/4}$$

4 pts

c) evaluate. At present

$$\frac{dT_p}{dL_\star} = \frac{1}{2}\cdot\frac{1}{4}(3.839\times10^{26}\text{ W})^{-3/4}\left[\frac{0.6}{\pi(1.496\times10^{11}\text{ m})^2\,5.67\times10^{-8}\text{ W m}^{-2}\text{K}^{-4}}\right]^{1/4}$$

$$= 1.60\times10^{-25}\text{ K/W}$$

Check units on this one? Non-trivial.

$$W^{-3/4}m^{-1/2}(W m^{-2}K^{-4})^{1/4} = W^{-3/4}\cancel{m^{-1/2}}W^{-1/4}\cancel{m^{1/2}}K$$
$$= K/W \text{ ok}$$

checking units is not required but is a useful sanity check.

5 pts d) Sun $L_\star = 0.7 L_\odot$ @ $t=0$
$2.2 L_\odot$ @ $t = 10.6$ Gyr

So take $\frac{dL_\star}{dt} \approx \frac{\Delta L_\star}{\Delta t} = \frac{(2.2-0.7)L_\odot}{10.6\times10^9\ \mathrm{yr}}$

$= \frac{1.5\times3.839\times10^{26}\ \mathrm{W}}{10.6\times10^9\ \mathrm{yr}} = 5.4\times10^{16}\ \mathrm{W/yr}$

$\frac{dT_P}{dt} = \frac{dT_P}{dL_\star}\cdot\frac{dL_\star}{dt} = 1.60\times10^{-25}\frac{\mathrm{K}}{\mathrm{W}} \times 5.4\times10^{16}\frac{\mathrm{W}}{\mathrm{yr}}$
$= 8.7\times10^{-9}\ \mathrm{K/yr}$

4 pts e) How long, at that rate, will it take for T_P to increase by 1 K?
$\Delta t = 1\ \mathrm{K} / 8.7\times10^{-9}\ \mathrm{K/yr} = 1.2\times10^{8}\ \mathrm{yr}$
120 Myr

It's not a big effect on human timescales, but on geologic timescales it can be important.

5 pts f) Find r such that $T_P = 0^\circ C$ (273 K) and $100^\circ C$ (373 K).
back in (a): $T_P{}^4 = \frac{L_\star(1-A)}{16\pi r^2 \sigma_{SB}} \Rightarrow r^2 = \frac{L_\star(1-A)}{16\pi\sigma_{SB}T_P{}^4}$

plugging in current solar luminosity we find

$r = 1.21\times10^{11}\ \mathrm{m} = 0.81$ Au and
$r = 6.46\times10^{10}\ \mathrm{m} = 0.43$ Au

5.5

Peak of Planck function

ok, Planck fn $I_\lambda = \dfrac{2hc^2 \lambda^{-5}}{e^{hc/\lambda kT} - 1}$

This will be easier if we write $X \equiv \dfrac{hc}{\lambda k_B T} = \dfrac{hc}{k_B T} \lambda^{-1}$

Then $I(X) = C \cdot \dfrac{X^5}{e^X - 1}$ with C = constants
aka $C X^5 (e^X - 1)^{-1}$.

$\dfrac{dI}{d\lambda} = 0$ at the peak. Also $\dfrac{dI}{d\lambda} = \dfrac{dI}{dx} \dfrac{dx}{d\lambda}$.

$$\frac{dI}{dx} = \frac{C 5x^4}{e^x - 1} + \frac{C x^5 (-1) e^x}{(e^x - 1)^2}$$

and $\dfrac{dx}{d\lambda} = \dfrac{hc}{k_B T} (-1) \lambda^{-2} = \dfrac{-hc}{\lambda k_B T} \cdot \dfrac{1}{\lambda} = -X \cdot X \dfrac{k_B T}{hc} = \dfrac{-k_B T}{hc} X^2$

so $dI/d\lambda = 0$ means

$$\left(\frac{C \cdot 5x^4}{e^x - 1} - \frac{C x^5 e^x}{(e^x - 1)^2} \right) \left(\frac{-k_B T}{hc} X^2 \right) = 0$$

well, clearly we aren't interested in $X = 0$ $(\lambda = \infty)$
so that must mean

$$\frac{5\cancel{x^4}}{\cancel{e^x - 1}} - \frac{\cancel{x^5} e^x}{(e^x - 1)^{\cancel{2}}} = 0 \qquad 5 - \frac{x e^x}{e^x - 1} = 0$$

$$\frac{5(e^x - 1) - x e^x}{e^x - 1} = 0$$

more →

$$5(e^x-1) - xe^x = 0$$

$$5e^x - 5 - xe^x = 0$$

$$-(5-x)e^x + 5 = 0 \quad \text{aka} \quad (x-5)e^x + 5 = 0$$

root of this, we are told, is $x_p \approx 4.965$ and also $x_p \equiv \frac{hc}{\lambda_p k_B T}$

so then $\lambda_p = \dfrac{hc}{x_p k_B T}$

$$\lambda_p = \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s} \times \overset{2.998}{\cancel{3.0}}\times10^{8}\ \text{m/s}}{4.965 \times 1.38\times10^{-23}\ \text{J/K}\cdot T}$$

$$= \frac{2.90\times10^{-3}}{T}\ \text{m}\cdot\text{K} = \frac{2.90\times10^{3}}{T}\ \mu\text{m}\cdot\text{K}$$