ASTR 210
HW 12 (extra credit)

Q12.1

2 pts a) 23.22: $\frac{dE}{dt} = -P(t)\frac{dV}{dt}$

use energy $E = uV = \frac{4\sigma_{SB}}{c}T^4 V$

and pressure $P = \frac{1}{3}u = \frac{4\sigma_{SB}}{3c}T^4$

properties of a photon "gas" w/ blackbody spectrum

$dE/dt = \frac{d}{dt}\left(\frac{4\sigma_{SB}}{c}T^4 V\right) = \frac{4\sigma_{SB}}{c}\left(4T^3 V\frac{dT}{dt} + T^4\frac{dV}{dt}\right)$

$\Rightarrow \cancel{\frac{4\sigma_{SB}}{c}}\left(4T^3 V\frac{dT}{dt} + T^4\frac{dV}{dt}\right) = -\cancel{\frac{4\sigma_{SB}}{3c}}T^4\frac{dV}{dt}$ ✓ (23.23)

2 pts b) $\Rightarrow 4T^3 V\frac{dT}{dt} + T^4\frac{dV}{dt} = -\frac{1}{3}T^4\frac{dV}{dt}$

$4V\frac{dT}{dt} + T\frac{dV}{dt} = -\frac{1}{3}T\frac{dV}{dt}$

$\left(\cancel{4}V\frac{dT}{dt} = -\frac{\cancel{4}}{3}T\frac{dV}{dt}\right)\cdot\frac{1}{VT}$

$\frac{1}{T}\frac{dT}{dt} = \frac{-1}{3V}\frac{dV}{dt}$ ✓ (23.24)

2 pts c) $V(t) = k\cdot a^3$ for some const k

$\frac{dV}{dt} = k\cdot 3a^2\frac{da}{dt}$ and $\frac{1}{V}\frac{dV}{dt} = \frac{k\cdot 3a^2}{k\cdot a^3}\frac{da}{dt} = \frac{3}{a}\frac{da}{dt}$

$\Rightarrow$ 23.24 becomes $\frac{1}{T}\frac{dT}{dt} = \frac{-1}{\cancel{3}}\cdot\frac{\cancel{3}}{a}\frac{da}{dt} = -\frac{1}{a}\frac{da}{dt}$ ✓ (23.25)

2 pts d) Note $\frac{d(\ln T)}{dt} = \frac{1}{T}\frac{dT}{dt}$ by chain rule

$\Rightarrow \frac{d(\ln T)}{dt} = \frac{-d(\ln a)}{dt}$ ✓ (23.26)

2 pts

e) $\int dt \cdot \frac{d(\ln T)}{dt} - \int \frac{d(\ln a)}{dt} \cdot dt \Rightarrow \ln T = -\ln a + \text{const}$
$= \ln\left(\frac{1}{a}\right) + \text{const}$

$\Rightarrow T = k \cdot a^{-1}$

Also if we are talking about a blackbody, we know that the spectrum peaks at $\lambda_p = \frac{2900\,\mu m \cdot K}{T}$ eq 8.4

if $T = k \cdot a^{-1}$ as universe expands, then $\lambda_p = \frac{2900\,\mu m \cdot K}{k} \cdot a$

so $\lambda_p \propto a^1$ as universe expands

2 pts

f) 23.28 : $\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{GM}{r(t)} + k$

use $M = \frac{4}{3}\pi r^3 \rho$ and $r = a(t) r_0$

$\frac{dr}{dt} = \dot{a} r_0$

$$\left[\frac{1}{2}(\dot{a} r_0)^2 = \frac{G \cdot \frac{4}{3}\pi \rho r^3}{r} + k = \frac{4}{3}\pi G \rho r^2 + k = \frac{4}{3}\pi G \rho a^2 r_0^2 + k\right] \cdot \frac{2}{r_0^2 a^2}$$

$$\frac{1}{2}\dot{a}^2 r_0^2 \cdot \frac{2}{r_0^2 a^2} = \frac{4}{3}\pi G \rho a^2 r_0^2 \cdot \frac{2}{r_0^2 a^2} + \frac{2k}{r_0^2 a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G \rho + \frac{2k}{r_0^2 a^2}$$ ✓ (23.32)

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Q12.1 cont'd

2 pts

g) 23.71: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left[u_r(t) + u_m(t) + u_\Lambda\right] - \frac{Kc^2}{r_{c,0}^2\, a^2}$

remember also $\frac{\dot{a}}{a} = H$

when $K=0$, universe has zero curvature and that implies

$$H^2 = \frac{8\pi G}{3c^2}\left[u_r + u_m + u_\Lambda\right] = \frac{8\pi G}{3c^2}\cdot u_c$$

which means $u_c = \frac{3c^2 H^2}{8\pi G}$ ✓ (23.73)

2 pts

h) Friedmann eqn $\dot{a} = H_0\left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}\, a^2\right]^{1/2}$ (24.35)

$$\ddot{a} = \frac{1}{2} H_0\left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}\, a^2\right]^{-1/2} \times$$

$$\left\{\Omega_{r,0}(-2a^{-3})\dot{a} + \Omega_{m,0}(-1\cdot a^{-2})\dot{a} + \Omega_{\Lambda,0}\cdot 2a\dot{a}\right\}$$

$$\ddot{a} = \frac{1}{2} H_0 \cancel{\dot{a}}\left(\frac{H_0}{\cancel{\dot{a}}}\right)\left\{-\frac{2\Omega_{r,0}}{a^3} - \frac{\Omega_{m,0}}{a^2} + 2\Omega_{\Lambda,0}\cdot a\right\}$$

$$= -H_0^2\left\{\frac{\Omega_{r,0}}{a^3} + \frac{\Omega_{m,0}}{2a^2} - \Omega_{\Lambda,0}\, a\right\}$$

✓ same as 24.36

Q12.2

Suppose $\Omega_{m,0} = 0.27$ and $\Omega_{\Lambda,0} = 0.73$

2 pts a) redshift of equality between matter and Λ?

