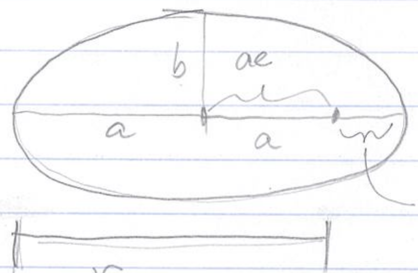
ASTR210 HW3 ©LYoung 2023

HW3 : Ch 3

3.1

a)




$r_{pe} = a - ae = a(1-e)$

$r_{ap} = a + ae = a(1+e)$


prove $e = \sqrt{1 - b^2/a^2}$

Hmm this looks like pythagoras.

working backwards, $e^2 = 1 - b^2/a^2$

$a^2e^2 = a^2 - b^2$

$a^2e^2 + b^2 = a^2$ still same info as

oh! Problem statement says that in this figure $r + r' = 2a$
but $r' = r$ so $r = a$.
($r + r' = 2a$ comes from the "string" definition of an ellipse.)

In the triangle $b^2 + a^2e^2 = r^2$ but $r^2 = a^2$
so $b^2 + a^2e^2 = a^2$

Now go backwards to $e = \sqrt{1 - b^2/a^2}$ as above

b) Satellite has perigee $R_\oplus + 300\,km$, apogee $R_\oplus + 3000\,km$. Find e.

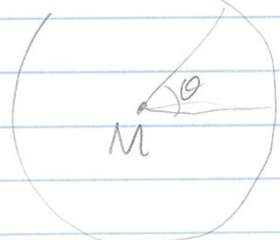
If $r_{pe} = a(1-e)$ and $r_{ap} = a(1+e)$,

$\frac{r_{pe}}{r_{ap}} = \frac{a(1-e)}{a(1+e)} = \frac{1-e}{1+e}$. Algebra: $r_{pe}(1+e) = r_{ap}(1-e)$

$r_{pe} + r_{pe} \cdot e = r_{ap} - r_{ap} \cdot e$ ; $(r_{pe} + r_{ap})e = r_{ap} - r_{pe}$

$$\Rightarrow e = \frac{r_{ap} - r_{pe}}{r_{ap} + r_{pe}} = \frac{(R_\oplus + 3000 - R_\oplus - 300)}{2R_\oplus + 3300} = \frac{2700\,km}{2 \cdot 6378 + 3300\,km} = \boxed{0.168}$$

(3.2) Circular keplerian orbit




Const. ang vel $\omega$

a) $\vec{r}(0) = R\hat{\imath}$

general position for some $\theta$ will be $R\cos\theta\,\hat{\imath} + R\sin\theta\,\hat{\jmath}$

and $\theta = \omega t$

so $\vec{r}(t) = R\cos\omega t\,\hat{\imath} + R\sin\omega t\,\hat{\jmath}$

b) $\vec{v} = d\vec{r}/dt = -R\omega\sin\omega t\,\hat{\imath} + R\omega\cos\omega t\,\hat{\jmath}$

$$|\vec{v}| = (\vec{v}\cdot\vec{v})^{1/2} = \left(R^2\omega^2\sin^2\omega t + R^2\omega^2\cos^2\omega t\right)^{1/2} = \left(R^2\omega^2\right)^{1/2} = R\omega$$

c) $\vec{a} = d\vec{v}/dt = -R\omega^2\cos\omega t\,\hat{\imath} - R\omega^2\sin\omega t\,\hat{\jmath}$

$$|\vec{a}| = (\vec{a}\cdot\vec{a})^{1/2} = \left(R^2\omega^4\cos^2\omega t + R^2\omega^4\sin^2\omega t\right)^{1/2} = R\omega^2$$

aka $v^2/R$

d) $\vec{a} = -\omega^2\vec{r}$. Acceleration is oriented radially inward.

e) Newton #2 is $F = ma$ or in this case $\frac{GMm}{R^2} = ma = m\frac{v^2}{R}$

f) $\frac{GM\cancel{m}}{R^{\cancel{2}}} = \frac{\cancel{m}v^2}{\cancel{R}}$

$v^2 = GM/R$ or $v = \sqrt{GM/R}$

g) orbital period is $P$. Circumference is $2\pi R$ so let's see:

$v = 2\pi R/P$.

