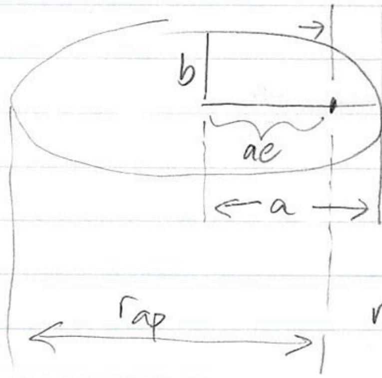
ASTR 210 HW 3

©LYoung
2024

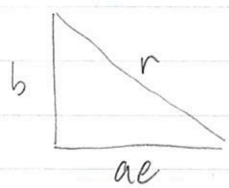
3.1

a)



$\leftarrow r_{pe} = a - ae = a(1-e)$

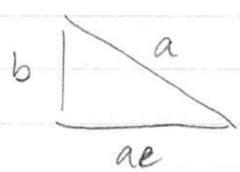
$r_{ap} = a + ae = a(1+e)$



prove $e = \sqrt{1 - b^2/a^2}$ using this triangle and $r + r' = 2a$

For the triangle shown in the picture, $r' = r$

Thus $r + r' = 2r = 2a \Rightarrow r = a$



pythagoras: $b^2 + a^2e^2 = a^2$

$$a^2e^2 = a^2 - b^2$$

$$e^2 = 1 - b^2/a^2$$

$$e = \sqrt{1 - b^2/a^2}$$

b) Zoozve $a = 0.7236$ AU

$e = 0.4101$

Earth orbit crossings may happen near aphelion

$$r_{ap} = a(1+e) = 0.7236 \text{ AU}(1.4101) = 1.0204 \text{ AU}.$$

As this value is > 1 AU, we know Zoozve crosses Earth's orbit.

Aside: since $\oplus$ orbit $a = 1.000$ AU and $e = 0.0167$, we know $\oplus - \odot$ distance varies from $(1 - 0.0167)$ AU to 1.0167 AU

3.2

a) 3.58 + 3.59 $\rightarrow$ 3.60

3.58: $K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{GMm}{L}\right)^2(1+e^2+2e\cos\theta)$

3.59: $U = -\frac{GMm}{r} = -\frac{(GM)^2m^3}{L^2}(1+e\cos\theta)$

$$E = K+U = \frac{(GM)^2m^3}{L^2}\frac{(1+e^2+2e\cos\theta)}{2} - \frac{(GM)^2m^3}{L^2}(1+e\cos\theta)$$

$$= \frac{(GM)^2m^3}{L^2}\left(\frac{1}{2}+\frac{e^2}{2}+\cancel{e\cos\theta}-1-\cancel{e\cos\theta}\right)$$

$$= \frac{(GM)^2m^3}{L^2}\left(\frac{e^2}{2}-\frac{1}{2}\right) = \frac{(GM)^2m^3}{2L^2}(e^2-1) = E$$

$\rightarrow$ 3.60 ok

b) 3.60 + 3.43 $\rightarrow E(M, m, a)$.

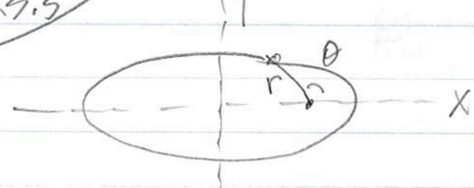
3.43: $\frac{L^2}{m^2} = GMa(1-e^2)$

3.60: $E = \frac{(GM)^2m^2m(e^2-1)}{2L^2} = \frac{(GM)^{\cancel{2}}m(e^2-1)}{2\cancel{G}Ma(1-e^2)}$

$$E = \frac{GMm(-1)}{2a} = \frac{-2GMm}{2a}.$$

©LYoung 2023

Q3.3



a) We are given: $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$ eq 3.67

Want $v(\theta) = \sqrt{\frac{GM}{a(1-e^2)}(1+2e\cos\theta+e^2)}$

using $r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$

maybe just plug $r(\theta)$ into $v(r)$?

$$v^2(r) = GM\left(\frac{2}{r} - \frac{1}{a}\right) = GM\left(\frac{2(1+e\cos\theta)}{a(1-e^2)} - \frac{1}{a}\right)$$

$$= GM\left(\frac{2(1+e\cos\theta)}{a(1-e^2)} - \frac{1-e^2}{a(1-e^2)}\right)$$

$$v^2 = GM\left(\frac{1+2e\cos\theta+e^2}{a(1-e^2)}\right) \rightarrow \text{ok} \checkmark$$

b) $dv/d\theta$ and $d^2v/d\theta^2$

$$v = (\text{stuff})(1+2e\cos\theta+e^2)^{1/2}$$

$$dv/d\theta = (\text{stuff})\tfrac{1}{2}(1+2e\cos\theta+e^2)^{-1/2}\,2e(-\sin\theta)$$

so $dv/d\theta = 0$ @ $\theta = 0, \pi$

$$d^2v/d\theta^2 = (\text{other stuff})\Big[(+\sin\theta)\left(\tfrac{+1}{2}\right)(1+2e\cos\theta+e^2)^{-3/2}(-\sin\theta)$$
$$+ (1+2e\cos\theta+e^2)^{-1/2}(-\cos\theta)\Big]$$

$$= C(1+2e\cos\theta+e^2)^{-1/2}\left[\frac{-\sin^2\theta}{(1+2e\cos\theta+e^2)} - \cos\theta\right]$$