$\Omega_{m,0} = \frac{u_{m,0}}{u_{c,0}}$ and $\Omega_{\Lambda,0} = \frac{u_\Lambda}{u_{c,0}}$. So when was $u_m(t) = u_\Lambda$?

$u_m(t) = u_{m,0}\, a^{-3}$ $\Rightarrow$ we want a_{eq} such that

$$\frac{u_{m,0}}{a_{eq}^3} = u_\Lambda \quad \text{or} \quad a_{eq}^3 = \frac{u_{m,0}}{u_\Lambda} = \frac{u_{m,0}}{u_{c,0}} \cdot \frac{u_{c,0}}{u_\Lambda} = \frac{\Omega_{m,0}}{\Omega_{\Lambda 0}}$$

$$a_{eq} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{0.27}{0.73}\right)^{1/3} = 0.718$$

Also $1 + z = 1/a$ so $z_{eq} = (1/a_{eq}) - 1 = 0.39$

2 pts b) Calc $\ddot{a}$ @ z_{eq}

from 24.36: $\ddot{a} = H_0^2\left[-\frac{\cancel{\Omega_{r,0}}}{\cancel{a^3}}^{\Omega_{r,0}=0} - \frac{\Omega_{m,0}}{2a^2} + \Omega_{\Lambda,0}\, a\right]$

$$\ddot{a}_{eq} = H_0^2\left[-\frac{0.27}{2(0.718)^2} + 0.73(0.718)\right] = H_0^2\left[0.262\right]$$

since that factor 0.262 is >0, the expansion was accelerating at that time.

2 pts c) Calculate H @ z_{eq}. we know $H = \dot{a}/a$

And since $\dot{a} = H_0\left[\frac{\cancel{\Omega_{r,0}}}{\cancel{a^2}}^{0} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}\, a^2\right]^{1/2}$, we can plug

$$a = 0.718 \Rightarrow H = \frac{\dot{a}}{a} = \frac{H_0\left[\frac{\Omega_{m,0}}{a_{eq}} + \Omega_{\Lambda,0}\, a_{eq}^2\right]^{1/2}}{a_{eq}}$$

Q12.2 cont'd

c) $$H(m-\Lambda \text{ equality}) = H_0 \frac{\left[\frac{0.27}{0.718} + 0.73(0.718)^2\right]^{1/2}}{0.718}$$

$$= 1.21\, H_0$$

if H_0 is $72 \frac{km/s}{Mpc}$ now, it would have been

$87.0 \frac{km/s}{Mpc}$ at $m-\Lambda$ equality.

2 pts

d) Current proper distance

comoving coord r is equal to current proper distance

$$r = l_p(t_0)$$

$$= c\int_{t_{emit}}^{t_0} \frac{dt}{a(t)}$$

Assume, for simplicity of the math, that $a(t) \propto t$ and $t_0 = 1/H_0$. To scale $a(t)$ correctly we must be assuming $a(t) = t/t_0$.

This means

$$r = l_p(t_0) = c\int_{t_{eq}}^{t_0} \frac{dt}{t/t_0} = c\int_{t_{eq}}^{t_0} \frac{dt}{t} \cdot t_0 = ct_0 \int_{t_{eq}}^{t_0} \frac{dt}{t}$$

$$= ct_0 \ln t \Big|_{t_{eq}}^{t_0} = ct_0 \ln\left(t_0/t_{eq}\right)$$

since $a_{eq} = t_{eq}/t_0$ in this model, the current proper dist. to an object @ z_{eq} is $ct_0 \ln\left(\frac{t_0}{t_0 a_{eq}}\right) = ct_0 \ln\left(1/a_{eq}\right)$. Then if $t_0 = 1/H_0$, current proper dist is $\frac{c \ln(1/a_{eq})}{H_0}$

cont'd →

for $H_0 = 72 \frac{km/s}{Mpc}$, $\frac{c \ln(1/a_{eq})}{H_0} = \frac{3\times10^5 km/s \cdot \ln(1/0.718)}{72 km/s/Mpc}$

$= 1381$ Mpc. current proper dist to $z_{eq} = 0.39$

Compare to simple application of Hubble law?

Non-rel version is $cz = H_0 \cdot d$ or $d = cz/H_0$

which gives $\frac{3\times10^5 km/s \ (0.39)}{72 km/s/Mpc} = 1625$ Mpc

Note. The value of 1381 Mpc is a bit different from what you'll get out of a cosmology calculator because it is oversimplified ($a \propto t$) to make the math easy.