ASTR 210 HW 6 ©2023 L Young

Q6.1

2 pts a) field in sunspot: P_s = gas P + magnetic P

$$P_s = \rho \frac{k_B T_s}{\mu m_p} + \frac{B_s^2}{2\mu_0}$$

2 pts b) photosphere: similar

$$P_p = \rho \frac{k_B T_p}{\mu m_p} + \frac{B_p^2}{2\mu_0}$$

4 pts c) $P_s = P_p$

$$\rho \frac{k_B T_p}{\mu m_p} + \frac{B_p^2}{2\mu_0} = \rho \frac{k_B T_s}{\mu m_p} + \frac{B_s^2}{2\mu_0}$$

solve for B_s

$$\frac{\rho k_B}{\mu m_p}(T_p - T_s) + \frac{B_p^2}{2\mu_0} = \frac{B_s^2}{2\mu_0}$$

$$B_s^2 = B_p^2 + \frac{2\mu_0 \rho k_B}{\mu m_p}(T_p - T_s)$$

5 pts d) $T_s = 4300\,K$, $T_p = 6100\,K$, $\rho = 3.5\times10^{-4}\ kg/m^3$, $\mu = 0.6$

$T_p - T_s = 1800\,K$

First set $B_p = 0$

$$B_s^2 = \frac{2 \cdot 4\pi\times10^{-7}\,W{\cdot}m \cdot 3.5\times10^{-4}\,kg/m^3 \cdot 1.38\times10^{-23}\,J/K\,(1800\,K)}{0.6\times1.67\times10^{-27}\,kg}$$

$B_s = 0.1477\,T$

Now set $B_p = 0.01\,T$. $B_s = \left[B_p^2 + \frac{2\mu_0 \rho k_B (T_p - T_s)}{\mu m_p}\right]^{1/2}$

$B_s = 0.148\,T$. For these values, ignoring B_p is OK.

answer must include some kind of comment

Q6.2 planetary temperatures

3 pts a) planet absorbs energy at a rate

$$W_p = \frac{L_\star}{4\pi r^2}(\pi R^2)(1-A).$$

eq 8.6 in text.

$r = 1\,\text{AU} = 1.496\times10^8\,\text{km}$
$= 1.496\times10^{11}\,\text{m}$

blackbody luminosity of planet is

$$L_p = 4\pi R^2 \sigma_{SB} T_p^4$$

in = out ; $$4\cancel{\pi} R^2 \sigma_{SB} T_p^4 = \frac{L_\star}{4\pi r^2}(\cancel{\pi} R^2)(1-A)$$

$$T_p^4 = \frac{L_\star(1-A)}{16\pi r^2 \sigma_{SB}}$$

$$T_p = \frac{1}{2}\left[\frac{L_\star(1-A)}{\pi r^2 \sigma_{SB}}\right]^{1/4}$$

3 pts b) $$dT_p/dL_\star = \frac{1}{2}\cdot\frac{1}{4}\left[\frac{L_\star(1-A)}{\pi r^2\sigma_{SB}}\right]^{-3/4}\frac{(1-A)}{\pi r^2 \sigma_{SB}}$$

$$= \frac{1}{2}\cdot\frac{1}{4} L_\star^{-3/4}\left[\frac{1-A}{\pi r^2 \sigma_{SB}}\right]^{1/4}$$

4 pts c) evaluate. At present

$$\frac{dT_p}{dL_\star} = \frac{1}{2}\cdot\frac{1}{4}(3.839\times10^{26}\,\text{W})^{-3/4}\left[\frac{0.6}{\pi(1.496\times10^{11}\,\text{m})^2\, 5.67\times10^{-8}\,\text{W}\,\text{m}^{-2}\,\text{K}^{-4}}\right]^{1/4}$$

$$= 1.60\times10^{-25}\,\text{K/W}$$

Check units on this one? Non-trivial.

$$\text{W}^{-3/4}\,\text{m}^{-1/2}(\text{W}\,\text{m}^{-2}\,\text{K}^{-4})^{-1/4} = \text{W}^{-3/4}\,\cancel{\text{m}^{-1/2}}\,\text{W}^{-1/4}\,\cancel{\text{m}^{1/2}}\,\text{K}$$

$$= \text{K/W} \quad \text{ok}$$

checking units is not required but is a useful sanity check.

5 pts d) Sun $L_\star = 0.7 L_\odot$ @ $t = 0$
$2.2 L_\odot$ @ $t = 10.6$ Gyr

So take $\frac{dL_\star}{dt} \approx \frac{\Delta L_\star}{\Delta t} = \frac{(2.2-0.7)L_\odot}{10.6\times10^9\,\mathrm{yr}}$

$$= \frac{1.5\times 3.839\times10^{26}\,\mathrm{W}}{10.6\times10^9\,\mathrm{yr}} = 5.4\times10^{16}\,\mathrm{W/yr}$$

$$\frac{dT_P}{dt} = \frac{dT_P}{dL_\star}\cdot\frac{dL_\star}{dt} = 1.60\times10^{-25}\,\frac{\mathrm{K}}{\mathrm{W}} \times 5.4\times10^{16}\,\frac{\mathrm{W}}{\mathrm{yr}}$$
$$= 8.7\times10^{-9}\,\mathrm{K/yr}$$

5 pts e) How long, at that rate, will it take for T_P to increase by $1\,K$?

$$\Delta t = 1\,\mathrm{K} / 8.7\times10^{-9}\,\mathrm{K/yr} = 1.2\times10^{8}\,\mathrm{yr}$$
120 Myr

It's not a big effect on human timescales, but on geologic timescales it can be important.