0 @ $\theta = 0, \pi$

$d^2V/d\theta^2 = (\text{positive stuff})(-\cos\theta)$ @ $\theta = 0, \pi$

so @ $\theta = 0$, $d^2V/d\theta^2 < 0$, velocity is max.

$\theta = \pi$, $d^2V/d\theta^2 > 0$, velocity is min.

c) perihelion $\theta = 0$. $V_{pe} = \sqrt{\frac{GM}{a(1-e^2)}(1+2e+e^2)} = \sqrt{\frac{GM(1+e)^2}{a(1+e)(1-e)}} = \sqrt{\frac{GM(1+e)}{a(1-e)}}$

Same thing will happen at aphelion but it'll be $1-2e+e^2$

$\Rightarrow V_{ap} = \sqrt{\frac{GM(1-e)}{a(1+e)}}$

d) Zoozve again

orbital speed @ aphelion from $V_{ap} = \sqrt{\frac{GM(1-e)}{a(1+e)}}$

$a = 0.7236$ AU, $e = 0.4101$

$$V_{ap} = \left(\frac{6.67\times10^{-11}\ kg^{-1}m^3s^{-2} \cdot 1.99\times10^{30}\ kg}{0.7236\ AU \times 1.496\times10^{11}\ m/AU} \frac{(1-0.4101)}{(1.4101)} \right)^{1/2}$$

$V_{ap} = 22.6$ km/s

Earth's orbital speed can be computed the same way with $e=0$ and $a = 1.00$ AU $\rightarrow V_\oplus = 29.7$ km/s

or you could also do $V_\oplus = \frac{\text{circumference}}{\text{period}} = \frac{2\pi \cdot 1\text{AU}}{1\text{yr}}$

Earth will be overtaking Zoozve if they get close.

Zoozve's speed is not high enough to keep it on a circular orbit.

©2023 L Young

3.4

a) Tab A.3 Jupiter radius $10.97\,R_\oplus = 10.97 \times 6.378\times10^6\,m$

mass $317.8\,M_\oplus = 317.8 \times 5.974\times10^{24}\,kg$

$$\text{density} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3\times317.8\times5.974\times10^{24}\,kg}{4\pi\,(10.97\times6.378\times10^6\,m)^3} = 1.32\times10^3\,\frac{kg}{m^3}$$

b) ice density 1000 kg/m³ fairly similar to Jupiter value

Roche Limit for Jupiter's moons is $2.44\left(\frac{\rho_{big}}{\rho_{small}}\right)^{1/3} R_{big}$

here ρ_{big} and R_{big} refer to Jupiter
ρ_{small} to the small body (ice)

$\Rightarrow$ Roche limit is $2.44\left(\frac{1320}{1000}\right)^{1/3} R_J = 2.68\,R_J$

ok to use 2.5 here

$\times\ 10.97\times6.378\times10^3\,km = 1.9\times10^5\,km$

c) Innermost moons have orbital radii 128,000 km
vs Roche Limit at 190,000 km (approx)
These moons are inside the Roche Limit and They are vulnerable to tidal disruption

d) Kepler's laws work: $P^2 = \frac{4\pi^2}{GM_J} a^3$

$$\Rightarrow P = 2\pi\left[\frac{(1.28\times10^8\,m)^3}{6.67\times10^{-11}\,m^3\,kg^{-1}\,s^{-2}\cdot 317.8\times5.974\times10^{24}\,kg}\right]^{1/2}$$

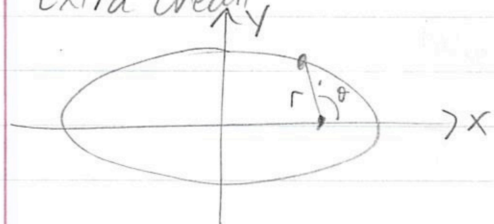
$P = 2.56\times10^4\,s$

$\times \frac{hr}{3600} = 7.1\ hr$

©LYoung
2023

3.5

Extra credit



$$x = ae + r\cos\theta$$
$$y = r\sin\theta$$

ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $b^2x^2 + a^2y^2 = a^2b^2$

$$y^2 = r^2\sin^2\theta\ ;\ x^2 = (ae + r\cos\theta)^2 = a^2e^2 + 2aer\cos\theta + r^2\cos^2\theta$$

