ASTR 210 HW 9

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9.1 Ionization and HII regions

3 pts a) Show that photons of energy $h\nu = 13.6\,eV$ fall in the Wien limit ($h\nu \gg kT$) for an O star with surface temp 43,650 K.

requirement: $kT \ll h\nu$ or $T < h\nu/k$.

$$h\nu/k = \frac{13.6\,eV \times 1.602\times10^{-19}\,J/eV}{1.381\times10^{-23}\,J/K} = 1.6\times10^{5}\,K.$$

O star in question has $T = 4.4\times10^4\,K$
which is certainly $< h\nu/k$.

Whether that qualifies as "$\ll$" depends on the accuracy you need.

10 pts b) Calculate $Q_\star$. It's related to L_ν (specific luminosity) as

$$Q_\star = \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu}\,d\nu.$$ Here also we use $L_\nu = 4\pi R^2 \cdot \pi I_\nu$

and $I_\nu = \text{Planck Fn} = \dfrac{2h\nu^3/c^2}{e^{h\nu/kT}-1}$.

For $h\nu \gg kT$, $e^{h\nu/kT} \gg 1 \longrightarrow I_\nu \approx \dfrac{2h\nu^3}{c^2} e^{-h\nu/kT}$.

Put that all together: $$Q_\star = \int_{\nu_0}^{\infty} \frac{4\pi R^2 \pi\, 2\cancel{h}\nu^{\cancel{3}\,2} c^{-2} e^{-h\nu/kT}}{\cancel{h}\cancel{\nu}}\,d\nu$$

$$Q_\star = \frac{4\pi R^2 \pi\, 2}{c^2}\int_{\nu_0}^{\infty} \nu^2 e^{-h\nu/kT}\,d\nu.$$ Sub $x = h\nu/kT$, $dx = \frac{h}{kT}\,d\nu$, $x_0 = h\nu_0/kT$.

$$Q_\star = \frac{8\pi^2R^2}{c^2}\int_{x_0}^{\infty}\left(\frac{x\,kT}{h}\right)^2 e^{-x}\left(\frac{kT}{h}\right)dx$$

$$= \frac{8\pi^2R^2}{c^2}\left(\frac{kT}{h}\right)^3\int_{x_0}^{\infty} x^2 e^{-x}\,dx.$$

hint says $\int x^2 e^{cx} dx = e^{cx}\left(\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}\right)$ and here $c = -1$.

$$\int x^2 e^{-x} dx = e^{-x}\left(\frac{x^2}{-1} - \frac{2x}{1} + \frac{2}{-1}\right) = e^{-x}(-x^2 - 2x - 2)$$

$$\int_{x_0}^{\infty} x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2)\Big|_{x_0}^{\infty} = 0 + e^{-x_0}(x_0^2 + 2x_0 + 2)$$

$$Q_\star = \frac{8\pi^2 R^2}{c^2}\left(\frac{kT}{h}\right)^3 e^{-h\nu_0/kT}\left(\left(\frac{h\nu_0}{kT}\right)^2 + 2\frac{h\nu_0}{kT} + 2\right)$$

Actually, upon reflection, it's probably less work to use $\int x^2 e^{cx} dx$ directly with $x \to \nu$ and $c \to -h/kT$.

Should end up in same place.

$$h\nu_0/kT = \frac{13.6\,eV \times 1.602\times10^{-19}\,J/eV}{1.381\times10^{-23}\,J/K \cdot 4.365\times10^4\,K} = 3.62$$

$$Q_\star = \frac{8\pi^2}{\underbrace{c^2}_{3.0\times10^8 m/s}}(10.7\times 6.96\times10^8 m)^2\left(\frac{1.381\times10^{-23}\times 4.365\times10^4\,J}{6.26\times10^{-34}\,J\cdot s}\right)^3 \times e^{-3.62}(3.62^2 + 2\cdot 3.62 + 2)$$

$Q_\star = 2.2\times10^{49}$ UV photons / sec.

4 pts c) $$R_S = \left[\frac{3}{4\pi}\frac{Q_\star}{\alpha n_e^2}\right]^{1/3} = \left[\frac{3}{4\pi}\frac{2.2\times10^{49}\,s^{-1}}{2.6\times10^{-19}\,m^3 s^{-1}(10^9\,m^{-3})^2}\right]^{1/3}$$

$= 2.7\times10^{16}$ m $= 0.88$ pc similar to distances between stars.

Note this value is similar to the corrected version of the text around eq 16.20 – see errata.

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9.2

3 pts a) $^4He = 4.0026032497\,u$

$^{12}C = 12.0\,u$

$u = 1.66053873\times10^{-27}\,kg$

$3\times {}^4He \rightarrow {}^{12}C$

$\Delta m = -12.0 + 3\times m(^4He) = 0.00781\,u$

$= 1.30\times10^{-29}\,kg$

$\Delta m\cdot c^2 = 1.166\times10^{-12}\,J$ for each reaction

energy per nucleon is $\frac{\Delta m\cdot c^2}{12} = 9.7\times10^{-14}\,J$

8 pts b) $0.3\,M_\odot$ of He burned at $25\,L_\odot$ will last how long?

