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6.1

4 pts a) Gravitational acceleration g is given by $\frac{GM}{R^2}$

For Titan this is $\frac{6.67\times10^{-11}\,m^3 kg^{-1} s^{-2} \cdot 1346\times10^{20}\,kg}{(2575\times10^3\,m)^2}$

$= 1.35\ m/s^2$

5 pts b) rule of thumb is $v_{rms} \lesssim \frac{v_{esc}}{6}$ to keep a species for 4.5 Gyr.

Can be rewritten as in text eq 8.19 - 8.23
planet w/ exobase temp T_{ex} and radius R_{ex} can retain species with

$$\mu \gtrsim \frac{54\, k_B T_{ex}}{g R_{ex} m_p}$$ assuming $R_{ex} \approx R$ ie atmosphere is thin

So for simplicity let's take Titan $T_{ex} \sim 94\,K$ and $R \approx R_{ex}$

then it can retain species with

$$\mu \gtrsim \frac{54 \cdot 1.38\times10^{-23}\, m^2 kg\, s^{-2} K^{-1} \cdot 94\,K}{1.35\ m/s^2 \cdot 2575\times10^3\,m \cdot 1.67\times10^{-27}\,kg}$$

$\mu \gtrsim 12$ which means Titan will hold CO_2 ($\mu = 44$) but not H_2.

3 pts c) Scale height is eq 9.15:

$$H = \frac{k_B T}{g \mu m_p}$$ for N_2, $\mu = 28$

$$H = \frac{1.38\times10^{-23}\,m^2 kg\, s^{-2} K^{-1} \cdot 94\,K}{1.35\ m/s^2 \cdot 28 \cdot 1.67\times10^{-27}\,kg} = 20.5\ km$$

more →

scale height of Titan's N_2 atmosphere is 20.5 km, which is quite a lot larger than Earth's ~8 km. It's because Titan's gravity is weaker, even though it's colder.

6.2 HSEq for Jupiter and Saturn

HSEq $dP/dr = -\frac{G M_r \rho}{r^2}$ and $M_r = 4\pi \int_0^r \rho(r) \cdot r^2 dr$

4 pts

a) If $\rho(r) = \bar{\rho} = \text{const}$

then $M_r = 4\pi \int_0^r \bar{\rho}\, r^2 dr = 4\pi \bar{\rho} \left[r^3/3 \right]_0^r = \frac{4\pi}{3} \bar{\rho}\, r^3$

as expected.

Thus $\frac{dP}{dr} = \frac{-G \frac{4\pi}{3} \bar{\rho}\, r^3 \bar{\rho}}{r^2} = -\frac{4\pi}{3} G \bar{\rho}^2 r$

aka $dP = -\frac{4\pi}{3} G \bar{\rho}^2\, r\, dr$

4 pts

b) $\int_{P_c}^{0} dP = -\frac{4\pi}{3} G \bar{\rho}^2 \int_0^R r\, dr$

$$P\Big|_{P_c}^{0} = -\frac{4\pi}{3} G \bar{\rho}^2 \cdot \frac{r^2}{2}\Big|_0^R$$

$$0 - P_c = -\frac{4\pi}{3} G \bar{\rho}^2 \cdot \frac{R^2}{2}$$

$$P_c = \frac{2\pi}{3} G \bar{\rho}^2 R^2$$

5 pts

c) evaluate with some #s

write $\bar{\rho} = \left(\frac{\bar{\rho}}{\rho_J}\right) \cdot \rho_J$ and $R = \left(\frac{R}{R_J}\right) \cdot R_J$

$R_J = 10.97 \times 6.371 \times 10^6$ m ; $M_J = 317.8 \times 5.974 \times 10^{24}$ kg

$\rho_J = \frac{M_J}{\frac{4}{3}\pi R_J^3} = \frac{3 M_J}{4\pi R_J^3} = 1.33 \times 10^3$ kg/m3

cont'd

$$P_c = \frac{2\pi}{3} G \left(\frac{\bar{\rho}}{\rho_J}\right)^2 \rho_J^2 \left(\frac{R}{R_J}\right)^2 R_J^2$$

$$= \frac{2\pi}{3} \cdot 6.67\times10^{-11}\, m^3 kg^{-1} s^{-2} \left(1.33\times10^3 \frac{kg}{m^3}\right)^2 (10.97\times 6.371\times10^6 m)^2$$

$$\times \left(\frac{\bar{\rho}}{\rho_J}\right)^2 \left(\frac{R}{R_J}\right)^2$$

$$= 1.2\times10^{12} \underbrace{m^3 m^{-6} m^2\, kg\, s^{-2}}_{N/m^2} \left(\frac{\bar{\rho}}{\rho_J}\right)^2 \left(\frac{R}{R_J}\right)^2$$

5 pts

d) $M_{sat}/M_J = \frac{95.16}{317.8}$ based on Table A.3

$R_{sat}/R_J = \frac{9.14}{10.97}$

$$\rho_{sat}/\rho_J = \frac{(M_{sat}/M_J)}{(R_{sat}/R_J)^3} \approx 0.52$$

$$P_{c,sat} = 1.2\times10^{12} \frac{N}{m^2} \left(\frac{M_{sat}/M_J}{(R_{sat}/R_J)^3}\right)^2 \left(\frac{R_{sat}}{R_J}\right)^2$$

$$= 1.2\times10^{12} \frac{N}{m^2} \frac{(M_{sat}/M_J)^2}{(R_{sat}/R_J)^4}$$

$$= 1.2\times10^{12} \frac{N}{m^2} \cdot \frac{(95.16/317.8)^2}{(9.14/10.97)^4} = 2.2\times10^{11}\, N/m^2$$