5 pts f) Find r such that $T_p = 0^\circ C$ (273 K) and $100^\circ C$ (373 K).

back in (a): $T_p^{\,4} = \frac{L_\star(1-A)}{16\pi r^2 \sigma_{SB}} \Rightarrow r^2 = \frac{L_\star(1-A)}{16\pi\sigma_{SB} T_p^{\,4}}$

plugging in current solar luminosity we find

$r = 1.21\times10^{11}\,\mathrm{m} = 0.81$ AU and
$r = 6.46\times10^{10}\,\mathrm{m} = 0.43$ AU

Q6.3

4 pts a) Gravitational acceleration g is given by $\frac{GM}{R^2}$

For Titan this is $\frac{6.67\times10^{-11}\,m^3 kg^{-1} s^{-2} \cdot 1346\times10^{20}\,kg}{(2575\times10^3\,m)^2}$

$= 1.35\ m/s^2$

5 pts b) rule of thumb is $v_{rms} \lesssim \frac{v_{esc}}{6}$ to keep a species for 4.5 Gyr.

Can be rewritten as in text eq 8.19 - 8.23
planet w/ exobase temp T_{ex} and radius R_{ex} can retain species with

$$\mu \gtrsim \frac{54\, k_B T_{ex}}{g R_{ex}\, m_p}$$ assuming $R_{ex} \approx R$ ie atmosphere is thin

So for simplicity let's take Titan $T_{ex} \sim 94\,K$ and $R \approx R_{ex}$

then it can retain species with

$$\mu \gtrsim \frac{54 \cdot 1.38\times10^{-23}\,m^2 kg\, s^{-2} K^{-1} \cdot 94\,K}{1.35\,m/s^2 \cdot 2575\times10^3\,m \cdot 1.67\times10^{-27}\,kg}$$

$\mu \gtrsim 12$ which means Titan will hold CO_2 ($\mu = 44$) but not H_2.

3 pts c) Scale height is eq 9.15:

$$H = \frac{k_B T}{g \mu m_p}$$ for N_2, $\mu = 28$

$$H = \frac{1.38\times10^{-23}\,m^2 kg\, s^{-2} K^{-1} \cdot 94\,K}{1.35\,m/s^2 \cdot 28 \cdot 1.67\times10^{-27}\,kg} = 20.5\ km$$

more →

scale height of Titan's N_2 atmosphere is 20.5 km, which is quite a lot larger than Earth's ~ 8 km. It's because Titan's gravity is weaker, even though it's colder.

again, answer should include some kind of comment

Q6.4

5 pts

Peak of Planck function

Ok, Planck fn $I_\lambda = \frac{2hc^2 \lambda^{-5}}{e^{hc/\lambda kT} - 1}$

This will be easier if we write $x \equiv \frac{hc}{\lambda k_B T} = \frac{hc}{k_B T} \lambda^{-1}$

Then $I(x) = C \cdot \frac{x^5}{e^x - 1}$ with C = constants
aka $C x^5 (e^x - 1)^{-1}$.

$\frac{dI}{d\lambda} = 0$ at the peak. Also $\frac{dI}{d\lambda} = \frac{dI}{dx} \frac{dx}{d\lambda}$.

$$\frac{dI}{dx} = \frac{C 5x^4}{e^x - 1} + \frac{C x^5 (-1) e^x}{(e^x-1)^2}$$

and $\frac{dx}{d\lambda} = \frac{hc}{k_B T} (-1) \lambda^{-2} = \frac{-hc}{\lambda k_B T} \cdot \frac{1}{\lambda} = -x \cdot x \frac{k_B T}{hc} = \frac{-k_B T}{hc} x^2$

so $dI/d\lambda = 0$ means

$$\left(\frac{C \cdot 5x^4}{e^x - 1} - \frac{C x^5 e^x}{(e^x-1)^2} \right) \left(\frac{-k_B T}{hc} x^2 \right) = 0$$

well, clearly we aren't interested in $x = 0$ ($\lambda = \infty$)
so that must mean

$$\frac{5\cancel{x^4}}{\cancel{e^x-1}} - \frac{x^{\cancel{5}} e^x}{(e^x-1)^{\cancel{2}}} = 0 \qquad 5 - \frac{x e^x}{e^x - 1} = 0$$

$$\frac{5(e^x - 1) - x e^x}{e^x - 1} = 0$$

more →

$$5(e^x-1)-xe^x=0$$

$$5e^x-5-xe^x=0$$

$$-(5-x)e^x+5=0 \quad \text{aka} \quad (x-5)e^x+5=0$$

root of this, we are told, is $x_p \approx 4.965$ and also $x_p \equiv \frac{hc}{\lambda_p k_B T}$

so then $\lambda_p = \dfrac{hc}{x_p k_B T}$

$$\lambda_p = \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s} \times \overset{2.998}{\cancel{3.0}}\times10^{8}\ \text{m/s}}{4.965 \times 1.38\times10^{-23}\ \text{J/K}\cdot T}$$

$$= \frac{2.90\times10^{-3}}{T}\ \text{m}\cdot\text{K} = \frac{2.90\times10^{3}}{T}\ \mu\text{m}\cdot\text{K}$$