$\tau = E/L$; here $E = \left(\begin{matrix}\#\text{He-burning}\\ \text{rxns}\end{matrix}\right)\times \Delta m\cdot c^2$

And $\left(\#\text{ He rxns}\right) = \frac{\text{total mass}}{\text{mass of } 3\ {}^4He}$

Thus $\tau = \frac{0.3\times1.99\times10^{30}\,kg}{(3\times4.0\times1.66\times10^{-27}\,kg)}\cdot\frac{1.17\times10^{-12}\,J}{25\times3.84\times10^{26}\,W}$

$\tau = 3.6\times10^{15}\,s = 1.2\times10^{8}\,yr$

5 pts c) $^{12}C \rightarrow {}^{28}Si$

At face value it's not obvious exactly how this is done, since we could in principle do $^{12}C + {}^{12}C + {}^4He \rightarrow {}^{28}Si$ or $^{12}C + 4\times{}^4He \rightarrow {}^{28}Si$ or some other variant. The precise value of the answer will depend on what you think the starting ingredients are.

For $^{12}C + {}^{12}C + {}^4He$ the starting mass is $24.0 + 4.002\ldots$ etc

$= 28.0026032497\,u$;

for $^{12}C + 4 \times {}^4He$ The starting mass is 28.010413 u
^{28}Si is 27.976926 u

So $^{12}C \rightarrow {}^{28}Si$ has Δm either 0.0257 u or 0.0335 u
meaning $\Delta E = 3.8 \times 10^{-12}$ J or 5.0×10^{-12} J

and $\frac{\Delta E}{\text{nucleon}}$ is 1.4×10^{-13} J or 1.8×10^{-13} J

$^{28}Si + {}^{28}Si \rightarrow {}^{56}Fe$
let's just do the one way for simplicity

$\Delta m = 55.934942 - 2(27.976926)\ u = 0.0189\ u$
$\Delta E = 2.82 \times 10^{-12}$ J
$\Delta E/\text{nucleon} = 5.0 \times 10^{-14}$ J

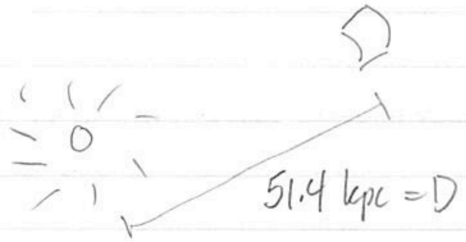
Compare to $4H \rightarrow {}^4He$, which released $\sim \frac{4.1 \times 10^{-12} \text{ J}}{4 \text{ nucleons}}$

$\sim 10^{-12}$ J/nucleon
The more advanced reactions generate less energy per nucleon than the original H burning did.

9.3 SN 1987a

note typo here. problem statement said 10^58 neutrinos.

5 pts a) 10^{57} neutrinos spread out over $\Delta t \sim 15$ s



$$F = \frac{L}{4\pi D^2}$$

Thus the neutrino flux @ Earth must have been

$$F = \frac{(10^{57}\ \nu_e / 15\text{s})}{4\pi\,(51.4\times10^3 \times 3.09\times10^{16}\text{m})^2} = 2.1\times10^{12}\ \nu_e/\text{s}\cdot\text{m}^2$$

5 pts b) $L_{bol} \sim L_V \sim 10^9 L_\odot$ assumed luminosity in optical photons.

Suppose it were at 10 kpc but with $A_V = 10$ mag extinction.

Chapter 13, eq 13.39 : $L/L_\odot = 10^{0.4(4.74 - M_{bol})}$

or $\frac{5}{2}\log_{10}(L/L_\odot) = 4.74 - M_{bol}$

gives $M_{bol} = 4.74 - \frac{5}{2}\log_{10}(10^9) = 4.74 - \frac{5}{2}\cdot 9 = -17.76$

Then assume $M_V = M_{bol} = -17.76$

apparent magnitude m_V?

m_V comes from $m - M = 5\log_{10}(D/10\text{pc}) + A$

i.e. $m_V = 5\log_{10}\left(\frac{10\times10^3\text{pc}}{10\text{pc}}\right) + M_V + A_V$

$= 5\cdot3 + -17.76 + 10 = 7.24$

If the limiting magnitude for detection is $m_V = +6$, this SN would be too faint to see behind that dust.

9.4

7 pts

What would be the rotation period of the Sun if it collapsed to a radius $R = 6000$ Km while conserving its angular momentum?

Text (eq 17.15) says ang. mom conservation is $v_0 r_0 = v_f r_f$

Derivation for solid objects w/ ang mom L, moment of inertia I in case you are interested.