5 pts

6.3 Mars temperatures

Table A.3: Mars has a semimajor axis of 1.524 AU and an orbital eccentricity of 0.0934.

perihelion = $a(1-e) = 1.524\,\text{AU}\,(1-0.0934) = 1.38\,\text{AU}$

aphelion = $a(1+e) = 1.524\,\text{AU}\,(1.0934) = 1.67\,\text{AU}$

Table 8.2: mars albedo ~ 0.16

from eq 8.10: $T_p \approx 279\,K\,(1-A)^{1/4}\left(\frac{r}{1\,\text{AU}}\right)^{-1/2}$

$$\text{mars } T_{peri} = 279\,K\,(1-0.16)^{1/4}\,(1.382)^{-1/2}$$
$$= 227\,K$$
$$T_{ap} = 279\,K\,(1-0.16)^{1/4}\,(1.666)^{-1/2}$$
$$= 207\,K.$$

6.4

3 pts a) Show g @ exobase $\approx$ g @ surface

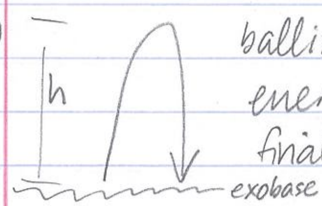
$$g = \frac{GM_\oplus}{r^2}$$ where r is 6371 km for the surface value of g and r is 6371 + 500 km for the exobase value

$$g_{surf} = \frac{GM_\oplus}{R_\oplus^2} = \frac{6.67\times10^{-11}\ kg^{-1}m^3s^{-2} \times 5.974\times10^{24}\ kg}{(6371\times10^3 m)^2}$$
$$= 9.8\ m/s^2$$

$$g_{ex} = \frac{GM_\oplus}{(R_\oplus + 500\ km)^2} = \frac{6.67\times10^{-11}\ kg^{-1}m^3s^{-2} \times 5.974\times10^{24}\ kg}{(6871\times10^3\ m)^2}$$
$$= 8.4\ m/s^2$$

Gravity at the exobase is about 86% as strong as at the surface. Assuming g = constant is going to give us answers that are ok at ~10% to 20% level.

4 pts b)



ballistic trajectories can be assumed to conserve energy, so we'll set initial kinetic energy = final potential energy

$$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gh \quad \Rightarrow \quad h = v^2/2g$$

4 pts c) Thermal equilibrium means the particles have an rms speed given by $v_{rms} = \sqrt{\frac{3kT}{m}} \Rightarrow v_{rms}^2 = \frac{3kT}{m}$

Now if $h = v^2/2g$, this means $h = \frac{3kT}{2mg} \sim \frac{kT}{mg}$

4 pts d) compute h reached for a N_2 molecule ($m = 28 m_p$) at 1000 K.

$$h \sim \frac{kT}{mg} = \frac{1.38 \times 10^{-23}\ m^2\,kg\,s^{-2}\,K^{-1} \cdot 1000\,K}{28 \times 1.67 \times 10^{-27}\,kg \times 9.8\ m/s^2}$$

$= 30$ km. or 35 Km if you use exobase g.

if you keep the factor of 3/2 from c you'll get 45 Km.

6.5 More HSEq - Singular Isothermal sphere

$$dP/dr = -\frac{GM_r \rho}{r^2} \quad \text{with} \quad M_r \equiv \int_0^r \rho(r)\, r^2 dr \cdot 4\pi$$

and $\rho(r) = k r^{-2}$

3 pts a) $M_r = \int_0^r k r^{-2} \cdot r^2 dr \cdot 4\pi = 4\pi k \int_0^r dr = 4\pi k r$

3 pts b) Ideal gas pressure is $P_{gas} = \frac{\rho k_B T}{\mu m_p}$

if $T = \text{const}$, Then $P_{gas} = \frac{k_B T}{\mu m_p} \cdot k r^{-2}$

$$\frac{dP}{dr} = \frac{k_B T}{\mu m_p}(-2 k r^{-3}) = -2 \frac{k_B T}{\mu m_p} k r^{-3}$$

4 pts c) $-2 \frac{k_B T k r^{-3}}{\mu m_p} = -G \frac{4\pi k r\, k r^{-2}}{r^2} = -4\pi G k^2 r^{-3}$

This will be satisfied everywhere (for all r) as long as The constants match up, i.e.

$$+\frac{\cancel{2} k_B T \cancel{k}}{\mu m_p} = +\overset{2}{\cancel{4}}\pi G k^{\cancel{2}}$$

$$k = \frac{k_B T}{2\pi G \mu m_p}, \quad \text{units: } \frac{J/K \cdot K}{m^3 kg^{-1} s^{-2}\, kg} = \frac{J}{m^3 s^{-2}} = \frac{kg\, m^{\cancel{2}} \cancel{s^{-2}}}{m^{\cancel{3}} \cancel{s^{-2}}} = kg/m$$

Check units? (Not required, but good practice.) ok!

if ρ has units of kg/m3 and r is in m

Then The units on k should be $[\rho][r^2] = \frac{kg}{m^3} \cdot m^2 = \frac{kg}{m}$