$P = 2\pi R/v$

$$P = \frac{2\pi R}{\sqrt{GM/R}} = \frac{2\pi R^{3/2}}{\sqrt{GM}} \Rightarrow P^2 = \frac{4\pi^2 R^3}{GM}$$

3.2 continued

h) 2 planets with periods 66.6 d and 98.2 d in orbit around a star of mass $1.4 M_\odot$
Find orbital radii. Show work! (The answers are in the paper I've linked.)

$$P^2 = \frac{4\pi^2 R^3}{GM} \rightarrow R^3 = \frac{P^2 \cdot GM}{4\pi^2} \rightarrow R = \left[\frac{P^2 \cdot GM}{4\pi^2}\right]^{1/3}$$

$$66.6\,d \times \frac{24 \times 3600\,s}{d} = 5.75 \times 10^6\,s$$

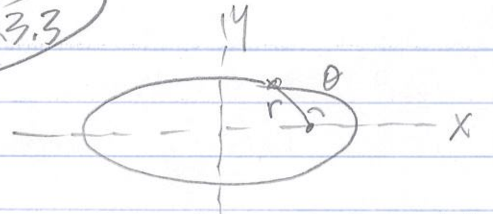
$$98.2\,d \times \frac{24 \times 3600\,s}{d} = 8.48 \times 10^6\,s$$

$$R_1 = \left[\frac{(5.75 \times 10^6\,s)^2 \cdot 6.67 \times 10^{-11}\,m^3 kg^{-1} s^{-2} \cdot 1.4 \cdot 1.99 \times 10^{30}\,kg}{4\pi^2}\right]^{1/3}$$

$$= \boxed{5.25 \times 10^{10}\,m} \times \frac{Au}{1.496 \times 10^{11}\,m} = \boxed{0.35\,Au}$$

$R_2$ = similar calc gives $\boxed{6.80 \times 10^{10}\,m}$ and $\boxed{0.45\,Au}$

Q3.3




a) we are given: $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$ eq 3.67

Want $v(\theta) = \sqrt{\frac{GM}{a(1-e^2)}(1+2e\cos\theta+e^2)}$

using $r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$

maybe just plug $r(\theta)$ into $v(r)$?

$$v^2(r) = GM\left(\frac{2}{r} - \frac{1}{a}\right) = GM\left(\frac{2(1+e\cos\theta)}{a(1-e^2)} - \frac{1}{a}\right)$$

$$= GM\left(\frac{2(1+e\cos\theta)}{a(1-e^2)} - \frac{1-e^2}{a(1-e^2)}\right)$$

$$v^2 = GM\left(\frac{1+2e\cos\theta+e^2}{a(1-e^2)}\right) \rightarrow \text{ ok } \checkmark$$

b) $dv/d\theta$ and $d^2v/d\theta^2$

$$v = (\text{stuff})(1+2e\cos\theta+e^2)^{1/2}$$

$$dv/d\theta = (\text{stuff})\tfrac{1}{2}(1+2e\cos\theta+e^2)^{-1/2}\,2e(-\sin\theta)$$

so $dv/d\theta = 0$ @ $\theta = 0, \pi$

$$d^2v/d\theta^2 = (\text{other stuff})\Big[(+\sin\theta)\left(\tfrac{+1}{2}\right)(1+2e\cos\theta+e^2)^{-3/2}(-\sin\theta)$$
$$+ (1+2e\cos\theta+e^2)^{-1/2}(-\cos\theta)\Big]$$

$$= C(1+2e\cos\theta+e^2)^{-1/2}\left[\frac{-\sin^2\theta}{(1+2e\cos\theta+e^2)} - \cos\theta\right]$$

$0$ @ $\theta = 0, \pi$

$d^2v/d\theta^2 = (\text{positive stuff})(-\cos\theta)$ @ $\theta = 0, \pi$

so @ $\theta = 0$, $d^2v/d\theta^2 < 0$, velocity is max.

$\theta = \pi$, $d^2v/d\theta^2 > 0$, velocity is min.

c) perihelion $\theta = 0$. $v_{pe} = \sqrt{\frac{GM}{a(1-e^2)}(1+2e+e^2)} = \sqrt{\frac{GM(1+e)^2}{a(1+e)(1-e)}} = \sqrt{\frac{GM(1+e)}{a(1-e)}}$

Same thing will happen at aphelion but it'll be $1-2e+e^2$

$\Rightarrow v_{ap} = \sqrt{\frac{GM(1-e)}{a(1+e)}}$

d) Next page...