$L = I\omega$ means $L_0 = I_0 \omega_0 = L_f = I_f \omega_f$

I is $\frac{2}{5} M r^2$ for a sphere, so $\frac{2}{5} M r_0^2 \omega_0 = \frac{2}{5} M r_f^2 \omega_f$

$\Rightarrow r_0^2 \omega_0 = r_f^2 \omega_f$. But rotation speed at equator of sphere is $v = r\omega$, hence $v_0 r_0 = v_f r_f$.

So anyway $P = \frac{2\pi}{\omega}$ or $\omega = 2\pi / P$.

$$r_0^2 \omega_0 = r_f^2 \omega_f \rightarrow \frac{r_0^2 \cdot 2\pi}{P_0} = \frac{r_f^2 \cdot 2\pi}{P_f} \rightarrow P_f = P_0 \left(\frac{r_f^2}{r_0^2} \right)$$

For the Sun, $P_0 = 25.4$ d near equator.
Quoted elsewhere as avg $P_\odot = 28$ d. $R_\odot = 6.96 \times 10^5$ km

Period if it contracted to $r_f = 6000$ km would be

$$P_f = 25.4\,d \left(\frac{6000 \text{ km}}{6.96 \times 10^5 \text{ km}} \right)^2 = 0.0019\ d$$
$$= 0.045\ h$$
$$= 2.7 \text{ min at equator}$$

ok to use 28 d too, will get 3.0 min

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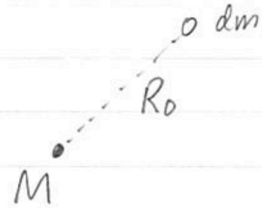
9.5 (extra credit) free fall

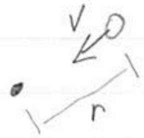
5 pts

a)



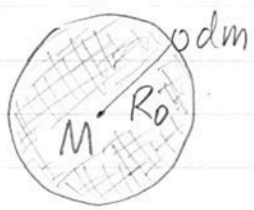
initially at rest

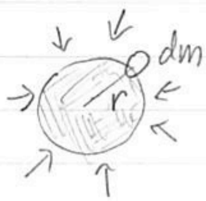
Grav. PE $= -\dfrac{GM\,dm}{R_0}$, $KE = 0$



later: Grav. PE $= -\dfrac{GM\,dm}{r}$, $KE = \dfrac{1}{2}\,dm\,v^2$

Note that the physics is exactly the same as that for M having finite size, as long as the collapse is self-similar and dm always stays at the outside edge





Anyway, energy conservation: $-\dfrac{GM\,\cancel{dm}}{R_0} = -\dfrac{GM\,\cancel{dm}}{r} + \dfrac{1}{2}\cancel{dm}\,v^2$

$v^2 = 2GM\left(\dfrac{1}{r} - \dfrac{1}{R_0}\right)$. In this case the negative root

is applicable, so $\quad v = -\left(2GM\left(\dfrac{1}{r} - \dfrac{1}{R_0}\right)\right)^{1/2} = \dfrac{dr}{dt}$.

5 pts

b) $v = dr/dt$

$$-(2GM)^{1/2}\,dt = \left(\frac{1}{r} - \frac{1}{R_0}\right)^{-1/2} dr$$

massage RHS: $\quad \dfrac{1}{r} - \dfrac{1}{R_0} = \dfrac{R_0 - r}{r R_0}$

$$\Rightarrow \text{RHS} = \left(\frac{r R_0}{R_0 - r}\right)^{1/2} dr = \left(\frac{r}{1 - r/R_0}\right)^{1/2} dr$$

let's use $r/R_0 \to x$ and $dr = R_0\,dx$ ...

our equation became $-(2GM)^{1/2}dt = \left(\frac{x R_0}{1-x}\right)^{1/2} R_0\, dx$

$$-(2GM)^{1/2}\int_0^{t_{ff}} dt = \int_1^0 \left(\frac{x R_0}{1-x}\right)^{1/2} R_0\, dx = R_0^{3/2}\int_1^0 \left(\frac{x}{1-x}\right)^{1/2} dx$$

$$-(2GM)^{1/2}\, t_{ff} = R_0^{3/2}\left(\frac{-\pi}{2}\right)$$

$$t_{ff} = \frac{R_0^{3/2}\pi}{2}\cdot\frac{1}{\sqrt{2GM}} = \frac{\pi}{2}\left(\frac{R_0^3}{2GM}\right)^{1/2}$$

Now let's write $M = \frac{4}{3}\pi R_0^3 \rho_0$

$$t_{ff} = \frac{\pi}{2}\left(\frac{\cancel{R_0^3}\, 3}{2G\, 4\pi \cancel{R_0^3} \rho_0}\right)^{1/2} = \left(\frac{3\pi^{\cancel{2}}}{4\cdot 2\cdot 4\cancel{\pi} G\rho_0}\right)^{1/2}$$

$$t_{ff} = \left(\frac{3\pi}{32 G\rho_0}\right)^{1/2}$$ as advertised