ellipse is: $b^2a^2e^2 + 2ab^2er\cos\theta + b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$

eliminate b in favor of a and e, as in Q 3.1a

$$e = \sqrt{1 - b^2/a^2} \text{ or } e^2 = 1 - \frac{b^2}{a^2} \text{ or } e^2a^2 = a^2 - b^2 \text{ or}$$
$$b^2 = a^2(1-e^2).$$

ellipse is: $\cancel{a^2}(1-e^2)a^2e^2 + 2a\cancel{a^2}(1-e^2)er\cos\theta + \cancel{a^2}(1-e^2)r^2\cos^2\theta$
$+ \cancel{a^2}r^2\sin^2\theta = a^2\cancel{a^2}(1-e^2)$

get rid of an a^2 everywhere

collect terms in r

$$(1-e^2)\cos^2\theta\, r^2 + \sin^2\theta\, r^2 + 2ae(1-e^2)\cos\theta\, r + a^2e^2(1-e^2) - a^2(1-e^2) = 0$$

$\cos^2\theta\, r^2 - e^2\cos^2\theta\, r^2 + \sin^2\theta\, r^2$

$$r^2 - e^2\cos^2\theta\, r^2 + 2ae(1-e^2)\cos\theta\, r + a^2(1-e^2)(e^2-1) = 0$$
$$\underbrace{(1-e^2\cos^2\theta)}_{A}r^2 + \underbrace{2ae(1-e^2)\cos\theta}_{B}\, r \underbrace{- a^2(1-e^2)^2}_{C} = 0$$

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} =$$

$$\frac{-2ae(1-e^2)\cos\theta \pm \left[4a^2e^2(1-e^2)^2\cos^2\theta + 4(1-e^2\cos^2\theta)a^2(1-e^2)^2\right]^{1/2}}{2(1+e\cos\theta)(1-e\cos\theta)}$$

work on the sqrt bit by itself awhile

$$\left[4a^2e^2(1-e^2)^2\cos^2\theta + 4(1-e^2\cos^2\theta)a^2(1-e^2)^2\right]$$

$$4a^2(1-e^2)^2\left[e^2\cancel{\cos^2\theta} + 1 - e^2\cancel{\cos^2\theta}\right]$$

$$\Rightarrow \text{sqrt bit is } \left[4a^2(1-e^2)^2\right]^{1/2} = 2a(1-e^2)$$

$$r = \frac{-\cancel{2}ae(1-e^2)\cos\theta \pm 2a(1-e^2)}{\cancel{2}(1+e\cos\theta)(1-e\cos\theta)} = \frac{+a(1-e^2)(-e\cos\theta \pm 1)}{(1+e\cos\theta)(1-e\cos\theta)}$$

$$r = \frac{a(1-e^2)\cancel{(-e\cos\theta+1)}}{(1+e\cos\theta)\cancel{(1-e\cos\theta)}} \quad \text{or} \quad \frac{+a(1-e^2)(-e\cos\theta-1)}{(1+e\cos\theta)(1-e\cos\theta)}$$

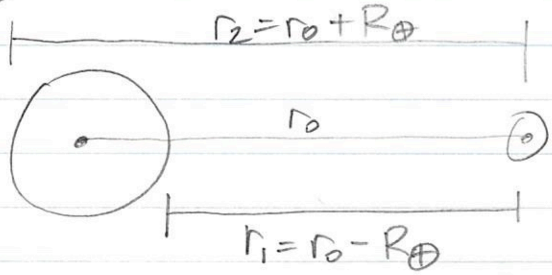
$$\downarrow$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

This one is always negative or otherwise yucky

©2023 LYoung

(3.6) extra credit



a) m @ center of $\oplus$

F_C by moon is $\dfrac{GM_M m}{r_0^2}$

b) m on near side of $\oplus$

F_R by moon is $\dfrac{GM_M m}{r_1^2} = \dfrac{GM_M m}{(r_0 - R_\oplus)^2}$

c) $$\Delta F = F_R - F_C = GM_M m \left(\frac{1}{r_0^2} - \frac{1}{(r_0 - R_\oplus)^2} \right)$$

$$= \frac{GM_M m}{r_0^2} \left(1 - \frac{1}{(1 - R_\oplus / r_0)^2} \right)$$

call $R_\oplus / r_0 \equiv x$ and $x \ll 1$.

use $f(x) \approx f(0) + x f'(0)$

$f(x) = (1-x)^{-2} \Rightarrow f(0) = 1$

$f'(x) = -2(1-x)^{-3} \Rightarrow f'(0) = -2$

$$\Delta F \approx \frac{GM_M m}{r_0^2} \left(1 - [1 + x(-2)] \right) = \frac{GM_M m}{r_0^2} \cdot 2 \frac{R_\oplus}{r_0}$$

$$\Delta F \approx \frac{2GM_M m R_\oplus}{r_0^3}$$