3.3 continued

d) Halley's comet $r_{pe} = 0.59\,\text{AU}$ , $r_{ap} = 35.14\,\text{AU}$

compute $a$ and $e$, find $V_{pe}$ and $V_{ap}$

$$a = \frac{r_{pe} + r_{ap}}{2} = \frac{35.14\,\text{AU} + 0.59\,\text{AU}}{2} = 17.87\,\text{AU} \times \frac{1.496\times10^{11}\,\text{m}}{\text{AU}}$$

$$= 2.67\times10^{12}\,\text{m}$$

$$e = \frac{r_{ap} - r_{pe}}{r_{ap} + r_{pe}} \text{ as in Q3.1b } = \frac{35.14 - 0.59}{35.14 + 0.59} = 0.967$$

$$V_{pe} = \left(\frac{GM(1+e)}{a(1-e)}\right)^{1/2} = \left(\frac{6.67\times10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2} \times 1.99\times10^{30}\,\text{kg}\,(1.967)}{2.67\times10^{12}\,\text{m}\,(1-0.967)}\right)^{1/2}$$

$$\boxed{= 54.4\,\text{km/s}}$$

$$V_{ap} = \text{similar but } \frac{(1-e)}{(1+e)} = \boxed{0.91\,\text{km/s}}$$

I don't care if they quote speeds in m/s or km/s

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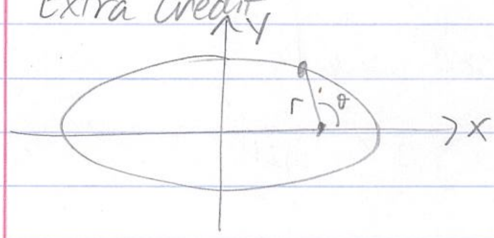
Ch 3.4

Extra credit




$$x = ae + r\cos\theta$$
$$y = r\sin\theta$$

ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $b^2x^2 + a^2y^2 = a^2b^2$

$y^2 = r^2\sin^2\theta$ ; $x^2 = (ae + r\cos\theta)^2 = a^2e^2 + 2aer\cos\theta + r^2\cos^2\theta$

ellipse is: $b^2a^2e^2 + 2ab^2er\cos\theta + b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$

eliminate b in favor of a and e, as in Q 3.1a

$e = \sqrt{1 - b^2/a^2}$ or $e^2 = 1 - \frac{b^2}{a^2}$ or $e^2a^2 = a^2 - b^2$ or $b^2 = a^2(1-e^2)$.

ellipse is: $\cancel{a^2}(1-e^2)a^2e^2 + 2a\cancel{a^2}(1-e^2)er\cos\theta + \cancel{a^2}(1-e^2)r^2\cos^2\theta + \cancel{a^2}r^2\sin^2\theta = a^2\cancel{a^2}(1-e^2)$

get rid of an $a^2$ everywhere

Collect terms in r

$$(1-e^2)\cos^2\theta\, r^2 + \sin^2\theta\, r^2 + 2ae(1-e^2)\cos\theta\, r + a^2e^2(1-e^2) - a^2(1-e^2) = 0$$

$\cos^2\theta\, r^2 - e^2\cos^2\theta\, r^2 + \sin^2\theta\, r^2$

$$r^2 - e^2\cos^2\theta\, r^2 + 2ae(1-e^2)\cos\theta\, r + a^2(1-e^2)(e^2-1) = 0$$

$$\underbrace{(1-e^2\cos^2\theta)}_{A}r^2 + \underbrace{2ae(1-e^2)\cos\theta}_{B}\, r \underbrace{- a^2(1-e^2)^2}_{C} = 0$$

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} =$$

$$\frac{-2ae(1-e^2)\cos\theta \pm \left[4a^2e^2(1-e^2)^2\cos^2\theta + 4(1-e^2\cos^2\theta)a^2(1-e^2)^2\right]^{1/2}}{2(1+e\cos\theta)(1-e\cos\theta)}$$

work on the sqrt bit by itself awhile

$$\left[4a^2e^2(1-e^2)^2\cos^2\theta + 4(1-e^2\cos^2\theta)a^2(1-e^2)^2\right]$$

$$4a^2(1-e^2)^2\left[e^2\cancel{\cos^2\theta} + 1 - e^2\cancel{\cos^2\theta}\right]$$

$$\Rightarrow \text{sqrt bit is } \left[4a^2(1-e^2)^2\right]^{1/2} = 2a(1-e^2)$$

$$r = \frac{-\cancel{2}ae(1-e^2)\cos\theta \pm \cancel{2}a(1-e^2)}{\cancel{2}(1+e\cos\theta)(1-e\cos\theta)} = \frac{+a(1-e^2)(-e\cos\theta \pm 1)}{(1+e\cos\theta)(1-e\cos\theta)}$$

$$r = \frac{a(1-e^2)\cancel{(-e\cos\theta+1)}}{(1+e\cos\theta)\cancel{(1-e\cos\theta)}} \text{ or } \frac{+a(1-e^2)(-e\cos\theta-1)}{(1+e\cos\theta)(1-e\cos\theta)}$$

$$\downarrow$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

This one is always negative or otherwise yucky